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Relative entropy method for measure-valued solutions

Theory of *weak solutions* to differential equation for various physical systems, including Navier–Stokes and Euler, seems quite complete. However recent years have delivered many new results on the level of measure-valued (mv) solutions (e.g. [3], [6], [5]). This shows that even though mv solutions are considered a very weak notion of solution, not carrying much information, they do play an important role in the analysis of physical systems.

The origins of the *relative entropy* method can be traced back to physics. The underlying principle behind it is the simple idea to measure in a certain way how much two evolutions of a given physical system, whose initial states are "close", differ and to investigate how this "distance" evolves in time. This framework is a useful tool in obtaining a variety of interesting analytical results. For instance it can be used to show uniqueness of solutions to a conservation law in the scalar case, while for many systems of equations it provides the so-called weak-strong uniqueness property, i.e. establishes uniqueness of classical solutions in a wider class of weak solutions.

The applications of the *relative entropy* framework was introduced in [1]. In our last work ([2]) particular uniqueness of an entropy solution was proven for a scalar conservation law, using the notion of measure-valued entropy solutions — this will be presented.

Remark. Scalar conservation law

$$\partial_t u(x,t) + \operatorname{div}_x f(u(x,t)) = 0, \quad \text{in } \mathbb{T}^d \times \mathbb{R}_+, \ u(x,0) = u_0(x), \qquad \text{in } \mathbb{T}^d. \tag{1}$$

Here $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ and u_0 is a given initial datum. The main ideas come from Tartar and DiPerna [4], who defined entropy mv solutions in the language of classical Young measures.

To show existence of a measure-valued entropy solution to (1) a parabolic approximate problem is considered. This generates a sequence of approximate solutions, which can be shown to be uniformly integrable. Thus one can see that there is going to be no concentration effect. However, other approximation schemes can be considered, which will not posses sufficient integrability. In fact we prove uniqueness in a wider class of mv solutions not necessarily corresponding to any approximation scheme.

References

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