Modele porozumienia z różnicą niecałkowitego rzędu (Fractional Consensus Systems)

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Opinion dynamics

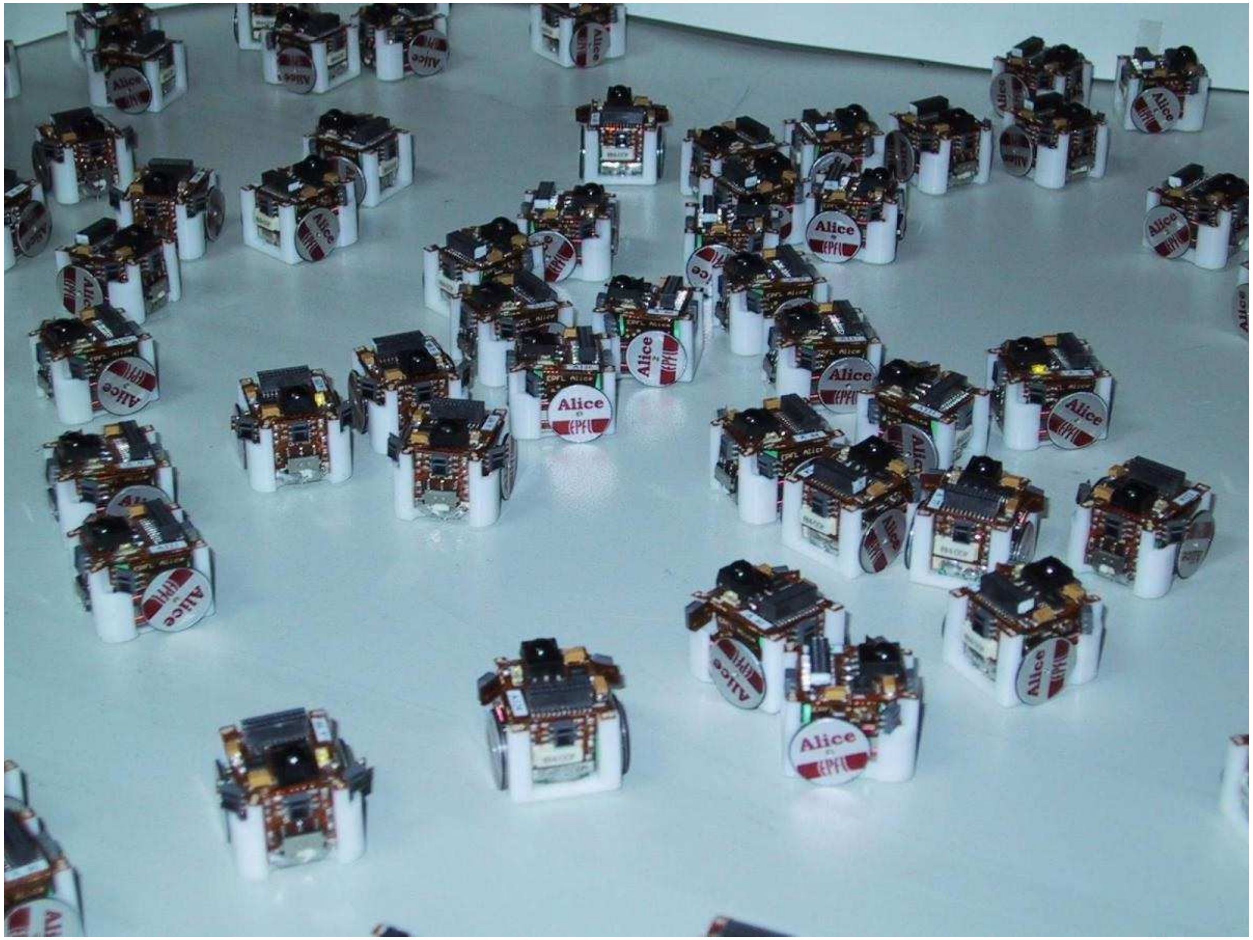
- Opinion dynamics is related with the formation of opinions in small or large groups of individuals, usually called agents or experts.

One can think about people

birds...

bats...

...or simple robots



What is important?

We can agree that:

- Mutual agreement, polarization into opinion clusters or consensus, are fundamental phenomena in social and natural systems.

Consensus

Opinion dynamics - mathematical approach

- In opinion dynamics one considers a set of agents where each holds an opinion from a certain opinion space. Agent may change his opinion when he gets aware of the opinions of others. A crucial point in modeling opinion dynamics resides in how to specify interactions between agents. From mathematical approach very successful seem to be models that include a *bounded confidence* constraint, so that agents do not interact with fellow agents if their opinions are too far apart. One of the best known mathematical models in opinion dynamics is **the Krause model** (called also Hegselmann and Krause model) that examines such situation.

The Krause model - classical case

The system consists of n agents and the opinion of agent i at time $t \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ is a real number $x_i(t)$, $i = 1, 2, \ldots, n$. Thus the state of the system is described by the vector

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n.$$

$x(t)$ is usually called the opinion profile at time t. In making up the opinion, at time $t + 1$, each agent takes into account opinions of those agents whose opinions are not too far from his own opinion at time t. In other words, only opinions of those agents he has confidence in.
Example: models of animals or robots with limited visibility.

The Krause model - classical case

Let $\epsilon_i > 0$ be the level of confidence employed by agent i, that is, agent i revises his opinion taking into account only the opinions of agents j such that $|x_j(t) - x_i(t)| < \epsilon_i.$ The Krause model relies on the idea of repeating averaging under bounded confidence:

$$x_i(t+1) = \frac{1}{\sum_{j:|x_j(t)-x_i(t)|<\epsilon_i} 1} \sum_{j:|x_j(t)-x_i(t)|<\epsilon_i} x_j(t).$$

We remark that the interaction topology associated with this model is changing with time and therefore this model is nonautonomous and nonlinear.

The Krause model - classical case

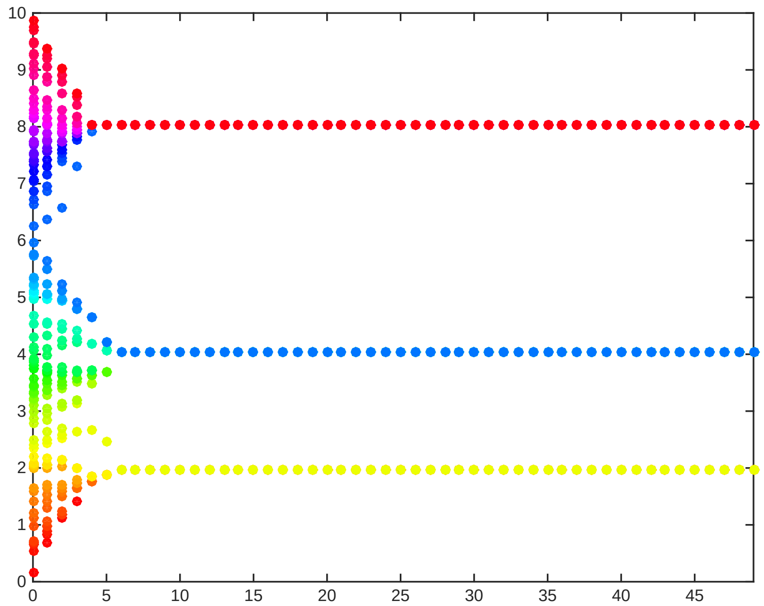


Figure 1: Time evolution of 100 agents opinions, according to the model. Initial opinions are chosen at random on an interval of length 10. In this case opinions converge to limiting values (clusters).

Sequence $\left(a_k^{(\alpha)}\right)_{k\in\mathbb{N}_0}$

Let $c \in \mathbb{R}$ and $\mathbb{N}_c := \{c, c+1, c+2, \ldots\}$. For $k \in \mathbb{N}_0$ we define

$$a_k^{(\alpha)} := \begin{cases} 1 & \text{for } k = 0 \\ (-1)^k \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for } k \in \mathbb{N}_1 \,. \end{cases}$$

$$a_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \,.$$

$$\begin{aligned} a_0^{(\alpha)} &:= 1 \,, \\ a_{k+1}^{(\alpha)} &:= \left(1 - \tfrac{\alpha+1}{k+1}\right) a_k^{(\alpha)} \,, \quad k \in \mathbb{N}_0 \,. \end{aligned}$$

Let $\alpha \in \mathbb{R}$.

The *Grünwald–Letnikov–type difference operator* Δ^α *of order* α *for a function* $x : \mathbb{N}_0 \to \mathbb{R}$ is defined by

$$(\Delta^\alpha x)(k) := \sum_{s=0}^{k} a_s^{(\alpha)} x(k-s) .$$

- For $\alpha = 0$ we get $(\Delta^0 x)(k) = x(k)$
- For $\alpha = 1$ we have that $\left(\Delta^1 x\right)(k) := x(k) - x(k-1)$.

The fractional order Hegselmann–Krause's type models

Let $\alpha \in [0,1)$ and $x_i : \mathbb{N}_0 \to [0,+\infty)$, $i \in N := \{1,\ldots,n\}$.

$x_i(t)$ $\leftarrow$ the assessment made by expert i at time $t \in \mathbb{N}_0$ of the nonnegative magnitude.

$I_i(\epsilon) := \{j \mid 1 \leq j \leq n, |x_i - x_j| < \epsilon\}$, $\epsilon > 0$

$|I_i(\epsilon)|$ $\leftarrow$ the number of elements of $I_i(\epsilon)$.

Consider the fractional order Hegselmann–Krause's type models:

$$\left(\Delta^\alpha x_i\right)(k+1) = \frac{\sum_{j \in I_i(\epsilon)} x_j(k)}{|I_i(\epsilon)|}, \quad i \in N \qquad (*)$$

with initial condition $x(0) \in [0,1]^n \subset \mathbb{R}^n$ that is the random vector.

The recurrence formula for solutions for each separate agent $i \in N$:

$$x_i(k+1) = \frac{\sum_{j \in I_i(\epsilon)} x_j(k)}{|I_i(\epsilon)|} + \sum_{s=0}^{k} \left| a_{k-s}^{(\alpha)} \right| x_i(s) \quad \text{for} \quad k \in \mathbb{N}_0\,.$$

It is easy to see that for $\alpha = 0$ one gets the classical Hegselmann–Krause's model, i.e.

$$x_i(k+1) = \frac{\sum_{j \in I_i(\epsilon)} x_j(k)}{|I_i(\epsilon)|}, \quad i \in N\,,$$

Properties of $(x_i(\cdot))$, $i \in N$

- Let $x_i(0) \geq 0$ for each $i \in N$, then

$$x_i(k) \geq 0,$$

for $i \in N, k \in \mathbb{N}$.

Properties of $(x_i(\cdot))$, $i \in N$

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- Let $x_i(0) \geq 0$ for each $i \in N$.

If for some $i, j \in N$ we have the relation

$$x_i(0) \leq x_j(0),$$

then for $k \in \mathbb{N}_1$:

$$\Downarrow$$

$$x_i(k) \leq x_j(k).$$

Definition of an ϵ–profile

An opinion profile $x = (x_1, \ldots, x_n)$ is called an ϵ–*profile* if there exists an ordering $x_{i_1} \leq x_{i_2} \leq \ldots \leq x_{i_n}$ of the components of x such that two adjacent components have a distance less or equal to ϵ, i.e.

$$x_{i_{k+1}} - x_{i_k} < \epsilon \quad \text{for all} \quad 1 \leq k \leq n-1 .$$

For an opinion profile $x = (x_1, \ldots, x_n)$ we say that there is a *split (or crack)* between agents i and j if $|x_i - x_j| \geq \epsilon$.

Definition of consensus

Let $\mathcal{A} = \{i_1, \ldots, i_s\} \subset N$ and $s < n$. The *consensus* with leaders from $\mathcal{A}$ of system $(*)$ is said to be achieved if, for each agent $i \in N$ there exists $j \in \mathcal{A}$ such that

$$\lim_{k \to \infty} |x_i(k) - x_j(k)| = 0$$

for any initial condition $x(0) = (x_1(0), \ldots, x_n(0))$.

If $\mathcal{A} = \{i_0\}$ and $i_0 \in N$, then we say that *system* $(*)$ *achieves a consensus*.

Theorem:

If the fractional-order Hegselmann-Krause–type model:

$$\left(\Delta^{\alpha} x_i\right)(k+1) = \frac{\sum_{j \in I_i(\epsilon)} x_j(k)}{|I_i(\epsilon)|}, \quad i \in N \qquad (*)$$

reaches a consensus, then the opinion profile $x(k)$ is an ϵ–profile for all $k \in \mathbb{N}_0$.

Corollary: If there exists $k \in \mathbb{N}_0$ such that the opinion profile $x(k)$ is non ϵ–profile, then the fractional-order Hegselmann-Krause–type model given by $(*)$ does not reach a consensus.

Sufficient condition for achieving a consensus by system $(*)$

Proposition: Let $x(0)$ be an ϵ-profile and $e(k) := x_n(k) - x_1(k)$, where $x_1(k) \leq x_2(k) \leq \ldots \leq x_n(k)$ for $k \in \mathbb{N}_0$. If in the fractional-order Hegselmann-Krause–type model

$$\left(\Delta^{\alpha} x_i\right)(k+1) = \frac{\sum_{j \in I_i(\epsilon)} x_j(k)}{|I_i(\epsilon)|}, \quad i \in N \qquad (*)$$

there is $k_0 \in \mathbb{N}_0$ such that $e(k_0) < \epsilon$, then system $(*)$ reaches a consensus.

Corollary: If $x_1(0) \leq x_2(0) \leq \ldots \leq x_n(0)$ and $x_n(0) - x_1(0) \geq n\epsilon$, then the fractional-order Hegselmann-Krause–type model given by $(*)$ does not achieve a consensus.

Theorem:

If $x(\cdot)$ evolves according to system $(*)$, then for every $i \in N$ there exists $\ell_i \in N$ such that $i \leq \ell_i \leq n$ and $x_i(k)$ converges to a limit $x_{\ell_i}(k)$, i.e. $\lim_{k\to\infty} |x_i(k) - x_{\ell_i}(k)| = 0$.

Moreover, for any $i, j \in N$ we have $x_{\ell_i}(\cdot) = x_{\ell_j}(\cdot)$ or $|x_{\ell_i}(k) - x_{\ell_j}(k)| \geq \epsilon$ for all $k \in \mathbb{N}$.

Systems with two agents

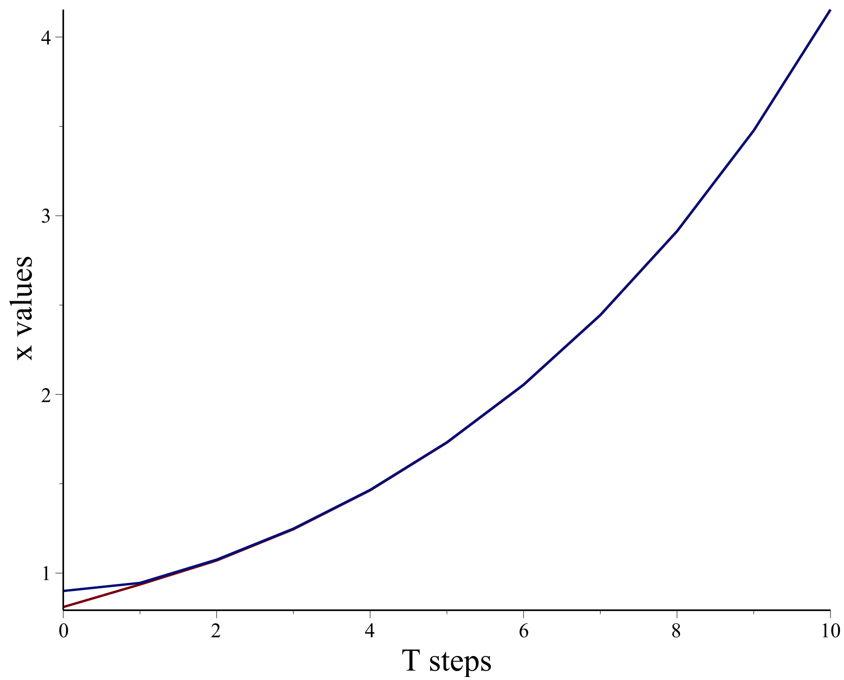
Proposition: Let $\epsilon > 0$ and $n = 2$. The system $(*)$ achieves the consensus if and only if

$$|x_1(0) - x_2(0)| < \epsilon,$$

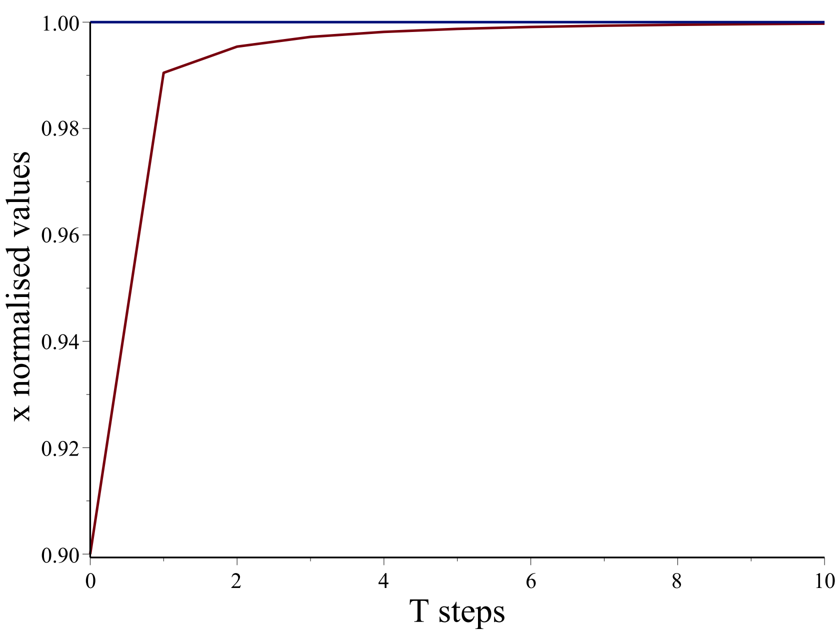
i.e. $(x_1(0), x_2(0))$ is an ϵ–profile.

Example

First we analyze the way to consensus for systems with order $\alpha = 0.1$ and with the difference $|x_1(0) - x_2(0)| < \epsilon$:.



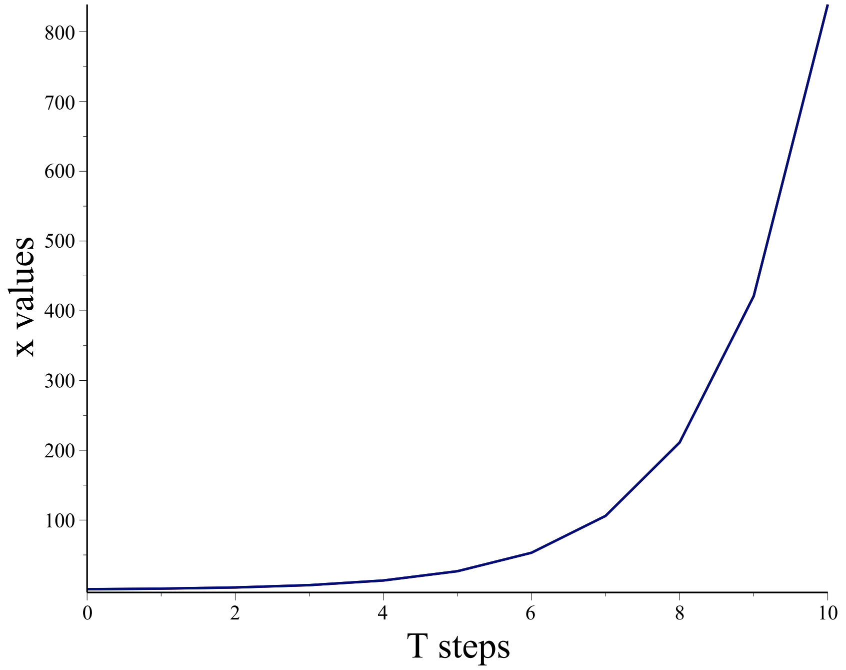
(a) $x = x(k)$



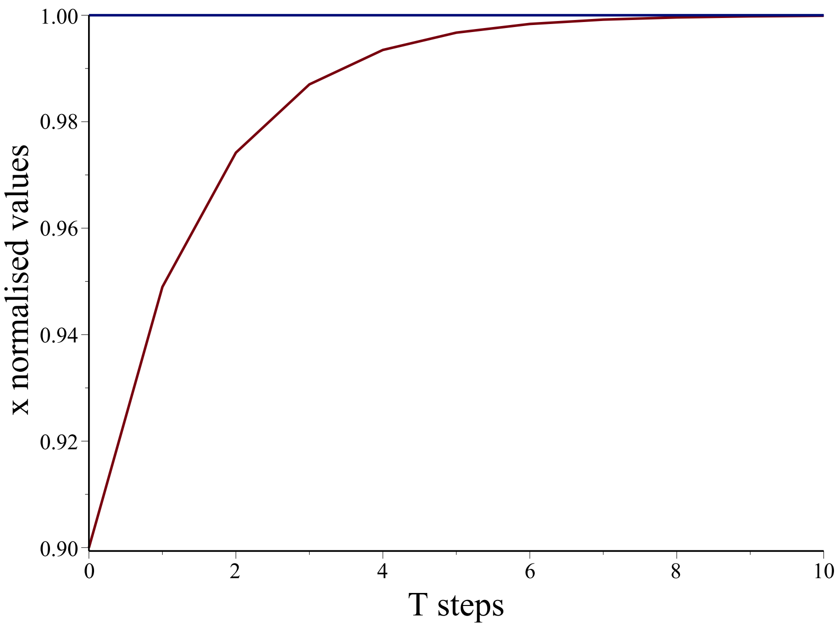
(b) $y = \dfrac{x(k)}{x_n(k)}$

Example

For higher orders there is no prestigious difference in the behaviour of solutions for both agents, see below the behaviour for $\alpha = 0.99$:



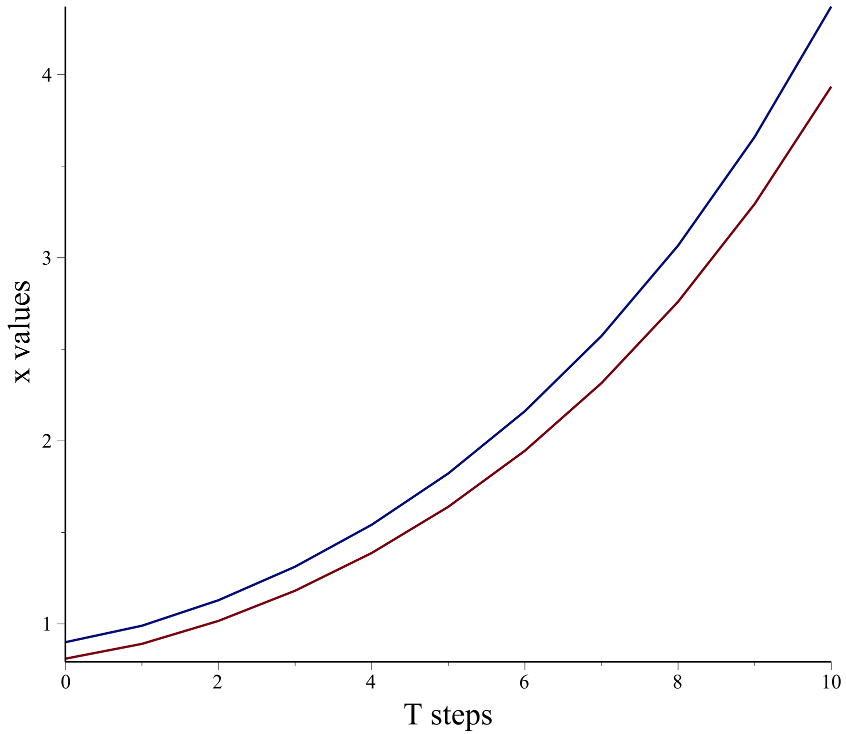
(c) $x = x(k)$



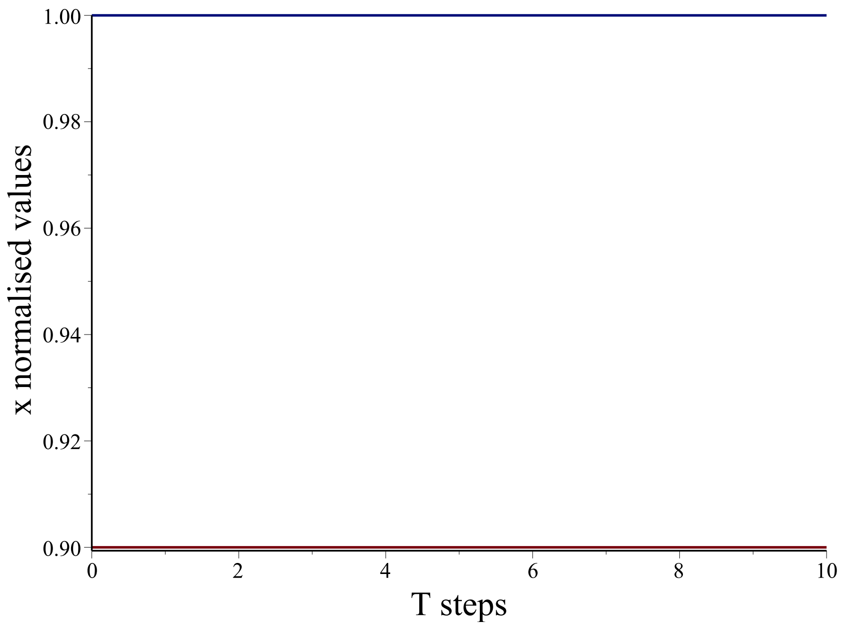
(d) $y = \frac{x(k)}{x_n(k)}$

Example

Now we choose again $\alpha = 0.1$ and the difference $|x_1(0) - x_2(0)| > \epsilon$. Then we see that trajectories do not converge to one leader.



(e) $x = x(k)$



(f) $y = \dfrac{x(k)}{x_n(k)}$

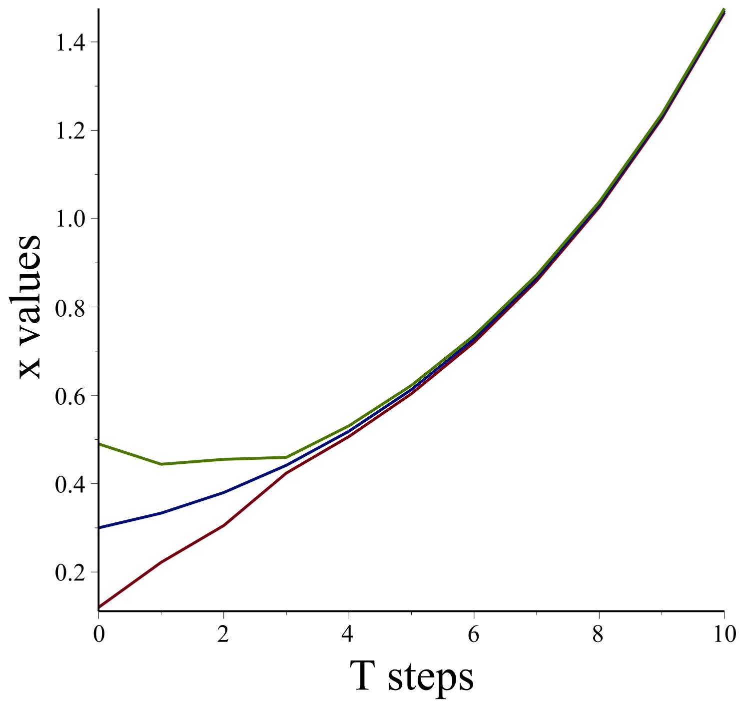
Proposition: If the opinion profile $x(0) = (x_1(0), x_2(0), x_3(0))$ is an ϵ–profile and

$$\boxed{\alpha + \frac{1}{2} < \frac{\epsilon}{e(0)}},$$

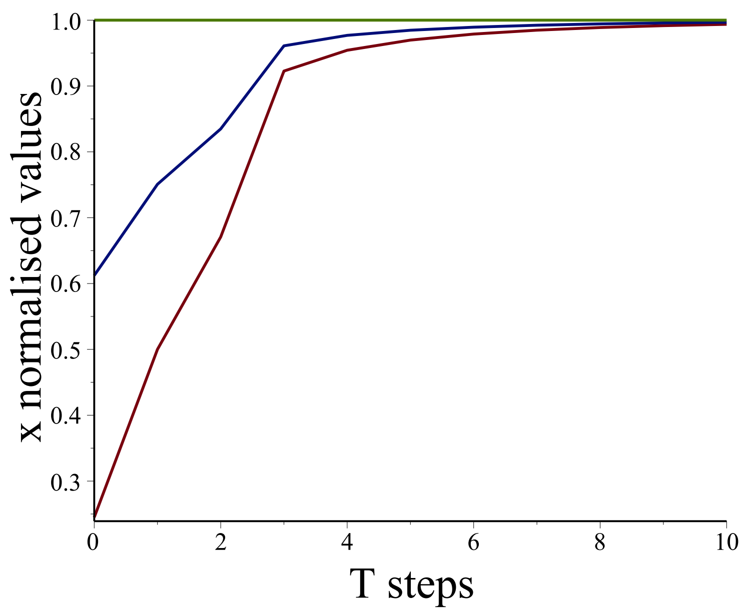
where $e(0) := x_3(0) - x_1(0)$, then the fractional-order Hegselmann-Krause–type model given by $(*)$ reaches a consensus for $n = 3$.

Example

Firstly, at we illustrate the way to reach a consensus for systems with order $\alpha = 0.1$ satisfying condition $\alpha + \frac{1}{2} < \frac{\epsilon}{e(0)}$ and with the opinion profile $x(0) = (x_1(0), x_2(0), x_3(0))$ being an $\epsilon = 0.2$–profile.



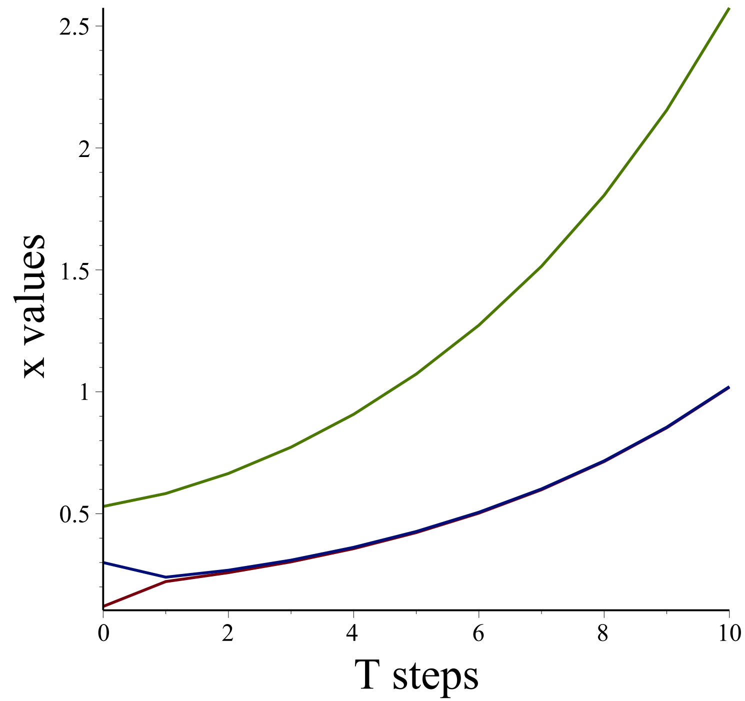
(g) $x = x(k)$



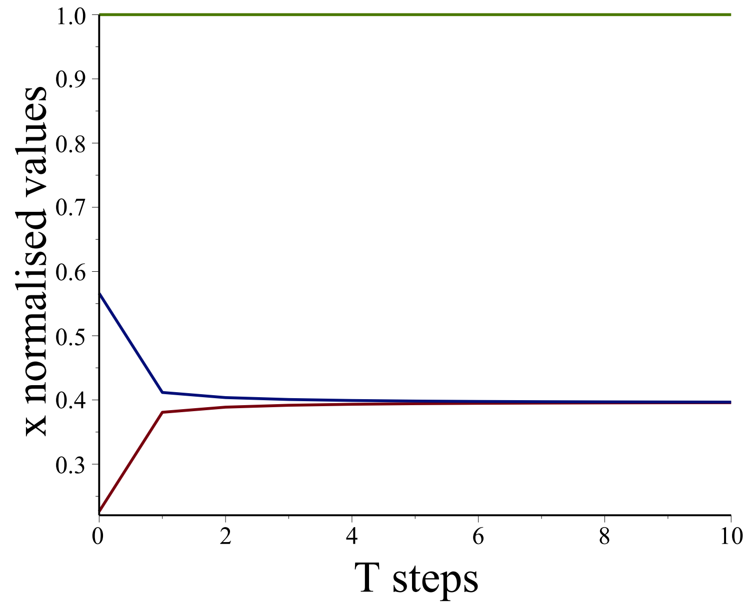
(h) $y = \frac{x(k)}{x_n(k)}$

Example

Let $\alpha = 0.1$, $|x_1(0) - x_2(0)| < \epsilon$ and $|x_2(0) - x_3(0)| > \epsilon$, where $\epsilon = 0.2$. We illustrate that there is no consensus for system $(*)$ with $n = 3$ agents since the necessary condition is not satisfied, i.e. $x(0)$ is not an ϵ–profile.



(i) $x = x(k)$



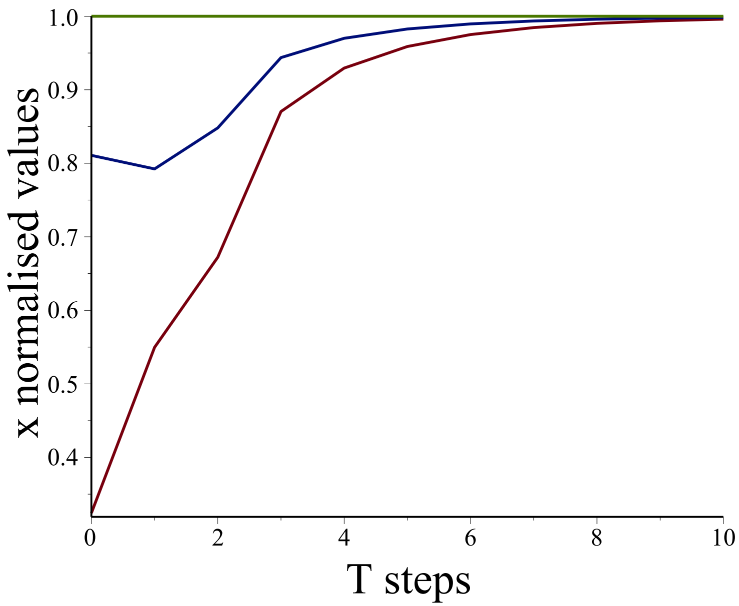
(j) $y = \frac{x(k)}{x_n(k)}$

Example

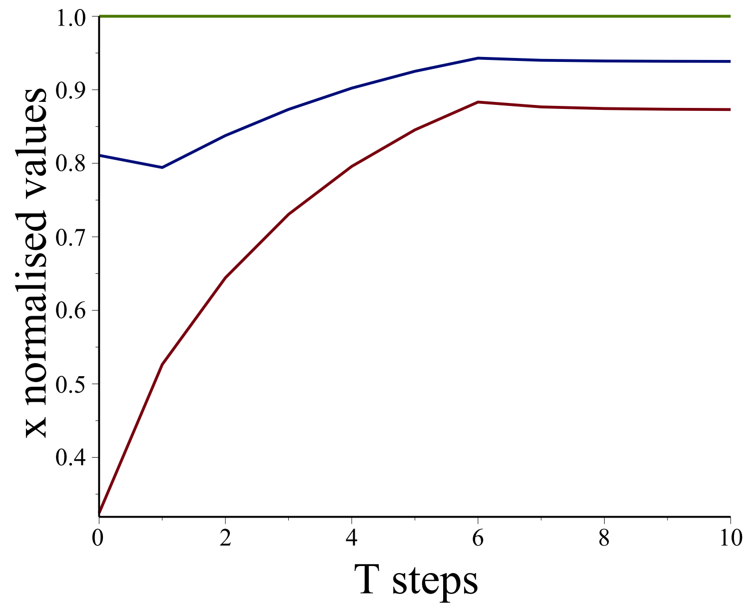
Now, we consider the situation when $\alpha + \frac{1}{2} < \frac{\epsilon}{e(0)}$ is not satisfied.

We illustrate the normalisations of the trajectories for $(*)$ with $n = 3$ agents which opinion profile

$x(0) = (0.12, 0.3, 0.37)$ is an ϵ–profile.



(k) $y = \frac{x(k)}{x_n(k)}, \alpha = 0.31$



(l) $y = \frac{x(k)}{x_n(k)}, \alpha = 0.45$

Corollary: If $x_3(0) - x_1(0) \geq 2\epsilon$, then $x(0) = (x_1(0), x_2(0), x_3(0))$ is not an ϵ–profile, then the fractional-order Hegselmann-Krause–type model given by $(*)$ does not achieve a consensus for $n = 3$.

Proposition: If the opinion profile $x(0) = (x_1(0), x_2(0), x_3(0))$ is an ϵ–profile and

$$\left(\alpha + \frac{1}{2}\right)^2 + \frac{\alpha(1-\alpha)}{2} < \frac{\epsilon}{e(0)},$$

where $e(0) := x_3(0) - x_1(0)$, then the fractional-order Hegselmann-Krause–type model given by $(*)$ reaches a consensus for $n = 3$.

Proposition: If the opinion profile $x(0) = (x_1(0), x_2(0), x_3(0))$ is an ϵ–profile and $e(0) = x_3(0) - x_1(0) \geq \epsilon$ with $\alpha \geq \frac{1}{2}$ then the fractional-order Hegselmann-Krause–type model given by $(*)$ does not reach a consensus for $n = 3$.

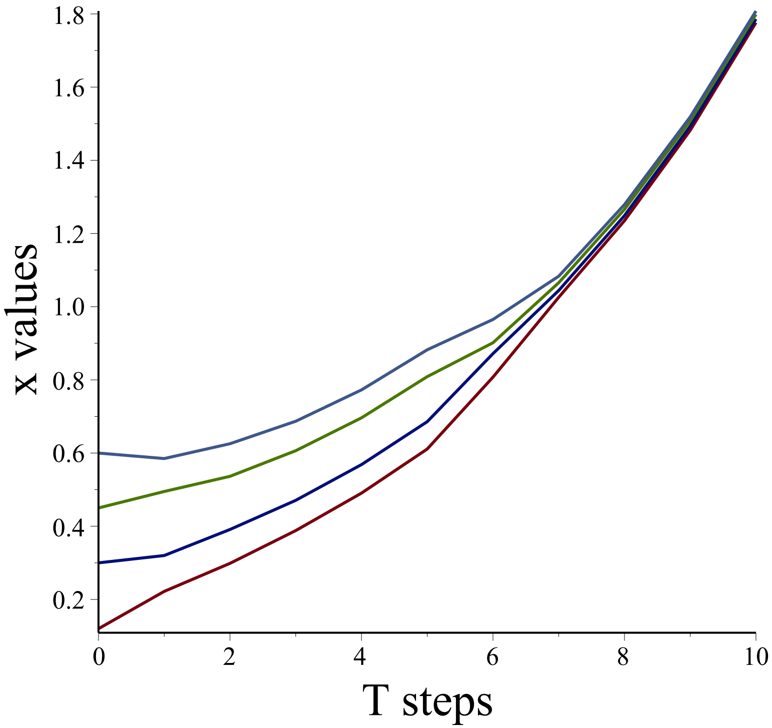
Four agents

Proposition: If $x(0)$ is an ϵ–profile, then there is $\alpha \geq 0$ that consensus is reached by system $(*)$ with $n = 4$ agents.

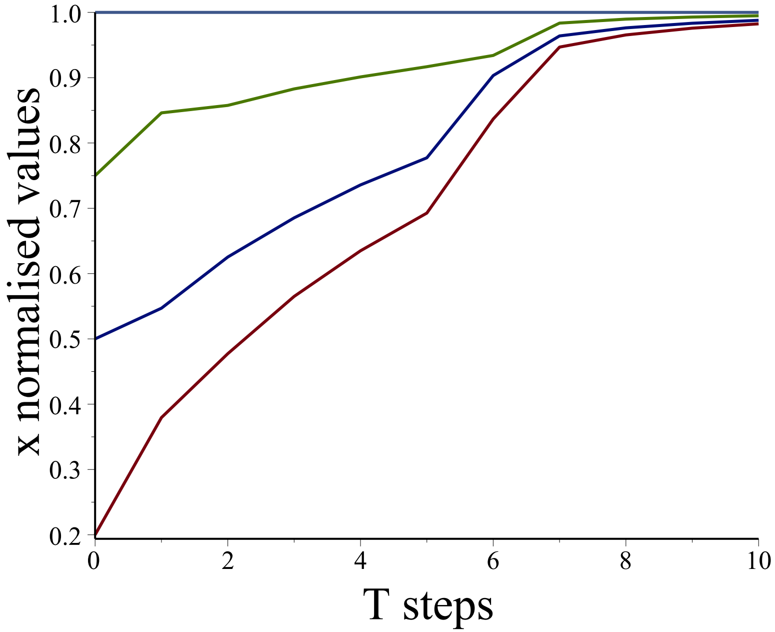
In the proof we have existence of sufficiently small order. However, using computer algebra systems (Maple) we calculated that for $k = 20$ steps we calculated that $\alpha < 0.0952970214384338$ involves consensus into the system.

Example

Now we consider situations for $n = 4$ agents. Direct trajectories and their normalisation for $\alpha = 0.1$ are ilustrated. Taking $x(0) = (0.12, 0.3, 0.45, 0.6)$ being $\epsilon = 0.2$–profile, we get that the consensus is reached for $\alpha = 0.1$.



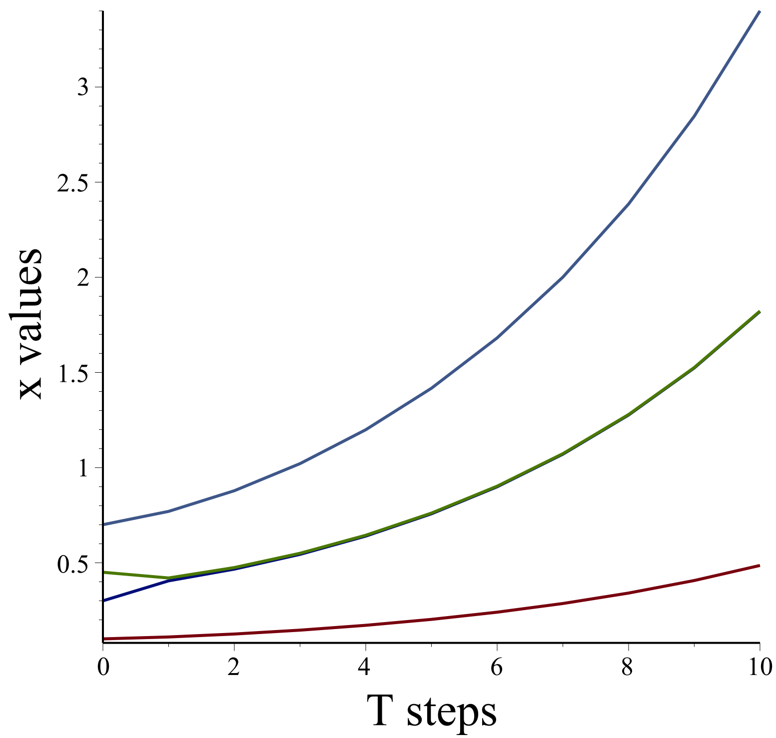
(m) $x = x(k)$



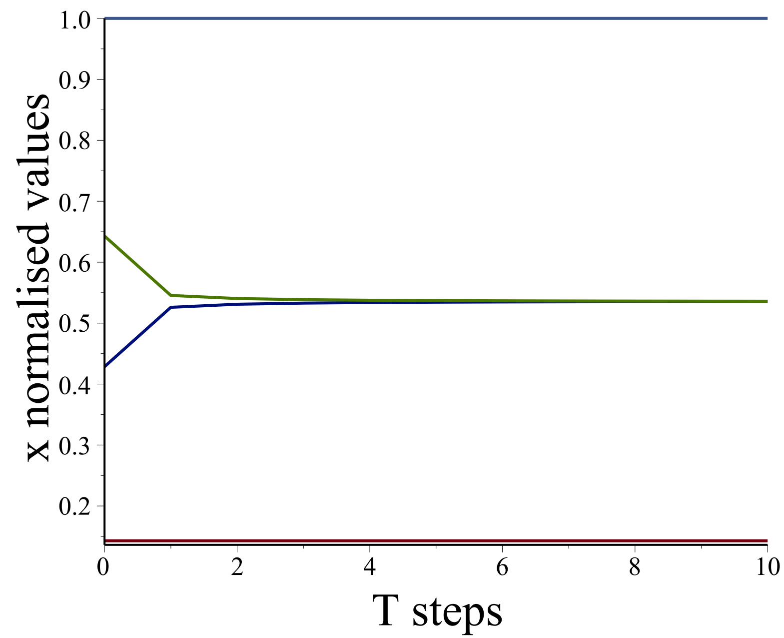
(n) $y = \frac{x(k)}{x_n(k)}$

Example

If $\alpha = 0.1$ is still preserved, but the ordered vector of initial conditions $x(0) = (0.1, 0.3, 0.45, 0.7)$ is changed and now it is not $\epsilon = 0.2$–profile, then the consensus is not achieved.



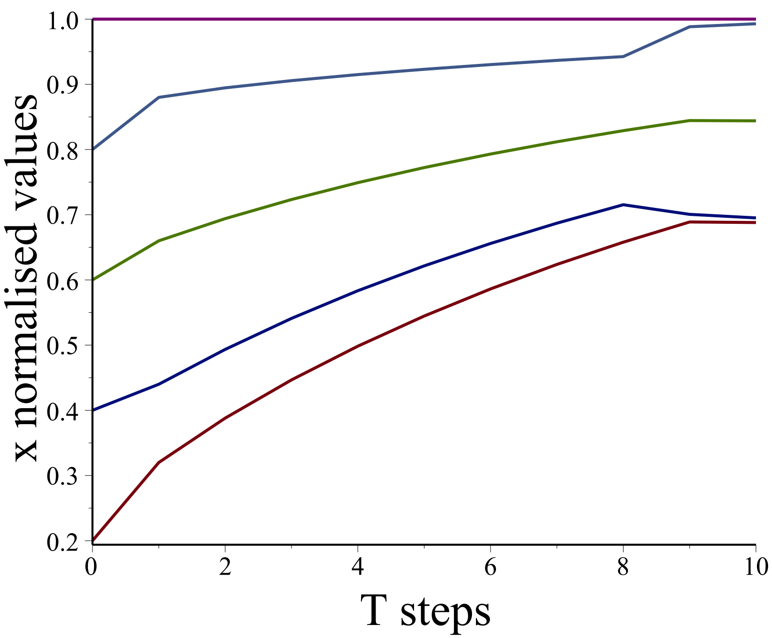
(o) $x = x(k)$



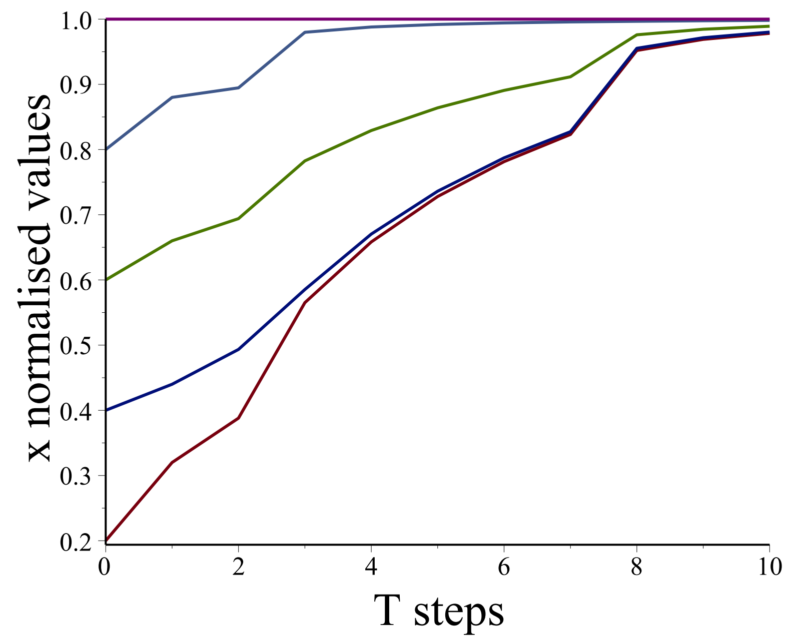
(p) $y = \frac{x(k)}{x_n(k)}$

Example

Now we present situations for $n = 5$ agents. The normalisations of trajectories are illustrated for order $\alpha = 0.1$ and $x(0) = (0.15, 0.3, 0.45, 0.6, 0.75)$ being $\epsilon = 0.2$–profile. If in the system with the same initial condition the tolerance of opinion is changed from $\epsilon = 0.2$ to $\epsilon = 0.25$, then the system reaches consensus with one leader.



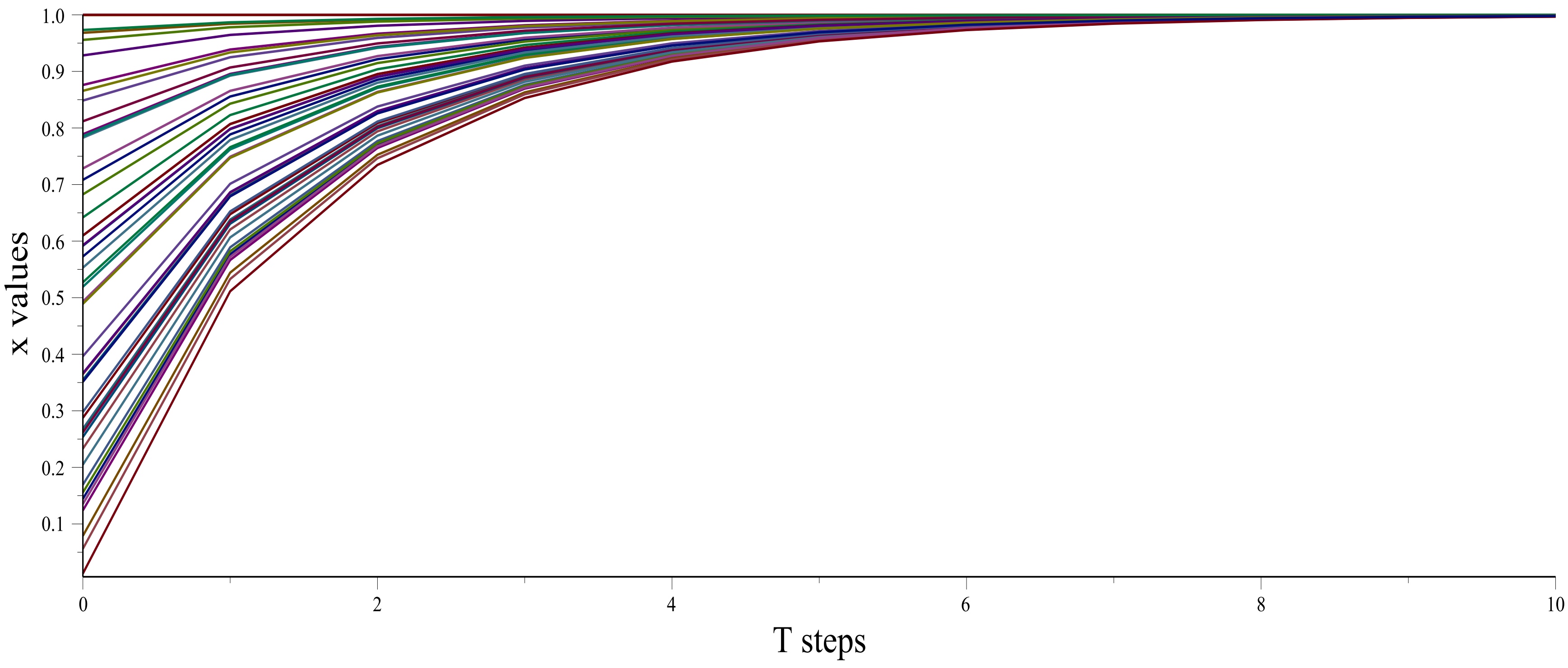
(q) $y = \dfrac{x(k)}{x_n(k)}, \epsilon = 0.2$



(r) $y = \dfrac{x(k)}{x_n(k)}, \epsilon = 0.25$

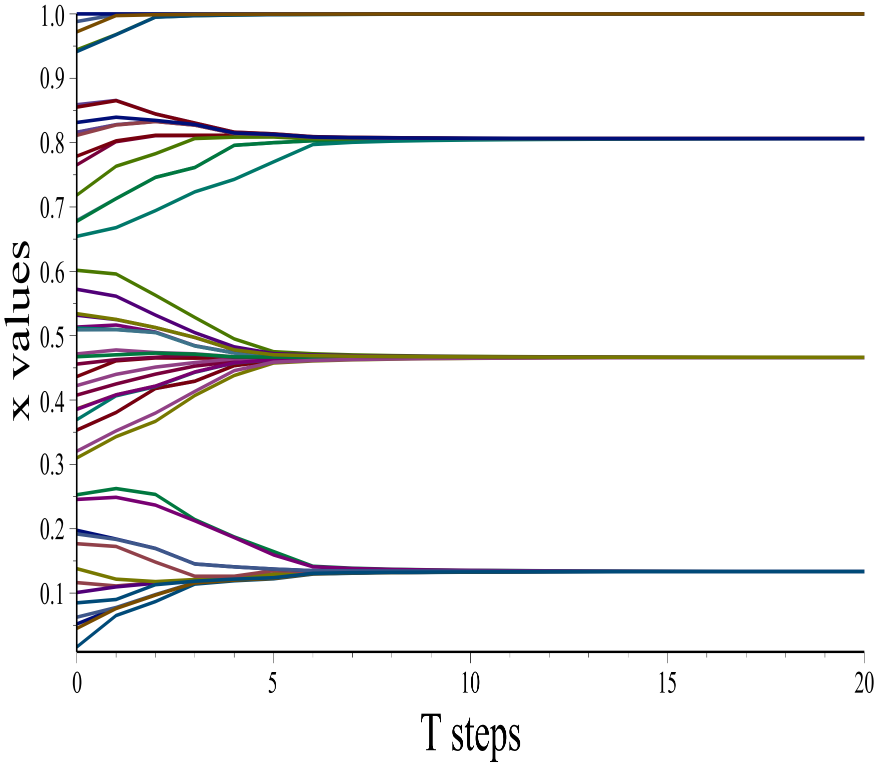
Example

Let us take the system with $n = 50$ agents. Direct trajectories and their normalisations are illustrated for $\alpha = 0.5$ and with the difference $e(0) = |x_{50}(0) - x_1(0)| < 1$. Then taking $\epsilon = 1$, we get $e(0) < 1$ and the one leader consensus is reached.

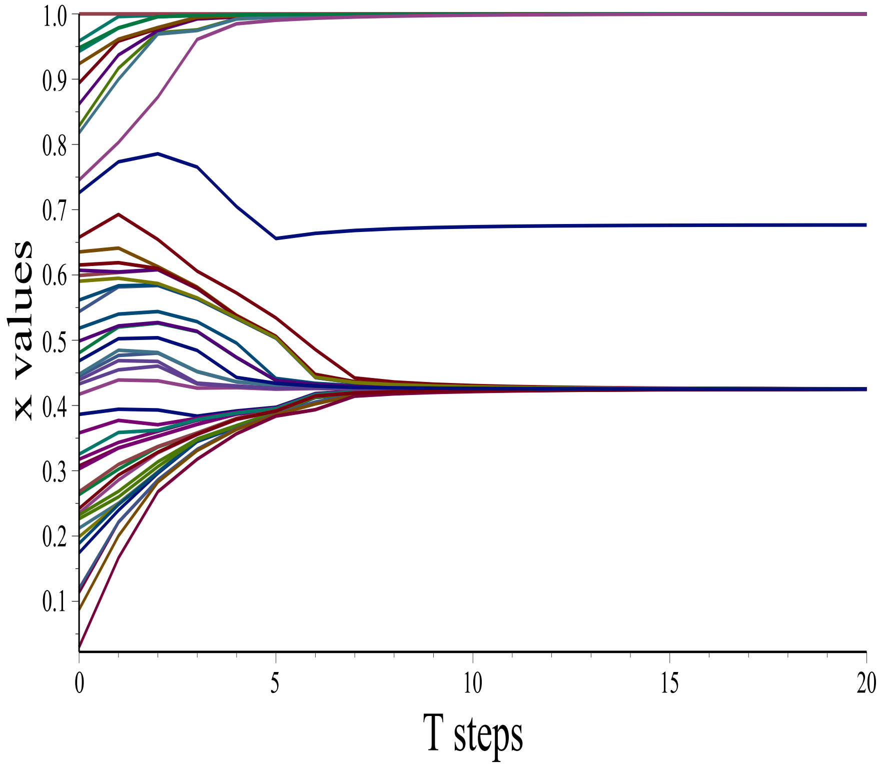


Example

Consider systems of the form (16) with $n = 50$ agents. Note that starting values $x_i(0) \in [0, 1]$. There is four leaders for $\epsilon = 0.1$ and three leaders for $\epsilon = 0.2$.



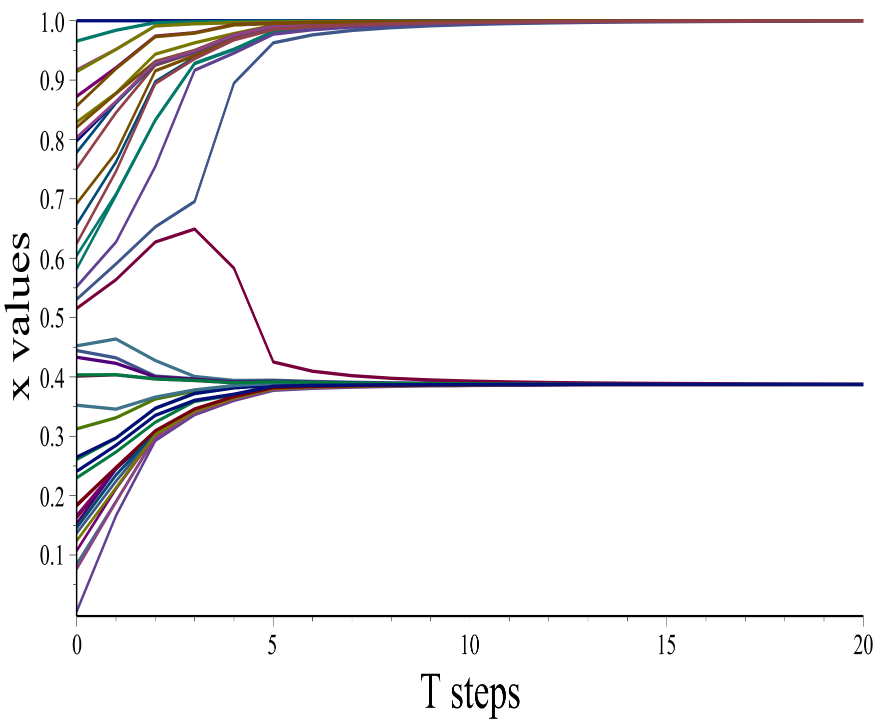
(s) $y = \frac{x(k)}{x_n(k)}, \epsilon = 0.1$



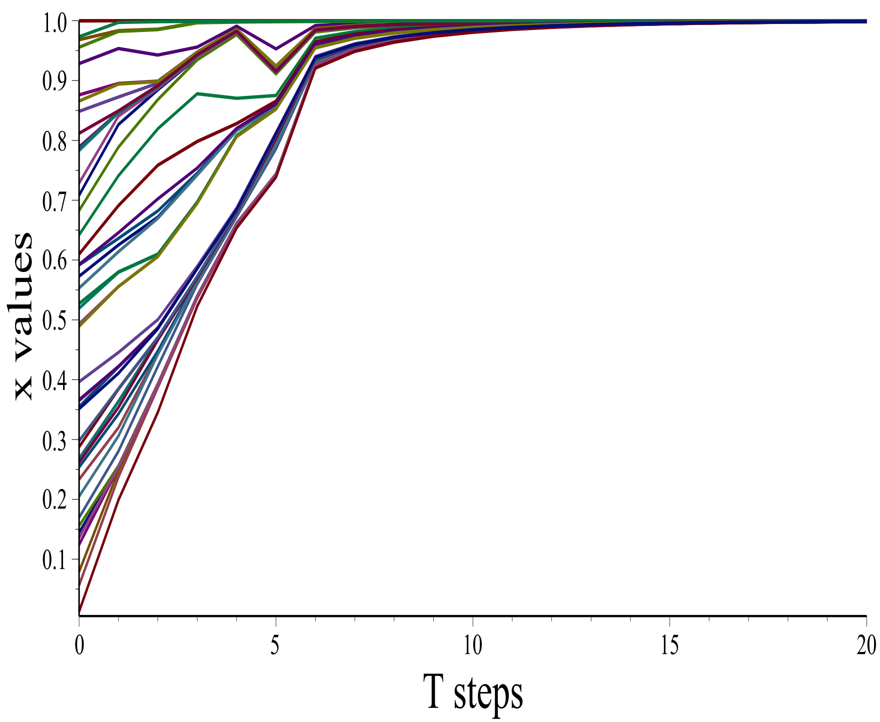
(t) $y = \frac{x(k)}{x_n(k)}, \epsilon = 0.2$

Example

Note that for $\alpha = 0.1$ and $\epsilon = 0.3$ we have two leaders, while for $\epsilon = 0.315$ the consensus is reached. Observe that we have to take more steps to see the limit behaviour with two leaders. Moreover, for $\epsilon > 0.315$ we have one leader.



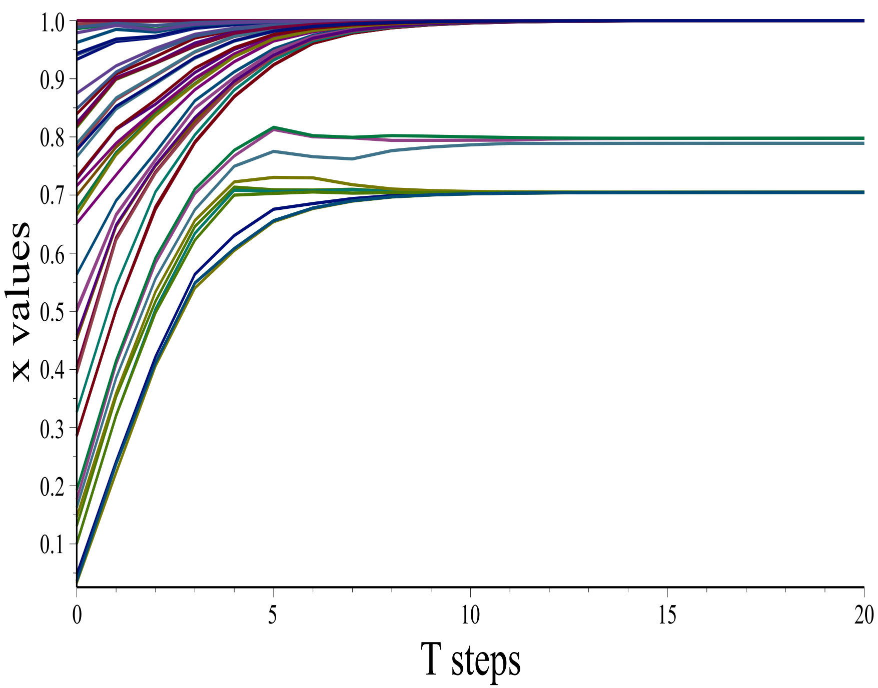
(u) $y = \frac{x(k)}{x_n(k)}, \epsilon = 0.3$



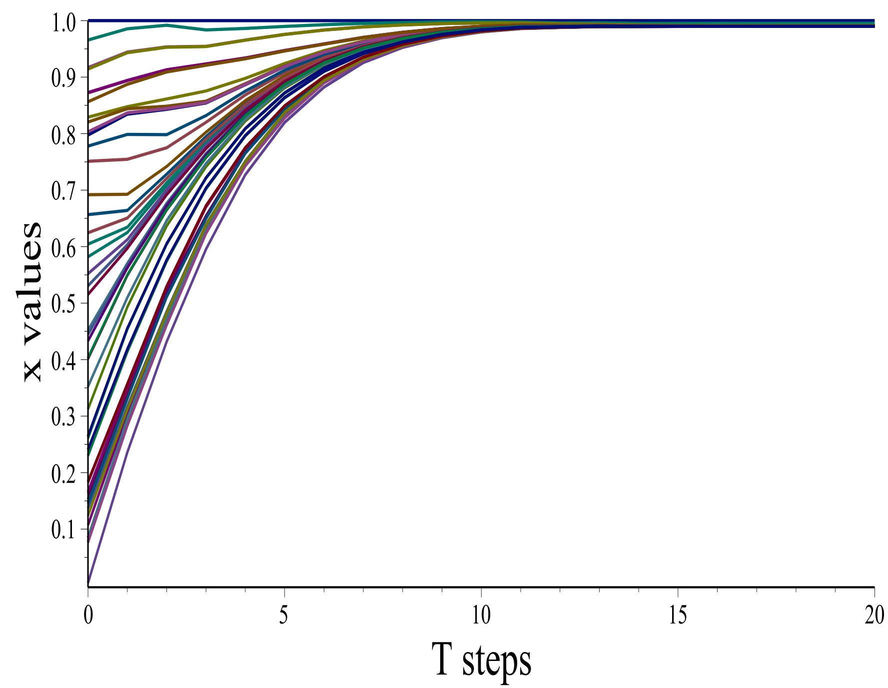
(v) $y = \frac{x(k)}{x_n(k)}, \epsilon = 0.316$

Example

For higher orders systems there is no many changes but to support systems with only few leaders we need to have a little greater ϵ. For example for $\alpha = 0.5$ we present graphs for $\epsilon = 0.6$ and $\epsilon = 0.61$ and we can have few leaders or only one leader, where it depends on the starting opinions.



(w) $y = \frac{x(k)}{x_n(k)}, \epsilon = 0.6$



(x) $y = \frac{x(k)}{x_n(k)}, \epsilon = 0.61$

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