$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad dB(s) = \begin{bmatrix} dB_1(s) \\ dB_2(s) \end{bmatrix}$$

and

$$\exp(tJ) = I + tJ + \frac{t^2}{2}J^2 + \cdots + \frac{t^n}{n!}J^n + \cdots \in \mathbf{R}^{2\times 2} .$$

Using that $J^2 = -I$ this can be rewritten as

$$X_1(t) = X_1(0)\cos(t) + X_2(0)\sin(t) + \int_0^t \alpha\cos(t-s)dB_1(s)$$
$$+ \int_0^t \beta\sin(t-s)dB_2(s) ,$$
$$X_2(t) = -X_1(0)\sin(t) + X_2(0)\cos(t) - \int_0^t \alpha\sin(t-s)dB_1(s)$$
$$+\beta\int_0^t \cos(t-s)dB_2(s) .$$

5.11. Hint: To prove that $\lim_{t\to 1}(1-t)\int_0^t \frac{dB_s}{1-s} = 0$ a.s., put $M_t = \int_0^t \frac{dB_s}{1-s}$ for $0 \le t < 1$ and apply the martingale inequality to prove that

$$P[\sup\{(1-t)|M_t|; t\in[1-2^{-n}, 1-2^{-n-1}]\} > \epsilon] \le 2\epsilon^{-2}\cdot 2^{-n} .$$

Hence by the Borel-Cantelli lemma we obtain that for a.a. ω there exists $n(\omega) < \infty$ such that

$$n \ge n(\omega) \Rightarrow \omega \notin A_n ,$$

where

$$A_n = \{\omega; \sup\{(1-t)|M_t|; t\in[1-2^{-n}, 1-2^{-n-1}]\} > 2^{-\frac{n}{4}}\} .$$

5.16. c) $X_t = \exp(\alpha B_t - \frac{1}{2}\alpha^2 t)\Big[x^2 + 2\int_0^t \exp(-2\alpha B_s + \alpha^2 s)ds\Big]^{1/2}$.

7.1. a) $Af(x) = \mu x f'(x) + \frac{1}{2}\sigma^2 f''(x); \quad f\in C_0^2(\mathbf{R})$.

b) $Af(x) = rxf'(x) + \frac{1}{2}\alpha^2 x^2 f''(x); \quad f\in C_0^2(\mathbf{R})$.

c) $Af(y) = rf'(y) + \frac{1}{2}\alpha^2 y^2 f''(y); \quad f\in C_0^2(\mathbf{R})$.

d) $Af(t,x) = \frac{\partial f}{\partial t} + \mu x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}; \quad f\in C_0^2(\mathbf{R}^2)$.

e) $Af(x_1,x_2) = \frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \frac{1}{2}e^{2x_1}\frac{\partial^2 f}{\partial x_2^2}; \quad f\in C_0^2(\mathbf{R}^2)$.

f) $Af(x_1,x_2) = \frac{\partial f}{\partial x_1} + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}x_1^2\frac{\partial^2 f}{\partial x_2^2}; \quad f\in C_0^2(\mathbf{R}^2)$.

g) $Af(x_1,\ldots,x_n) = \sum_{k=1}^n r_k x_k\frac{\partial f}{\partial x_k} + \frac{1}{2}\sum_{i,j=1}^n x_i x_j\Big(\sum_{k=1}^n \alpha_{ik}\alpha_{jk}\Big)\frac{\partial^2 f}{\partial x_i\partial x_j};$
$f\in C_0^2(\mathbf{R}^n)$.

7.2. a) $dX_t = dt + \sqrt{2}\,dB_t$.

b) $dX(t) = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ cX_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \alpha X_2(t) \end{bmatrix} dB_t$.

c) $dX(t) = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} 2X_2(t) \\ \ln(1+X_1^2(t)+X_2^2(t)) \end{bmatrix} dt + \begin{bmatrix} X_1(t) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}$.

(Several other diffusion coefficients are possible.)

7.4. a), b). Let $\tau_k = \inf\{t > 0; B_t^x = 0 \text{ or } B_t^x = k\};\ k > x > 0$ and put

$$\rho_k = P^x[B_{\tau_k} = k] \ .$$

Then by Dynkin's formula applied to $f(y) = y^2$ for $0 \le y \le k$ we get

$$E^x[\tau_k] = k^2 p_k - x^2 \ . \tag{S1}$$

On the other hand, Dynkin's formula applied to $f(y) = y$ for $0 \le y \le k$ gives

$$kp_k = x \ . \tag{S2}$$

Combining these two identities we get that

$$E^x[\tau] = \lim_{k\to\infty} E^x[\tau_k] = \lim_{k\to\infty} x(k-x)) = \infty \ . \tag{S3}$$

Moreover, from (S2) we get

$$P^x[\exists t < \infty \text{ with } B_t = 0] = \lim_{k\to\infty} P^x[B_{\tau_k} = 0] = \lim_{k\to\infty} (1-p_k) = 1 \ , \tag{S4}$$

so $\tau < \infty$ a.s. P^x.

7.18. c) $p = \dfrac{\exp(-\frac{2bx}{\sigma^2}) - \exp(-\frac{2ab}{\sigma^2})}{\exp(-\frac{2b^2}{\sigma^2}) - \exp(-\frac{2ab}{\sigma^2})}$.

8.1. a) $g(t,x) = E^x[\phi(B_t)]$.

b) $u(x) = E^x[\int\limits_0^\infty e^{-\alpha t}\psi(B_t)dt]$.

8.12. $dQ(\omega) = \exp(3B_1(T) - B_2(T) - 5T)dP(\omega)$.

9.1. a) $dX_t = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ \beta \end{bmatrix} dB_t$.

b) $dX_t = \begin{bmatrix} a \\ b \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dB_t$.

c) $dX_t = \alpha X_t dt + \beta dB_t$.

d) $dX_t = \alpha dt + \beta X_t dB_t$.

e) $dX_t = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} \ln(1+X_1^2(t)) \\ X_2(t) \end{bmatrix} dt + \sqrt{2} \begin{bmatrix} X_2(t) & 0 \\ X_1(t) & X_1(t) \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}$.

9.3. a) $u(t,x) = E^x[\phi(B_{T-t})]$.

b) $u(t,x) = E^x[\psi(B_t)]$.