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Gabor and Wavelet Frames

Geometry and Applications

This thesis is submitted as the partial fulfillment
of the requirements for the degree of
DOCTOR OF PHILOSOPHY
in MATHEMATICS

Thesis advisor: Prof. Janusz Grabowski

Warsaw, 30th June 2001

GABOR AND WAVELET FRAMES. GEOMETRY AND APPLICATIONS.

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Abstract. The theory of frames was born in 1952 as a part of the non-harmonic analysis. However, its expansive growth in late 80's is due to the perspective of real-life applications.

In the thesis we deal first with the abstract theory of the orbits of a single vector under the action of unitary transformation group that are tight frames, named here *coherent tight frames.* The relations with the algebraic aspects of the problem are concerned. In particular, we use the representation theory to analyze phenomena observed recently by researchers in Gabor and wavelet analysis, in particular, fiberization technique. We find a partial characterization of vectors generating coherent tight frames. We apply the theory to obtain some new results for discrete subgroups of Heisenberg group. Under a certain technical assumption we give the representation formula for the frame operator in the discrete wavelet case.

In the second part of the thesis we deal with the question how tight frames are related to orthonormal bases. We study closely a geometry of Gabor tight frames with bound 2 and this of Wilson basis constructed in 1992. The geometrical characterization of the functions generating Wilson basis is given. We discuss this outcome confronted with P.G. Casazza's result about a representation of a frame by means of orthonormal bases. We complete with the simple argument that Wilson bases are unconditional in Banach spaces related to the group action. These spaces are known in the literature as *coorbit* or *modulation spaces.*

We apply this approach to obtain results on the values of exact frame bounds for Gabor and wavelet systems. We illustrate theorems with a numerical presentation. We generalize also the Littlewood-Paley type inequalities introduced for the systems in 1993 by C. Chui and X. Shi. In the thesis also a new insight into the fiberization approach developed in 1995-1998 by A. Ron and Z. Shen is given.

Date: June 30, 2001.

1991 *Mathematics Subject Classification.* Primary 42C15; Secondary 46E15.

Key words and phrases. Hilbert spaces, orthonormal bases, Heisenberg group, wavelets, coherent frames.

Thesis advisor: Prof. Janusz Grabowski. The thesis is supported by State Committee of Scientific Research KBN grant 2P03A 03916.

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A magnitudine enim speciei et creature cognoscibiliter poterit creator horum videri. Sap. 13,5

"For from the number and the beauty of creations one can recognize their Creator by resemblance." *Wisdom 13,5*

1. Introduction

The concept of a frame was introduced by R. Duffin and A. Schaeffer in 1952 [33] and it is a generalization of the notion of unconditional (Riesz) basis to the case of dependent vectors. Since then frames were extensively studied, especially from 80's, when their importance for applications became clear. Riesz bases are isomorphic images of orthonormal systems, i.e., the system $(f_n)_{n\in\mathbb{N}}$ is a `Riesz basis` if for any sequence $(a_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N})$ it satisfies

$$c\sum_{n\in\mathbb{N}} |a_n|^2 \le \left\| \sum_{n\in\mathbb{N}} a_n f_n \right\|^2 \le C \sum_{n\in\mathbb{N}} |a_n|^2 .$$

To introduce the concept of the frame, we shall use its reformulation, namely its dual version:

$$c\,\|f\|^2 \le \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \le C\,\|f\|^2 . \tag{1.1}$$

Then `a frame` is a system of vectors $(f_n)_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ if (1.1) holds for any $f \in \mathcal{H}$ with some positive constants c and C. If $c = C$, then the frame is called `tight` and is closely related to orthonormal systems; in particular, an orthonormal system is a tight frame and the orthogonal projection image of an orthonormal system is a tight frame.

In this context one can raise the following question:

What can we say more about the frame and its approximation properties if all vectors of the frame are 'of the same type' ?

More precisely, let G be a countable group of unitary mappings on a Hilbert space $\mathcal{H}$. For a vector $u \in \mathcal{H}$ consider the orbit

$$\{gu : g \in G\} .$$

When this set is a frame? When this set is a tight frame? Or more quantitatively: what are the values of constants? What is the influence of the linear properties of operators from G and the algebraic structure of G itself? The aim of this thesis is to take an attempt to answer these questions.

Remark 1. *The author considers the group of linear mappings instead of a more standard approach where an abstract algebraic group $\mathfrak{G}$ and a linear representation π of $\mathfrak{G}$ are used. This choice, in the author's opinion, will increase the comfort of the reader in some parts of the thesis.*

This approach to frame properties links problems of approximation theory with geometrical and algebraical questions. The second point is that this framework is suitable for special cases on which the interest of this thesis focuses, namely Gabor and wavelet systems. Both cases are very interesting from a pure-mathematical point of view and very useful for applications.

Definition 1. *Given $a, b \in \mathbb{R}_+$ and a function $u \in L^2(\mathbb{R})$, define*

$$u_{mn}(x) = e^{2\pi iamx}u(x-nb).$$

If the system $(u_{mn})_{m,n\in\mathbb{Z}}$ is a frame in $L^2(\mathbb{R})$, it is called `a Gabor frame`.

Definition 2. *Given $a, b \in \mathbb{R}_+$ $(a > 1)$ and a function $u \in L^2(\mathbb{R})$, define*

$$u_{jk}(x) = a^{j/2}u(a^j x - kb).$$

If the system $(u_{jk})_{j,k\in\mathbb{Z}}$ is a frame in $L^2(\mathbb{R})$, it is called `a wavelet frame`.

Note that the orbit, which are obtained under unitary action of a group, are called in mathematical physics the *coherent states* (see [2], [66] etc.).

First concepts of Gabor analysis were introduced in the telecommunication engineering field in 1946 [51]. The idea was the following: Let us take two functions from $L^2(\mathbb{R})$, f -'signal' and g -'window'. Instead of applying the standard Fourier transform

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi ix\xi}dx\ ,$$

let us consider the Fourier transform of the product of f with the translated u

$$\mathcal{G}f(p,q) = \int_{\mathbb{R}} f(x)\,\overline{u(x-q)}e^{-2\pi ixp}dx.$$

If u is properly chosen (smooth and reasonably decaying functions will do), $\mathcal{G}$ is invertible and is an isometry, if $\|u\| = 1$. So we can use $\mathcal{G}$ instead of $\mathcal{F}$. The main difference between them is their behaviour after discretization. Note that if we use only $\mathcal{F}f(n)$ to reconstruct f, we shall get a periodic function, so we loose much. On the other side in the Gabor case we have frame reconstruction procedures, because $(u_{mn})_{m,n\in Z}$ is very often a frame. For example, in the case of $u(x) = e^{-x^2/2}$, being a Gaussian function, the only limitation is $ab < 1$. (see [82] for details). The price to pay is a two-dimensionality of the Gabor transform.

Wavelets have their origin in the Haar system (1910) and in Calderon-Zygmund theory in 40's, but their expansive growth started only in late 80's. Let us consider a function $f \in L^2(\mathbb{R})$ and the system

$$f_{jk}(x) = 2^{j/2}f\left(2^j x - k\right),\ j,k \in \mathbb{Z}.$$

The problem was to find orthonormal bases of that type, the Haar system being the only and discontinuous example for 70 years. In 1980-83 Stromberg [84] constructed a linear spline function with exponential decay whose translates and dilations were an orthonormal basis. Thus he is a for-runner of modern wavelets.

Nevertheless, it was believed that there is no smooth solutions with reasonable decay as in the case of Gabor system (see Section 4.1). The reality was different. Looking for a proof of non-existence of such solutions, Y. Meyer found in 1986 an orthonormal wavelet basis in a Schwartz class. Then other constructions followed: Battle-Lemarié [9], [70], Daubechies [29] and many others.

However, all constructions are complicated and for sure they do not exhaust all wavelet orthonormal bases. The results of my research suggest that 'almost every' function with a quite wide range of parameters a and b gives a frame with useful analysis and reconstruction procedures, compression effects, etc., while it is non-trivial to find a function which gives an orthonormal basis and the choice heavily depends on the values of a and b.

In the domain of frames the interest of researchers concentrates on the following subjects. (References given here hardly cover the results in the domain and serve rather as examples.)

Atomic decompositions of Banach spaces were studied in [39], [40], [22], [61], with relation to group representations. Not only subgroups, but also irregular subsets of groups were the subject of research under the name of irregular sampling, see [53], [46], [43], [69], [10], [81].

In the series of papers [38], [39], [40], [41], [42], [44], new Banach spaces were introduced and investigated. They are closely related to group representation and are called coorbit spaces or **modulation spaces**.

Unconditional bases are closely related to frames. They appear to be contained in frames as subsequences [16] and can be constructed from frames as in [32], [44], [57]. Orthonormal bases of wavelets are unconditional bases in many important Banach spaces, see [75], [92] and references given there.

The estimated or optimal values constants appearing in (1.1) for $L^2(\mathbb{R})$, i.e., **of frame bounds**, were the subject of studies from late 80's. Exact values were known only for special functions and for special values of parameters a and b ([62], [64], [45], [63], [28], [85], [79], [78]). They are also important in the algorithms used for reconstruction of the function from its coefficients. For instance, the convergence rates of these algorithms depend on their values.

Frames in other spaces than $L^2(\mathbb{R})$, being weighted Hilbert spaces or Banach spaces, were studied in [41], [52]. In [13] it was shown that for weighted $L^2(\mathbb{R})$ no frame in the sense of definition (1.1) exists. So one needs a concept of the Banach frame. We defer the interested reader to [55] for details.

The frame operator is a self-adjoint operator related with the frame. Representation formulae for the frame operator and questions of its invertibility in different Banach spaces are subjects of [86], [62], [41] and [21].

An important advantage of the research done in this field is that computational algorithms for Gabor and wavelet frames are efficient and of low complexity, which has a great impact on why engineers are interested in this domain. This point of view is represented in [32], [28], [63], [54], [47], [42].

The results of the thesis are of a threefold type: **First**, we investigate dependencies between geometry of vectors, algebraic and topological properties of groups in the case when the orbit of this vector under the action of the group is a frame or a tight frame. The strong links were found with the classical theory of unitary representations, their direct integral decomposition and the commuting properties of the algebra of representation operators. The research in this domain concentrates on three questions: 1. for which group representations the coherent tight frames do exist? 2. for a given group representation, which vectors generate the coherent tight frames, if any? 3. which operators preserve the tight frame generating vectors? The first question is studied in [49], [50], while the third was the subject of two monographic articles [26], [58]. In this thesis we shall concentrate on the second one.

Second, we are interested in a geometric structure of the Gabor tight frame with bound 2 (we consider here so-called normalized Gabor tight frame with a norm-one atom) and the possibility to extract an orthonormal basis from it. We exploit in this purpose Wilson basis proposed by Daubechies-Jaffard-Journé [32] and check what are the other solutions of that type, proving, at least partially, the conjecture stated in [91]. We also show the geometric nature of the function that generates a Wilson orthonormal basis. There is also a short discussion with the result of P.G. Casazza [18] that the tight frame can be represented by a combination of two orthonormal bases if and only if it is a Riesz basis. We also check the unconditionality of the obtained bases in non-Hilbert spaces, namely coorbit spaces and related to them Bargmann spaces of holomorphic functions. The latter are of importance in the mathematical physics, since the canonical commutation rules between the position and momentum operators are implemented there.

This unconditionality result was proved in [57] and [44]. The approach in the thesis, presented already in [91], is simpler and gives a shorter proof of unconditionality of Gröchenig-Walnut explicit basis in Bargmann spaces for all $1 \leq p < \infty$. The full original proof relies on much of the theory of coorbit spaces and counts about 70 pages, while here we can present it in Chapter 4 where it is based on some result about Bargmann spaces from [61].

Third, we develop the approach to the frame problems by means of direct integral decompositions of unitary representations. The algebraic properties of the group operators show to have the significant influence on the form of the set of *tight frame vectors* - vectors whose orbits under the action of the group are tight frames. The frame operator can be also decomposed to much simpler operators acting on the 'fibers' in the direct integral. These 'fiber' spaces are somehow *"infinitesimal"* invariant subspaces; observe that there are representations which have no irreducible subrepresentations (see the beginning of Chapter 3 for an example).

This approach is closely related to the ideas from works of A. Ron and Z. Shen [77] and [79] and of Feichtinger-Gröchenig as in [41], Thm. 3.2. There is also a relevance with the papers of P.G. Casazza (see [19] and references there). The application of the approach extends also some known results, e.g., Littlewood-Paley type inequalities [24].

The optimal frame bounds are known only for special functions and for special values of discretization parameters (comp. [28], [63], [85], [79]). Using the direct integral decomposition method, we find optimal frame bounds for a wide class of functions. The formulae obtained here are simple and of low computational cost. Numerical examples serve as an illustration of the method. Some of them exemplify non-trivial and recent results from the frame theory, e.g., Seip-Wallstén theorem about the Gabor frame generated by a Gaussian function stating that the Gaussian function generates a Gabor frame if and only if $ab < 1$ [82].

In this setting many important results can be seen directly and generalized, e.g., estimates of the frame bounds by I. Daubechies [28], criteria for invertibility of the frame operator by D. Walnut [86], by P.G. Casazza - O. Christensen [21] and by B. Deng - W. Schempp - C. Xiao - Z. Wu [35] and the relation between the frame bounds and the zero-th Walnut's coefficient by C.K. Chui and X. Shi [24], referred to in the literature as Littlewood-Paley type inequalities or a 'diagonal result'.

Using the recent results about quasi-affine systems with integer dilations [78], [25], we obtain some results for the wavelet frames, too.

Much of the results of the thesis was published or presented in [91], [89], [88], as well as on the conference lectures. The algorithms were derived to compute the lower and upper frame bounds for Gabor and wavelet frames. The numerical values coincide with the results presented in the literature [28], [63].

1.1. **Organization of the thesis.** The Chapter 2 contains a basic material and the main concepts appearing thereafter in the thesis: a general definition of frame, a direct integral of Hilbert spaces and some information about the representation types.

In Chapter 3 we consider in detail problems of *coherent frames*, i.e., frames which are orbits of one vector under the action of a unitary mapping group. On the basis of classical representation theory and the direct integral decomposition we develop a study of *'fiberization phenomenon'* observed in a series of papers by A. Ron and Z. Shen [79], [77], [78] and further discussed in works [17], [19], [41].

The main topic of Chapter 4 is Wilson basis. Its construction is justified by the results for the direct integral decomposition; the characterization of functions which generate Wilson bases is presented and expressed in the geometrical language. Also the result of its unconditionality is proved in the spirit of [91].

In Chapter 5 we apply results of previous chapters to the exact bounds problems for Gabor frames. This chapter also includes a discussion when the assumptions taken are satisfied and to what extent they are necessary. In this Chapter we also present numerical results of the author's work. This material is published in the paper [89].

The frame bounds as a pair of real numbers determine an interval. Its finiteness is guaranteed by the frame condition. The frame bounds estimates give its 'upper cover', while the exact bounds yield its 'minimal cover'. In Chapter 6 we check what subinterval it must *contain*. To this purpose we use results from [24] and their generalization.

Notation 1. *All items (theorems, propositions, lemmas, remarks etc.) are numbered sequentially with the number of chapter and the number of section.*

Notation 2. *G stands for a locally compact, secondary countable group. $\mathcal{H}$ is a separable Hilbert space. π and ρ are strongly continuous unitary representations of G in $B(\mathcal{H})$. $\mathfrak{A}_G$ is the von Neumann algebra generated by the operators $g \in G$ (closed in the strong operator topology). $\mathcal{D}_G$ is the Banach algebra generated by these operators, but closed in the norm topology.*

Acknowledgement 1. *I would like to thank greatfullly Prof. Janusz Grabowski, my thesis advisor, for a constant encouragement, fruitful discussions and for the excellent work atmosphere.*

Special thanks are due to Prof. Hans G. Feichtinger, Vienna University, for an introduction into a field of Gabor analysis, for his hospitality during my stays in Vienna and for valuable discussions and hints.

I thank with pleasure Prof. Przemysław Wojtaszczyk for very interesting discussions and remarks, especially in the 'unconditional' part of the thesis, which owes him much.

To Prof. D. Walnut for drawing my attention to papers of P. Auscher and other helpful remarks.

To Prof. P.G. Casazza for the remarks about the convergence of Walnut's representation and for an access to the preprints of his unpublished results.

To Dr. A.J.E.M. Janssen for reading some of my papers and critical remarks which improved the quality of the 'exact bound' part of the thesis.

I would like also to acknowledge discussions and correspondence with Prof. Prof. S.T. Ali, O. Christensen, J.-P. Gazeau, K. Gröchenig, S. Jaffard and K.Seip. I thank Prof. K. Gröchenig for an access to the manuscript of his monograph on Gabor analysis and Prof. D. Larson for the pre-copy of his monograph "Frames, Bases and Group Representations".

I would like to thank Prof. Aleksander Pełczyński, Polish Academy of Sciences, for a possibility to present results of this thesis on his seminar and for discussions that took place there.

I thank Prof. Peter Michor, the director of the Erwin Schrödinger Institute, who encouraged me to discuss some problems of the thesis with guests of ESI.

It is a pleasure to mention here talks with and remarks of my colleagues Dr. K. Woźniakowski, Dr. Norbert Kaiblinger, Vienna University, and Dr. Hartmut Führ, Technical University, Munich.

I thank heartily to my closest for their understanding and support during the time of writing this thesis.

And to you, Ania, my love, for your presence and the change you have made in my life.

Acknowledgement 2. *The part of numerical computations for this thesis was performed with the use of computational facilities of Interdisciplinary Center for Mathematical and Computational Modelling, Warsaw University under direction of Prof. M. Niezgódka, to whom I thank for a kind help.*

The research for the thesis was financially supported by State Committee for Scientific Research KBN grant 2P03A 03916.

2. Preliminaries

2.1. **Frames.** Let (X, μ) be a measurable space and Φ be a *measurable* mapping of X into the Hilbert space $\mathcal{H}$, i.e. for $(e_i)_{i\in\mathbb{N}}$ being an orthonormal basis of $\mathcal{H}$ the function $\langle\Phi(x), e_i\rangle$ is measurable on X in the usual sense for all $i \in \mathbb{N}$.

Definition 3. *The measurable mapping $\Phi : X \to \mathcal{H}$ is* `a frame` *if and only if there exist positive constants c and C such that for any $y \in\mathcal{H}$ the following holds:*

$$c\,\|y\|^2 \le \int_X |\langle y, \Phi(x)\rangle|^2 \;d\mu(x) \le C\,\|y\|^2 . \tag{2.1}$$

The frame is called tight if $c = C$.

Example 1. *Observe that this definition covers the case introduced in the Introduction (1.1) for $X = \mathbb{N}$ and for μ being the counting measure.*

Example 2. *Further, let us assume that G is a group of unitary mappings (locally compact) in a Hilbert space $\mathcal{H}$. Then we take $X = G$, $d\mu = dg$ a left Haar measure on G (unique up to a constant factor) and for some fixed $u \in\mathcal{H}$ let Φ be:*

$$\Phi(g) = gu$$

where $g \in G$.

Definition 4. *For a frame Φ we consider the* `analysis operator` *$I_\Phi : \mathcal{H} \to L^2(X)$ being*

$$I_\Phi y(x) = \langle y, \Phi(x)\rangle_{\mathcal{H}}$$

and the **synthesis operator***, adjoint to* I_Φ*,* $I_\Phi^* : L^2(X) \to H$ *defined by*

$$I_\Phi^* a = \int_X a(x)\, \Phi(x)\, d\mu(x).$$

Definition 5. *The* **frame operator** $S_\Phi =^{df} I_\Phi^* I_\Phi : \mathcal{H} \to \mathcal{H}$ *, being the composition of the analysis and synthesis operators, is defined according to the formula*

$$S_\Phi y = \int_X \langle y, \Phi(x) \rangle\, \Phi(x)\, d\mu(x).$$

Thus, S_Φ depends on the frame Φ. Observe that we can define the frame operator also for a wider class of systems.

Definition 6. *The measurable mapping* Φ *is* **admissible** *if the analysis operator* $I_\Phi : \mathcal{H} \to L^2(X)$ *is defined and bounded.*

Example 3. *If* Φ *is defined as in Example 2,* Φ *is* **admissible** *implies that the vector* $u \in \mathcal{H}$ *is* **admissible***, i.e.,*

$$\int_G |\langle u, gu \rangle|^2\, dg < \infty.$$

Indeed, $I_\Phi x(g) = \langle x, gu \rangle_{\mathcal{H}}$ *and* $\|I_\Phi x\|^2_{L^2(G)} = \int_G |\langle x, gu \rangle|^2\, dg \le C\, \|x\|^2_{\mathcal{H}}$ *for any* $x \in \mathcal{H}$*, in particular for* $x = u$*, so* $\int_G |\langle u, gu \rangle|^2\, dg \le C\, \|u\|^2_{\mathcal{H}} < \infty$.

From orthogonality relations for locally compact groups follows that if G acts irreducibly, then Φ is admissible *if and only if* $\int_G |\langle u, gu \rangle|^2\, dg < \infty$.

If the system is admissible, the synthesis operator is also bounded and the frame operator S_Φ is well defined. If S_Φ is an isomorphism of $\mathcal{H}$, then Φ is a frame, because $\langle Sy, y \rangle = \int_X |\langle y, \Phi(x) \rangle|^2\, d\mu(x)$. The norm of S_Φ and the inverse of the norm of S_Φ^{-1} are the upper and the lower bound for this frame, respectively.

Remark 2. *In the case of Example 2, the frame operator commutes with any* $g' \in G$*. Indeed,*

$$\begin{aligned} S_\Phi g' y = \int_G \langle g'y, gu \rangle\, gu\, dg &= \int_G \langle y, (g')^{-1} gu \rangle\, gu\, dg = \\ &= \int_G \langle y, g_1 u \rangle\, g' g_1 u\, d(g' g_1) = \\ &= g' \int_G \langle y, g_1 u \rangle\, g_1 u\, dg_1 = g' S_\Phi y. \end{aligned}$$

The last-but-one equality follows from the left-invariance of Haar measure.

If Φ is a frame, $S^{-1}\Phi$ is a frame, too. This frame is called `dual` because of the following identity

$$f = \int_X \left\langle f, S^{-1}\Phi(x) \right\rangle\, \Phi(x) d\mu(x) = \int_X \langle f, \Phi(x) \rangle\, S^{-1}\Phi(x) d\mu(x).$$

It is easy to see that $S^{-1/2}\Phi$ is a tight frame with bound 1. Indeed,

$$\begin{aligned} f &= S^{-1/2} S S^{-1/2} f = S^{-1/2} \int_X \left\langle S^{-1/2} f, \Phi(x) \right\rangle\, \Phi(x) = \\ &= \int_X \left\langle f, S^{-1/2}\Phi(x) \right\rangle\, S^{-1/2}\Phi(x). \end{aligned}$$

The operator $S^{-1/2}$, like S, is a self-adjoint isomorphism of $\mathcal{H}$.

If the system Φ is a *tight* frame, then the related frame operator $S_\Phi = I_\Phi^* I_\Phi = \lambda\,\mathrm{Id}$ is a scalar multiple of identity and the constant is the tight frame bound. Then $P = \lambda^{-1} I_\Phi I_\Phi^* : L^2(X) \to L^2(X)$ is an orthogonal projection on the range $I_\Phi(\mathcal{H})$. Indeed

$$\begin{aligned} P^2 &= (\lambda^{-1}II^*)^2 = \lambda^{-2}II^*II^* = \lambda^{-1}II^* = P, \\ P^* &= (\lambda^{-1}II^*)^* = \lambda^{-1}II^* = P, \\ PL^2(X) &= II^*L^2(X) = I(\mathcal{H}). \end{aligned}$$

The last identity holds under the assumption that $\Phi(X)$ spans $\mathcal{H}$.

2.2. **Heisenberg group.** Heisenberg group appears in the mathematical physics in the context of canonical commutation rules between the position and momentum operators and in the context of Heisenberg Uncertainty Principle. It is a Lie group being a central extension of additive group $\mathbb{R}^2$. In this thesis we shall concentrate on the reduced Heisenberg group according to the following

Definition 7. *The* `Heisenberg group` *is*

$$\mathbf{H} = \left(\mathbb{R}^2 \times \mathbb{S}^1, \circ\right)$$

with the multiplication $\circ$ *given by*

$$(p,q,t) \circ (p',q',t') = (p+p', q+q', t+t'+pq' \mod 1)\,,$$

where $\mathbb{S}^1$ *is regarded as* $\mathbb{R}/\mathbb{Z}$. *The unit of this group is* $(0,0,0)$ *and the inverse of* (p,q,t) *is* $(-p,-q,-t+pq \mod 1)$.

Remark 3. *We use here the additive notation for* $\mathbb{S}^1$ *instead of more standard multiplicative notation. In opinion of the author the first one is more comfortable here.*

Definition 8. *The* `Schrödinger representation` *is a representation of the Heisenberg group in* $L^2(\mathbb{R})$*:*

$$(p,q,t)\, f(x) = e^{2\pi i t} e^{2\pi i p x} f(x-q). \tag{2.2}$$

Schrödinger representation implements canonical commutation rules known from mathematical physics. The representation is irreducible and faithful [48]. Because of the multiplicative action of the third parameter we will write sometimes (p,q) for $(p,q,0)$. For a function $f \in L^2(\mathbb{R})$ the vectors $(p,q,t)\,f$ are called *coherent states* of f. The operators M_a and T_b defined by:

$$M_a f(x) := e^{2\pi i a x} f(x), \tag{2.3}$$

$$T_b f(x) := f(x-b), \tag{2.4}$$

are isometries of $L^2(\mathbb{R})$.

Assume that $ab = p/q$ is rational. There is a subgroup in $\mathbf{H}$ which is denoted by $Gab(a,b)$ and equal as a set to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_q$. It is closely related with the von Neumann algebra generated by M_a and T_b. In fact,

$$(k,l,\frac{m}{q}) = e^{2\pi i m p/q} M_a^k T_b^l \in U\left(L^2(\mathbb{R})\right)$$

is a representation of $Gab(a,b)$ in $\mathcal{H} = L^2(\mathbb{R})$.

If ab is irrational, one considers a subgroup in $\mathbf{H}$ being

$$Gab(a,b) = (\mathbb{Z}^2 \times \{mab \mod 1 : m \in \mathbb{Z}\}\,,\cdot)$$

with the multiplication

$$(k,l,t)\cdot(k',l',t') = (k+k', l+l', t+t'+lk'ab \bmod 1).$$

Again

$$(k,l,m) = e^{2\pi imab} M_a^k T_b^l \in U\left(L^2(\mathbb{R})\right)$$

is a representation of $Gab(a,b)$ in $\mathcal{H} = L^2(\mathbb{R})$.

Example 4. *The system $(M_{\frac{m}{2}} T_n \varphi)_{m,n\in\mathbb{Z}}$ for $\varphi(x) := e^{-\pi x^2/2}$ is a frame in $L^2(\mathbb{R})$. Indeed, let us consider the **admissible** system $(g_{mn})_{m,n\in\mathbb{Z}} = (M_{ma} T_{nb} g)_{m,n\in\mathbb{Z}}$ and f, h compactly supported functions on $\mathbb{R}$. A.J.E.M. Janssen proved (* [62]*, prop. 2.6 and 2.8, for summary see* [64]*, p. 59) that the following representation formula holds in the weak sense*

$$\langle Sf, h\rangle = \sum_{k,m} \langle\, g,\ (kb^{-1}, ma^{-1})\ g\rangle\ \left\langle (kb^{-1}, ma^{-1})\ f, h\right\rangle. \tag{2.5}$$

Moreover, if

$$\sum_{k,m} \left|\langle\, g,\ (kb^{-1}, ma^{-1})\ g\rangle\right| < \infty, \tag{2.6}$$

the above representation is unconditional. In the case $g = \varphi$, $a = \frac{1}{2}$, and $b = 1$, the coefficients $a_{km} := \left|\langle\, \varphi,\ (kb^{-1}, ma^{-1})\ \varphi\rangle\right| = e^{-\pi(k^2+m^2)}$ form certainly the convergent series, the condition (2.6) is satisfied and the representation (2.5) turns into

$$Sf = \sum_{k,m} a_{km}\ (k, 2m) f.$$

with unconditional convergence. We can easily estimate that

$$a_{00} = 1 > \sum_{(k,m)\neq(0,0)} e^{-\pi(k^2+m^2)} \approx 0,18$$

and hence, since $(k, 2m)$ are isometries in this space, S and S^{-1} are bounded in $L^2(\mathbb{R})$.

2.3. Affine group.

Definition 9. Aff $= (\mathbb{R}_+ \times \mathbb{R}, \circ)$ *is the **affine group** equipped with the multiplication*

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 \cdot a_2, a_1 \cdot b_2 + b_1).$$

In this group the unit is (1,0) and the inverse of (a,b) is $\left(a^{-1}, -b/a\right)$. There is a unitary group representation U of Aff on $L^2(\mathbb{R})$ by dilations and translations, i.e.,

$$U(a,b) f(x) = a^{1/2} f(ax - b).$$

The left Haar measure μ for Aff is $a^{-2} da db$ and the right is $\mu_r = a^{-1} da db$.

One considers also the systems

$$\left(U(a^j, bk) g\ :\ j,k \in \mathbb{Z}\right).$$

The most detailed study has concentrated on the case $a = 2$, $b = 1$. However, we will follow the common tendency to consider arbitrary values of a and b. We will sometimes use the notation $D = D_a = U(a,0)$ for $f(\cdot) \mapsto a^{1/2} f(a\cdot)$ and $T = T_b = U(0,b)$ for $f(\cdot) \mapsto f(\cdot - b)$.

Proposition 1. ***(Corollary 3.3.6,*** [60]***)*** *U possesses cyclic elements, but is not irreducible.*

In [60] also an example of a non-cyclic element is constructed. One can see it this way: if we decompose $L^2(\mathbb{R})$ into the direct sum $H^2_+(\mathbb{R}) \oplus H^2_-(\mathbb{R})$, where

$$H^2_\pm(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset \mathbb{R}_\pm\}$$

then any of these subspaces is an irreducible subrepresentation of U. Thus the cyclic vectors are all these which have non-zero projections on both subspaces, while all vectors of the spaces $H^2_\pm$ are non-cyclic.

2.4. **Direct Integral.** The direct integral theory is well-known in mathematical physics and has found many comprehensive presentations: A.A. Kirillov's book [65], K. Maurin's book [73], G.W. Mackey's [72] and many others.

Let X be a locally compact separable topological space with a positive measure μ and $(H_x)_{x\in X}$ be a family of separable Hilbert spaces.

Definition 10. *Let $(e_i)_{i\in\mathbb{N}}$ be a family of functions on X having values in the appropriate spaces $(H_x)_{x\in X}$. $(e_i)_{i\in\mathbb{N}}$ is a* ***pervasive sequence*** *if*

i) for any $x \in X$ and $i \in \mathbb{N}$ the vector $e_i(x) \in H_x$;

ii) $\int_X \|g(x)\|^2 \, d\mu(x) < \infty$ for any finite linear combination g of e_i's;

iii) for $\mu-$almost all x the set of all $e_i(x)$ spans H_x.

It is known ([72], lemma on p.91) that such a sequence exists if and only if the function $X \ni x \mapsto \dim H_x$ is measurable on X, so in particular when the spaces H_x are of the same dimension.

Definition 11. *The* ***direct integral*** *of the family of Hilbert spaces $(H_x)_{x\in X}$ is the space*

$$\mathcal{H} = \int_X H_x \, d\mu(x)$$

of all functions u on X assuming values in H_x for all $x \in X$, which are ***measurable****, i.e., for a pervasive sequence $(e_i)_{i\in\mathbb{N}}$ all functions $x \mapsto \langle u(x), e_i(x)\rangle$ are measurable in the usual sense for any $i \in \mathbb{N}$, being defined up to a μ-null set and satisfying:*

$$\int_X \|u(x)\|^2_{H_x} d\mu(x) < \infty.$$

The natural norm of this space is

$$\|u\| = \left(\int_X \|u(x)\|^2_{H_x} d\mu(x)\right)^{1/2}$$

and the inner product is given by

$$\langle u, v\rangle := \int_X \langle u(x), v(x)\rangle_{H_x} d\mu(x).$$

Remark 4. *A direct sum is a special case of a direct integral when X is a finite set and μ is the counting measure.*

It is known that with this inner product $\mathcal{H}$ is a Hilbert space.

Example 5. *Let us consider $X = [0, t]$, μ - the Lebesgue measure on $[0, t]$, i.e., $\mu = dx$, and $H_x = l^2(\mathbb{Z})$ for all $x \in X$. Then, $\mathcal{H} = L^2\left([0, t], dx, l^2(\mathbb{Z})\right)$. By means of the isometry E, defined as*

$$(Eh)(x, n) = h(x + nt),$$

one can identify $L^2(\mathbb{R})$ *with* $\mathcal{H}$*. Hence we get the direct integral decomposition*

$$L^2(\mathbb{R}) = \int_{[0,t]} l^2\left(\mathbb{Z}\right)\, dx.$$

Example 6. *Let us consider* $X = [0,1] \times [0, \frac{1}{2}]$*,* μ *- the Lebesgue measure on* X*, i.e.,* $\mu = dxdt$*, and* $H_x = \mathbb{C}^2$*, with* $e_i(x) = e_i$ *being a standard orthonormal basis for* $\mathbb{C}^2$*. Then,* $\mathcal{H} = L^2\left([0,1] \times [0, \frac{1}{2}], dxdt, \mathbb{C}^2\right)$*. By means of the isometry* E *defined by*

$$Ef(x,t) := \left(\sum_n f(x+n)e^{-2\pi int}, \sum_n f(x+n)e^{-2\pi in(t+\frac{1}{2})}\right), \tag{2.7}$$

one can identify $L^2(\mathbb{R})$ *with* $\mathcal{H}$*. Hence we get the decomposition of* $L^2(\mathbb{R})$ *into the direct integral*

$$L^2(\mathbb{R}) = \int\int_{[0,1]\times[0,\frac{1}{2}]} \mathbb{C}^2\, dx.$$

Indeed, by Parseval's identity we get

$$\|Ef\|^2_{\mathcal{H}} =$$

$$= \int_{[0,1]}\int_{[0,\frac{1}{2}]} \left(\left|\sum_n f(x+n)e^{-2\pi int}\right|^2 + \left|\sum_n f(x+n)e^{-2\pi in(t+\frac{1}{2})}\right|^2\right) dtdx =$$

$$= \int_{[0,1]}\int_{[0,1]} \left(\left|\sum_n f(x+n)e^{-2\pi int}\right|^2\right) dtdx =$$

$$= \int_{[0,1]} \left\|\left(f(x+n)\right)_{n\in\mathbb{Z}}\right\|^2_{l^2(\mathbb{Z})} dx = \|f\|^2_{L^2(\mathbb{R})}.$$

Let us consider the operators acting in the Hilbert space $\mathcal{H}$ represented in the form of the direct inetgral $\int H_x\, d\mu(x)$. One distinguishes a class which acts, in some sense, on each of the 'fiber' spaces separately.

In the next chapters we shall use the following setting to deal with the decomposable operators. Let G be a locally compact group of unitary transformations of a separable Hilbert space $\mathcal{H}$ and let G have a countable basis. Given K - an abelian subgroup of G, consider the Banach algebra $\mathcal{D}_K$ generated by the operators from K, its maximal ideal space $\mathfrak{M}\left(\mathcal{D}_K\right)$ with Gelfand measure and topology and the direct integral diagonalizing the operators from K:

$$\mathcal{H} = \int_{\mathfrak{M}(\mathcal{D}_K)} H_z d\mu(z). \tag{2.8}$$

Consider a an element of $\mathcal{H}$ and $g \in K$. The direct integral yields the correspondence between the elements of $\mathcal{H}$ and functions on $\mathfrak{M}\left(\mathcal{D}_K\right)$ assuming values in H_z. Thus ($\overline{\mathcal{H}}$ being a Hilbert space containing all H_z as the subspaces)

$$\begin{aligned} a &\rightsquigarrow \widetilde{a} : \mathfrak{M}\left(\mathcal{D}_H\right) \to \overline{\mathcal{H}},\ \widetilde{a}\left(z\right) \in H_z, \\ g &\rightsquigarrow \widetilde{g} : \mathfrak{M}\left(\mathcal{D}_H\right) \to B\left(\overline{\mathcal{H}}\right),\ \widetilde{g}\left(z\right) \in End\left(H_z\right). \end{aligned}$$

Definition 12. *If* f *is a measurable essentially bounded function* $f \in L^\infty(X,\mu)$ *and there exists a family of operators* $\widetilde{T}(x) \in L(H_x)$ *such that*

$$\widetilde{(Th)}(x) = \widetilde{T}(x)\widetilde{h}(x) = f(x)\widetilde{h}(x),$$

then the operator T is `diagonal` *in the Hilbert space* $\mathcal{H}=\int H_x d\mu(x)$.

Definition 13. *If f is a continuous function on X and there exists a family of operators $\widetilde{T}(x)\in L(H_x)$ such that*

$$\widetilde{(Th)}(x)=\widetilde{T}(x)\widetilde{h}(x)=f(x)\widetilde{h}(x),$$

then the operator T is `continuously diagonal`.

Definition 14. *An operator $T\in B(\mathcal{H})$ is* `decomposable` *if there exists a family of operators $\widetilde{T}(x)\in L(H_x)$ such that*

$$\widetilde{(Th)}(x)=\widetilde{T}(x)\widetilde{h}(x)$$

for a.e. $x\in X$, all the functions

$$X\ni x\mapsto\langle\widetilde{T}(x)e_i(x),e_j(x)\rangle$$

are measurable, and the function

$$X\ni x\mapsto\|\widetilde{T}(x)\|_{H_x}$$

is essentially bounded.

Lemma 1. *(*[65]*, sec.4.5) Given a direct integral $\mathcal{H}=\int_X H_x d\mu(x)$ and let T be a decomposable operator. Then for any $u\in\mathcal{H}$, $Tu\in\mathcal{H}$ and*

$$\|Tu\|\leq ess\sup_{x\in X}\left(\|\widetilde{T}(x)\|\right)\|u\|.$$

Moreover,

$$\|T\|=\operatorname{ess}\sup_{x\in X}\left(\|\widetilde{T}(x)\|\right).$$

Example 7. *Every continuously diagonal operator is diagonal. Every diagonal operator is decomposable.*

Example 8. *Let us consider a direct integral decomposition of the representation operators for the case of Schrödinger representation and parameters $a=\frac{1}{2}$ and $b=1$. The representation operators are decomposable in the direct integral from Example 6.*

$$M_{\frac{1}{2}}f(x)=e^{\pi ix}f(x),\qquad M_{\frac{1}{2}}:(x,t)\mapsto\begin{pmatrix}0 & e^{\pi ix}\\ e^{\pi ix} & 0\end{pmatrix},$$

$$T_1f(x)=f(x-1),\qquad T_1:(x,t)\mapsto\begin{pmatrix}e^{-2\pi it} & 0\\ 0 & -e^{-2\pi it}\end{pmatrix}.$$

These fiber representations are irreducible for each (x,t) and any two of them are inequivalent.

2.5. **Groups and representations.** The representation is of `type I` if any of its primary subrepresentations contains an irreducible subrepresentation.

The group is `tame` if all its representations are of type I. In the opposite case, the group is `wild`. The commutative groups, compact groups, connected semisimple groups and linear algebraic groups are tame. The discrete groups are tame or wild. A countable discrete group is tame if and only if it contains a commutative subgroup of finite index. There is also an example of a wild Lie solvable group.

A representation π is `infinite` provided that π is equivalent to the direct sum of its infinitely many copies. A representation π is `finite` in the contrary case.

The representations π and ρ are `quasi-equivalent`, if no subrepresentation of π is disjoint from the representation ρ and vice versa,

A representation π is of `type II` if π is quasi-equivalent to some finite ρ and no subrepresentation of π is of type I.

A representation π is of `type III` if a quasi-equivalence of π and ρ implies an equivalence of π and ρ.

A representation is of type III if and only if it has no infinite subrepresentations.

If π is a representation of type II, then π is infinitely divisible.

If G is a countably infinite, discrete group, having no finite non-trivial conjugacy classess, then its left regular representation L is primary, finite and of type II.

Let G_0 be the normal subgroup of all elements belonging to finite conjugacy classes. If G/G_0 is infinite, then the regular representation of G is of type II.

The regular representation of a locally compact unimodular group has no type III part.

3. Unitary Action of a Group and Tight Frames

3.1. **Admissibility and related notions.** Let G be a locally compact group with a countable basis of unitary transformations of a separable Hilbert space $\mathcal{H}$ and let G have a countable basis. Given K - an abelian subgroup of G, consider the appropriate direct integral diagonalizing the operators from K:

$$\mathcal{H} = \int_{\mathfrak{M}(\mathcal{D}_K)} H_z d\mu(z). \tag{3.1}$$

Consider a an element of $\mathcal{H}$. The direct integral yields correspondence between the elements of $\mathcal{H}$ and functions on $\mathfrak{M}(\mathcal{D}_K)$ assuming values in H_z. Thus ($\overline{\mathcal{H}}$ being a Hilbert space containing all H_z as the subspaces)

$$\begin{array}{lll} a & \rightsquigarrow & \widetilde{a} : \mathfrak{M}(\mathcal{D}_H) \to \overline{\mathcal{H}},\ \widetilde{a}(z) \in H_z, \\ g & \rightsquigarrow & \widetilde{g} : \mathfrak{M}(\mathcal{D}_H) \to B\left(\overline{\mathcal{H}}\right),\ \widetilde{g}(z) \in End\left(H_z\right). \end{array}$$

Definition 15. *Define the set of admissible vectors for the measurable space* $(X,\ \mu)$ *as*

$$Adm\left(X, \mathcal{H}\right) = \left\{ u \in \mathcal{H} : \int_X \left|\langle u, \Phi(x)\rangle\right|^2 d\mu(x) < \infty \right\}.$$

Definition 16. *Define the set of admissible vectors for the group* G *as*

$$Adm\left(G, \mathcal{H}\right) = \left\{ u \in \mathcal{H} : \int_G \left|\langle u, gu\rangle\right|^2 dg < \infty \right\}.$$

Definition 17. *Let* $Fr_{\lambda,\mu}\left(G, \mathcal{H}\right)$ *be the set of frame generating vectors with lower bound* λ *and upper bound* μ

$$Fr_{\lambda,\mu}\left(G, \mathcal{H}\right) = \left\{ u \in \mathcal{H} : \forall y \in \mathcal{H}\ \ \lambda \left\|y\right\|^2 \le \int_G \left|\langle y, gu\rangle\right|^2 dg \le \mu \left\|y\right\|^2 \right\}.$$

The set of frame generating vectors is

$$Fr\left(G, \mathcal{H}\right) = \cup_{\lambda,\mu>0} Fr_{\lambda,\mu}\left(G, \mathcal{H}\right),$$

and let the set of tight frame generating vectors with the fixed frame bound λ *be*

$$TFr_\lambda\left(G, \mathcal{H}\right) = \left\{ u \in \mathcal{H} : \forall y \in \mathcal{H}\ \ \int_G \left|\langle y, gu\rangle\right|^2 dg = \lambda \left\|y\right\|^2 \right\}.$$

Remark 5. *Let G acts irreducibly in $\mathcal{H}$. If the set of admissible vectors is nonempty, then it is dense in $\mathcal{H}$.*

Example 9. *([1], p. 153-154) The function $u \in L^2(\mathbb{R})$ is admissible for the affine group if and only if*

$$u \in Adm\left(Aff, H_+^2(\mathbb{R})\right) \Longleftrightarrow \int_o^\infty \frac{\left|\widehat{u}(\xi)\right|^2}{\xi} d\xi < \infty.$$

Definition 18. *Let us consider a homogenous space G/K with a canonical surjection from G and a fixed section σ. Let further dv be a $G-$invariant measure for G/K. Define the set of admissible vectors with respect to a homogenous space G/K in the Hilbert space H_{z_o} as*

$$Adm\left(G/K, H_{z_o}\right) = \left\{ u \in H_{z_o} : \int_{G/K} \left| \left\langle \widetilde{u}(z_o), \widetilde{\sigma(v)}(z_o)\, \widetilde{u}(z_o) \right\rangle \right|^2 dv < \infty \right\}$$

Definition 19. *Define a to be essentially admissible* mod K *if and only if*

$$ess \sup_{z \in \mathfrak{M}(\mathcal{D}_K)} \int_{G/K} \left\langle \cdot, \widetilde{\sigma(v)}(z)\, \widetilde{a}(z) \right\rangle \left\langle \widetilde{\sigma(v)}(z)\, \widetilde{a}(z), \cdot \right\rangle dv < \infty.$$

Definition 20. *Define the set of vectors essentially admissible mod K as*

$$EssAdm\left(G/H, \mathcal{H}\right) = \left\{ u \in \mathcal{H} : ess \sup_{z \in \mathfrak{M}(\mathcal{D}_K)} \int_{G/K} \left| \left\langle \widetilde{u}(z), \widetilde{\sigma(v)}(z)\, \widetilde{u}(z) \right\rangle \right|^2 d\sigma(v) < \infty \right\}.$$

For fixed and admissible $a \in \mathcal{H}$ we consider the related frame operator

$$S_a = \int_G \langle \cdot, ga \rangle\, ga\, dg.$$

The operator equality should be understood in the weak sense.

Theorem 1. *If $a \in \mathcal{H}$ is essentially admissible mod K, the operator S_a is bounded and decomposable with respect to the direct integral (3.1).*

Proof. Recall the theorem 8.4.1. from [65]:

Theorem 2. *Suppose that the group G is locally compact and has a countable basis and also the Hilbert space $\mathcal{H}$ is separable. Then an arbitrary commutative C^*-subalgebra $\mathcal{D}_o$ of $\mathfrak{A}_G^{'}$ gives a decomposition of the representation G as a direct integral*

$$g = \int_{\mathfrak{M}(\mathcal{D}_o)} \widetilde{g}(z) d\mu(z) \quad for\ all\ g \in G,$$

where μ is the Gelfand measure of $\mathfrak{M}(\mathcal{D}_o)$.

Analogously we show that if the operator commutes with the von Neumann algebra $\mathfrak{A}$, then it decomposes with respect to any of its C^*-subalgebras. As we have observed in Remark 2, the frame operator S_a for any admissible $a \in \mathcal{H}$ commutes with $g \in G$, so also with the whole $\mathfrak{A}_G$, thus belonging to $\mathfrak{A}_G^{'}$. So it has a direct integral decomposition with respect to any commutative C^*-subalgebra in $\left(\mathfrak{A}_G^{'}\right)^{'} = \mathfrak{A}_G$. In particular, the Banach algebra generated by the abelian subgroup K of G can serve us to this aim. Now the assertion of the theorem follows. ∎

Theorem 3. *If $a \in \mathcal{H}$ is essentially admissible mod K, and the operators $\widetilde{S_a}(z)^{-1}$ are uniformly bounded from below by any positive number, then Ga is a frame [$a \in Fr\,(G,\mathcal{H})$] and its upper, resp. lower, frame bounds are:*

$$M = ess \sup_{z\in\mathfrak{M}(\mathcal{D}_K)} \left\|\widetilde{S_a}(z)\right\|,$$

$$m = ess \inf_{z\in\mathfrak{M}(\mathcal{D}_K)} \left\|\widetilde{S_a}(z)^{-1}\right\|^{-1}.$$

Proof. follows from Lemma 1. □

Remark 6. *Notify that these bounds are exact.*

Theorem 4. *If Ga is a frame in $\mathcal{H}$ and K is an arbitrary abelian subgroup of G and we consider a direct integral $\mathcal{H} = \int_{\mathfrak{M}(\mathcal{D}_K)} H_z$, then*

- *a is essentially admissible* mod K,
- *S_a is bounded and $\widetilde{S_a}(z)$ are uniformly bounded by $\|S_a\|$,*
- *S_a^{-1} is bounded and $\widetilde{S_a}(z)^{-1}$ are uniformly bounded by$\left\|S_a^{-1}\right\|$,*
- *The quantities*

$$M = \|S_a\| = ess \sup_{z\in\mathfrak{M}(\mathcal{D}_H)} \left\|\widetilde{S_a}(z)\right\|,$$

$$m = \left\|S_a^{-1}\right\|^{-1} = ess \inf_{z\in\mathfrak{M}(\mathcal{D}_H)} \left\|\widetilde{S_a}(z)^{-1}\right\|^{-1}$$

are the exact frame bounds for Ga in $\mathcal{H}$.

Theorem 5. *The system Ga is a tight frame in $\mathcal{H}$ [$a \in TFr(G,\mathcal{H})$] if and only if for almost each $z \in \mathfrak{M}\,(\mathcal{D}_K)$ the operator*

$$\widetilde{S_a}(z) = \lambda(z) Id_{H_z}$$

and the function $\lambda : \mathfrak{M}\,(\mathcal{D}_K) \to \mathbb{C}$ is almost everywhere constant.

Proof. follows from the fact that the identity operator has the only decomposition (unique up to the null-measure set) to the family of identity operators of appropriate spaces H_z. □

There is another version of this theorem

Theorem 6. *The system Ga is a tight frame in $\mathcal{H}$ [$a \in TFr(G,\mathcal{H})$] if and only if for almost each $z \in \mathfrak{M}\,(\mathcal{D}_K)$ the system $\widetilde{G}(z)\widetilde{a}(z)$ is a tight frame for H_z [$\widetilde{a}(z) \in TFr(\widetilde{G}(z), H_z)$] and the bounds of these frames are essentially equal.*

3.2. **Formulas.** The notions described in the previous section satisfy the following relations. Some of them are obvious and some of them follow from non-trivial theorems appearing in the sequel. They also summarize some reults from the previous section. As in previous section H is a separable Hilbert space, G is a unitary transformations group in H, G is locally compact and has a countable basis.

$$\begin{aligned} u \in Fr\,(G,\mathcal{H}) \quad &\Longleftrightarrow \\ \Longleftrightarrow \quad &\exists \lambda,\mu > 0\ : u_z \in Fr_{\lambda,\mu}\,(G/K, H_z)\ \ for\ almost\ all\ z \in \mathfrak{M}\,(\mathcal{D}_K) \end{aligned}$$

$$u \in TFr\,(G,\mathcal{H}) \iff$$
$$\iff \underset{K-abelian}{\exists\,(\forall)\,K \leq G}\; \exists \lambda > 0\ : u_z \in TFr_\lambda\,(G/K, H_z)\ \ for\ almost\ all\ z \in \mathfrak{M}\,(\mathcal{D}_K)\,.$$

3.3. **Orthogonality relations - first step.** Let $\mathcal{H}$ be a separable Hilbert space and let G be a locally compact group of unitary operators in $\mathcal{H}$ and let G has a countable basis of topology, i.e., is metrizable. For the unimodular locally compact groups (finite groups, compact groups, and the Heisenberg group belong to this class) we have

Theorem 7. ***(Sec. 9.3, Problem 2,*** [65] ***)*** *Let $\mathcal{H}$ be a separable Hilbert space and let G be a unimodular locally compact group of unitary mappings in $\mathcal{H}$. If G acts irreducibly in $\mathcal{H}$ and a belongs to the set of admissible vectors, then there exists a constant c depending on G, dg and $\mathcal{H}$ such that*

$$\int_G \langle\cdot, ga\rangle\, ga\ dg = c\,\langle a,a\rangle\, Id_{\mathcal{H}}. \tag{3.2}$$

The convergence of the integral is in the weak sense.

According to the definition, $c\langle x,x\rangle$ is the frame bound of this system. The identity (3.2) is in mathematical physics referred to as *'resolution of identity'* and it is an important property of *'coherent states'* - systems of functions arising under the action of a group. See the monograph [66] or the survey article [2] for the detailed study on the subject.

Remark 7. *If there exists one admissible vector, then the set of such vectors is dense in $\mathcal{H}$. When the group is unimodular, as in Theorem 7, this set coincides with all $\mathcal{H}$. Also for the affine group, which is not unimodular, (3.2) holds, but not for all $x \in \mathcal{H}$.*

Theorem 8. *Duflo-Moore'75 (Thm. 8.2.1 in* [1]*) Let $\mathcal{H}$ be a separable Hilbert space and let G be a locally compact group of unitary mappings in $\mathcal{H}$. If G acts irreducibly in $\mathcal{H}$ and a belongs to the set of admissible vectors, then there exists a densely defined operator C (possibly unbounded) such that*

$$\int_G \langle\cdot, ga\rangle\,\langle ga, y\rangle\, dg = \langle Ca, Ca\rangle\, Id_{\mathcal{H}}.$$

Proposition 2. *Let $\mathcal{H}$ be a separable Hilbert space and let G be a unimodular locally compact group of unitary mappings in $\mathcal{H}$. Let G act irreducibly in $\mathcal{H}$ and let a belong to the set of admissible vectors. If $J : \mathcal{H} \to \mathcal{H}$ is a unitary mapping, then*

$$\int_G \langle\cdot, ga\rangle\, JgJ^{-1}a\ dg\ = c\,\langle a, Ja\rangle\ J.$$

Proposition 3. *Let $\mathcal{H}$ be a separable Hilbert space and let G be a unimodular locally compact group of unitary mappings in $\mathcal{H}$. Let G act irreducibly in $\mathcal{H}$ and let a belong to the set of admissible vectors. Let $J : \mathcal{H} \to \mathcal{H}$ be a unitary mapping and let $\mathcal{H}_+ = \mathcal{H} \oplus \mathcal{H}$ and any vector $a \in \mathcal{H}_+$ has a decomposition into*

$$a = (a_1, a_2)\,, \quad a_i \in \mathcal{H}.$$

Let G' be a group of unitary mappings in $\mathcal{H}_+$ such that for any $g \in G$:

$$g' = \begin{pmatrix} g & 0 \\ 0 & JgJ^{-1} \end{pmatrix}.$$

Then the frame operator S related with the system $G'a$ has the following matrix representation in the orthogonal sum $\mathcal{H}_+ = \mathcal{H} \oplus \mathcal{H}$

$$S = \begin{pmatrix} c\langle a_1, a_1\rangle Id & c\langle a_1, Ja_2\rangle J \\ c\langle Ja_1, a_2\rangle J^{-1} & c\langle a_2, a_2\rangle Id \end{pmatrix}.$$

Thus the system $G'a$ is a tight frame iff $c\langle a_1, a_1\rangle = c\langle a_2, a_2\rangle$ and $\langle Ja_1, a_2\rangle = \langle a_1, Ja_2\rangle = 0$.

Remark 8. *The theorem is certainly valid also for any finite number of summands with the obvious modifications.*

Remark 9. *Let us consider the orthogonal projection $P : \mathcal{H}_+ \to \mathcal{H}$ such that $Pa = a_1 + a_2$. In this special case the system $PG'a$ is a tight frame with bound λ if and only if $c\langle a_1, a_1\rangle = \lambda$ and $\langle Ja_1, a_1\rangle = 0$ and J is different from identity.*

Is it possible that a type I factor being a direct sum of infinitely (but countably many) copies of irreducible representation has a non-empty $TFr(G, \mathcal{H})$? Note that if such a vector existed, his projections onto all summands would be of the same length, so his length will be infinite - **contradiction**.

Conclusion 1. *(H. Führ): For Ga to be a tight frame it is necessary that a number of irreducible equivalent summands in the direct integral decomposition is almost everywher finite.*

Since the existence of tight frame generating vector and a *frame* generating vector is equivalent, the same condition is necessary for Ga to be a frame.

3.4. **The motivation.** When a unitary representation acts irreducibly, the frame operator related to any admissible vector is a scalar multiple of identity. (Since a group and a Haar measure are fixed throughout this chapter, the frame operator depends only on a vector from the representation space, what is denoted by a subscript.)

For unimodular groups the term 'admissible' can be relaxed - the set of the admissible vectors coincides with the Hilbert space $\mathcal{H}$. For non-unimodular groups acting irreducibly it is at least dense in $\mathcal{H}$. The second thing related to that is square-integrability, i.e., the existence of at least one admissible vector. When switching from the group to its subgroup and from the space to an invariant subspace, square-integrablility is preserved.

In turn, reducible representations, e.g., of subgroups of Heisenberg group in $L^2(\mathbb{R})$ and of finite groups in finite-dimensional spaces, require a use of a direct sum of irreducible subrepresentations (finite groups) and a decomposition into a direct integral (infinite case).

Sometimes, the representation has no irreducible subrepresentation, e.g., if any invariant subspace decomposes to a direct sum of 2 invariant subspaces. The example is the real additive group $\mathbb{R}$ acting unitarily in $L^2(\mathbb{R})$ by

$$(t)f(x) = e^{2\pi itx} f(x),$$

where all invariant subspaces are of the form $\mathcal{H}_E = \{f \in L^2(\mathbb{R}) : supp\, f \subset E\}$. Thus the use of direct direct sum is impossible and the other tool as direct integral is needed.

In the theory of decomposable operators it is settled (see, e.g., [72] - section 2.5, p. 92-124) that given a von Neumann algebra $\mathcal{R}$, $\mathcal{D}$ being its center, let $\mathcal{D}_o$ be a Banach commutative algebra dense in $\mathcal{D}$ in the strong operator topology, one decomposes any operator from $\mathcal{R}$ with respect to the space $\mathfrak{M}(\mathcal{D}_o)$ of maximal ideals in $\mathcal{D}_o$. The Hilbert space $\mathcal{H}$ is then the direct integral of spaces $\mathcal{H}_x$ indexed by elements $x \in \mathcal{M}(\mathcal{D}_o)$.

As $\mathcal{D}$ and $\mathcal{R}$ are von Neumann algebras (i.e., $\mathcal{R} = \mathcal{R}^{''}$), then

$$\mathcal{D} = Z(\mathcal{R}) = \mathcal{R} \cap \mathcal{R}' = \mathcal{R}' \cap \mathcal{R}'' = Z(\mathcal{R}')$$

and the same space $\mathcal{M}(\mathcal{D}_0)$ can serve for the decomposition of operators from $\mathcal{R}'$.

Choosing $\mathcal{R} = \mathfrak{A}_G$, i.e., the von Neumann algebra generated by the representation operators and closed in the strong operator topology, observe that the representation operators and frame operators are decomposable in the same direct integral with respect to $\mathcal{M}(\mathcal{D}_o)$.

In this chapter we deal first with irreducible representations in unimodular and non-unimodular cases, then with arbitrary representations for finite groups, to end with a locally compact non-abelian group and its strongly continuous unitary representation.

3.5. **Finite & compact groups - first examples.**

Corollary 1. *The orbit of every vector under irreducible action of the compact group is a tight frame. The frame bound is related to the dimension of the representation.*

Remark 10. *All finite groups are compact. The number $\frac{\#G}{\dim H}$ is known to be an integer for an irreducible representation (see* [65] *Sec.11.1 Thm.4). In* [76] *it was shown that for a nilpotent group the Haar measure can be normalized so as to get that all frame bounds are integers.*

Example 10. *Consider the group $SU(2)$ of unitary transformations of $\mathbb{C}^2$ with the determinant equal to 1. $SU(2)$ is a non-cyclic group acting irreducibly on $\mathbb{C}^2$ and transitively on sphere. The orbit of the vector $(1,0)$ is then the whole sphere and it is obviously a tight frame with bound $\frac{1}{2}$.*

Example 11. *The case of rotations, i.e., the cyclic group of operators $G \simeq \mathbb{Z}_n$ being multiplications of the complex numbers $\mathbb{C}$ by numbers:*

$$e^{2\pi i \frac{k}{n}} \; for \; k = 0, 1, ..., n-1;$$

This group action is irreducible and for any vector $x \in \mathbb{C}$ the relevant frame operator S is equal to:

$$Sy = \sum_{k \in \mathbb{Z}_n} y \, \overline{x \, e^{2\pi i \frac{k}{n}}} x \, e^{2\pi i \frac{k}{n}} = \left(\sum_k e^{-2\pi i \frac{k}{n}} \, e^{2\pi i \frac{k}{n}} \right) y \, \overline{x} \, x = n \, y \, |x|^2.$$

The frame bound is then $n|x|^2$.

Example 12. *The case of rotations, i.e., the group of operators $G \simeq \mathbb{Z}_n$ acting in $\mathbb{C}^2$ as matrices:*

$$\begin{pmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{pmatrix}$$

for $k = 0, 1, ..., n-1$. One can check directly that the orbit of the vector (1,0) under the action of this group is a tight frame (see e.g. [90]*). However, this action is reducible, because it is a two-dimensional representation of the abelian group isomorhic to $\mathbb{Z}_n$. Observe also that the bound of this frame is $\frac{n}{2}$. More generally, we get the tight frame for any vector with real coordinates. The eigenvectors for these matrices are (1,i) and (1,-i). The orthogonal projections of each vector on the eigenvectors Have equal lengths.*

Example 13. *The case of the symmetric group S_3 acting on $\mathbb{C}^3$ by permutation of indices. The hyperplane*

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_2 + z_3 = 0\}$$

is an invariant subspace and the representation restricted to it is irreducible. Let

$$u = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right) \quad \text{and} \quad v = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right).$$

The orbit of u consists of three vectors, each of them repeated twice, while the orbit of v consists of three pairs of a vector and its opposite. So they are not isometric. Since the representation is irreducible, both orbits are tight frames. Thus, we get an example of two non-isometric orbits that are tight frames for a non-abelian group with an irreducible action.

3.6. **Orthogonality relations - second step.** We shall consider now

Notation 3. *Let us denote for the use of the next theorem the abstract algebraic locally compact group by $\mathfrak{G}$, its element by $\mathfrak{g}$ and the Haar measure on it $d\mathfrak{g}$.*

Theorem 9. *If two groups of unitary mappings of the separable Hilbert space $\mathcal{H}$, being G_1 and G_2, are the homomorphic images of the same abstract algebraic locally compact group $\mathfrak{G}$ under the homomorphisms φ_1 and φ_2, respectively, if further both these groups act irreducibly in $\mathcal{H}$ and*

$$\int_{\mathfrak{G}} |\langle x, \varphi_i(\mathfrak{g})a\rangle|^2 \, d\mathfrak{g} < \infty$$

for some $a \in \mathcal{H}$, all $x \in \mathcal{H}$ and $i = 1, 2$, then

$$\int_{\mathfrak{G}} \langle x, \varphi_1(\mathfrak{g})a\rangle \langle \varphi_2(\mathfrak{g})a, x\rangle \, d\mathfrak{g} = 0.$$

Remark 11. *The constant c in (3.2) is in the compact case equal to the ratio of the volume of the group in the sense of Haar measure to the dimension of representation. In the non-compact case, both of this quantities are infinite, but their ratio c remains finite. The quantity c^{-1} is called a* formal dimension *of the representation* [65], *p.139.*

It is well known in the representation theory that the orthogonality relations we have used above have their counterparts for non-unimodular groups established by Duflo & Moore in 1976 ([37] or [15]). They are expressed in terms of unbounded, densely defined operators.

Theorem 10. *(Thm. 8.2.1 & Cor. 8.2.2 in* [1]*) Let G act irreducibly in $\mathcal{H}$. There exists a unique, positive, self-adjoint, invertyible operator C in $\mathcal{H}$, whose domain is dense in $\mathcal{H}$ and equla to the set of admissible vectors. If a and $a\prime$ be any nonzero admissible vectors and y, u be any two vectors in $\mathcal{H}$, then*

$$\int_G \langle y, ga\rangle \langle ga', u\rangle = \langle Ca', Ca\rangle \langle y, u\rangle.$$

Let us reformulate this result quoted also in Prop. 4.2 from [15] for the regular representation in terms of Haar measure instead of the operators:

Theorem 11. *Let f_i, h_i $(i = 1, 2)$ be vectors in a subspace V of $L^2(G)$ on which the left regular representation acts irreducibly. Then there exists a constant c such that:*

$$\langle Sh_1, f_1\rangle := \int_G \langle h_1, L_g h_2\rangle \langle L_g f_2, f_1\rangle\, dg = c\langle f_2, h_2\rangle_{L^2(G,\mu_r)} \langle h_1, f_1\rangle_{L^2(G,dg)},$$

where dg and μ_r are respectively the left and the right Haar measure on G.

3.7. **Toy model.** Let G be a finite group of unitary operators in the finite-dimensional $\mathcal{H}$ and let

$$\mathcal{H} = \oplus_{i\in I} \oplus_{j\in J_i} H_{ij}$$
$$g = \oplus_{i\in I} \oplus_{j\in J_i} g_{ij}$$

be its decomposition in a direct sum of subspaces G acts irreducibly when restricted to. Let P_{ij} be an orthogonal projection of $\mathcal{H}$ onto H_{ij}. We shall group them in the equivalence classes, i.e., $\exists_{J\in L(\mathcal{H})}\ \forall_{g\in G}\ JgP_{i_1j_1}J^{-1} = gP_{i_2j_2}$ if and only if $i_1 = i_2$. For each equivalence class and a set J_i fix some $j_0 \in J_i$ - an arbitrrary element of this set. Then for any $j \in J_i$ let $L^i_j : H_{ij} \to H_{ij_o}$ be a unitary operator such that $\forall_{g\in G}\ L^i_j gP_{ij} \left(L^i_j\right)^{-1} = gP_{ij_o}$. Since the action of G is irreducible on each of H_{ij} then from Schur's lemma any L^i_j is uniquely defined up to a constant of modulus 1.

Theorem 12. *For the vector $a \in H$ the orbit Ga is a tight frame(or, equivalently,*

$$\langle Sy, u\rangle := \int_G \langle y, ga\rangle\langle ga, u\rangle dg$$

is a scalar multiple of identity) if and only if, for any $i \in I$ and for any $j, j' \in J_i$,

$$c_i\langle L^i_j a_{ij}, L^i_{j'} a_{ij'}\rangle_H = \lambda\delta_{jj'}$$

holds.

Notation 4. *c_i is a constant in the orthogonality relations for gP_{ij} that is irreducible in H_{ij} and its type of equivalence does not depend on j. c_i depends only on the equivalence class of a representation. It follows from the proof that $c_i = \frac{\#G}{dimH_{ij}}$. It is known in the standard representation theory that it is an integer.*

Proof. Applying subsequently decomposition of $\mathcal{H}$ into the above described direct sum and the orthogonality relations for unimodular groups [76], we get:

$$\langle Sy, u\rangle := \sum_{g\in G} \langle y, ga\rangle \ \langle ga, u\rangle =$$
$$= \sum_{g\in G} \sum_{i,j} \langle y_{ij}, gP_{ij}a_{ij}\rangle_{H_{ij}} \sum_{i',j'} \langle gP_{i'j'}a_{i'j'}, u_{i'j'}\rangle_{H_{i'j'}} =$$
$$= \sum_{i,j} \sum_{j'} c_i \ \langle L^i_{j'} a_{ij'}, L^i_j a_{ij}\rangle_{H_{ij_0}} \ \langle L^i_{j'} y_{ij'}, L^i_j u_{ij}\rangle_{H_{ij_0}},$$

where c_i is the constant in ortgogonality relations. We are interested in the case when the above expression is equal to $\lambda \sum_{ij}\langle y_{ij}, u_{ij}\rangle$. It happens if and only if for any $i \in I$ and for any $j, j' \in J_i$

$$c_i\langle L^i_j a_{ij}, L^i_{j'} a_{ij'}\rangle_{H_{ij_0}} = \lambda\delta_{jj'},$$

□

Consequences of this fact are that for a tight frame generating vector:
1. The lengths of projections to the equivalent subrepresentations are the same.
2. The lengths of projections to the non-equivalent subrepresentations are inversely proportional to the quantities $\frac{\#G}{dim H_{ij}}$ for appropriate subrepresentations.
3. The projections to the equivalent subrepresentations are orthogonal after the use of the equivalence operator. We shall show in the sequel that much of this holds for the arbitrary locally compact groups.

3.7.1. *Finite-discrete Heisenberg group - example.*

Example 14. *Consider the group $G = (\mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_3, \cdot)$ with the multiplication $\cdot$ defined by:*

$$(k, l, m) \cdot (p, q, r) = (k + p \bmod 3, l + q \bmod 6, m + r + lp \bmod 3).$$

The finite-discrete analogue of Schrödinger representation is the representation π acting in $\mathbb{C}^{12} = l^2(\mathbb{Z}_{12})$ by means of operators M and T (ε is a 6th degree root of unity):

$$\pi(k, l, m) := \varepsilon^{2m} T^k M^l.$$

The operators T and M, which are analogues of translation and modulation operators in the finite-discrete case, are defined by

$$Ta(n) = a(n + 4 \bmod 12), \tag{3.3}$$

$$Ma(n) = \varepsilon^n a(n), \quad n \in \mathbb{Z}_{12}. \tag{3.4}$$

They satisfy the following relations:

$$TM = \varepsilon^4 MT, \tag{3.5}$$

$$T^3 = M^6 = \mathrm{Id}. \tag{3.6}$$

In this case $Z(G) = \{Id, -Id\}$. Also $\mathfrak{A}_{z(G)} = Z(\mathfrak{A}_G) = \{\lambda \mathrm{Id} : \lambda \in \mathbb{C}\}$.

One checks that T preserves the subspaces:

$$V_0 = l^2(\{0, 4, 8\}),$$
$$V_1 = l^2(\{1, 5, 9\}),$$
$$V_2 = l^2(\{2, 6, 10\}),$$
$$V_3 = l^2(\{3, 7, 11\}).$$

The subspaces can be identified with the canonical $\mathbb{C}^3$.Let P_i be the orthogonal projection on V_i. Moreover, in these subspaces T has eigenvectors (expressed in the coordinates of the appropriate V_i):

$$(1, 1, 1),$$
$$(1, \varepsilon^2, \varepsilon^4),$$
$$(1, \varepsilon^4, \varepsilon^2)$$

M permutes these vectors up to a factor with modulus 1. Thus, V_i are subspaces of irreducible subrepresentations of G. One notes that $T_0 = TP_0$ and $M_0 = MP_0$ being the restrictions of T and M to the subspace V_0 read as

$$T_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^4 & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}.$$

Their counterparts in V_2 *read as*

$$T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}.$$

The operator

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

establishes a unitary equivalence between (T_0, M_0, V_0) *and* (T_2, M_2, V_2) *and also between* (T_1, M_1, V_1) *and* (T_3, M_3, V_3). *Thus,*

$$\pi = \pi_0 \oplus \pi_1 \oplus \pi_2 \oplus \pi_3,$$

where π_i *is the restriction of* π *to* V_i *and* $\pi_0 \sim \pi_2$ *and* $\pi_1 \sim \pi_3$. *To see that* π_0 *and* π_1 *are non-equivalent, it is enough to observe that*

$$M_0^3 = \mathrm{Id} \quad \text{and} \quad M_1^3 = -\mathrm{Id}.$$

If J *establishes the equivalence between* (T_0, M_0, V_0) *and* (T_1, M_1, V_1), *then*

$$JM_0 = M_1 J, \quad JM_0^3 = M_1^3 J, \quad J = -J, \quad J = 0.$$

From Theorem 12 one has that $(\varepsilon^{2m} T^k M^l a)_{(k,l,m)\in G}$ *(or, equivalently,* $(T^k M^l a)_{(k,l)\in \mathbb{Z}_3\times\mathbb{Z}_6}$*) is a tight frame in* $\mathbb{C}^{12}$ *if and only if*

$$a = (x, u, y, v) \in V_0 \oplus V_1 \oplus V_2 \oplus V_3,$$

while

$$\langle x, Jy\rangle = \langle u, Jv\rangle = 0$$

and $\|x\| = \|y\| = \|u\| = \|v\| = 1$. *In other words,* a *satisfies the following system of equations:*

$$|a_0|^2 + |a_4|^2 + |a_8|^2 = 1,$$

$$|a_1|^2 + |a_5|^2 + |a_9|^2 = 1,$$

$$|a_2|^2 + |a_6|^2 + |a_{10}|^2 = 1,$$

$$|a_3|^2 + |a_7|^2 + |a_{11}|^2 = 1,$$

$$a_0\overline{a_{10}} + a_4\overline{a_2} + a_8\overline{a_6} = 0,$$

$$a_1\overline{a_{11}} + a_5\overline{a_3} + a_9\overline{a_7} = 0.$$

The geometry of this set is very interesting. It can be considered as the set of the unit length vectors in a product of two tangent bundles of spheres $S^5 \in \mathbb{C}^3$. *The behaviour of the representation operators and the form of the tight frame generating vectors corresponds to the case* $ab = \frac{2}{3}$ *in the Gabor rational setting.*

3.8. **Locally compact group.** We deal in this section with a locally compact group G, which is secondary countable, has a strongly continuous unitary representation π in a separable Hilbert space $\mathcal{H}$.

Given any direct integral decomposition $\mathcal{H} = \int_X H_x d\mu(x)$ and for a decomposable $T \in B(\mathcal{H})$, we claim that:
1) $\|T\| = \operatorname{ess\,sup}_x \|T_x\|$,

2) if T is invertible, $\|T^{-1}\|^{-1} = \operatorname{ess\,inf}_x \|T_x^{-1}\|^{-1}$,

3) $T = \lambda\,\mathrm{Id}$ iff $T_x = \lambda\,\mathrm{Id}_{\mathcal{H}_x}$ a.e. for $x \in X$.

Let $\mathfrak{A}_G$ be a factor and $\mathcal{D}_{max}$ be a maximal commutative subalgebra in $\mathfrak{A}'_G$; $\mathcal{D}_o$ be a Banach algebra strongly dense in $\mathcal{D}_{max}$. Then G decompose with respect to $\mathfrak{M}(\mathcal{D}_0)$ into equivalent and irreducible components (cf. Mautner theorem in Appendix 7.2).

For any $x \in \mathfrak{M}(\mathcal{D}_0)$ let L^x be a unitary intertwinning operator (unique up to a scalar multiple with module 1) between π_x and π_{x_0} for some fixed $x_o \in \mathfrak{M}(\mathcal{D}_o)$ acting on the appropriate fibers of integral decomposition. One can even show that there is one decomposable operator L which acts on all fibers as L^x (cf. Theorem 2.7 [72] + observe that both functions $x \mapsto \pi_x$ and $x \mapsto \pi_{x_0}$ are measurable - the latter as a constant function). Observe that for the representation of type I the factor coming from the central decomposition can contain only countably many copies of π_{x_o}. Denote the spaces of subrepresentations π_x by H_x. In these terms we have:

Proposition 4. *If $\mathfrak{A}_G$ is a factor and one (so all) π_x is square-integrable on the respective fiber, then the frame operator S_a is bounded and equals to*

$$S_a = \int_{\mathfrak{M}(\mathcal{D}_0)^2} c_x \ \langle L^{x'}\widetilde{a}(x'), L^x\widetilde{a}(x)\rangle_{H_{x_o}} \ \left[L^{x'}\right]^{-1} L^x \ dx\, dx'.$$

Proof. Note that since $\mathfrak{A}_G$ is a factor, its center is trivial and the decomposition with respect to $\mathfrak{M}(\mathcal{D}_o)$ is into equivalent and irreducible representations. For the factors of type I the space $\mathfrak{M}(\mathcal{D}_o)$ is discrete and countable.

$$\langle Sy, u\rangle = \int_G \langle y, ga\rangle \ \langle ga, u\rangle \, dg =$$
$$= \int_{\mathfrak{M}(\mathcal{D}_0)} \int_{\mathfrak{M}(\mathcal{D}_0)} \int_G \langle \widetilde{y}(x), \widetilde{g}(x)\widetilde{a}(x)\rangle_{H_x} \, dx \ \langle \widetilde{g}(x')\widetilde{a}(x'), \widetilde{u}(x')\rangle_{H_{x'}} \, dx' =$$
$$= \int_{\mathfrak{M}(\mathcal{D}_0)^2} c_x \ \langle L^{x'}\widetilde{a}(x'), L^x\widetilde{a}(x)\rangle_{H_{x_o}} \ \langle L^x\widetilde{y}(x), L^{x'}\widetilde{u}(x')\rangle \, dx\, dx',$$

where c_x is a constant appearing in orthogonal relations Thm. 7 and K_x is the unitary equivalence operator between (π_x, H_x) and (π_{x_0}, H_{x_0}) ($x_0 \in \mathfrak{M}(\mathcal{D}_0)$ is fixed); K_x is unique up to the scalar multiple of modulus 1. The quantity c_x is an analogue of $\frac{\#G}{dim H_{ij}}$ for a locally compact group.

The part in parantheses follows from the fact that all subrepresentations are equivalent. □

The classical result ([65], [72]) in the representation theory is that for *tame* groups one has a decomposition into the direct integral of a very natural form.

Start with the decomposition along $\mathcal{Z}_0$ being a C^*-subalgebra of $Z(\mathfrak{A}) = \mathfrak{A} \cap \mathfrak{A}'$ closed in the norm operator topology and dense in $Z(\mathfrak{A})$ in the strong operator topology ($\mathfrak{A} := \mathfrak{A}_G$) and $\mathfrak{M}(\mathcal{Z}_0)$ being its maximal ideal space with a basic measure μ.

Theorem 13. ***(Thm. 8.3, p.128,*** [65]***)*** *Let G be a locally compact tame group of unitary operators acting in a separable Hilbert space and with a countable basis. Then there exists such a direct integral with respect to the space $\mathfrak{M}(\mathcal{Z}_0)$*

$$\mathcal{H} = \int_{\mathfrak{M}(\mathcal{Z}_0)} \bigoplus_{k=1}^{n(z)} P_k^z \, H_z \; d\mu(z)$$

that all elements $g \in G$ can be decomposed as

$$g = \int_{\mathfrak{M}(\mathcal{Z}_0)} \bigoplus_{k=1}^{n(z)} P_k^z \, \widetilde{g}(z) \; d\mu(z),$$

where the representations $P_k^z \widetilde{g}(z)$ are irreducible in $P_k^z H_z$. Moreover, $P_k^z \widetilde{g}(z)$ is equivalent to $P_{k'}^{z'} \widetilde{g}(z')$ if and only if $z = z'$, so the integral decomposition is into primary and strongly disjoint components.

Remark 12. *The 'moreover' part of this theorem follows from Mauttner result (see Chapter 8, Appendix).*

Using this theorem we shall find the form of the frame operator, when the algebra $\mathfrak{A}_G$ is not a factor. We study the two-level decomposition: 1. by a center of the algebra - into the factors (cf. von Neumann theorem in Appendix 7.2); 2. by a maximal commutative subalgebra of the commutant - into the irreducible representations (cf. Mautner theorem in Appendix 7.2).

For each $z \in \mathfrak{M}(\mathcal{Z}_o)$ the representations $P_k^z \widetilde{g}(z)$ and $P_{k;}^z \widetilde{g}(z)$ are equivalent. For each equivalence class fix some $j_o \leq n(z)$. Then for any $j \leq n(z)$ let $L_k^z : P_k^z H_z \to P_{j_o}^z H_z$ be a unitary operator (unique up to the multiplicative constant of modulus 1) such that

$$\forall_{g \in G} \; L_k^z \widetilde{g}(z) P_k^z \, (L_k^z)^{-1} = \widetilde{g}(z) P_{j_o}^z.$$

Then we argue as in the preceding section to obtain:

Theorem 14. *If $a \in \mathcal{H}$, and $\widetilde{a}(z)$ is admissible on H_z for almost all z's, the operator S_a is bounded and decomposable with respect to the direct integral (3.1) and*

$$\widetilde{S_a}(z) = \sum_{k=1}^{n(z)} \sum_{k'=1}^{n(z)} c_z \langle L_{k'}^z P_{k'}^z \widetilde{a}(z), L_k^z P_k^z \widetilde{a}(z) \rangle_{H_{k_o z}} \;\; P_{k'}^z \, [L_{k'}^z]^{-1} \, L_k^z P_k^z.$$

The integral converges weakly.

Proof. Decomposability with respect to $\mathfrak{M}(\mathcal{Z}_0)$ follows from Theorem 13.

$$\begin{aligned}
\langle Sy, u\rangle \quad : \quad &= \int_G \langle y, ga\rangle \ \langle ga, u\rangle \, dg \\
&= \int_G \int_{\mathfrak{M}(\mathcal{Z}_o)} \sum_{k=1}^{n(z)} \langle P_k^z \widetilde{y}(z),\ P_k^z \widetilde{g}(z) P_k^z \widetilde{a}(z)\rangle_{P_k^z H_z}\, dz \ \times \\
&\quad \times \int_{\mathfrak{M}(\mathcal{Z}_o)} \sum_{k'=1}^{n(z')} \langle P_{k'}^z \widetilde{g}(z') P_{k'}^z \widetilde{a}(z'),\ P_{k'}^z \widetilde{u}(z')\rangle_{P_{k'}^z H_{z'}}\, dz' \\
&= \int_{\mathfrak{M}(\mathcal{Z}_o)} \sum_{k=1}^{n(z)} \sum_{k'=1}^{n(z)} c_z \langle L_{k'}^z P_{k'}^z \widetilde{a}(z), L_k^z P_k^z \widetilde{a}(z)\rangle_{H_{k_o z}} \times \\
&\qquad\qquad \times \langle L_k^z P_k^z \widetilde{y}(z), L_{k'}^z P_{k'}^z \widetilde{u}(z)\rangle_{H_{k_o z}}.
\end{aligned}$$

The last equality follows from orthogonality relations for irreducible square-integrable equivalent/ non-equivalent representations. □

Comment to the table 3.1. It was proved in [34] and [64] that if $\mathfrak{A} := \mathfrak{A}_{Gab(a,b)}$, then $\mathfrak{A}' = \mathfrak{A}_{Gab(b^{-1},a^{-1})}$.

TABLE 1. **Algebraic properties of the operator representation algebra.**

	algebraic relations	consequences
Gabor ($ab = 1$)	$\mathfrak{A}$ is commutative and has a cyclic vector $\Rightarrow \mathfrak{A} = \mathfrak{A}'$, $\mathfrak{A} = \mathfrak{A}' = Z(\mathfrak{A}) = \mathcal{D}_{max}$	any irred. (1-dim.) happens once; all are inequivalent.
Gabor ($ab = 1/N$)	$\mathfrak{A}'$ is commutative, $\mathfrak{A}' \subset \mathfrak{A}$, $\mathfrak{A}' = Z(\mathfrak{A}) = \mathcal{D}_{max}$	any irred. (N-dim.) happens once; all are inequivalent.
Gabor $Gab(a,b)$ ($ab = p/q$)	$\mathfrak{A} = \mathfrak{A}_{Gab(a,b)}$, $\mathfrak{A}' = \mathfrak{A}_{Gab(b^{-1},a^{-1})}$, $Z(\mathfrak{A}) = \mathfrak{A}_{Gab(aq,bq)}$, $\mathcal{D}_{max} = \mathfrak{A}_{Gab(b^{-1},bq)}$, $\mathcal{D}_{max} : Z(\mathfrak{A}) = p$, for $aq = pb^{-1}$.	any irred. (q-dim.) happens p times.
Gabor $Gab(a,b)$ ($ab \notin$)	$Z(G) = \{e\}$, $Z(\mathfrak{A}) \neq \{\mathrm{Id}\}$, $\mathcal{D}_{max} \geq \mathfrak{A}_{Gab(b^{-1},0)}$	primary repr. (∞-dim.) of II type [72] Cor. p. 62

3.9. **Classification of tight frames.** Now the time came to study the characterization of these vectors whose orbits are tight frames. Start with a factor case. The factor coming from the representation of type I can contain only countably many copies of π_{x_0}.

Theorem 15. *Assume that $\mathfrak{A}_G$ is a factor of type I. Then Ga is a tight frame if and only if*

$$\langle L_k a_k, L_{k'} a_{k'}\rangle = \lambda\delta_{kk'}.$$

Given a locally compact group of unitary operators from $L(\mathcal{H})$, consider $\mathfrak{A}_G$ - the von Neumann algebra generated by the group G and closed in the strong topology. Denote its center by $Z(\mathfrak{A}_G)$ and choose $\mathcal{Z}_0$ an arbitrary Banach algebra which is dense in $Z(\mathfrak{A}_G)$. Since $Z(\mathfrak{A}_G)$ and $\mathcal{Z}_0$ are commutative, $\mathfrak{M}(\mathcal{Z}_0)$ - the maximal ideal space for $\mathcal{Z}_0$ is the base space of the decomposition of $\mathcal{H}$ into the direct integral $\int H_M\ dM$,which diagonalizes $Z(\mathfrak{A}_G)$ and makes the operators from $\mathfrak{A}_G$ decomposable. Moreover, if we restrict G to the two spaces H_M and $H_{M'}$, where $M \neq M'$, the representations $G\,|_{H_M}$ and $G\,|_{H_{M'}}$ are not *quasi-equivalent*, i.e., any of their subrepresentations are not equivalent (comp. Mautner in [Mackey]). This construction is called a central decomposition. It depends obviously on the choice of the Banach algebra $\mathcal{Z}_0$.

Theorem 16. *Let $a \in EssAdm(G/K, \mathcal{H})$. Then Ga is a tight frame for $\mathcal{H}$ if and only if $[\widetilde{g}(z)\widetilde{a}(z)]_{g\in G}$ are tight frames for H_z for any $z \in \mathfrak{M}(\mathcal{Z}(\mathfrak{A}_G))$ with the same frame bound.*

From theorem 14 follows

Theorem 17. *For an essentially admissible vector $a \in\mathcal{H}$ the system Ga is a tight frame in $\mathcal{H}$ if and only if*

$$c_z\langle L_k^z P_k\widetilde{a}(z), L_{k'}^z P_{k'}\widetilde{a}(z)\rangle = \lambda\delta_{kk'}$$

for almost all z and all k, k'.

Example 15. *Fix $t \in \mathbb{R}$ and G be a cyclic group of unitary transformations of $\mathbb{C}$ being multiplications by complex numbers*

$$e^{2\pi int} \quad \text{for } n \in \mathbb{Z}.$$

No vector is admissible for this group. Indeed,

$$\sum_{n\in\mathbb{Z}} |\langle e^{2\pi int}z, z\rangle|^2 = \sum_{n\in\mathbb{Z}} |z|^4 = +\infty.$$

Example 16. *Let G be a group of unitary transformations of $L^2(\mathbb{T})$ being multiplications by characters*

$$\chi_n(t)f(t) = e^{2\pi int}f(t).$$

For this group action admissible vectors form a dense subspace. Indeed,

$$(3.7)\qquad \sum_{n\in\mathbb{Z}} |\langle\chi_n f, f\rangle_{L^2(\mathbb{T})}|^2 = \sum_{n\in\mathbb{Z}}\langle f, \chi_n f\rangle_{L^2(\mathbb{T})}\langle\chi_n f, f\rangle_{L^2(\mathbb{T})} =$$

$$\sum_{n\in\mathbb{Z}}\langle|f|^2, \chi_n\rangle_{L^2(\mathbb{T})}\langle\chi_n, |f|^2\rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} |f|^4 = \|f\|^4_{L^4(\mathbb{T})}.$$

The last-line equality follows from Parseval's identity and $L^4(\mathbb{T})$ is dense in $L^2(\mathbb{T})$.

The above examples show that a square-integrable representation (Example 2) can happen not to have this property on fibers (Example 1). Square-integrability can be replaced with the weaker condition, to obtain another version of orthogonality relations.

Notation 5. *Below and in the sequel v stands for an element of a homogenous space G/H and is identified with an element of G by means of a Borel section σ. We write $\sigma(v)$.*

Theorem 18. *([59], Thm. 9.2.2. in [1])*

If a non-zero $a \in \mathcal{H}$ satisfies:

$$d_a := \int_{G/H} |\langle \sigma(v)a, a\rangle|^2 dv < \infty$$

for a certain closed abelian subgroup K of G and assume that there exists dv - a $G-$invariant measure in G/K, then

$$S_a = \int_{G/K} \langle \cdot, \sigma(v)a\rangle\, \sigma(v)a\; dv = d_a \|a\|^{-2}\ \mathrm{Id}_{\mathcal{H}}.$$

This version completely suffices for our purposes. For the representation from Example 15 this condition is trivial:

$$|\langle z, z\rangle|^2 = |z|^4 < \infty.$$

In Example 16 the representation is not irreducible. After the decomposition into the direct integral of irreducible representations (and non-equivalent ones) the condition becomes:

$$\int_X |f|^4 < \infty,$$

where $X = \mathbb{T} = \widehat{H} = \widehat{G}$ is a decomposition base space.

The following fact is very useful to handle with 'too big' groups, i.e., these that have an infinite abelian subgroup. Let H be an abelian and discrete group of unitary operators and $\mathcal{D}_H$ be the Banach algebra generated by the operators from H. Let further $\mathfrak{M} := \mathfrak{M}(\mathcal{D}_H)$ be the maximal ideal space for $\mathcal{D}_H$. Observe that the elements of $\mathfrak{M}$ can be identified with the elements of $\widehat{H}$. Observe further that the decomposition along the base space $\mathfrak{M}$ diagonalizes H. Let $1_{\mathfrak{M}} : \widehat{H} \to \{0, 1\}$ be a characteristic function of $\mathfrak{M}$ and let then φ_M be a standard identification between maximal ideals in $\mathcal{D}_H$ and linear-multiplicative functionals on this algebra.

Lemma 2. *Under the above assumptions if $F, G \in L^2(\mathfrak{M}, dM)$, then*

$$\int_H dh \int_{\mathfrak{M}} \overline{\varphi_M(h)}\; F(M)\; dM \int_{\mathfrak{M}} \varphi_{M'}(h)\; \overline{G(M')}\; dM' = \int_{\mathfrak{M}} F(M)\; \overline{G(M)}\; dM.$$

Proof.

$$\begin{aligned} L &= \int_H \int_{\widehat{H}} \overline{\chi(h)}\; F(\chi)\; 1_{\mathfrak{M}}(\chi)\; d\chi \times \int_{\widehat{H}} \chi'(h)\; \overline{G(\chi')}\; 1_{\mathfrak{M}}(\chi') d\chi'\; dh = \\ &= \int_H \int_{\widehat{H}} F(\chi)\; \overline{\chi(h)}\; 1_{\mathfrak{M}}(\chi)\; d\chi \quad \overline{\int_{\widehat{H}} G(\chi')\; \overline{\chi'(h)}\; 1_{\mathfrak{M}}(\chi')\; d\chi'}\; dh = \\ &= \int_H \widehat{F_{\mathfrak{M}}}(h)\; \overline{\widehat{G_{\mathfrak{M}}}(h)}\; dh \quad = \int_{\widehat{H}} F_{\mathfrak{M}}(\chi)\; \overline{G_{\mathfrak{M}}(\chi)}\; d\chi = \\ &= \int_{\widehat{H}} 1_{\mathfrak{M}}(\chi) F(\chi) \overline{G(\chi)}\; d\chi = \int_{\mathfrak{M}} F(\chi) \overline{G(\chi)}\; d\chi = P. \end{aligned}$$

□

In the special case we can do something to reduce 'the size' of the group.

Theorem 19. *Let G be unimodular, $Z(\mathfrak{A}_G) = \mathfrak{A}_{Z(G)}$ and let there exists a vector a that is essentially admissible mod $Z(G)$, i.e.,*

$$c_a := ess \sup_{k,z} \int_{G/Z(G)} |\langle P_k^z \widetilde{\sigma(v)}(z) P_k^z \widetilde{a}\,(z)\,, P_k^z \widetilde{a}\,(z)\rangle_{P_k^z H_z}|^2 \, dv < \infty,$$

where dv stands for a G-invariant measure on $G/Z(G)$. Then the frame operator S_a is equal to:

$$S_a = \int_{\mathfrak{M}(\mathcal{Z}_0)} \sum_{k=1}^{n(z)} \sum_{k'=1}^{n(z)} c_z \, \langle L_{k'}^z P_{k'}^z \widetilde{a}(z), L_k^z P_k^z \widetilde{a}(z)\rangle \, P_{k'}^z \, [L_{k'}^z]^{-1} \, L_k^z P_k^z \widetilde{y}(z) dz.$$

where the operators L_k^z have the same meaning as in theorem 14. Hence Ga is a tight frame if and only if

$$c_z \, \langle L_{k'}^z P_{k'}^z \widetilde{a}(z), L_k^z P_k^z \widetilde{a}(z)\rangle \equiv \lambda \delta_{kk'}$$

for almost all $z \in \mathfrak{M}(\mathcal{Z}_0)$ and a certain $\lambda \in \mathbb{C}$.

Proof. Note that the decomposition along $\mathcal{D}_{Z(G)}$ coincides with the Mauttner central decomposition. It follows from the assumption $Z(\mathfrak{A}_G) = \mathfrak{A}_{Z(G)}$. Consider $K = Z(G)$ being a normal abelian subgroup of G. For the proof we need first Fubini's theorem for the groups G and K. It is known ([1], sec. 4.1) that it can be done for any closed subgroup. Thus, we have

$$\int_G f(g) \, dg = \int_{K\backslash G} \int_K f(h\sigma(v)) \, dh \, dv.$$

The whole expression is:

$$\int_{K\backslash G} \int_K \int_{\mathfrak{M}(\mathcal{Z}_0)} \sum_{k=1}^{n(z)} \langle P_k^z \widetilde{y}(z), P_k^z \widetilde{h\sigma(v)}(z) P_k^z \widetilde{a}(z)\rangle \, dz \, dv \, dh \times$$

$$\times \int_{\mathfrak{M}(\mathcal{Z}_0)} \sum_{k'=1}^{n(z')} \langle P_{k'}^z \widetilde{h\sigma(v)}(z') P_{k'}^z \widetilde{a}(z'), \; P_{k'}^z \widetilde{u}(z')\rangle \, dz' \, dv \, dh.$$

To the integral over $K = Z(G)$ and two integrals over $\mathfrak{M}(\mathcal{Z}_o)$ apply Lemma 2 to obtain

$$\int_{\mathfrak{M}(\mathcal{Z}_o)} \sum_{k=1}^{n(z)} \sum_{k'=1}^{n(z')} \langle P_k^z \widetilde{y}(z), P_k^z \widetilde{\sigma(v)}(z) P_k^z \widetilde{a}(z)\rangle \langle P_{k'}^z \widetilde{\sigma(v)}(z) P_{k'}^z \widetilde{a}(z), \; P_{k'}^z \widetilde{u}(z)\rangle \, dz.$$

Now let us apply Theorem 18 to $K = Z(G)$ and get

$$S_a = \int_{\mathfrak{M}(\mathcal{Z}_0)} \sum_{k=1}^{n(z)} \sum_{k'=1}^{n(z)} c_z \langle L_{k'}^z P_{k'}^z \widetilde{a}(z), L_k^z P_k^z \widetilde{a}(z)\rangle \langle L_k^z P_k^z \widetilde{y}(z), \; L_{k'}^z P_{k'}^z \widetilde{u}(z)\rangle dz.$$

□

3.10. **Non-unimodular case.** In the non-unimodular case the only difference is that one needs to put a modular function Δ in Fubini's theorem. It is obvious that $K = Z(G) \leq ker\Delta$. Then by the standard theory (see, e.g., [65] sec.9.1)

$$\int_G f(g)\, dg = \int_{K\backslash G} \int_K f(hv)\, dh\, \Delta(v)\, dv.$$

For the affine group the situation is more complicated, since $Z(\mathrm{Aff}) = \{(1,0)\}$. There are two irreducible subspaces: $H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ (cf. section 2.3). For $a \in H^2_+(\mathbb{R})$,

$$\int_{\mathrm{Aff}} \langle \cdot, ga\rangle\langle ga, \cdot\rangle dg = \langle Wa, Wa\rangle \,\mathrm{Id} = \int_+ |\widehat{a}(t)|^2 t^{-1} dt \,\mathrm{Id}_{H^2_+(\mathbb{R})} = c_+(a)\,\mathrm{Id}_{H^2_+(\mathbb{R})}.$$

Analogously, for $a \in H^2_-(\mathbb{R})$

$$\int_{\mathrm{Aff}} \langle \cdot, ga\rangle\langle ga, \cdot\rangle dg = \int_+ |\widehat{a}(-t)|^2 t^{-1} dt \,\mathrm{Id}_{H^2_-(\mathbb{R})} = c_-(a)\,\mathrm{Id}_{H^2_+(\mathbb{R})}.$$

The weight t^{-1} is related with the modular function of the affine group. To have a tight frame in $\mathcal{H} = L^2(\mathbb{R})$ one takes $c_+(a) = c_-(a)$ as in [60]. In particular, any real-valued function generates a tight frame. The above considerations lead to the conclusion that one should rather consider $H^2_+(\mathbb{R}, |t|^{-1}dt)$ than $H^2_+(\mathbb{R}, dt)$.

3.10.1. *Wavelet case.* In the case of wavelet frames one considers a system

$$D^j T^k g, \qquad j,k \in \mathbb{Z},$$

where $D = D_a$ $(a > 1)$ and $T = T_b$ (cf. Section 2.3). The question is the same: when the system of vectors $(D^jT^kg)_{j,k\in\mathbb{Z}}$ is a (tight) frame?

We analyze first the frame operator. $\mathbb{Q}_a$ are fractions with an integer numerator and a power of a in the denominator. Define some technical notions to handle with the set.

Definition 21. *Let $\alpha \in \mathbb{Q}_a/b$. Let J_α be a set of integers*

$$J_\alpha = \left\{ j \in \mathbb{Z} : \exists_{n\in\mathbb{Z}}\ \alpha = na^j b^{-1} \right\}.$$

If a is integer, J_α is a half-axis in $\mathbb{Z}$. If a is a p-th root of an integer, J_α has gaps of length $p-1$. Otherwise, J_α is a one element set.

Definition 22. *Let*

$$c_\alpha(y) := \sum_{j\in J_\alpha} \overline{\widehat{g}(a^{-j}y)}\widehat{g}(a^{-j}(y+\alpha)). \tag{3.8}$$

Remark 13. *These coefficients resemble the Ron-Shen affine product (cf.* [78]*).*

The theorem below applies to integer as well as to non-integer dilations.

Theorem 20. *For $g \in L^2(\mathbb{R})$ let S be the frame operator related to the system $(D^jT^kg)_{j,k\in\mathbb{Z}}$ and $\widehat{S} = \mathcal{F}S\mathcal{F}^{-1}$ be its Fourier counterpart. If $(D^jT^k\mathcal{F}^{-1}(|\mathcal{F}g|))_{j,k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then*

$$\widehat{S}\ \widehat{f} \quad = \sum_{\alpha\in\mathbb{Q}_a/b} c_\alpha(y)\ \widehat{f}(y+\alpha).$$

Proof. Let us assume first that $\widehat{f}$ is bounded function with compact support on $\mathbb{R}$. Note further that switching to the frequency domain, we obtain

$$\mathcal{F}\, D^j\, T^k \quad = \quad M_{kba^{-j}}\, D^{-j}\, \mathcal{F}.$$

By Plancherel theorem in $L^2(\mathbb{R})$,

$$\langle f, D^jT^k g\rangle = \int_{\mathbb{R}} \widehat{f}(\xi)\, \overline{M_{kba^{-j}}\left(D^{-j}\widehat{g}\right)(\xi)}\, d\xi =$$

$$\int_0^{a^jb^{-1}} \left(\sum_{n\in} \widehat{f}\left(\xi + na^jb^{-1}\right) \overline{\left(D^{-j}\widehat{g}\right)\left(\xi+na^jb^{-1}\right)} \right) \; e^{-2\pi i kba^{-j}\xi}\, d\xi.$$

From Parseval's identity for the system $\left(e^{-2\pi ikba^{-j}\xi}\right)_{k\in\mathbb{Z}}$ in $L^2(0, a^jb^{-1})$ (we sum up over k):

$$\sum_{k\in\mathbb{Z}}\langle f, D^jT^kg\rangle\langle D^jT^kg, f\rangle =$$

$$= \int_0^{a^jb^{-1}} a^jb^{-1} \left(\sum_{n\in\mathbb{Z}} \widehat{f}(\xi+na^jb^{-1})\overline{(D^{-j}\widehat{g})(\xi+na^jb^{-1})}\right)\times$$

$$\times\left(\sum_{n'\in\mathbb{Z}} \overline{\widehat{f}(\xi+n'a^jb^{-1})}(D^{-j}\widehat{g})(\xi+n'a^jb^{-1})\right)d\xi.$$

Putting $y = \xi + n'a^jb^{-1}$ and renumbering n, we transform it to

$$\int_{\mathbb{R}} b^{-1} \sum_{n\in\mathbb{Z}} \widehat{f}\left(y+na^jb^{-1}\right) \overline{\widehat{g}\left(a^{-j}y\right)}\, \widehat{g}\left(a^{-j}y + nb^{-1}\right) \overline{\widehat{f}(y)}\, dy.$$

Replacing na^jb^{-1} by $\alpha \in \mathbb{Q}_a/b$, one gets

$$\int_{\mathbb{R}} b^{-1} \sum_{\alpha\in a^jb^{-1}\mathbb{Z}} \widehat{f}(y+\alpha)\widehat{g}(a^{-j}y)\widehat{g}(a^{-j}y+a^{-j}\alpha)\overline{\widehat{f}(y)}dy.$$

Having summed the expression over $j \in \mathbb{Z}$, we interchange the sums. This is justified since the following quantity is finite (Note that in the below calculation we use the assumption about $\widehat{f}$, B being its bound and *supp* $\widehat{f} = [-N, N]$.):

$$\left|\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}} b^{-1} \sum_{\alpha\in a^jb^{-1}\mathbb{Z}} \widehat{f}(y+\alpha)\widehat{g}(a^{-j}y)\widehat{g}(a^{-j}y+a^{-j}\alpha)\overline{\widehat{f}(y)}dy\right| \leq$$

$$\leq \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}} b^{-1} \sum_{\alpha\in a^jb^{-1}\mathbb{Z}} \left|\widehat{f}(y+\alpha)\right| \left|\widehat{g}(a^{-j}y)\right| \left|\widehat{g}(a^{-j}y+a^{-j}\alpha)\right| \left|\overline{\widehat{f}(y)}\right| dy \leq$$

$$\leq \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}} b^{-1} \sum_{\alpha\in a^jb^{-1}\mathbb{Z}} B^2\chi_{[-N,N]}(y)\chi_{[-N-\alpha,N-\alpha]}(y) \left|\widehat{g}(a^{-j}y)\right| \left|\widehat{g}(a^{-j}y+a^{-j}\alpha)\right| dy.$$

The finiteness of the number follows from the assumption about $\left(D^jT^k\mathcal{F}^{-1}\left(|\mathcal{F}g|\right)\right)_{j,k\in\mathbb{Z}}$ being a frame - one can repeat all the above calculations from the very beginnning for this system; thus one obtains the number under consideration and it is finite because the starting sum

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f, D^jT^k\mathcal{F}^{-1}\left(|\mathcal{F}g|\right)\rangle\langle D^jT^k\mathcal{F}^{-1}\left(|\mathcal{F}g|\right), f\rangle$$

was finite due to the frame condition for $\mathcal{F}^{-1}(|\mathcal{F}g|)$. So we can rearrange the sums arbitrarily. Let us order $\mathbb{Q}_a/b$ by the order of maximal points of appropriate J_α sets. When two numbers are not distinguished by it, then put them in the standard order of real axis. After the rearrangement we have

$$\langle \widehat{S}\ \widehat{f}, \widehat{f}\rangle = \int_{\mathbb{R}} b^{-1} \sum_{\alpha\in\mathbb{Q}_a/b} \left[\sum_{j\in J_\alpha} \overline{\widehat{g}(a^{-j}y)}\widehat{g}(a^{-j}(y+\alpha))\right] \widehat{f}(y+\alpha)\ \overline{\widehat{f}(y)}\ dy.$$

□

3.11. **Two equivalent representations.** This section is a certain extension of Remark 9 to the case of representations having a decomposition into a direct integral. The results of the section will be used in Chapter 4 to describe the phenomenon of Wilson basis. Since the facts are mainly reformulation of the results from previous sections we only sketch their pfoofs. Let G be a locally compact group of unitary mappings in the separable Hilbert space $\mathcal{H}$ and let G have a countable basis. In this section we take the simplifying assumption that a decomposition into non-equivalent subrepresentations is simultaneously into irreducible ones.

Let us consider the operator resembling the frame operator in the form, but related to the action of two equivalent representations, namely, the operator $S_{G,J,a} := \int_G \langle\cdot, ga\rangle\ JgJ^{-1}a\ dg$, where the integral convergence is in the weak sense. The operator J establishes the equivalence of representations.

Lemma 3. *If $\mathfrak{A}_K$ is a commutative subalgebra in $\mathfrak{A}_G$, then $S_{G,J,a}$ decomposes with the basis $\mathfrak{M}(\mathcal{D}_K)$.*

Lemma 4. *If J is a unitary operator and $J\mathfrak{A}J^{-1}\subset\mathfrak{A}$, then J is a composition of a decomposable operator (denote it J_z) with the basis $\mathfrak{M}(\mathcal{D}_K)$ and a measure isomorhism M of $\mathfrak{M}(\mathcal{D}_K)$ ([72], Thm. 2.7).*

Theorem 21. *Assume that K is a subgroup of $Z(G)$. The operator $S_{G,J,a}$ decomposes with respect to $\mathfrak{M}(\mathcal{D}_K)$ as*

$$S_{G,J,a} = \int_{K\backslash G}\int_{\mathfrak{M}(\mathcal{D}_K)} \langle\cdot,\, v_z a_z\rangle \left\langle v_z\, J_z^{-1} a_{M^{-1}(z)}, \cdot\right\rangle dz\, dv\, J^{-1}.$$

Proof. follows from the lemma above and observation that

$$\left(\widetilde{J^{-1}a}\right)(z) = \widetilde{J^{-1}}(z)\ \widetilde{a}\left(M^{-1}(z)\right).$$

□

Theorem 22. *For the unimodular group G the decomposition of the operator $S_{G,J,a}$ J is given by*

$$\left(\widetilde{S_{G,J,a}J}\right)(z) = c_z \left\langle \widetilde{a}(z),\ \widetilde{J^{-1}}(z)\ \widetilde{a}\left(M^{-1}(z)\right)\right\rangle.$$

Proof. follows from the orthogonality relations for homogenous space. □

Corollary 2. *The operator $S_{G,J,a}$ J is zero if and only if the expression*

$$\left\langle \widetilde{J}(z)\ \widetilde{a}(z), \widetilde{a}\left(M^{-1}(z)\right)\right\rangle \equiv_z 0$$

is identically 0 for almost all z.

4. Wilson basis

4.1. **Introduction.** The subject of this chapter is the Wilson basis phenomenon discussed from the direct-integral-decomposition point of view. In the case of the abelian Gabor group $\mathbb{Z}^2$ corresponding to the case $b = a^{-1}$ and to the commutative algebra generated by the operators M_a and $T_{a^{-1}}$it is known that functions which generate an orthonormal basis lack smoothness or good decay properties (Balian, Low, Battle). The idea of Daubechies, Jaffard, Journe was to take twice as many vectors as needed. Then one combines the suitably chosen combinations of pairs of the vectors and expects to obtain an orthonormal basis. This idea can work when the Gabor group has an abelian subgroup of index 2 and it is the case for $2b = a^{-1}$.

Theorem 23. [32] *If the Gabor system $\left(M_{1/2}^m T_1^n g\right)$ with $\|g\|_2 = 1$ is a tight frame with bound 2 and g is real-valued, then the system $(b_{lm})_{l\geq 0, m\in\mathbb{Z}}$, defined by*

$$b_{0m}(x) = e^{2\pi imx} g(x), \tag{4.1}$$

$$b_{lm}(x) = 2^{-1/2}\left(e^{\pi imx} g(x-l) + (-1)^{l+m} e^{\pi imx} g(x+l)\right), \tag{4.2}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

In preliminaries we summarize some facts about the suitable decomposition of the representation operators, useful information about the coorbit and Bargmann spaces and some technical lemmas to be used later on plus the remark that a tight frame of norm-one vectors with bound 1 is an orthonormal basis.

After Daubechies, Jaffard and Journé work the Wilson bases rceived more and more attention. Feichtinger, Gröchenig and Walnut pointed the important unconditional properties of Wilson basis in coorbit and Bargmann spaces for $p < \infty$. Gröchenig and Walnut constructed later a Riesz unconditional bases for Bargmann scale of spaces using the reproducing kernel of the Bargmann space evaluated at the certain set of points. Let us recall the definition of these spaces. The `Bargmann space` $A^p(\mathbb{C})$ is defined for $1 \leq p < \infty$ as a space

$$A^p(\mathbb{C}) := \left\{ f - \text{entire} \ : \ \int_{\mathbb{C}} \left| f(z) e^{-\pi|z|^2/2} \right|^p d\lambda(z) \ < \infty \right\}$$

endowed with the natural norm

$$\| f \|_{A^p(\mathbb{C})} = \left(\int_{\mathbb{C}} \left| f(z) e^{-\pi|z|^2/2} \right|^p d\lambda(z) \right)^{1/p},$$

where λ is the Lebesgue measure of the complex plane. Gröchenig-Walnut theorem states that:

Theorem 24. [57] *The system of vectors:*

$$u_{m0} = e^{-\frac{\pi}{2}m^2} e^{i\pi zm}\,(z),$$
$$u_{ml} = 2^{-1/2} e^{-\frac{\pi}{2}\left|\frac{m}{2}+il\right|^2} \left(e^{-i\pi z(\frac{m}{2}-il)} + (-1)^{l+m} e^{-i\pi z(\frac{m}{2}+il)} \right),$$

for $m \in$ and $l \geq 1$, is an unconditional basis in $A^p(\mathbb{C})$ for $1 \leq p < \infty$.

Observe that $e^{i\pi w_{ml} z}$ is a reproducing kernel of the point w_{ml}.

In the meantime P. Auscher showed a second orthonormal basis for the Gabor tight frame with bound 2, namely:

$$d_{m0}(x) = e^{\pi i(2m+1)x} g(x), \tag{4.3}$$

$$d_{ml}(x) = 2^{-1/2}\left(e^{\pi imx} g(x-l) - (-1)^{l+m} e^{\pi imx} g(x+l)\right), \tag{4.4}$$

(replacing + by − and $2m$ by $2m+1$ in the previous formulas).

He also characterized the functions, both constructions work for, in the form of the following condition:

$$E_k(x) = \sum_{n\in\mathbb{Z}} (-1)^n \overline{g(x-k-n/2-1/2)}\, g(-x-n/2) = 0 \quad a.e. \tag{4.5}$$

He did it assuming a sufficient decay of the function. The recent results of P.G. Casazza and M. Lammers show that it is a redundant assumption.

The original contribution of the chapter is to show the proofs of the constructions:

1) of orthonormal Wilson basis by I. Daubechies, S. Jaffard, J-L. Journé;

2) of its ONB-companion found by P. Auscher;

3) of an unconditional basis in the Bargmann scale of spaces by K. Gröchenig and D. Walnut.

The proofs presented here are shorter and simpler than their original versions and are based on some elementary knowledge of Bargmann and coorbit spaces summoned in the preliminaries for the comfort of the reader.

The second original contribution of the chapter is to give a geometric sense of Auscher's condition. Namely, there is an isometric mapping Z from $L^2(\mathbb{R})$ to functions from $L^2(B)\oplus L^2(B)$, where $B=(0,1)\times(0,1/2)$. Thus the value of Zf at the point of B is a two-coordinate vector. The new version of Auscher's condition says that f generates a Wilson basis if and only if $Z\Re f$ and $Z\Im f$ are colinear for each argument from B. In particular, if $Z\Im f \equiv 0$ identically, then f generates a Wilson basis. So the real-valued f acts perfectly well. $\mathfrak{A}_H \le C(\mathfrak{A}_G)$.

Section 4.1 contains some preliminary information about coorbit and Bargmann scale of spaces, suitable decompositions of representation operators and frame operator, a classical result about unitary equivalence of representations and some technical lemmas. Section 4.2 contains Wilson basis construction with the geometric version of Auscher's condition. Next one establishes the unconditionality of Gröchenig-Walnut basis in the Bargmann scale of spaces.

4.2. **Preliminaries.** In this chapter we shall use Schrödinger representation defined in the Chapter 2. Let us recall that **p**-coorbit space is a linear subspace of tempered distributions defined by

$$Co(L^{\mathbf{p}})_\varphi = \left\{ f\in\mathcal{S}'(\mathbb{R}) : \int_{\mathbb{R}^2} |\langle f, (p,q)\,\varphi\rangle|^{\mathbf{p}}\, dpdq < \infty \right\}$$

endowed with a natural norm

$$\|f\|_{Co(L^{\mathbf{p}})_\varphi} = \left(\int_{\mathbb{R}^2} |\langle f, (p,q)\,\varphi\rangle|^{\mathbf{p}}\, dpdq \right)^{1/\mathbf{p}}.$$

A priori the space depends on a choice of function $\varphi\in\mathcal{S}(\mathbb{R})$. The equivalence of the norms for different φ's is non-trivial [39]. In the sequel we use the coorbit spaces for one fixed $\varphi(x)=e^{-\pi x^2/2}$. Denote $\varphi_{ml}(x) := [\left(\frac{m}{2},l\right)\varphi](x) = e^{\pi i m x}\varphi(x-l)$ for any $l,m\in\mathbb{Z}$.

Proposition 5. *For φ and φ_{ml} defined above it holds that*
$$\left|\langle (p,q)\,\varphi, \varphi_{ml}\rangle_{L^2(\mathbb{R})}\right| = \exp\left(-\tfrac{\pi}{4}(l-q)^2 - \pi\left(p-\tfrac{m}{2}\right)^2\right).$$

Proof. Resolving definitions of (p,q) and φ_{ml}, one gets that the left-hand side is equal to:

$$L = \int_{\mathbb{R}} e^{2\pi i p x}\varphi(x-q)\overline{e^{\pi i m x}\varphi(x-l)}dx = \int_{\mathbb{R}} e^{2\pi i (p-\frac{m}{2})x}\varphi(x-q)\varphi(x-l)dx.$$

Now check that

$$E = \frac{1}{2}(x-q)^2+\frac{1}{2}(x-l)^2 = x^2-(q+l)\,x+\frac{1}{2}\left(q^2+l^2\right) = (x-(q+l))^2+\frac{1}{4}\left(q-l\right)^2,$$

and

$$\varphi(x-q)\;\varphi(x-l) = e^{-\pi E} = e^{-\frac{\pi}{4}(q-l)^2}e^{-\pi(x-(q+l))^2}.$$

Let us recall that $\int_{\mathbb{R}} e^{2\pi i x\xi}e^{-\pi x^2}dx = e^{-\pi\xi^2}$. Thus we come to

$$\left|\int_{\mathbb{R}} e^{2\pi i(p-\frac{m}{2})x}\,e^{-\pi(x-(q+l))^2}dx\right| = \left|\int_{\mathbb{R}} e^{2\pi i(p-\frac{m}{2})(y+q+l)}e^{-\pi y^2}dy\right| =$$

$$= \left|\int_{\mathbb{R}} e^{2\pi i(p-\frac{m}{2})y}e^{-\pi y^2}dy\right| = e^{-\pi(p-\frac{m}{2})^2}$$

(by substitution $y = x-(q+l)$). Eventually, $L = e^{-\frac{\pi}{4}(q-l)^2}e^{-\pi(p-\frac{m}{2})^2}$. □

Proposition 6. *The function* $C(x) := \sum_{n\in\mathbb{Z}} e^{-a(x-nb)^2}$ *expands in the Fourier series in* $L^2(0,b)$ *as* $C(x) = \sqrt{\frac{\pi}{ab^2}}\sum_{m\in\mathbb{Z}} e^{-\frac{\pi}{ab^2}m^2}e^{2\pi i\frac{x}{b}m}$. *In particular,* $\sup_x C(x) \le \sqrt{\frac{\pi}{ab^2}}\sum_{m\in\mathbb{Z}} e^{-\frac{\pi}{ab^2}m^2}$.

Proof. $C(x)$ is b-periodic, so we calculate the appropriate Fourier coefficients by

$$b^{-1}\int_0^b \left[\sum_{n\in\mathbb{Z}} e^{-a(x-nb)^2}\right] e^{-2\pi i\frac{x}{b}m}dx = b^{-1}\int_{\mathbb{R}} e^{-ay^2}e^{-2\pi i\frac{y}{b}m}dy =$$

$$= \sqrt{\frac{\pi}{ab^2}}\int_{\mathbb{R}} e^{-\pi z^2}e^{-2\pi i z\sqrt{\frac{\pi}{ab^2}}m}dz = \sqrt{\frac{\pi}{ab^2}}e^{-\frac{\pi}{ab^2}m^2}.$$

(Substitutions $y = x-nb$, $z = \sqrt{\frac{a}{\pi}}y$.) □

Proposition 7. *The quantity* $P_0\left(p,q\right) = \sum_{m,l\in\mathbb{Z}}\left|\langle(p,q)\,\varphi,\varphi_{ml}\rangle_{L^2(\mathbb{R})}\right|$ *has a finite supremum over* $p,q\in\mathbb{R}$.

Proof. By Proposition 1 the sum

$$P_0\left(p,q\right) = \sum_{m,l\in\mathbb{Z}} e^{-\frac{\pi}{4}(q-l)^2}e^{-\pi(p-\frac{m}{2})^2} = \left(\sum_{l\in\mathbb{Z}} e^{-\frac{\pi}{4}(q-l)^2}\right)\left(\sum_{m\in\mathbb{Z}} e^{-\pi(p-\frac{m}{2})^2}\right).$$

The sums in the product are of the type analyzed in Proposition 2 for $a=\pi$, $b=\frac{1}{2}$ and $a=\frac{\pi}{4}$, $b=1$, respectively. Thus by Proposition 2

$$\sup_{p,q} P_0\left(p,q\right) = \sup_p\left(\sum_{m\in\mathbb{Z}} e^{-\pi(p-\frac{m}{2})^2}\right)\times\sup_q\left(\sum_{l\in\mathbb{Z}} e^{-\frac{\pi}{4}(q-l)^2}\right) \le$$

$$2\left(\sum_{m\in\mathbb{Z}} e^{-4\pi m^2}\right)\times 2\left(\sum_{l\in\mathbb{Z}} e^{-4\pi l^2}\right) = 4\left(\sum_{m\in\mathbb{Z}} e^{-4\pi m^2}\right)^2.$$

The value of the last constant is approximately equal to 4 ± 10^{-5} . □

Proposition 8. *The norm of φ in $Co(L^1)_\varphi$ is equal to* $\|\varphi\|_{Co(L^1)_\varphi} = 2$.

Proof. From Proposition 5 for $m = l = 0$ we have

$$\|\varphi\|_{Co(L^1)_\varphi} = \int\limits_{\mathbb{R}^2} \left|\langle \pi\left(\mathbf{p},\mathbf{q}\right)\varphi, \varphi_{ml}\rangle_{L^2(\mathbb{R})}\right| dpdq = \int\limits_{\mathbb{R}} e^{-\frac{\pi}{4}q^2} dq \int\limits_{\mathbb{R}} e^{-\pi p^2} dp = 2.$$

(Recall that $\int\limits_{\mathbb{R}} e^{-\pi p^2} dp = 1$.) □

Proposition 9. *For $\lambda \in l^{\mathbf{p}}\left(\mathbb{Z}^2\right)$ and the system $\left(\varphi_{ml}\right)_{m,l\in\mathbb{Z}}$ defined as above the following inequality holds for $1 \le \mathbf{p} \le \infty$*

$$\left\|\sum_{m,l\in\mathbb{Z}} \lambda_{ml}\varphi_{ml}\right\|_{Co(L^{\mathbf{p}})_\varphi} \le C \left\|\lambda\right\|_{l^{\mathbf{p}}(\mathbb{Z}^2)}.$$

Proof. Note first that (r, s) are isometries of $Co(L^1)_\varphi$ for any $r, s \in \mathbb{R}$, so by Proposition 8 $\|\varphi_{ml}\|_{Co(L^1)_\varphi} = \|\varphi\|_{Co(L^1)_\varphi} = 2$. Thus, for $\mathbf{p} = 1$ we have from the triangle inequality that

$$\left\|\sum_{m,l\in\mathbb{Z}} \lambda_{ml}\varphi_{ml}\right\|_{Co(L^1)_\varphi} \le \sum_{m,l\in\mathbb{Z}} \left|\lambda_{ml}\right| \, \left\|\varphi_{ml}\right\|_{Co(L^1)_\varphi} \le 2\left\|\lambda\right\|_{l^1}.$$

For $\mathbf{p} = \infty$ we have

$$\left\|\sum_{m,l\in\mathbb{Z}} \lambda_{ml}\varphi_{ml}\right\|_{Co(L^\infty)_\varphi} = \sup_{p,q}\left|\left\langle \sum_{m,l\in\mathbb{Z}} \lambda_{ml}\varphi_{ml}, \pi\left(\mathbf{p},\mathbf{q}\right)\varphi\right\rangle_{L^2(\mathbb{R})}\right| =$$

$$\sup_{p,q}\left|\sum_{m,l\in\mathbb{Z}} \lambda_{ml}\left\langle \varphi_{ml}, (p,q)\,\varphi\right\rangle_{L^2(\mathbb{R})}\right| \le \sup_{m,l}\left|\lambda_{ml}\right| \sup_{p,q}\left|\left\langle \varphi_{ml}, (p,q)\,\varphi\right\rangle_{L^2(\mathbb{R})}\right|$$

$$= \left\|\lambda\right\|_{l^\infty(\mathbb{Z}^2)} \sup_{p,q} P_0\left(p,q\right) \le C\left\|\lambda\right\|_{l^\infty(\mathbb{Z}^2)}.$$

Interpolating between the two latter, we obtain the desired inequality. (That it is possible see section 4.4.1). □

Proposition 10. *(see [64], section 1.5) The mapping $Z_\lambda f(t,u) = \lambda^{1/2}\sum_{k\in\mathbb{Z}} f(\lambda(t-k))e^{2\pi iku}$ is unitary from $L^2(\mathbb{R})$ to the space Q of quasi-periodic functions*

$$Q = \left\{F \in L^2_{loc}(\mathbb{R}^2) : F(t+1,u) = e^{2\pi iu}F(t,u) \text{ and } F(t,u+1) = F(t,u)\right\}$$

endowed with a norm

$$\left\|F\right\|_Q = \left(\int_{[0,1]^2} \left|F(t,u)\right|^2 d(t,u)\right)^{1/2}.$$

Proposition 11. *Let $\mathfrak{A} = (M_a, T_{a^{-1}})$ be a commutative von Neumann algebra generated by the operators M_a and $T_{a^{-1}}$. Then $Z_{a^{-1}}$ is the identification of $L^2(\mathbb{R})$ with $Q \simeq L^2(0,1)^2$ which diagonalizes the algebra $\mathfrak{A}$. In particular,*

$$Z_{a^{-1}}\left[M_a f\right](t,u) = e^{2\pi i t}\ \left[Z_{a^{-1}} f\right]\ (t,u),$$
$$Z_{a^{-1}}\left[T_{a^{-1}} f\right](t,u) = e^{-2\pi i u}\ \left[Z_{a^{-1}} f\right]\ (t,u).$$

Proof. follows by direct calculations. □

4.3. **Balian-Low theorem.** It would be very convenient to have a nice orthonormal basis (no redundancy) of Gabor type based on a function g such that both g and $\widehat{g}$ have good decay properties (expansions with good phase space localization). Unfortunately, such an orthonormal basis does not exist. A theorem stated by Balian [5] and Low [71] asserts that a Gabor system of functions can only constitute an orthonormal basis if either

$$\int x^2 \left|g\left(x\right)\right|^2 dx = \infty \quad \text{or} \int \xi^2 \left|\widehat{g}\left(\xi\right)\right|^2 d\xi = \infty.$$

Balian's and Low's proofs contain a technical gap that was filled afterwards as reported in [28]; a much simpler proof was subsequently found by G. Battle [8]. Even if the orthonormality, but not the "basis" requirement, is given up, the same conclusion holds, as shown by the extension of Battle's argument in [31]. Both the original proof and Battle's proof of the Balian-Low theorem rely heavily on the special structure of the Gabor system. We might therefore wonder whether there exist other bases with phase space localization (see e.g. [14]).

However, for the Gabor case the orthonormality, or, what is weaker, the existence of numerically stable expansions with *nonredundant* functions is incompatible with good phase space localization. In the sequel we shall see that the crucial point here is non-redundancy.

Example 17. *Let G be abelian. Then all tight frames are isometric up to the constant. In particular, if one coherent tight frame Gx is an orthonormal basis, then all coherent tight frames are orthonormal bases up to a scalar multiple. Indeed, $G = Z(G)$ and given $k \in G$ and the tight frame $(hy)_{h\in G}$, let $(ha)_{h\in G}$ be another tight frame with a frame bound λ. Then we have (the last-but-one equality follows from the fact that abelian groups are unimodular):*

$$\mu \left\langle y, ky\right\rangle = \int_G \left\langle y, ha\right\rangle \left\langle ha, ky\right\rangle dh = \int_G \left\langle h^* y, a\right\rangle \left\langle a, kh^* y\right\rangle dh =$$
$$= \int_G \left\langle k^* a, hy\right\rangle \left\langle hy, a\right\rangle d\left(h^{-1}\right) = \lambda \left\langle k^* a, a\right\rangle = \lambda \left\langle a, ka\right\rangle.$$

Thus, all tight frames for abelian group are isometric up to the constant.

Example 18. *Let G be finite. Then no irreducible representation of dimension bigger than 1 can give an orthonormal basis as an orbit. Indeed, let Gx be an orthonormal basis. Then $\sum_{g\in G} gx$ is an invariant vector and π is not irreducible.*

4.4. **Daubechies-Jaffard-Journé Theorem.** Daubechies-Jaffard-Journé theorem describes a way of getting an orthonormal basis from a Gabor tight frame with bound 2. It was invented to circumvent the theorem of Balian and Low saying, in short words, that one cannot obtain a tight frame with bound 1 from a function, which is smooth and has a smooth Fourier transform. The power of Daubechies-Jaffard-Journé theorem comes from the fact that a function which is smooth together with its Fourier transform *can* generate a tight frame with bound 2, while *cannot* generate a tight frame with bound 1.

Our proof of Daubechies-Jaffard-Journé theorem shows that orthonormality of Wilson basis follows from the two conditions: 1. the equation for Gabor tight frame and 2. the symmetry-antisymmetry conditions implied by the fact that the generating function is real-valued. The main point of the proof is to show *two* orthonormal bases. Each of their elements is a linear combination of two vectors of the Gabor tight frame. We conjecture that these are all bases of this type. Let us call them Wilson-type bases. Then the following question arises: How many Wilson-type bases can one obtain from a Gabor tight frame with bound n? We shall try to give a partial answer to this question formulated in [91]. This classification is a first step to solve K. Gröchenig problem: *Can one construct an analogue of Wilson basis for the Gabor tight frame with bound 3*, i.e., for $\left(M_{m/3}T_n g\right)_{m,n\in\mathbb{Z}}$[56] ?

To get the general solution and a classification of all Wilson bases of this type we need also an isometric classification of Gabor tight frames with bound 2. The following theorem concerning the construction of Wilson basis is due to I. Daubechies, S. Jaffard and J.-L. Journé [32]. Then it was modified by P. Auscher using some decay assumptions [4]. In this thesis and also in [91] the decay conditions are relaxed and the proof is of purely geometrical nature. In particular, we get the characterization of all functions generating Wilson bases in the language of their geometric properties.

Consider two representations of $G = Gab(\frac{1}{2}, 1) \ni (m, l, k)$ (The third component acts as a multiplication - cf. sec.2.2 and is omitted in the sequel.):

$$(m,l) = M_{\frac{m}{2}}T_l \qquad J(m,l)J^{-1} = (-M_{\frac{1}{2}})^m(-T_{-1})^l.$$

Let $\mathfrak{A} := \mathfrak{A}_G = \overline{gen\{g : g \in G\}}^{strong}$. In the case $a = \frac{1}{2}$, $b = 1$, the center $Z(\mathfrak{A})$ is simultaneously the maximal commutative subalgebra in the commutant $\mathfrak{A}'$ and the decomposition into irreducible representations is simultaneously a decomposition into factors. Moreover, the same integral can be applied, since $J\mathfrak{A}J^{-1} \subset \mathfrak{A}$ and the space of the integral is $\mathfrak{M}(Z(\mathfrak{A}))$.

Zak transform

$$f \mapsto Zf(x_1, x_2) = \left(\sum_n f(x_1+n)e^{2\pi i n x_2}, \sum_n (-1)^n f(x_1+n)e^{2\pi i n x_2}\right),$$

$$x_1 \in [0,1), \quad x_2 \in [0, 1/2),$$

yields an identification of $L^2(\mathbb{R})$ with the direct integral $L^2([0,1)\times[0,\frac{1}{2}), \mathbb{C}^2)$, which transforms the representation operators $M_{\frac{1}{2}}$, T_1(and $JM_{\frac{1}{2}}J^{-1}JT_1J^{-1}$) to the decomposable operators from $L^2([0,1)\times[0,\frac{1}{2}), \mathrm{End}(\mathbb{C}^2))$, which act irreducibly on the fibers by the formulae:

$$M_{\frac{1}{2}}f(x) = e^{\pi i x}f(x), \qquad M_{\frac{1}{2}} = (1,0) \;:\; (x_1,x_2) \mapsto \begin{pmatrix} 0 & e^{\pi i x_1} \\ e^{\pi i x_1} & 0 \end{pmatrix},$$

$$T_1 f(x) = f(x-1), \qquad T_1 = (0,1) \;:\; (x_1,x_2) \mapsto \begin{pmatrix} e^{2\pi i x_2} & 0 \\ 0 & -e^{2\pi i x_2} \end{pmatrix},$$

$$-M_{\frac{1}{2}}f(x) = -e^{\pi i x}f(x), \qquad -M_{\frac{1}{2}} = J(1,0)J^{-1} \;:\; (x_1,x_2) \mapsto \begin{pmatrix} 0 & -e^{\pi i x_1} \\ -e^{\pi i x_1} & 0 \end{pmatrix},$$

$$-T_{-1}f(x) = -f(x+1), \qquad -T_{-1} = J(0,1)J^{-1} \;:\; (x_1,x_2) \mapsto \begin{pmatrix} -e^{-2\pi i x_2} & 0 \\ 0 & e^{-2\pi i x_2} \end{pmatrix}.$$

The representations g_{z_1,z_2} and $\left(JgJ^{-1}\right)_{w_1,w_2}$ are equivalent if and only if

$$(z_1,z_2) = (w_1, \frac{1}{2} - w_2)$$

and the intertwinning operator is

$$J_{(z_1,z_2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we conclude that the equivalence operator J is definded by means of its fiber action $J_{(z_1,z_2)}$ and a measure isomorphism in the base space $F(w_1,w_2) := (w_1, \frac{1}{2} - w_2)$.

Let $L = \{(m,l) \in \mathbb{Z}^2 : l \geq 0\}$. Let $c(m,l)$ be the normalization constants equal to 2^{-1}, if $(m,l) = J(m,l)J^{-1}$, and $2^{-\frac{1}{2}}$, otherwise. If $(m,l) + J(m,l)J^{-1} = 0$, remove (m,l) from the index set L. It concludes with $L = 2\mathbb{Z} \times \{0\} \cup \{(m,l) \in \mathbb{Z}^2 : l > 0\}$. After this preparation one can formulate:

Theorem 25. [32] *Let* $((m,l)g)_{m,l\in\mathbb{Z}}$ *be a tight frame and* $\|g\|_2 = 1$. *Then the system*

$$b_{ml} = c(m,l)\left((m,l)g + J(m,l)J^{-1}g\right),$$

with $(m,l) \in L$, *is an orthonormal basis in* $L^2(\mathbb{R})$ *if and only if the points* $Z(\Re g)(w)$ *and* $Z(\Im g)(w)$ *are colinear for any* w *in the space* $\mathbb{C}^2$.

Proof. Observe that a system of norm-one vectors is orthonormal if and only if the related frame operator is Id ([28], [32]). Thus let us consider the frame operator of (b_{lm}) as

$$\text{(4.6)} \quad Vf = 2^{-2} \sum_{(m,l)\in\mathbb{Z}^2} \langle f, (m,l)g + J(m,l)J^{-1}g\rangle \left((m,l)g + J(m,l)J^{-1}g\right) =$$

$$\text{(4.7)} \quad = 2^{-2}(S_\pi f + S_\rho f + 2Qf).$$

$S_{Gab(1/2,1),g}$ and $S_{JGab(1/2,1)J^{-1},g}$ are the frame operators of appropriate representations; both are equal 2Id (since the frames are tight). The first summation is not only over L, because constant has been changed. To finish it is enough to find that

$$Q = \sum_{(m,l)\in\mathbb{Z}^2} \langle \cdot, (m,l)g\rangle J(m,l)J^{-1}g = 0.$$

From the Corollary 2 and the discussion before this theorem it is equivalent to the condition

$$\text{(4.8)} \quad \left\langle Jg_{(w_1,w_2)}, g_{(w_1,1/2-w_2)}\right\rangle \equiv 0$$

for all w_1, w_2. In terms of Zak transform this is (Z_j is the jth coordinate of Z)

$$Z_1 g(w)\overline{Z_1 g(F(w))} = Z_2 g(w)\overline{Z_2 g(F(w))}.$$

Note that $w = (w_1,w_2)$ and $F(w) = (w_1, 1/2 - w_2)$. Decompose g into real and imaginary parts ($g = f + ih$). For real-valued functions one has

$$\overline{Z_j f(F(w))} = Z_{3-j} f(w), \qquad j = 1,2\,,$$

and

$$\overline{Z_j(f+ih)(F(w))} = Z_{3-j}f(w) - iZ_{3-j}h(w).$$

The condition (4.8) translates into

$$Z_2f(w)Z_1h(w) = Z_1f(w)Z_2h(w).$$

Hence the points $Zf(w)$ and $Zh(w)$ are colinear in $\mathbb{C}^2$ for any w. The special case, when this condition holds is $Z_1h = Z_2h \equiv 0$, so $h = 0$ or g is real-valued. □

Thus we have come to the geometric version of the equivalent condition given by P. Auscher [3].

Remark 14. *Our proof of this theorem allows us to say that orthonormality of Wilson basis follows from the system of equations satisfied by the tight frame and symmetry - antisymmetry condition (4.8) expressed in the terms of the rank 2 operator. The general theory of direct integral decomposition tells that for the two equivalent representations, which have got the same algebra generated by the representation operators, their equivalence operator can be decomposed into the equivalence operators between the fiber representations with the only addition that the base space of one representation is permuted by an isomorphism in the sense of measure. In the case discussed in this section the equivalence operator is related to the pair of the family of operators*

$$J_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for all $x \in [0,1) \times [0,1/2)$ *and the isomorphism* F *of the base space (comp.* [72] *thm. 2.7 and its proof). Note that the same proof as above works for the case* $[(m,l) - J(m,l)J^{-1}]g$*. What one needs to check it is also* $Q = 0$.

4.5. **Discussion with Casazza's result about a sum of at least 3 ONB.** Recently P.G. Casazza [18] showed that a tight frame is a sum of 2 orthonormal bases if and only if it is a Riesz basis. So the arbitrary tight frame can be represented only as a sum of 3 ONB and this representation (this is also shown in [18]. In the previous sections we have derived two orthonormal bases from a Gabor tight frame and one can reconstruct it from them. The seeming contradiction is easy to relax. Casazza represents a tight frame x_i by the sum $ae_i + bf_i$ with constant coefficients a, b for all i (e_i and f_i are orthonormal bases), while in our case the tight frame x_i has the form: $(e_i + f_i)_i \cup (e_i - f_i)_i$.

4.6. **Unconditionality.**

4.6.1. *Bargmann and coorbit spaces.* We shall denote in this section the Gaussian function by $\varphi(x) := e^{-\pi x^2/2}$. By building the Wilson-type basis we shall come to the unconditional basis in Bargmann space in terms of reproducing kernels. This construction is due to K. Gröchenig and D. Walnut [57]. The version of the proof which is presented here is published in [91].

Definition 23. *The Bargmann transform (see* [7]*) is a unitary mapping of* $L^2(\mathbb{R})$ *onto* $A^2(\mathbb{C})$*, defined by the following formula*

$$Bf(z) = 2^{1/4} e^{-\pi z^2/2} \int_{\mathbb{R}} e^{2\pi xz - \pi x^2} f(x)\ dx.$$

Definition 24. *The reproducing kernel for the Bargmann space $A^2(\mathbb{C})$ is a function e_w satisfying*

$$\langle f, e_w \rangle_{A^2(\mathbb{C})} = f(w).$$

The function e_w is given by the formula:

$$e_w(z) = e^{-i\pi z\overline{w}}.$$

One can directly check that

$$B\left(e^{2\pi ipx}\varphi(x-q)\right) = e^{-\pi|p+iq|^2/2} \quad e_{-(p+iq)}\left(z\right).$$

Observe that Bargmann transform intertwines the action of Schrödinger representation on $Co(L^p)$ with isometric translations on $A^p(\mathbb{C})$[57].

Proposition 12. *If $1 \leq p \leq q \leq \infty$, then $Co(L^p)_{\varphi_0} \subset Co(L^q)_{\varphi_0}$ and the injection is continuous.*

Proposition 13. *If $1 \leq p \leq q \leq \infty$ and $0 \leq \vartheta \leq 1$, then*

$$\left(Co(L^p)_{\varphi_0}, Co(L^q)_{\varphi_0}\right)_{\vartheta,r} = Co(L^r)_{\varphi_0},$$

where $\frac{1}{r} = \frac{1-\vartheta}{p} + \frac{\vartheta}{q}$, i.e., $Co(L^p)_{\varphi_0}$ are a complex scale of interpolation.

Proposition 14. *The linear span of $\left((p,q)\,\varphi_0\right)_{\mathbf{p},\mathbf{q}\in\mathbb{R}}$ is dense in $Co(L^p)_{\varphi_0}$ for $1 \leq p < \infty$.*

These facts can be obtained from general theory of coorbit spaces or deduced from the results for Bargmann spaces [61] and translated into terms of coorbit spaces by means of Bargmann transform, since we shall use these facts only for the Gaussian function denoted in the further text by $\varphi(x) := e^{-\pi x^2/2}$.

4.6.2. *Wilson basis in coorbit and Bargmann spaces.* Let us consider the system:

$$x_{m0} = M_m\varphi, \tag{4.9}$$

$$x_{ml} = 2^{-1/2}\left(M_{\frac{m}{2}}T_l\varphi + (-1)^{l+m}M_{\frac{m}{2}}T_{-l}\varphi\right), \tag{4.10}$$

where $m \in \mathbb{Z}$ and $l \geq 1$. The theorem below is the central point of this section.

Theorem 26. [57] *The system (x_{ml}) is an unconditional basis in $Co(L^p)$.*

This assertion follows from the Lemma given below.

Lemma 5. *$(x_{ml})\,,\left(2S^{-1}x_{ml}\right)$ are biorthogonal Riesz bases in $L^2(\mathbb{R})$. Vectors of both systems belong to $Co(L^p)$ for $1 \leq p \leq \infty$ and*

$$\left\|\sum \lambda_{ml}x_{ml}\right\|_{Co(L^p)} \leq C\left\|\lambda\right\|_{l^p},$$

$$\left\|\sum \lambda_{ml}\,S^{-1}x_{ml}\right\|_{Co(L^p)} \leq C\left\|\lambda\right\|_{l^p}.$$

Proof. We have seen in Example 4 that $\left(M_{\frac{m}{2}}T_n\varphi\right)$ is a frame. The same argument shows that the related frame operator S is invertible in $Co(L^p)$ (time-frequency shifts are isometries for these spaces, too). Thus, S^{-1} is continuous in $Co(L^p)$ for $1 \leq p \leq \infty$. Thus $\left(S^{-1/2}\varphi_{mn}\right)$ is a tight frame. Since φ is real-valued, it follows from Walnut's representation that $S^{-1/2}\varphi$ is real-valued, too. It is easy to check that $2^{1/2}S^{-1/2}\varphi$ has norm 1. From commutativity of $\pi\left(\frac{m}{2},n\right)$ with $S^{-1/2}$, the system $\left(S^{-1/2}x_{ml}\right)$ is a Wilson system for $S^{-1/2}\varphi$. Therefore, $\left(2^{1/2}S^{-1/2}x_{ml}\right)$

is an orthonormal basis in $L^2(\mathbb{R})$ from Theorem 25. For the Hilbert part of the statement we have:

$$\left\langle x_{ml}, 2S^{-1}x_{m'l'}\right\rangle = \left\langle 2^{1/2}S^{-1/2}x_{ml}, 2^{1/2}S^{-1/2}x_{m'l'}\right\rangle. \tag{4.11}$$

As stated above, $\left(2^{1/2}S^{-1/2}x_{ml}\right)$ is an orthonormal basis in $L^2(\mathbb{R})$, and $2^{-1/2}S^{1/2}$ and $2^{1/2}S^{-1/2}$ are isomorphisms of $L^2(\mathbb{R})$ mapping this basis to (x_{ml}), $\left(2S^{-1}x_{ml}\right)$, respectively. Biorthogonality follows from (4.11).

Vectors x_{ml} belong to all $Co(L^p)$, because they are finite sums of isometric images of one vector. Since S^{-1} is continuous in these spaces, the same holds for $2S^{-1}x^{ml}$.

Let us note first that

$$\left\|\sum \lambda_{ml}x_{ml}\right\|_{Co(L^p)} \leq 2\left\|\sum \lambda_{i(m,l)j(m,l)}\varphi_{ml}\right\|_{Co(L^p)}.$$

From triangle inequality and Proposition 4.4.2,

$$\left\|\sum \lambda_{ml}\varphi_{ml}\right\|_{Co(L^1)(\varphi)} \leq \|\varphi\|_{Co(L^1)(\varphi)}\|\lambda\|_{l^1} \leq C\|\lambda\|_{l^1}. \tag{4.12}$$

From Proposition 4.4.1,

$$\left\|\sum \lambda_{ml}\varphi_{ml}\right\|_{Co(L^\infty)(\varphi)} = \sup_{p,q}\left|\left\langle \sum \lambda_{ml}\varphi_{ml}, (p,q)\varphi\right\rangle\right| \leq$$

$$\leq \|\lambda\|_{l^\infty}\sup_{p,q}\sum\left|\langle (p,q)\varphi, \varphi_{ml}\rangle\right| \leq C\|\lambda\|_{l^\infty}. \tag{4.13}$$

Interpolating (4.12) and (4.13), we obtain the desired inequality. The second one follows from the continuity of S^{-1}.

□

Proof. Completion of the proof of theorem 26: Consider two operators from the scale of the spaces $l^p(\mathbb{Z}^2)$ into the scale of spaces $Co(L^p)$:

$$T_0 : e_{ml} \mapsto x_{ml}, \tag{4.14}$$

$$T_1 : e_{mn} \mapsto 2S^{-1}x_{ml}. \tag{4.15}$$

Thus T_0^* maps $Co(L^q)$ into $l^q(\mathbb{Z}^2)$, where $q = p^*$. The biorthogonality of the systems gives $\mathrm{Id} = T_1T_0^*$. Since both operators are continuous, we obtain a continuous factorization of Id through l^p, which implies unconditionality of the basis. □

According to the recent and general result of H. Feichtinger and K. Gröchenig [41], it is clear that S is invertible in all $Co(L^p)$. In this special case, which is the subject of Gröchenig-Walnut theorem, we show it with much less work. If we now apply to the system (x_{ml}) the Bargmann transform, we get

$$u_{m0} = Bx_{m0} = e^{-\frac{\pi}{2}m^2}e_{-m}(z)$$

$$u_{ml} = Bx_{ml} = 2^{-1/2}e^{-\frac{\pi}{2}\left|\frac{m}{2}+il\right|^2}e_{-\left(\frac{m}{2}+il\right)}(z) + 2^{-1/2}(-1)^{l+m}e^{-\frac{\pi}{2}\left|\frac{m}{2}-il\right|^2}e_{-\left(\frac{m}{2}-il\right)}(z),$$

$$u_{ml} = 2^{-1/2}e^{-\frac{\pi}{2}|w_{m,l}|^2}\left(e_{-w_{m,l}}(z) + (-1)^{l+m}e_{-\overline{w_{m,l}}}(z)\right),$$

where $w_{m,l} = \frac{m}{2} + il$.

Since B is an isometry of $Co(L^p)$ onto $A^p(\mathbb{C})$, we get

Theorem 27. [57] *The system of vectors:*

$$u_{m0} = e^{-\frac{\pi}{2}m^2} e^{i\pi z m}(z),$$
$$u_{ml} = 2^{-1/2} e^{-\frac{\pi}{2}|w_{m,l}|^2}\left(e^{-i\pi z \overline{w_{m,l}}} + (-1)^{l+m} e^{-i\pi z w_{m,l}}\right),$$

for $m \in Z$ and $l \geq 1$, is an unconditional basis in $A^p(\mathbb{C})$ for $1 \leq p < \infty$.

5. Frame bounds

5.1. **Walnut's representation.** For a function $g \in L^2(\mathbb{R})$, let us define Walnut's coefficients to be (k is an integer, $x \in [0, b]$):

$$c_k(x) = \sum_{n\in\mathbb{Z}} g(x - nb)\, \overline{g(x - nb + a^{-1}k)}.$$

If it is necessary to avoid ambiguity, we emphasize the dependence of c_k on g writing $c_k(g)(x)$.Observe that each of these coefficients is almost everywhere finite. Indeed,

$$|c_k(g)(x)| \leq c_k(|g|)(x),$$

$$\begin{aligned}\int_0^b c_k(|g|)(x)dx &= \int_0^b \sum_{n\in\mathbb{Z}} |g|(x - nb)\, \overline{|g|(x - nb + a^{-1}k)}\, dx = \\ &= \int_{\mathbb{R}} |g(y)|\, \left|\overline{g(y + a^{-1}k)}\right|\, dy \leq \|g\|^2_{L^2(\mathbb{R})} < \infty.\end{aligned}$$

The non-negative measurable function whose integral is finite is finite almost everywhere.

In the context of Gabor frames Walnut's coefficients are important because of the following fact. Assume that $g \in Adm\left(Gab_{a,b}, L^2(\mathbb{R})\right)$. For the Gabor frame operator realted to the system $Gab_{a,b}g$ the following representation formula holds for all compactly supported and bounded functions f on $\mathbb{R}$and with unconditional convergence for almost all $x \in [0, b]$ [86]:

$$\langle Sf, f\rangle = a^{-1} \sum_{k\in\mathbb{Z}} \left\langle c_k(x) f(x + ka^{-1}), f(x)\right\rangle_{L^2(\mathbb{R})}. \tag{5.1}$$

In fact, one obtains it by means of the decomposition into a direct integral along the algebra $\mathfrak{A}(T_{a^{-1}}) \leq \mathfrak{A}_G{}'$. The proof of this theorem can be found in [86], [41] or [91].

5.2. **Matrix representation of frame operator - rational oversampling.** We shall now present a theorem revealing exact frame bounds. These are their optimal values, which is an advantage with comparison of known results. The formulae for exact bounds is of no bigger computational complexity than known formulae for estimations. To formulate the theorem, let us introduce some notation.

If a slight condition is imposed on g's decay (like $g(x) \leq \frac{C}{(1+|x|)^{1+\varepsilon}}$), $c_k(x)$ are bounded almost everywhere. Also from the condition (5.6) below and from

Cauchy-Schwarz inequality the boundedness of $c_k(x)$ follows. Denote $A_k(x,\lambda)$ a $p\times p$ matrix with the entries

$$A_k(x,\lambda)_{ij} = a^{-1}c_{pk+i-j}(x+(i-j)a^{-1})\lambda^k. \tag{5.2}$$

In the proof of the theorem below we will use Walnut's representation theorem [86] which states that the frame operator S for $(M_a^mT_b^ng)_{m,n\in\mathbb{Z}}$ is a 'combination' of 'translates' by $a^{-1}k$ with 'coefficients' c_k.

Theorem 28. *Let us assume that ab is rational (not necessarily the inverse of an integer) and $ab=\frac{p}{q}$. For $(M_a^mT_b^ng)_{m,n\in\mathbb{Z}}$ being a frame the exact frame bounds are:*

$$\begin{aligned} m &= \inf_{x\in[0,a^{-1}]}\ \inf_{\lambda\in\mathbb{S}^1}\left\|\left[\sum_k A_k(x,\lambda)\right]^{-1}\right\|^{-1}, \\ M &= \sup_{x\in[0,a^{-1}]}\ \sup_{\lambda\in\mathbb{S}^1}\left\|\sum_k A_k(x,\lambda)\right\|. \end{aligned}$$

In particular, $m>0$ and $M<\infty$.

Remark 15. *This method involves a two-dimensional extremum, but entries of A_k are easy to compute. They are expressed in terms of function itself and not its transform, what simplifies formulae and reduces a computational cost. Recently, the matrix approach to exact bounds by means of Zak transform was developed, see e.g.* [93], [64]. *This representation coincides with that which one can get by the methods presented in Chapter 3. Also Walnut's representation can be obtained as an application of the decomposition into direct integral along the algebra $\mathcal{C}(M_a^n)_{n\in\mathbb{Z}}$.*

Proof. Let us consider the frame operator S of the system $(M_a^mT_b^ng)_{m,n\in\mathbb{Z}}$. Recall that $g\in Adm\left(Gab_{a,b},L^2(\mathbb{R})\right)$. If it is a positive isomorphism, $(M_a^mT_b^ng)_{m,n\in\mathbb{Z}}$ is a frame and the smallest and the largest points of the spectrum of frame operator S coincide with the optimal values of the frame bounds for $(M_a^mT_b^ng)_{m,n\in\mathbb{Z}}$.

Let us study the decomposition of $L^2(\mathbb{R})$ into a direct integral

$$L^2(\mathbb{R}) = \int_{[0,a^{-1}]} l^2(\mathbb{Z})_x\ dx$$

defined by the formula $(f)_x = \left(f\left(x+a^{-1}k\right)\right)_{k\in\mathbb{Z}}$. It is easy to see that $l^2(\mathbb{Z})_x$ are 'invariant subspaces' for S (comp. (5.1)). Therefore we can reduce the problem of looking for the spectrum of S to studying the spectrum of S acting on $l^2(\mathbb{Z})_y$ (cf. section 2.4 and section 3.5). For $h\in l^2(\mathbb{Z})_y$ we have

$$(S_yh)(l) = a^{-1}\sum_{k\in\mathbb{Z}} c_k(y+la^{-1})h(k+l),$$

where S_y is the restriction of S to $l^2(\mathbb{Z})_y$. Then

$$\sup\sigma(S) = \sup_y\ \sup\ \sigma(S_y), \tag{5.3}$$

$$\inf\sigma(S) = \inf_y\ \inf\ \sigma(S_y). \tag{5.4}$$

The similar argument can be found in [41] Thm. 3.2.

Obviously, $c_k(x)$ are b-periodic. Therefore also

$$c_k(y+la^{-1}) = c_k(y+(l \bmod p)a^{-1}).$$

Let us decompose each of $l^2(\mathbb{Z})_y$ into the direct sum by means of the following identification J :

$$\mathcal{H} = l^2(\mathbb{Z})_y = l^2(p\mathbb{Z}) \oplus l^2(p\mathbb{Z}) \oplus \ldots \oplus l^2(p\mathbb{Z}). \tag{5.5}$$

$$l^2(\mathbb{Z}) \ni (a_n) \xrightarrow{J} \begin{pmatrix} (a_{kp})_{k\in\mathbb{Z}} \\ (a_{kp+1})_{k\in\mathbb{Z}} \\ \cdots \\ (a_{kp+p-1})_{k\in\mathbb{Z}} \end{pmatrix}$$

It is easy to see that after this decomposition and identification S_y is equivalent to the matrix of operators

$$S_y = \begin{pmatrix} \sum_{m\in\mathbb{Z}} c_{pm}(y)T^{pm} & \cdots & \cdots & \sum_{m\in\mathbb{Z}} c_{pm-p+1}(y+(p-1)a^{-1})T^{pm} \\ \sum_{m\in\mathbb{Z}} c_{pm+1}(y)T^{pm} & \ddots & \cdots & \sum_{m\in\mathbb{Z}} c_{pm-p+2}(y+(p-1)a^{-1})T^{pm} \\ & \ddots & \ddots & \\ \sum_{m\in\mathbb{Z}} c_{pm+p-1}(y)T^{pm} & \cdots & \cdots & \sum_{m\in\mathbb{Z}} c_{pm}(y+(p-1)a^{-1})T^{pm} \end{pmatrix}$$

with the summations over $m \in \mathbb{Z}$ and T^p being a translation operator in $l^2(p\mathbb{Z})$. Note that any of entry operators in this matrix are bounded, since the decomposition (5.5) is into finite number of summands and if $P_i : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection on ith summand of $\mathcal{H}$, it has norm 1. The $(i,j)-$entry of matrix S_y is

$$(S_y)_{ij} = P_j S_y P_i.$$

Then

$$(S_y)_{ij} = \sum_{m\in\mathbb{Z}} c_{pm+i-j}(y+ja^{-1})T^{pm}.$$

Since each of the operators $(S_y)_{ij}$ is a convolution, their spectral points can be obtained by Fourier transform and the spectrum is equal to

$$\sigma\left((S_y)_{ij}\right) = \left\{ \sum_{m\in\mathbb{Z}} c_{pm+i-j}(y+ja^{-1})\lambda^m : \ |\lambda| = 1 \right\}.$$

The curve $\sum_{m\in\mathbb{Z}} c_{pm+i-j}(y+ja^{-1})\lambda^m$ is an essentially bounded function on $\mathbb{T}$. The norm $\left\|(S_y)_{ij}\right\|$ and the quantity $\left\|(S_y)_{ij}^{-1}\right\|^{-1}$ are the endpoints of the interval $\sigma\left((S_y)_{ij}\right)$. The same thing holds also for the whole matrix S_y:

$$\sigma(S_y) = \left[\left\|S_y^{-1}\right\|^{-1}, \|S_y\|\right] = \bigcup_{|\lambda|=1} \sigma\left[\left(\sum_m c_{pm+i-j}(y+ja^{-1})\lambda^m\right)_{i,j=0..p-1}\right] =$$

$$= \bigcup_{|\lambda|=1} \left[\left\|\left[\left(\sum_m c_{pm+i-j}(y+ja^{-1})\lambda^m\right)_{i,j=0..p-1}\right]^{-1}\right\|^{-1}, \left\|\left(\sum_m c_{pm+i-j}(y+ja^{-1})\lambda^m\right)_{i,j=0..p-1}\right\|\right].$$

Indeed, $\widehat{(a*b)} = \widehat{a}\cdot\widehat{b}$ for $a,b \in l^2(\mathbb{Z})$ where $\widehat{\ }: l^2(\mathbb{Z}) \to L^2(\mathbb{T})$. By the analogous argument the matrix of convolution operators is by Fourier transform equivalent to the operator of multiplication by a matrix function with entries being Fourier transforms of convolution sequences, i.e.,

$$\begin{bmatrix} a*\cdot & b*\cdot \\ c*\cdot & d*\cdot \end{bmatrix}^{\widehat{}} = \begin{bmatrix} \widehat{a} & \widehat{b} \\ \widehat{c} & \widehat{d} \end{bmatrix}.$$

and the assertion follows from (5.3) and (5.2). □

5.3. **Integer oversampling.** For the case of integer oversampling ($p = 1$) we can do better:

Theorem 29. *Let $(ab)^{-1} = N \in N$. If for a given real-valued function $g \in L^2(R)$ the conditions*

$$\infty > \text{ess}\sup \; c_k(x) \geq \text{ess}\inf \; c_k(x) > 0 \; a.e. \tag{5.6}$$

$$|c_1(x)| \geq \sum_{k>1} k^2 |c_k(x)| \; a.e. \, , \tag{5.7}$$

are satisfied, then the exact frame bounds for $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ are given by (denote $h(x) := sgn\, c_1(x)$)

$$m = a^{-1}\text{ess}\inf_x \left(\sum_{k\in Z} (-h(x))^k c_k(x) \right), \tag{5.8}$$

$$M = a^{-1}\text{ess}\sup_x \left(\sum_{k\in Z} h(x)^k c_k(x) \right). \tag{5.9}$$

Remark 16. *It is easy to check that if one of the inequalities in the condition (5.6) is not satisfied, then $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ is not a frame and $m \leq 0$ or $M = \infty$, respectively. The condition (5.6) implies by Cauchy-Schwartz inequalities that the same inequalities are satisfied by other c_k's. If the condition (5.6) holds and $m \leq 0$, then the system $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ is not a frame, either. The detailed study on the necessity of these assumptions is given in Sec. 5.5. The criterion ess $\inf c_0(x) > \sum_{k\neq 0}$ ess $\sup |c_k(x)|$ from [86] guarantees that $m > 0$. It can be checked that, if $c_k(x)$ is not identically 0 only for $|k| \leq 2$, then the condition (5.7) in Theorem 29 is necessary. The condition (5.7) can be interpreted as a criterion for a function flatness. For example if $a = b = \frac{1}{2}$, the function $\frac{1}{C+x^4}$ satisfies the condition (5.7) as long as $C \leq C_0$, with C_0 lying between $3,74$ and $3,77$.*

Proof. Observe that in the integer oversampling case all matrices considered above are scalars and $A_k(x,\lambda) = a^{-1}c_k(x)\lambda^k$. Observe further that $c_k(x) = \overline{c_k(x)}$. We have

$$\begin{aligned}\sigma\left(S_y\right) &= \overline{\bigcup_{|\lambda|=1}\left\{\sum_k c_k(y)\lambda^k\right\}}\\ &= \overline{\left\{c_0(y)+2\sum_{k\geq 1}c_k(y)\Re(\lambda^k) : |\lambda|=1\right\}}\\ &= \overline{\left\{c_0(y)+2\sum_{k\geq 1}c_k(y)\cos kt : t\in[0,2\pi]\right\}},\end{aligned}$$

where $\lambda = e^{it}$. We shall consider

$$f(t) = c_0(y) + 2\sum_{k\geq 1} c_k(y)\cos kt.$$

To fix attention, let us assume that $c_1(y) > 0$ and prove the following:

Lemma 6. *If $t \in [0, \pi]$, then $f'(t) \leq 0$.*

From Lemma 6 the assertion of Theorem 29 follows easily. Indeed,

$$\begin{aligned} f_{max} &= f(0) = \sum_k c_k(y),\\ f_{min} &= f(\pi) = \sum_k (-1)^k c_k(y).\end{aligned}$$

Then $m = \inf\sigma(S) = \inf_y \inf\sigma(S_y) = \inf_y \inf\left(\sum_k (-1)^k c_k(y)\right)$ and analogously for M. If, in turn, $c_1(y) < 0$, we get that derivative is non-negative and extrema are realized at the ends of the interval but in the switched order. □

Proof. of Lemma 6. We show that

$$f'(t) = -2\sum_{k>0} kc_k(y)\sin kt \leq 0.$$

Let us first justify the inequality $|\sin kt| \leq k\sin t$ by dividing the interval $[0, \pi]$ into the three subintervals

$$\left[0, \frac{\pi}{2k}\right], \quad \left[\frac{\pi}{2k}, \pi - \frac{\pi}{2k}\right], \quad \left[\pi - \frac{\pi}{2k}, \pi\right]$$

and proving the inequality for each of them. From this inequality and the assumption 5.7, we get

$$\begin{aligned}\sum_{k>0} kc_k(y)\sin kt &= c_1(y)\sin t + \sum_{k>1} kc_k(y)\sin kt\\ &\geq \sum_{k>1} k\left|c_k(y)\right| k\sin t + \sum_{k>1} kc_k(y)\sin kt\\ &\geq \sum_{k>1}\left(k\left|c_k(y)\right|\left|\sin kt\right| + kc_k(y)\sin kt\right) \geq 0.\end{aligned}$$

□

5.4. **Admissibility condition for integer oversampling.** Let us investigate now properties of the condition (5.7) for small a and b. For some functions this condition is local in the following sense:

Theorem 30. *If $\varphi : \mathbb{R} \to \mathbb{C}$ is:*
1) a continuous nonnegative function with compact support,
or
2) a function e^{-rx^2} for some $r > 0$,
or
3) a function $e^{-r|x|}$ for some $r > 0$,
then for a and b small enough the condition (5.7) holds.

Proof. The condition (5.7) reads as

$$c_1(x) \geq \sum_{k>1} k^2 c_k(x)$$

which is equivalent to

$$b \sum_n g(x - nb) \overline{g(x - nb + a^{-1})} \geq b \sum_{k>1} k^2 \sum_n g(x - nb) \overline{g(x - nb + ka^{-1})}. \tag{5.10}$$

Taking the limit with b tending to 0, we have

$$\int_{\mathbb{R}} g(x) g\left(x + a^{-1}\right) dx \geq \sum_{k>1} k^2 \int_{\mathbb{R}} g(x) g\left(x + ka^{-1}\right) dx. \tag{5.11}$$

It is straightforward that if (5.11) holds with the strong inequality, then there exists b small enough that (5.10) holds, too. This follows from the positivity of the considered function.

It is easy to see, that if g has a compact support contained in $[-N, N]$, then the condition (5.7) is satisfied for $a \in \left(0, \frac{2}{N}\right]$ and b small enough. For the function e^{-rx^2}, the inequality (5.11) turns into the condition

$$e^{-\frac{r}{2a^2}} \int_{\mathbb{R}} e^{-2r\left(y + \frac{1}{2a}\right)^2} dy \geq \sum_{k>1} k^2 e^{-\frac{r}{2}\left(\frac{k}{a}\right)^2} \int_{\mathbb{R}} e^{-2r\left(y + \frac{1}{2a}\right)^2} dy$$

being equivalent to

$$1 \geq \sum_{k>1} k^2 \left(e^{-\frac{r}{2a^2}}\right)^{k^2 - 1}$$

and certainly valid for a which are small enough.

In the case of the exponential function, the inequality (5.11) reads as

$$e^{-ra^{-1}} \left(a^{-1} + \frac{1}{r}\right) \geq \sum_{k>1} k^2 e^{-ra^{-1}k} \left(a^{-1}k + \frac{1}{r}\right).$$

Let us denote $z := e^{-ra^{-1}}$. Summing up the power series in z we get:

$$a^{-1} \left(16z^2 + \frac{11z^3 - 4z^2 - z}{(1 - z)^4}\right) \leq \frac{1}{r} \left(2z - \frac{z(1 + z)}{(1 - z)^3}\right),$$

Since $z = e^{-ra^{-1}}$, the term $a^{-1} = -\frac{1}{r} \ln z$. For z close to 0 the left-hand side is of order $\frac{1}{r} z \ln z$ and so negative, while the right-hand side is of order $\frac{1}{r} z$ and positive.

Thus, for z small enough, the right-hand side dominates and the inequality (5.11) holds also in the third case of the considered ones. $\square$

5.5. **Necessary conditions for exact bounds.** The theorem below is a corollary from the proof of Theorem 29.

Theorem 31. *If for* $(ab)^{-1} \in \mathbb{N}$ *the following conditions hold*

$$0 < ess\ \inf\ c_0(x) \le ess\ \sup c_0(x) < \infty\ a.e.\ , \tag{5.12}$$

$$|c_1(x)| \ge \sum_{k>1} k^2 |c_k(x)|\ \ a.e.\ , \tag{5.13}$$

$$\inf_x \sum_k \left[-sgn\, c_1(x)\right]^k c_k(x) > 0\ , \tag{5.14}$$

$$\sup_x \sum_k \left[sgn\, c_1(x)\right]^k c_k(x) < \infty, \tag{5.15}$$

then $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ *is a frame with the bounds given by (5.8) and (5.9).*

The objective of this section is to consider the necessary conditions for $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ to be a frame and to have optimal bounds as given in Theorem 29.

Theorem 32. *If one of the assumptions (5.12), (5.14), or (5.15) in Theorem 31 does not hold, then* $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ *is not a frame.*

Proof. If (5.12) does not hold, then it follows from the discussion following Theorem 2.5 [28] or Proposition 4.1.4 [60] that $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ cannot be a frame. If (5.14) does not hold, then the frame operator S is not positive, because $m(a,b)$ is its (at least asymptotic) spectrum point. So $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ is not a frame. In case of (5.15) the analogous reasoning shows that S is not bounded. Hence, we get a contradiction with the frame condition for $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ which requires that S is an isomorphism. $\square$

We have shown that for being a frame, the assumptions (5.12), (5.14), and (5.15) are necessary. Let us check now what is a necessary condition, if we know additionally the optimality of frame bounds. The answer is the following:

Theorem 33. *If* $(M_a^m T_b^n g)_{m,n\in\mathbb{Z}}$ *is a Gabor frame and the optimal frame bounds are given by (5.8) and (5.9), then*

$$|c_1(x)| \ge -2h(x) \sum_{k>1} c_{2k+1}(x) + 2\left|\sum_{k>1} c_{2k}(x)\right| \quad a.e.\ .$$

Remark 17. *The assertion of Theorem 33 shows that the magnitude of* $c_1(\cdot)$ *in comparison with the further* c_k*'s is essential in getting the optimal bounds. Moreover, one can easily see that, if* $c_k(x) \equiv 0$ *for* $|k| > 2$*, then a necessary condition to get optimal bounds by the formulae (5.8) and (5.9) is*

$$|c_1(x)| \ge 4\,|c_2(x)| \quad a.e.\ .$$

Henceforth, the weight k^2 *is also relevant.*

To prove the theorem, we need first Proposition 17 considering relations between c_k's and frame bounds.

Proof. Let us fix $x \in [0, b]$. Substitute

$$A(x) = \sum (-sgn\, c_1(x))^k c_k(x)$$

and

$$B(x) = \sum [sgn\, c_1(x)]^k c_k(x) \quad .$$

The following calculation justifies the assertion (The first inequality is the thesis of Proposition 17):

$$A(x) := \sum [-sgn\, c_1(x)]^k c_k(x) \leq c_0(x) - |c_1(x)|,$$

$$c_0(x) - 2|c_1(x)| + 2c_2(x) - 2sgn\, c_1(x) c_3(x) + 2c_4(x) - ... \leq c_0(x) - |c_1(x)|,$$

$$|c_1(x)| \geq 2(c_2(x) + c_4(x) + ...) - 2sgn\, c_1(x)(c_3(x) + c_5(x) + ...).$$

Analogously, for the upper bound, we get

$$|c_1(x)| \geq -2(c_2(x) + c_4(x) + ...) - 2sgn\, c_1(x)(c_3(x) + c_5(x) + ...).$$

Hence,

$$|c_1(x)| \geq 2|c_2(x) + c_4(x) + ...| - 2sgn\, c_1(x)(c_3(x) + c_5(x) + ...) \quad .$$

□

6. Estimates for frame bounds

6.1. **Littlewood-Paley type inequalities.** The result of C.K. Chui and X. Shi, namely [24] Thm. 2, describes relations between the 0th coefficient in Walnut's representation

$$c_0(x) = \sum_{n \in \mathbb{Z}} |g(x - nb)|^2$$

or in the wavelet representation formula (3.8)

$$c_0(y) := \sum_{j=-\infty}^{\infty} |\widehat{f}(a^{-j}y)|^2.$$

and the appropriate frame bounds. In the affine case they are similar to the classical Littlewood-Paley inequalities and act in the literature under the name of Littlewood-Paley type inequalities or 'diagonal result'. The latter name is derived from the fact that in the matrix of the frame operator (the infinite-dimensional one) coefficients $c_0(x) = \sum |g(x - nb)|^2$ appear on the diagonal. By the decomposition methods one finds a simpler proof to this result and the possibility of a generalisation.

Proposition 15. [24] *Let $(M_a^m T_b^n g)_{m,n \in \mathbb{Z}}$ be a Gabor frame with the bounds A and B. Then*

$$A \leq a^{-1} c_0(x) \leq B \quad a.e. \quad .$$

Proof. From Theorem 16 we infer that

$$A \, \mathrm{Id}_{l^2(\mathbb{Z})} \leq S_x \leq B \, \mathrm{Id}_{l^2(\mathbb{Z})}.$$

TABLE 2. Frame bounds for the system $(M_a^m T_b^n f)_{m,n\in\mathbb{Z}}$ for the Gaussian function $f(x) = \pi^{-1/4}e^{-x^2/2}$

			$ab = 1/4$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
0.50	0.9094	1.2025	7.0905	7.0905
1.00	3.8530	3.8530	4.1469	4.1469
2.00	3.3219	3.3219	4.6788	4.6788
3.00	1.4271	1.4271	6.7719	6.7719
			$ab = 3/8$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
1.00	1.7605	1.7696	3.5727	3.5724
1.50	2.5002	2.5521	2.8331	2.7809
2.00	2.2095	2.2245	3.1243	3.1192
2.50	1.5767	1.6000	3.7759	3.7758
			$ab = 1/2$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
1.00	0.4543	0.6011	3.5456	3.5456
2.00	1.5752	1.6001	2.4251	2.4251
3.00	0.7127	0.7134	3.3868	3.3868
			$ab = 3/4$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
1.00	-0.8786	0.0275	3.5452	3.5449
1.50	0.2448	0.4001	2.4217	2.3637
2.00	0.5792	0.8322	2.0876	1.8178
2.50	0.5551	0.7737	2.1211	1.8998
			$ab = 0.95$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
1.50	-0.3162	0.0590	2.4215	2.3633
2.00	0.0334	0.1552	2.0720	1.8054
2.50	0.0939	0.1823	2.0189	1.6713
3.00	0.0826	0.1538	2.0750	1.8123
3.50	0.0559	0.1037	2.2171	2.0799
4.00	0.0311	0.0575	2.4314	2.3756

Consider the standard unit vector basis (e_k) in $l^2(\mathbb{Z})_x$ and observe that $\left[c_k(x+\cdot a^{-1})\right] \in l^2(\mathbb{Z})$. Then

$$\langle S_x e_m, e_m\rangle = \langle a^{-1}\sum c_k(x+\cdot a^{-1})T^k e_m, e_m\rangle =$$
$$= a^{-1}\sum c_k(x+ma^{-1})T^k e_m(m) = a^{-1}c_0(x+ma^{-1})$$

for the arbitrary x and m. □

Remark 18. *In the proof we have only used that A and B are the frame bounds for S_x.*

Therefore, we get:

TABLE 3. Frame bounds for the system $(M_a^m T_b^n f)_{m,n\in\mathbb{Z}}$ with the exponential function $f(x) = e^{-|x|}$

			$ab = 1/4$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
1.00	2.5995	2.7160	6.0563	6.0563
1.50	2.6653	2.6855	6.7812	6.7812
2.00	2.1787	2.1828	8.3255	8.3255
			$ab = 3/16$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
0.50	2.0144	2.8882	8.8735	8.4876
1.00	4.2025	4.4920	7.3385	7.0300
1.50	3.7238	3.7757	8.8716	8.8385
2.00	2.9379	2.9672	11.0677	11.0647
			$ab = 1/8$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
1.00	6.7573	6.7598	10.5543	10.5543
1.50	5.6343	5.6344	13.2588	13.2588
2.00	4.4115	4.4115	16.5970	16.5970

TABLE 4. Frame bounds for the system $(M_a^m T_b^n f)_{m,n\in\mathbb{Z}}$ with the inverse polynomial function $f(x) = \frac{1}{1+x^4}$

			$ab = 1/8$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
0.25	6.9347	7.7693	19.7224	19.7196
0.75	13.1973	13.2050	13.4579	13.4574
1.00	12.0363	12.0396	14.6408	14.6408
2.00	8.0036	8.0038	16.1124	16.1124
			$ab = 1/2$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
0.75	-0.0304	0.7157	6.6943	6.6656
1.00	1.3639	1.5804	5.3053	5.2231
1.50	2.7515	2.8631	3.8836	3.8836
2.00	1.8842	1.8949	4.1447	4.1247
			$ab = 3/4$	
b	Lower (A)	Lower ex. (m)	Upper (B)	Upper ex. (M)
1.00	-0.9601	0.2844	5.4063	4.9655
1.25	-0.0839	0.8012	4.5624	3.9939
1.50	0.8352	1.2912	3.5881	3.3104
1.75	1.1227	1.5431	3.1196	2.8965
2.00	0.9216	1.3085	3.0976	2.8738

Proposition 16. *If $A(x)$ and $B(x)$ are the smallest and largest spectral points for S_x in $l^2(\mathbb{Z})$, then*

$$A(x) \leq a^{-1} c_0(x) \leq B(x) \quad a.e.\ .$$

We have also the following generalization.

Proposition 17. *If $A(x)$ and $B(x)$ are the smallest and largest spectral points for S_x in $l^2(Z)$, then*

$$A(x) \leq \frac{a^{-1}}{2}\left(c_0(x) + c_0\left(x + na^{-1}\right) - 2\left|c_n(x)\right|\right),$$
$$B(x) \geq \frac{a^{-1}}{2}\left(c_0\left(x\right) + c_0\left(x + na^{-1}\right) + 2\left|c_n\left(x\right)\right|\right).$$

Proof. Denote, as in Proposition 15, $d_k(l) := c_k(x + la^{-1})$. Observe then that

$$\langle S_x e_n, e_0\rangle = a^{-1}\langle \sum_k d_k T^k e_n, e_0\rangle = a^{-1}\sum_k d_k(0)T^k e_n(0) = a^{-1}d_n(0) = a^{-1}c_n(x)$$

and

$$c_{-n}(x + na^{-1}) = \overline{c_n(x)}.$$

Replacing e_0 with $f = e_0 + \lambda e_n$, $|\lambda| = 1$ in the proof of Proposition 15 and optimizing over λ, we get

$$\langle S_x f, f\rangle = \langle S_x e_0, e_0\rangle + |\lambda|^2\langle S_x e_n, e_n\rangle$$
$$+\lambda\langle S_x e_n, e_0\rangle + \overline{\lambda}\langle S_x e_0, e_n\rangle,$$

$$a\langle S_x f, f\rangle = c_0(x) + c_0(x + na^{-1})|\lambda|^2 + \lambda c_n(x) + \overline{\lambda c_n(x)},$$

$$\frac{a}{1+|\lambda|^2}\langle S_x f, f\rangle = \frac{c_0(x) + c_0(x + na^{-1})}{2} + \Re\left(\lambda c_n(x)\right),$$

hence the assertion follows. □

In the completely analogous way we show, using a "wavelet representation" from Theorem 20:

Proposition 18. *For an integer a and $\alpha \in \mathbb{Q}_a/b$ let $A(x)$ and $B(x)$ are the smallest and largest spectral points for S_x in $l^2(\mathbb{Z})$. Then*

$$A\left(x\right) \leq (2a)^{-1}\left[c_0\left(x\right) + c_0\left(x + \alpha\right) - 2\left|c_\alpha\left(x\right)\right|\right]$$
$$B\left(x\right) \geq (2a)^{-1}\left[c_0\left(x\right) + c_0\left(x + \alpha\right) + 2\left|c_\alpha\left(x\right)\right|\right]$$

6.2. **Sufficient conditions for frame.** The first sufficient conditions for Gabor and wavelet frames were presented by I. Daubechies in [28]. As for the existence of an upper frame bound, there was made certain research effort to establish sufficient and necessary conditions (see e.g. [20]). However, we shall concentrate in this chapter on the problem of a lower frame bound. For simplicity, we consider here only the lower frame bound conditions, since the upper's one follows immediately from another assumptions. The condition from [28] reads in our language as:

$$\inf_x |c_0(x)| - \sum_{k\neq 0}\sup_x c_k(|g|)(x) > 0,$$

where $c_k(|g|)$ is the kth Walnut's coefficient for the function $|g|$. In the case of a positive g it coincides with its refinement that appeared in [60] and [86], namely

Theorem 34. *If*

$$\inf_x |c_0(x)| - \sum_{k\neq 0} \sup_x |c_k(x)| > 0,$$

then $(M_a^m T_b^n f)_{m,n\in\mathbb{Z}}$ *is a frame.*

The further progress was reported recently in [21] and [35]:

Theorem 35. *Any of the two following conditions implies an existence of the lower frame bound*

$$\inf_x |c_0(x)| - \sup_x \sum_{k\neq 0} |c_k(x)| > 0, \tag{6.1}$$

$$\inf_x \left[|c_0(x)| - \sum_{k\neq 0} |c_k(x)| \right] > 0. \tag{6.2}$$

7. Appendix. Banach and von Neumann algebras

7.1. **Commutative C^*-algebras.** Let B be a Banach space with the multiplication of elements $\cdot : B \times B \to B$, which is continuous with respect to both variables in the norm topology, and an antilinear involution $* : B \to B$ satisfying $\|x\| = \|x^*\|$. Then B is a Banach algebra with involution. If, additionally, $\|x^*x\| = \|x\|^2$ for all $x \in B$, then B is a C^*-`algebra`. If $xy = yx$ for any $x, y \in B$, B is commutative. For commutative Banach algebras $I \subset B$ is an `ideal` if, for any $x \in B$, $xI \subset I$ holds. Denote by $\mathfrak{M}(B)$ the space of all maximal ideals in B (`spectrum` of B). For a maximal ideal M the quotient B/M is the field of complex scalars. The space $\mathfrak{M}(B)$ can be identified with the space of all continuous homomorphisms of B onto

Indeed, for a continuous homomorphism $\chi : B \to \mathbb{C}$ the set $\chi^{-1}(0)$ is a maximal ideal. Conversely, the canonical projection B/M is a character on B. The `Gelfand transform` relates the element $x \in B$ with the function $\widehat{x}$ on $\mathfrak{M}(B)$ defined by:

$$\widehat{x}(\chi) = \chi(x).$$

The space $\mathfrak{M}(B)$ is equipped with the weakest topology under which all functions $\widehat{x}$ are continuous, i.e., `Gelfand topology`. The inverse Gelfand transform maps continuous functions on the spectrum of commutative C^*-algebra B onto elements of B (so-called *functional calculus* or `spectral theorem`). For a general commutative Banach algebra this is not the case.

Let us assume that B is an algebra of (not all) operators in a Hilbert space $\mathcal{H}$. (So B is represented in $\mathcal{H}$). The Gelfand transform of identity induces the decomposition of the Hilbert space into the direct integral indexed by points of the spectrum $\mathfrak{M}(B)$ of B. In this integral all elements of B are diagonal and decomposable.

7.2. **Von Neumann algebras.** Let N be a set of bounded operators acting in the Hilbert space $\mathcal{H}$. The `commutant` of N is the set of all bounded operators from $B(\mathcal{H})$ commuting with the elements of N:

$$N' := \{B \in B(\mathcal{H}) : BN - NB = 0\}.$$

The set N' is a subalgebra of $B(\mathcal{H})$ containing the unit. Denoting $(M')' = M''$, one has $M' = M'''$ and $M \subset M''$. A C^*-algebra $\mathfrak{A}$ of operators acting in the Hilbert space is a `von Neumann algebra` if one of the following equivalent conditions holds:

1. $\mathfrak{A}$ contains the unit and is weakly closed;
2. $\mathfrak{A}$ contains the unit and is strongly closed;
3. $\mathfrak{A} = \mathfrak{A}''$.

If $\mathfrak{A}$ is a von Neumann algebra, $\mathfrak{A}$ and $\mathfrak{A}'$ are commutants of each other. Their common center (the elements of $\mathfrak{A}$ commuting with all others elements of $\mathfrak{A}$) is $\mathfrak{A} \cap \mathfrak{A}'$.

Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{R}$ be a subalgebra in the algebra $B(\mathcal{H})$ of all bounded operators acting in $\mathcal{H}$, $\mathcal{D}$ be its commutative subalgebra, and $\mathcal{D}_0$ be a subalgebra in $\mathcal{D}$. With a given direct integral of Hilbert spaces one links the algebras of decomposable, diagonal, and continuously diagonal operators (cf. section 2.4). The triplet $\mathcal{R} \geq \mathcal{D} \geq \mathcal{D}_0$ can be represented as the algebras of decomposable, diagonal and continuously diagonal operators if and only if $\mathcal{R}$ is closed in the strong operator topology, $\mathcal{D}$ is the center of $\mathcal{R}$ and $\mathcal{D}_0$ is closed in the norm topology and is dense in $\mathcal{D}$ in the strong topology. (Equivalently, the strong topology can be replaced with the weak one in this formulation.) Once this set of conditions is true, the base space of the direct integral

$$\int_X H_x d\mu(x)$$

is $X = \mathfrak{M}(\mathcal{D}_0)$ - the maximal ideal space for the algebra $\mathcal{D}_0$ and μ is the `basic measure`.

Von Neumann theorem. If $\mathcal{D}$, being an algebra of diagonal operators, is a center of $\mathcal{R}$ or, what is the same, of $\mathcal{R}'$, then the decomposition is into a direct integral of primary representations. Thus, the fiber algebras are factors (i.e., have trivial centers).

Mautner theorem. If $\mathcal{D}$, being an algebra of diagonal operators, is a maximal commutative subalgebra in the commutant $\mathcal{R}'$, then the decomposition is into a direct integral of irreducible representations.

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