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CHARACTERIZATION OF WILSON SYSTEMS FOR GENERAL LATTICES

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The Wilson orthonormal basis was constructed in 1991 by Daubechies, Jaffard and Journé using combinations of elements of Gabor tight frame with redundancy 2. In 1994, Auscher gave a characterization of the atoms for which the Wilson system is an orthonormal basis. Recently, Kutyniok and Strohmer generalized the notion of the Wilson system to the lattices whose generator matrix is in Hermite normal form.

We extend their result to the full characterization of Wilson orthonormal bases on the general lattice of volume 1/2. Moreover, we generalize this result to other forms of Wilson systems differing from the classical one by the appropriate sign modification.

Keywords: Gabor frame; tight frame; Wilson orthonormal basis; symplectic lattices; metaplectic representation.

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1. Introduction

Wilson systems were introduced in 1991 by Daubechies, Jaffard and Journé as the way to obtain an orthonormal basis from the elements of Gabor tight frame whose generating atom would have both good time — and frequency — localization.[7] From Balian–Low Theorem[2,16,3] it is known that the generating atom of any Gabor system at the critical density, i.e. in the only case when the Gabor system itself can be an orthonormal basis, cannot have both good time — and frequency — localization. Thus, the process of choosing the system of double redundancy and replacing the pairs of its elements by their linear combinations yielded effective reduction of the underlying tight frame into the orthonormal basis. In Ref. 7 also the example was given of the smooth and fast decaying function for which the obtained Wilson system was an orthonormal basis. All such atoms were characterized in 1994 by Auscher.[1] The effect of reducing the frame bound of the underlying Gabor tight frame when passing to the Wilson system by factor 2 was observed also for higher even redundancies.[5,6]

Motivated by applications for OFDM-QAM coding, Strohmer and Kutyniok generalized the notion of Wilson system to the lattices whose generator matrix is in Hermite normal form.[15] They proved that if Fourier transform of the image of an atom under metaplectic representation operator related to the lattice generator is real-valued, then the appropriately defined Wilson system is an orthonormal basis for $L^2(\mathbb{R})$. From their proof one can infer this result for all lattices of volume $1/2$.

In the present paper we extend Kutyniok–Strohmer result to the full characterization of the atoms for which Wilson system on such lattice is an orthonormal basis for $L^2(\mathbb{R})$. Moreover, this result covers for instance the case when the sign sequence $(-1)^{m+n}$ in the classical Wilson system definition is replaced with $(-1)^m$.

The paper is organized as follows: in Sec. 2 we introduce the necessary notation and the properties of symplectic matrices and metaplectic representation; in Sec. 3 the definition of Wilson system is introduced and the results for rectangular lattices are summarized, while in Sec. 4 we demonstrate how to extend the characterization from Ref. 17 to the case of the general lattices. We provide also the examples of a function for which the modified Wilson system is an orthonormal basis in the cases of rectangular and hexagonal lattices as well as new pairings resulting from our approach in the rectangular case.

2. Preliminaries

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are, respectively, the set of all natural, integer, real, and complex numbers. The set of all matrices $k \times k$ with real entries is $M_k(\mathbb{R})$. The general linear group is the subset of invertible matrices in $M_k(\mathbb{R})$ being denoted by $GL(k, \mathbb{R})$ and the special linear group being its subset with determinant 1 by $SL(k, \mathbb{R})$. We refer to the entries of the matrix $X \in M_k(\mathbb{R})$ in ith row and jth column by X_{ij}. In the forthcoming presentation we limit ourselves to the case of one-dimensional Heisenberg group and Schrödinger representations. For the validity of the results in higher dimension cf. Remark 4.1. Let us also distinguish three families of matrices in $SL(2, \mathbb{R})$:

$$\mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{N}_a = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \quad \mathcal{D}_b = \begin{bmatrix} b & 0 \\ 0 & \frac{1}{b} \end{bmatrix}.$$

and introduce notation $A' := \mathcal{J}^{-1} A \mathcal{J}$.

Heisenberg group $\mathbb{H}$ (cf. Ref. 9, p. 19 and Ref. 10, Definition 9.1.2) is the set $\mathbb{R}^3$ equipped with the multiplication $\circ$:

$$(p, q, t) \circ (p', q', t') = \left(p + p', q + q', t + t' + \frac{1}{2}(pq' - qp')\right).$$

For $\mathbf{p} = (p, q) \in \mathbb{R}^2$ we shall denote an element (p, q, t) of $\mathbb{H}$ by $(\mathbf{p}, t)$. It is known that for an arbitrary matrix $A \in SL(2, \mathbb{R})$ a map $\alpha_A(\mathbf{p}, t) = (A\mathbf{p}, t)$ is an automorphism of $\mathbb{H}$ (cf. Ref. 9, Theorem 1.22, p. 21). Let us denote also by $\varepsilon(\mathbf{p}) = pq$,

$\delta_A(\mathbf{p}) = \varepsilon(A\mathbf{p}) - \varepsilon(\mathbf{p})$, $\kappa_A(\mathbf{p}) := e^{-\pi i \delta_A(\mathbf{p})}$, for $m, n \in \mathbb{Z}$ let $\lambda_A(m,n) := \frac{\kappa_A(m/2,n)}{\kappa_A(m/2,-n)}$. Denote by $I(p,q) = (p,-q)$. Then $\lambda_A(m,n) = e^{\pi i(\delta_A(I(m/2,n)) - \delta_A(m/2,n))}$ and also $\varepsilon(I\mathbf{p}) = -\varepsilon(\mathbf{p})$.

The symmetric quadratic form ε can be identified with matrix

$$Q = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix},$$

while the form $\delta_A(I\mathbf{p}) - \delta_A(\mathbf{p})$ with the matrix $I^T A^T Q A I - A^T Q A + 2Q$.

Proposition 2.1. *For all $m, n \in \mathbb{Z}$ and for any $A \in SL(2, \mathbb{R})$ it holds that $\lambda_A(m,n) = e^{-2\pi i A_{12} A_{21} mn}$. In particular, $\lambda_A(m,n) = 1$ for all $m, n \in \mathbb{Z}$ if and only if $A_{12}A_{21} \in \mathbb{Z}$.*

Note that for a matrix A being in Hermite normal form it holds that $A_{21} = 0$, so $\lambda_A(m,n) = 1$ for all $m, n \in \mathbb{Z}$.

Proof. The symmetric quadratic form $\varepsilon \circ A$ can be identified with matrix $C = A^T Q A$ whose anti-diagonal entries $C_{12} = C_{21} = (A_{11}A_{22} + A_{12}A_{21})/2$. One verifies that for an arbitrary $B \in M_2(\mathbb{R})$ in the matrix $I^T B I - B$ only anti-diagonal terms do not vanish and are equal to $-2B_{21}$ and $-2B_{12}$, respectively. Plugging in $B = A^T Q A$ together with the fact that A has determinant 1 yields that the matrix defining the quadratic form $\delta_A(I\mathbf{p}) - \delta_A(\mathbf{p})$ is equal to $-4A_{12}A_{21}Q$. The assertion follows. $\square$

Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary operator defined as

$$\mathcal{F}\, f(\xi) \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i x \xi}\, dx.$$

Unitary operators T_p and M_q in $L^2(\mathbb{R})$, called *modulation* and *translation*, respectively, are defined for $p, q \in \mathbb{R}$ as

$$T_p h(x) = h(x-p), \quad M_q h(x) = e^{2\pi i q x} h(x), \quad \text{for any } h \in L^2(\mathbb{R}).$$

Let unitary operator J in $L^2(\mathbb{R})$ be defined as

$$Jh(x) = h(x - 2[x] + 1) \quad \text{for all } h \in L^2(\mathbb{R}),$$

where $[x]$ is the largest integer smaller than x.

Schrödinger representation ρ_S — a unitary representation of $\mathbb{H}$ in $L^2(\mathbb{R})$ — is defined as (cf. Ref. 10, Example 9.2.1, p. 182, see also Ref. 9, Sec. 1.3, p. 19)

$$\rho_S(p,q,t) f = e^{2\pi i t} e^{-\pi i p q}\, M_q\, T_{-p}\, f.$$

To shorten the notation we shall write $\pi(\mathbf{p})$ for $M_p T_q$.

For $B \in SL(2, \mathbb{Z})$, we shall call $B(\mathbb{Z}/2 \times \mathbb{Z})$ a *lattice* and *Gabor system* $\mathcal{G}^B(f)$ of redundancy 2 shall be defined as $(\pi(B(m/2,n)f))_{m,n \in \mathbb{Z}}$. Let $\mathcal{S}^B$ be the set of all such normalized $f \in L^2(\mathbb{R})$ that $\mathcal{G}^B(f)$ is a tight frame. When B is omitted, we shall understand that $B = \mathrm{Id}$, i.e. we deal with the standard rectangular lattice.

The representations ρ_S and $\rho_S \circ \alpha_A$ are equivalent and the equivalence is established by the image of A under *metaplectic representation* (cf. Ref. 9, Sec. 4.2, pp. 177–179) denoted by $\mu(A)$. The equivalence between the representations ρ_S and $\rho_S \circ \alpha_A$ translates into

$$M_{q'}T_{-p'} = \kappa_A(p,q)\,\mu(A)\,M_q\,T_{-p}\,\mu(A)^{-1}$$

where $\mathbf{p}' = (p', q') = A\mathbf{p}$. Expressed in terms of π this relation reads as follows:

$$\pi(A\mathbf{p}) = \kappa_{A'}(\mathbf{p})\,\mu(A')\,\pi(\mathbf{p})\,\mu(A')^{-1}. \tag{2.1}$$

Note that the seeming discrepancy between the last two formulas is due to the different order and signs of coordinates in the Heisenberg group.

The metaplectic representation μ is a mapping from $SL(2,\mathbb{R})$ into unitary operators of $L^2(\mathbb{R})$ becoming a representation when we allow ambiguity of sign up to ± 1, or when we consider a double-valued covering of $SL(2,\mathbb{R})$ known as the metaplectic group.

As each element of $SL(2,\mathbb{R})$ can be decomposed into the product of the matrices $\mathcal{J}$, $\mathcal{N}_a$, $\mathcal{D}_b$, the operators $\mu(A)$ are the products of the respective operators (cf. Ref. 9, pp. 177–179):

$$\mu(\mathcal{J})f(x) = \mathcal{F}^{-1}f(x), \quad \mu(\mathcal{N}_a)f(x) = e^{2\pi i a x^2}f(x), \quad \mu(\mathcal{D}_b)f(x) = |b|^{1/2}f(bx).$$

Let $Z : L^2(\mathbb{R}) \to L^2([0,1)^2)$ be the Zak transform with the parameter 2 (for the detailed discussion of properties and applications of Zak transform see, for instance, Refs. 12–14 and 18) defined as:

$$Zf(t,\omega) = 2^{1/2}\sum_{n\in\mathbb{Z}} f(2(t-n))e^{2\pi i n\omega}$$

with the following quasi-periodicity properties:

$$Zf(t+1,\omega) = e^{2\pi i\omega}Zf(t,\omega), \quad Zf(t,\omega+1) = Zf(t,\omega).$$

Zak transform has the following properties for operators M_1 and T_2:

$$Z[M_1 f](t,\omega) = e^{4\pi i t}Zf(t,\omega), \quad Z[T_2 f](t,\omega) = e^{-2\pi i\omega}Zf(t,\omega).$$

One directly verifies that

$$Z\,[Jf]\,(t,\omega) \;=\; Zf(\zeta t, 1-\omega),$$

for $t,\omega \in [0,1]$, where $\zeta t = t - [2t] + \frac{1}{2}$, and that ζ cycles the points t and $t+\frac{1}{2}$ for any $t \in \left[0,\frac{1}{2}\right]$.

Let us define an isomorphism Φ between $L^2(\mathbb{R})$ and $L^2([0,1)\times[0,1/2),\mathbb{C}^2)$ by

$$\Phi f(t,\omega) = \left(Zf(t,\omega), Zf\left(t+\frac{1}{2},\omega\right)\right). \tag{2.2}$$

Note that Φ is a particular case of Piecewise Zak Transform.[18]

3. Rectangular Lattices

The theorem below combines Proposition 5.2[7] and Theorem 5.5.[1]

Theorem 3.1. *Let* $f \in L^2(\mathbb{R})$, $\|f\| = 1$. *If* $(M_m T_{n/2} f)_{m,n\in\mathbb{Z}}$ *is a tight frame in* $L^2(\mathbb{R})$, *then the system composed of* $(M_{2m} f)_{m\in\mathbb{Z}}$ *and*

$$[2^{-1/2}\left(M_m T_{n/2} f + (-1)^{m+n} M_{-m} T_{n/2} f\right)]_{n\geq 1, m\in\mathbb{Z}} \tag{3.1}$$

is an orthonormal basis in $L^2(\mathbb{R})$ *if and only if*

$$E_k(x) = \sum_{n\in\mathbb{Z}} (-1)^n \overline{f(x-k-n/2-1/2)} f(-x-n/2) = 0 \tag{3.2}$$

for almost all $x \in [0, 1/2)$. *In particular, if* $\mathcal{F}f$ *is real-valued, the condition* (3.2) *is satisfied. Moreover, the system composed of* $(M_{2m+1} f)_{m\in\mathbb{Z}}$ *and*

$$[2^{-1/2}(M_m T_{n/2} f - (-1)^{m+n} M_{-m} T_{n/2} f)]_{n\geq 1, m\in\mathbb{Z}} \tag{3.3}$$

is an orthonormal basis in $L^2(\mathbb{R})$ *if and only if the same condition holds.*

We shall be using in the sequel the following definition of a Wilson system:

Definition 3.1. Given $f \in \mathcal{S}$ and for each choice of $\alpha, \beta, \gamma \in \{0, 1\}$ the (α, β, γ)-Wilson system $\mathcal{W}_{\alpha,\beta,\gamma}(f) = V_0 \cup V_1$, where

$$V_0 = \begin{cases} (M_{m+\gamma/2} f)_{m\in\mathbb{Z}} & \alpha = 1 \\ (M_{m/2} f)_{m\in\mathbb{Z}} & \alpha = 0, \gamma = 0 \\ \emptyset & \alpha = 0, \gamma = 1, \end{cases}$$

$$V_1 = (v_{mn})_{m\in\mathbb{Z}, n>0},$$

$$v_{mn} = 2^{-1/2}(M_{m/2} T_n f + (-1)^{m\alpha+n\beta+\gamma} M_{m/2} T_{-n} f).$$

Note that the classical Wilson system (3.1) is obtained as $\mathcal{F}\mathcal{W}_{1,1,0}(f)$ and (3.3) as $\mathcal{F}\mathcal{W}_{1,1,1}(f)$. Let us consider the subset $\mathcal{V}_{\alpha,\beta,\gamma}$ of $\mathcal{S}$ consisting of these f that $\mathcal{W}_{\alpha,\beta,\gamma}(f)$ is an orthonormal basis in $L^2(\mathbb{R})$.

We shall summarize Theorems 2.1 and 4.1, and Corollaries 5.1 and 6.1.[17]

Theorem 3.2. *Let* $f \in \mathcal{S}$. *Then* $f \in \mathcal{V}_{\alpha,\beta,\gamma}$ *if and only if for all* $m, n \in \mathbb{Z}$

$$\langle M_{\frac{\beta}{2}} T_{1-\alpha} J f, M_m T_{2n} f\rangle = 0.$$

Example 3.1. Let us start with a Gabor tight frame atom $f = S^{-1/2}\varphi$, $\varphi(x) = 2^{1/4} e^{-\pi x^2}$ being the Gaussian function and S the frame operator related to $\mathcal{G}(\varphi)$. Then define a function g by means of Zak transform:

$$Zg(t,\omega) = \begin{cases} Zf(t,\omega) & \text{for } (t,\omega) \in [0,1/2) \times [0,1/2), \\ Zf(t,\omega) & \text{for } (t,\omega) \in [1/2,1) \times [0,1/2), \\ \overline{Zf(t,1-\omega)} & \text{for } (t,\omega) \in [0,1/2) \times [1/2,1), \\ -\overline{Zf(t,1-\omega)} & \text{for } (t,\omega) \in [1/2,1) \times [1/2,1). \end{cases}$$

Isomorphism Φ defined in (2.2) diagonalizes the operators $(M_m T_{2n})_{m,n\in\mathbb{Z}}$ and one verifies that the characterization condition in Theorem 3.2 is equivalent to

$$\langle \Phi[Jg](t,\omega), \Phi g(t,\omega)\rangle_{\mathbb{C}^2} = 0$$

for almost all $(t,\omega) \in [0,1/2) \times [0,1)$ and that the system $\mathcal{W}_{1,0,0}g$ by means of Theorem 3.2 is indeed an orthonormal basis for $L^2(\mathbb{R})$.

To ascertain that g as f is a Gabor tight frame atom it is enough to verify that $\|\Phi(f)\|^2 = \|\Phi(g)\|^2$ which holds because $|Zg| = |Zf|$ for almost all $(t,\omega) \in [0,1) \times [0,1)$ by definition. As $Zf(t,1-\omega) = \overline{Z\bar{f}(t,\omega)}$, one can alternatively define g by

$$Zg(t,\omega) = \begin{cases} Zf(t,\omega) & \text{for } (t,\omega) \in [0,1/2) \times [0,1/2), \\ Zf(t,\omega) & \text{for } (t,\omega) \in [1/2,1) \times [0,1/2), \\ Z\bar{f}(t,\omega) & \text{for } (t,\omega) \in [0,1/2) \times [1/2,1), \\ -Z\bar{f}(t,\omega) & \text{for } (t,\omega) \in [1/2,1) \times [1/2,1). \end{cases}$$

4. General Lattices

In this section we define a Wilson system for all choices α, β, γ for the general lattice and characterize the atoms for which such a Wilson system is an orthonormal basis for $L^2(\mathbb{R})$. In the sequel we shall be using the notation $\pi(\mathbf{p})$ rather than $M_p T_q$ used previously.

Definition 4.1. A Wilson system $\mathcal{W}^B_{\alpha,\beta,\gamma}(f)$ for a lattice $B(\mathbb{Z}/2 \times \mathbb{Z})$ is the union $V_0 \cup V_1$, where

$$V_0 = \begin{cases} (\pi(B(m+\gamma/2,0))f)_{m\in\mathbb{Z}} & \alpha = 1 \\ (\pi(B(m/2,0))f)_{m\in\mathbb{Z}} & \alpha = 0,\ \gamma = 0 \\ \emptyset & \alpha = 0,\ \gamma = 1, \end{cases} \tag{4.1}$$

$$V_1 = (v_{mn})_{m\in\mathbb{Z},n>0}, \tag{4.2}$$

$$v_{mn} = 2^{-1/2}(\pi(B(m/2,n))f + \lambda_{B'}(m,n)(-1)^{m\alpha+n\beta+\gamma}\pi(B(m/2,-n))f), \tag{4.3}$$

Note that the above definition coincides with Definition 3.1 when $B = \mathrm{Id}$. The below proposition is a core of the argument in Theorem 2.5[15] which we extend from $\mathcal{W}^B_{1,1,0}(f)$ to the arbitrary choice of (α,β,γ).

Proposition 4.1. *Let $f \in \mathcal{S}^B$. Then $\mathcal{W}^B_{\alpha,\beta,\gamma}(f)$ is isometric with $\mathcal{W}_{\alpha,\beta,\gamma}(\mu(B')^{-1}f)$.*

Proof. By the definition of metaplectic representation μ (see Ref. 9, p. 177, compare (2.1) above)

$$\pi(B(m/2,n)) = \kappa_{B'}(m/2,n)\mu(B')M_{m/2}T_n\mu(B')^{-1}. \tag{4.4}$$

Applying it to (4.3), we obtain by Proposition 2.1 that

$$\begin{aligned} v_{mn} &= 2^{-1/2}(\kappa_{B'}(m/2,n)\mu(B')M_{m/2}T_n\mu(B')^{-1}f \\ &\quad + \kappa_{B'}(m/2,-n)\lambda_{B'}(m,n)(-1)^{m\alpha+n\beta+\gamma}\mu(B')M_{m/2}T_{-n}\mu(B')^{-1}f) \\ &= 2^{-1/2}\kappa_{B'}(m/2,n)\mu(B')\left(M_{m/2}T_n + (-1)^{m\alpha+n\beta+\gamma}M_{m/2}T_{-n}\right)\mu(B')^{-1}f \end{aligned}$$

and analogously for the elements of V_0. Since $\mu(B')$ is unitary, system $\mathcal{W}^B_{\alpha,\beta,\gamma}(f)$ is isometric to $\mathcal{W}_{\alpha,\beta,\gamma}(\mu(B')^{-1}f)$. □

The below useful corollary follows immediately from the above proposition.

Corollary 4.1. *Let $f \in \mathcal{S}^B$. Then $f \in \mathcal{V}^B_{\alpha,\beta,\gamma}$ if and only if $\mu(B')^{-1}f \in \mathcal{V}_{\alpha,\beta,\gamma}$.*

Theorem 2.5[15] can be restated as:

Theorem 4.1. *Let $f \in \mathcal{S}^B$. If $\mu(B')^{-1}f$ is real-valued, then $f \in \mathcal{V}^B_{1,1,0}$.*

Proof. By Proposition 4.1 the system $\mathcal{W}^B_{1,1,0}(f)$ is isometric to $\mathcal{W}_{1,1,0}(\mu(B')^{-1}f)$, which by Theorem 3.1 is an orthonormal basis for $L^2(\mathbb{R})$ from the assumption about real-valuedness of $\mu(B')^{-1}f$. □

By Proposition 4.1 and using Theorem 3.2 we are able to obtain the characterization of $\mathcal{V}^B_{1,1,0}$ together with similar results for $\mathcal{V}^B_{\alpha,\beta,\gamma}$.

Theorem 4.2. *Let $f \in \mathcal{S}^B$. Then $f \in \mathcal{V}^B_{\alpha,\beta,\gamma}$ if and only if for all $m,n \in \mathbb{Z}$*

$$\left\langle \mu(B')J\mu(B')^{-1}f, \pi\left(B(m-\beta/2, 2n+\alpha-1)\right)f\right\rangle = 0.$$

Proof. Since $\mathcal{W}^B_{\alpha,\beta,\gamma}(f)$ is isometric with $\mathcal{W}_{\alpha,\beta,\gamma}(\mu(B')^{-1}f)$, by Theorem 3.2 such a system is an orthonormal basis if for all $m,n \in \mathbb{Z}$

$$\langle M_{\frac{\beta}{2}}T_{1-\alpha}J\mu(B')^{-1}f, M_mT_{2n}\mu(B')^{-1}f\rangle = 0$$

or

$$\langle J\mu(B')^{-1}f, M_{-\frac{\beta}{2}}T_{\alpha-1}M_mT_{2n}\mu(B')^{-1}f\rangle = 0,$$

which is equivalent to

$$\langle J\mu(B')^{-1}f, M_{m-\frac{\beta}{2}}T_{2n+\alpha-1}\mu(B')^{-1}f\rangle = 0. \tag{4.5}$$

Again using (4.4), we get by linearity of B and unitarity of $\mu(B')$

$$\left\langle J\mu(B')^{-1}f, \mu(B')^{-1}\pi\left(B\left(m-\frac{\beta}{2}, 2n+\alpha-1\right)\right)f\right\rangle = 0,$$

$$\left\langle \mu(B')J\mu(B')^{-1}f, \pi\left(B\left(m-\frac{\beta}{2}, 2n+\alpha-1\right)\right)f\right\rangle = 0. \quad □$$

The result in Theorem 4.2 embraces the case of the hexagonal lattice discussed as an example in Ref. 15 and also yields new facts about Wilson systems on the rectangular lattice. Indeed, notice that picking symplectic B that preserves the lattice $\mathbb{Z}/2 \times \mathbb{Z}$ we obtain different pairings of time-frequency shifts that are also orthonormal bases for $L^2(\mathbb{R})$.

Example 4.1. Consider g defined in Example 3.1 and the hexagonal lattice $B(\mathbb{Z}/2 \times \mathbb{Z})$ for

$$B = \begin{bmatrix} \frac{1}{h} & \frac{1}{4h} \\ 0 & h \end{bmatrix}, \quad B' = \begin{bmatrix} h & 0 \\ -\frac{1}{4h} & \frac{1}{h} \end{bmatrix}, \quad h = \frac{\sqrt[4]{3}}{\sqrt{2}}.$$

As $B' = \mathcal{N}_{-\frac{1}{4h^2}} \mathcal{D}_h = \mathcal{N}_{-\frac{1}{2\sqrt{3}}} \mathcal{D}_{\frac{\sqrt[4]{3}}{\sqrt{2}}}$, one finds $\mu(B')$ to be equal

$$\mu(B') = \mu(\mathcal{N}_{-\frac{1}{2\sqrt{3}}})\mu(\mathcal{D}_{\frac{\sqrt[4]{3}}{\sqrt{2}}}),$$

$$\mu(B')g(x) = \sqrt{\frac{\sqrt[4]{3}}{\sqrt{2}}} e^{-\pi i x^2/\sqrt{3}} g\left(\frac{\sqrt[4]{3}}{\sqrt{2}} x\right).$$

Then by Theorem 4.2 $\mu(B')g \in \mathcal{V}^B_{1,0,0}$ and $\mathcal{W}^B_{1,0,0}(g)$ is an orthonormal basis in $L^2(\mathbb{R})$.

Example 4.2. Let us start again with a Gabor tight frame atom $f = S^{-1/2}\varphi$ from Example 3.1. Consider

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

Note that B preserves the lattice $\mathbb{Z}/2 \times \mathbb{Z}$ and that in this case

$$\mu(B') = \mathcal{F}\mu(\mathcal{N}_2)\mathcal{F}^{-1}.$$

Let $g(x) = \mu(B')f(x)$. Definition 4.1 of the Wilson system in the case of non-rectangular lattices yields that the system $\mathcal{W}^B_{1,1,0}(g)$ composed of $(M_m T_{2m} g)_{m\in\mathbb{Z}}$ and

$$[2^{-1/2}\left(M_{m/2}T_{m+n}g + (-1)^{m+n}M_{m/2}T_{m-n}g\right)]_{n\geq 1, m\in\mathbb{Z}} \tag{4.6}$$

is an orthonormal basis in $L^2(\mathbb{R})$.

Example 4.3. Using the same function f as in the previous example, consider

$$B = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

Again B preserves the lattice $\mathbb{Z}/2 \times \mathbb{Z}$ and

$$\mu(B') = \mu(\mathcal{N}_{-1/2}).$$

Let

$$g(x) = \mu(B')f(x) = e^{-\pi i x^2} S^{-1/2}\varphi(x).$$

Definition 4.1 yields in this case the system $\mathcal{W}^B_{1,1,0}(g)$ composed of $(M_m g)_{m\in\mathbb{Z}}$ and

$$[2^{-1/2}\left(M_{m/2+n/2}T_n g + (-1)^{m+n} M_{m/2-n/2}T_{-n}g\right)]_{n\geq 1, m\in\mathbb{Z}} \tag{4.7}$$

that is an orthonormal basis in $L^2(\mathbb{R})$. So one can pick the time-frequency shifts that are symmetric with respect to the point $M_{m/2}$ and the obtained Wilson system can still be an orthonormal basis for $L^2(\mathbb{R})$, but the interval that connects them is no longer perpendicular to the axis referring to the modulations.

Remark 4.1. Theorem 4.2 holds in the higher dimensions d as well with the only limitation that the lattice generating matrix B has to be symplectic of the order d i.e. $B \in Sp(d,\mathbb{R})$. The definition of the symplectic lattice for which the statements hold i.e. $B(\mathbb{Z}/2 \times \mathbb{Z})$ is a bit different from one in the literature, where the lattice is considered *symplectic* if it is of the form $\alpha A\mathbb{Z}^2$, where A is symplectic and $\alpha \neq 0$ (see Ref. 10, Definition 9.4.2, p. 198). One can however easily see that for lattices of volume 1/2 these settings can be switched picking $A = B\ \mathrm{diag}(1/\sqrt{2}, \sqrt{2})$ and $\alpha = 1/\sqrt{2}$.

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References

1. P. Auscher, Remarks on the local Fourier bases, in *Wavelets: Mathematics and Applications*, eds. J. Benedetto and M. Frazier, Studies in Advanced Mathematics (CRC Press, Boca Raton, 1994), pp. 203–218.
2. R. Balian, Un principe d'incertitude fort en theorie du signal ou en mecanique quantique, *C. R. Acad. Sci. Ser. 2* **292**(20) (1981) 1357–1362.
3. G. Battle, Heisenberg proof of Balian–Low theorem, *Lett. Math. Phys.* **15** (1988) 175–177.
4. K. Bittner, Linear approximation and reproduction of polynomials by Wilson bases, *J. Fourier Anal. Appl.* **8**(1) (2002) 85–108.
5. H. Bölcskei, K. Gröchenig, F. Hlawatsch and H. G. Feichtinger, Oversampled Wilson expansions, *IEEE Signal Process. Lett.* **4**(4) (1997) 106–108.

6. H. Bölcskei, K. Gröchenig, F. Hlawatsch and H. G. Feichtinger, Discrete-time Wilson expansions, in *Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis*, Paris, France (1996), pp. 525–528.
7. I. Daubechies, S. Jaffard and J.-L. Journé, A simple Wilson orthonormal basis with exponential decay, *SIAM J. Math. Anal.* **22** (1991) 554–573.
8. H. G. Feichtinger, K. Gröchenig and D. Walnut, Wilson bases and modulation spaces, *Math. Nachr.* **155** (1992) 7–17.
9. G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. of Math. Studies, Vol. 122 (Princeton University Press, Princeton, 1989).
10. K. Gröchenig, *Foundations of Time-Frequency Analysis*, Applied and Numerical Harmonic Analysis (Birkhaüser, Boston, 2001).
11. K. Gröchenig and D. Walnut, A Riesz basis for Bargmann–Fock space related to sampling and interpolation, *Ark. Math.* **30** (1992) 283–295.
12. A. J. E. M. Janssen, Bargmann transform, Zak transform, and coherent states, *J. Math. Phys.* **23** (1982) 720–731.
13. A. J. E. M. Janssen, The Zak transform: A signal transform for sampled time-continuous signals, *Philips J. Res.* **43** (1988) 23–69.
14. A. J. E. M. Janssen, The duality condition for Weyl–Heisenberg frames, in *Gabor Analysis and Algorithms*, eds. H. G Feichtinger and Th. Strohmer (Birkhäuser, 1998), pp. 33–84.
15. G. Kutyniok and T. Strohmer, Wilson bases for general time-frequency lattices, *SIAM J. Math. Anal.* **37** (2005) 685–711.
16. F. Low, Complete sets of wave-packets, in *A Passion for Physics: Essays in Honor of Geoffrey Chew*, eds. C. De Tar *et al.* (World Scientific, 1985), pp. 17–22.
17. P. Wojdyłło, Modified orthonormal Wilson bases, *Samp. Theory Signal Image Process.* **6** (2007) 223–235.
18. M. Zibulski and Y. Zeevi, Discrete multi-window Gabor type transforms, *IEEE Trans. Signal Proc.* **45** (1997) 1428–1442.