International Journal of Wavelets, Multiresolution and Information Processing

WILSON SYSTEM FOR TRIPLE REDUNDANCY

PIOTR WOJDYŁŁO

Institute of Mathematics, Polish Academy of Sciences,
Sniadeckich 8, 00-956 Warszawa 10, PO Box 21, POLAND
pwoj@impan.gov.pl

Received 1 December 2009
Revised 20 March 2010

A Wilson orthonormal basis was constructed in 1991 by I. Daubechies, S. Jaffard and J.-L. Journé from Gabor tight frame elements, when the redundancy of the Gabor system is 2. In 1994 P. Auscher gave a characterization of the atoms for which the Wilson system is an orthonormal basis. Afterwards, K. Gröchenig posed a question whether the construction of an orthonormal Wilson basis is possible for Gabor tight frame of redundancy 3. We give a partial positive answer to this question constructing in this case a Wilson system being a tight frame with bound 1.

Keywords: Gabor frame; tight frame; Wilson system; Wilson orthonormal basis; representation theory.

AMS Subject Classification: 42C15, 47B40, 46E30

1. Introduction

In Gabor analysis due to Heisenberg Uncertainty Principle and Balian-Low Theorem the orthonormal bases obtained under the action of regular lattice of time-frequency shifts cannot have good localization properties and the tight frames are an often picked substitute for orthonormal bases. But the link is even stronger. In 1991 I. Daubechies, S. Jaffard and J.-L. Journé from elements of Gabor tight frame with redundancy 2 constructed a system whose elements were the combination of two symmetric time–frequency shifts.[5] For the atoms whose Fourier transform is real-valued the system was proved to be an orthonormal basis in $L^2(\mathbf{R})$ and the exemplary constructed atom had good time and frequency localization properties. In 1994 P. Auscher under mild decay assumptions gave a characterization of the atoms for which the system is an orthonormal basis.[1] The theorem below combines Proposition 5.2 from Ref. 5 and Theorem 5.5 from Ref. 1. Recall that T_a is the translation (time–shift) and M_b is the modulation (frequency–shift) operator.

Theorem 1.1. *Let $f \in L^2(\mathbf{R})$, $\|f\| = 1$. If $(M_m T_{n/2} f)_{m,n\in\mathbf{Z}}$ is a tight frame in $L^2(\mathbf{R})$, then the system composed of $(M_{2m} f)_{m\in\mathbf{Z}}$ and*

$$\left[\, 2^{-1/2} \, \left(M_m \, T_{n/2} \, f \, + \, (-1)^{m+n} \, M_{-m} \, T_{n/2} \, f \right)\right]_{n\geq 1,\, m\in\mathbf{Z}} \tag{1.1}$$

is an orthonormal basis in $L^2(\mathbf{R})$ *if and only if for all* $k \in \mathbf{Z}$ *and for almost all* $x \in [0, 1/2)$

$$E_k(x) \;=\; \sum_{n\in\mathbf{Z}} \;(-1)^n \;\overline{f\;(x-k-n/2-1/2)}\;f\;(-x-n/2) \;=\; 0. \tag{1.2}$$

Moreover, the system composed of $(M_{2m+1}f)_{m\in\mathbf{Z}}$ *and*

$$\Big[\,2^{-1/2}\;\big(\,M_m\,T_{n/2}\,f \;-\; (-1)^{m+n}\;M_{-m}\,T_{n/2}\,f\,\big)\Big]_{n\geq 1,\; m\in\mathbf{Z}} \tag{1.3}$$

is also an orthonormal basis in $L^2(\mathbf{R})$ *if and only if (1.2) is satisfied. In particular, if* $\mathcal{F}f$ *is real-valued, the condition (1.2) is satisfied.*

The extensive study lead, in particular, to the results on the unconditionality of Wilson bases in Bargmann spaces. In 1992 H.G. Feichtinger, K. Gröchenig, and D. Walnut[6] showed that the Wilson orthonormal basis with an atom related to a Gaussian function is unconditional in the coorbit spaces $Co(L^p)$ and subsequently K. Gröchenig, and D. Walnut[11] strengthened it by considering the Wilson basis with an atom being a Gaussian function. The so obtained Wilson basis is certainly not orthonormal, but is still a Riesz basis and also unconditional in the coorbit spaces $Co(L^p)$. Equivalently, its image under Bargmann transform is unconditional in Fock-Bargmann spaces. A simplified proof of this result can be found in Ref. 24.

In 1996-1997 H. Bölcskei, K. Gröchenig, F. Hlawatsch, and H.G. Feichtinger constructed the analogue of the Wilson system for the Gabor tight frame with even redundancy $2N$ for $N \in \mathbf{Z}$ being under certain condition a tight frame with the frame bound reduced by factor 2.[3 4]

Later on, K. Gröchenig posed an inspiring and challenging question, whether there exists a Wilson basis (or its analogue) for the case of redundancy 3 (Ref. 10 pp. 168–169). K. Bittner demonstrated that polynomials are reproduced by Wilson bases and described the rate of linear approximation in these bases.[2] G. Kutyniok and Th. Strohmer generalized the notion of Wilson system to the lattices whose generator matrix is in Hermite normal form.[16] Some modifications of the classical Wilson system defined in (1.1) were introduced in Ref. 23 together with the characterization of the atoms $f \in L^2(\mathbf{R})$, for which the system is an orthonormal basis in $L^2(\mathbf{R})$. This result covers for instance the case when the sign sequence $(-1)^{m+n}$ in (1.1) is replaced with $(-1)^m$. These modifications can be carried over in the similar manner for the general lattices of the form $B(1/2\mathbf{Z} \times \mathbf{Z})$, where B has determinant 1, yielding also the appropriate characterizations.[21]

In the present paper we construct, using some specific symplectic matrix A of order 3, a system whose elements are the combinations of the time–frequency shifts with redundancy 3. The time–frequency shifts selection relies on the iterated action of matrix B derived from matrix A. The image of matrix A under the metaplectic representation provides an equivalence operator of the underlying Gabor representations which is in turn used to prove the tight frame condition.

Thus, we give a partial positive answer to K. Gröchenig's question about an orthonormal Wilson basis for Gabor tight frame of redundancy 3, constructing some Wilson-like tight frame with bound 1 instead. If norms of all its elements were equal 1, the system would be an orthonormal basis, which however we do not resolve whether being true or not. To the aim of the objective construction let us assume that $\|f\| = 1$ and that the system $\big(M_m \, T_{n/3} \, f \,\big)_{m,n \in \mathbf{Z}}$ is a tight frame with bound 3 and that

$$\kappa_1(m,n) = e^{\pi i(m+2n)m/3}, \quad \kappa_2(m,n) = e^{\pi i n(2m+n)/3}.$$

Theorem 1.2. *The system composed of f and of*

$$3^{-1/2} \, \big(M_m \, T_{n/3} \, f \, + \, \kappa_1(m,n) \, M_{-m-n} \, T_{m/3} \, f \, + \, \kappa_2(m,n) M_n \, T_{(-m-n)/3} \, f \,\big) \quad (1.4)$$

for $m \geq 0$, $n > 0$ is a tight frame with bound 1 if and only if for $k = 1, 2$ and for all $m, n \in \mathbf{Z}$

$$\big\langle \, f \, , \, \mu(A)^k \, M_{3m} \, T_n \, f \big\rangle = 0, \quad (1.5)$$

where $\mu(A)$ is the image under the metaplectic representation of symplectic matrix

$$A = \begin{bmatrix} 0 & -\frac{1}{3} \\ 3 & -1 \end{bmatrix}. \quad (1.6)$$

In Section 2 we describe necessary notions, assumptions, and the equivalence yielded via the metaplectic representation. Then in Sections 3 and 4 we give the proof of Theorem 1.2 whose first part (Section 3) relies on the decomposition of the operators M_1 and $T_{1/3}$ in the direct integral and the structure of operator $\mu(A)$ up to some almost nowhere vanishing function, while the second (Section 4) consists of computation of frame operator W of system (1.4) and uses some combinatorial argument mimicking to some extent P. Auscher's approach[1] with summing up over orbits of B action together with the obtained in the previous section structural property of $\mu(A)$ that yields independence of the expansions with respect to $(\, M_{3m} \, T_n \, \mu(A)^k)_{m,n \in \mathbf{Z}, k=0,1,2}$ and validity of characterization condition (1.5). In the conclusion of Section 4 we give a parametrization of the set of functions satisfying condition (1.5) to demonstrate that it is not empty and Theorem 1.2 is not void.

2. Preliminaries

$\mathbf{N}, \mathbf{Z}, \mathbf{R}, \mathbf{C}$ are respectively the set of all natural, integer, real, and complex numbers. Let $\{x\}$ for $x \in \mathbf{R}$ be a fractional part of x. The set of all matrices $k \times k$ with real entries is $M_k(\mathbf{R})$. The general linear group is the subset of invertible matrices in $M_k(\mathbf{R})$ being denoted by $GL(k, \mathbf{R})$ and the special linear group being its subset with determinant 1 by $SL(k, \mathbf{R})$. The symplectic group of all matrices preserving standard non-degenerate skew–symmetric form $\sigma : \mathbf{R}^{2k} \times \mathbf{R}^{2k} \to \mathbf{R}$ is denoted by $Sp(k, \mathbf{R})$. Note that $Sp(k, \mathbf{R}) \subset GL(2k, \mathbf{R}) \subset M_{2k}(\mathbf{R})$ and that $Sp(1, \mathbf{R}) = SL(2, \mathbf{R})$. The group of unitary matrices in $\mathbf{C}^n$ shall be denoted by $U(n)$.

Note also that for $k = 1$ form σ is defined by $\sigma(x, y) = \langle x, \mathcal{J}y\rangle$, where

$$\mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

$L^p(X, F)$ is the linear space of p-integrable/essentially bounded/measurable F-valued functions on measurable space X depending on whether $1 \le p < \infty$, $p = \infty$, or $p = 0$. If F is a finite–dimensional Banach space equipped with norm $\|\cdot\|_F$, space $L^p(X, F)$ for $1 \le p < \infty$ is endowed with the natural norm $\|s\|_{L^p(X,F)} = \left(\int_X \|s(x)\|_F^p dx\right)^{1/p}$ and $\|s\|_{L^\infty(X,F)} = \operatorname{ess\,sup} \|s(x)\|_F$. For convenience, we shall also use the notation $L^0(X, F)$ when F is just a subset of the linear space, e.g. $L^0(X, U(3))$. If F is a Hilbert space, $L^2(X, F)$ is a direct integral of Hilbert spaces and if $\Xi : L^2(\mathbf{R}) \to L^2(X, F)$ is a Hilbert space isomorphism, bounded operator $Q : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ is decomposable in this direct integral if

$$\Xi[Q] \;:=\; \Xi\, Q\, \Xi^{-1} \;\in\; L^\infty(\; X,\; End(F)\;). \tag{2.1}$$

In the capacity of isomorphism between $L^2(\mathbf{R})$ and the appropriate direct integrals we shall be using mappings based on Zak Transform and Piecewise Zak Transform.[12] [13] [14] [25] [26] The notions of decomposability and direct integral are far more general than the definition presented here, but for our purposes the above formulation suffices and we defer the interested reader to the monographs dealing with the topic.[18] [15] [8]

One-dimensional *Heisenberg group* H_1 (Ref. 7 p. 19, Ref. 10 Definition 9.1.2) is the set $\mathbf{R}^2 \times \mathbf{R}$ equipped with the multiplication $\circ$:

$$(p, q, t) \circ (p', q', t') = (p + p', q + q', t + t' + \frac{1}{2}(pq' - qp')).$$

For a symplectic matrix $A \in Sp(1, \mathbf{R}) = SL(2, \mathbf{R})$ a symplectic map $\alpha_A(p, q, t) = (p', q', t)$, where

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix},$$

is an automorphism of H_1 (Ref. 7, Theorem 1.22, p. 21).

Fourier transform $\mathcal{F} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ is a unitary operator defined as

$$\mathcal{F}\, f(\xi) \;=\; \int_{\mathbf{R}} f(x)\, e^{-2\pi i x\xi}\, dx,$$

Unitary operators T_p and M_q in $L^2(\mathbf{R})$, called *translation* and *modulation* respectively, are defined for $p, q \in \mathbf{R}$ as

$$T_p h(x) = h(x - p), \quad M_q h(x) = e^{2\pi i q x} h(x), \quad \text{for any} \quad h \in L^2(\mathbf{R}).$$

One-dimensional *Schrödinger representation* ρ_S – a unitary representation of H_1 in $L^2(\mathbf{R})$ – is defined as (Ref. 10, Example 9.2.1 p. 182, see also Ref. 7, Sec. 1.3, p. 19)

$$[\rho_S(p, q, t) f](x) = e^{2\pi i t}\, e^{\pi i p q}\, M_q\, T_{-p}\,.$$

For convenience we shall sometimes use

$$\mathbf{t} := \begin{pmatrix} t \\ \omega \end{pmatrix}, \ \mathbf{a} := \begin{pmatrix} a \\ b \end{pmatrix}, \ \mathbf{a}' := \begin{pmatrix} a' \\ b' \end{pmatrix}, \ \mathbf{p} := \begin{pmatrix} p \\ q \end{pmatrix}, \ \mathbf{p}' := \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

The representations ρ_S and $\rho_S \circ \alpha_A$ are equivalent and the equivalence is established by the operator $\mu(A)$, where the map $\mu : Sp(1, \mathbf{R}) \to U(L^2(\mathbf{R}))$ is a double-valued unitary representation of $Sp(1, \mathbf{R})$ called *metaplectic representation* (Ref. 7, Sec. 4.2, pp. 177-179).

Lemma 2.1. *For any symplectic matrix A let $B = \mathcal{J} A \mathcal{J}^{-1}$ and $\mathbf{a}' = B\mathbf{a}$. Then*

$$e^{-\pi i a' b'} \ M_{a'} T_{b'} = e^{-\pi i a b} \ \mu(A) \ M_a \, T_b \ \mu(A)^{-1}.$$

Proof. Let $\mathbf{p}' = A\mathbf{p}, \mathbf{a} = \mathcal{J}\mathbf{p}, \mathbf{a}' = \mathcal{J}\mathbf{p}'$. From the equivalence explained above[7] we obtain

$$\mu(A) \ e^{\pi i p q} \, M_q \, T_{-p} \ \mu(A)^{-1} \ = \ e^{\pi i p' q'} \ M_{q'} \, T_{-p'}$$

and

$$e^{-\pi i a b} \ \mu(A) \ M_a \, T_b \ \mu(A)^{-1} \ = \ e^{-\pi i a' b'} \ M_{a'} \, T_{b'} \ ,$$

where $B = \mathcal{J} A \mathcal{J}^{-1}$ as required. □

Corollary 2.1. *In the notation of Theorem 1.2:*

$$\begin{aligned} \kappa_1(m,n) \ M_{-m-n} \, T_{m/3} \ &= \ \mu(A) \ M_m \, T_{n/3} \ \mu(A)^{-1} \ , \\ \kappa_2(m,n) \ M_n \, T_{(-m-n)/3} \ &= \ \mu(A)^2 \ M_m \, T_{n/3} \ \mu(A)^{-2}. \end{aligned}$$

Proof. Let us define the consecutive elements of the orbit under action of $B = \mathcal{J} A \mathcal{J}^{-1}$ as:

$$\begin{pmatrix} m_i \\ \frac{n_i}{3} \end{pmatrix} = B^i \begin{pmatrix} m \\ \frac{n}{3} \end{pmatrix}, \tag{2.2}$$

As

$$B = \begin{bmatrix} -1 & -3 \\ \frac{1}{3} & 0 \end{bmatrix} \qquad B^2 = \begin{bmatrix} 0 & 3 \\ -\frac{1}{3} & -1 \end{bmatrix} \qquad ,$$

we have that

$$\begin{pmatrix} m_1 \\ \frac{n_1}{3} \end{pmatrix} = \begin{pmatrix} -m-n \\ \frac{m}{3} \end{pmatrix}, \qquad \begin{pmatrix} m_2 \\ \frac{n_2}{3} \end{pmatrix} = \begin{pmatrix} n \\ \frac{-m-n}{3} \end{pmatrix}$$

Let us denote $\kappa \begin{pmatrix} a \\ b \end{pmatrix} = e^{-\pi i a b}$ and verify that

$$\kappa \begin{pmatrix} m \\ \frac{n}{3} \end{pmatrix} = e^{-\pi i m n/3}, \quad \kappa \begin{pmatrix} m_1 \\ \frac{n_1}{3} \end{pmatrix} = e^{-\pi i(-m-n)m/3}, \quad \kappa \begin{pmatrix} m_2 \\ \frac{n_2}{3} \end{pmatrix} = e^{-\pi i n(-m-n)/3}$$

Since

$$\kappa \begin{pmatrix} m_i \\ \frac{n_i}{3} \end{pmatrix} M_{m_i} T_{n_i/3} = \kappa \begin{pmatrix} m \\ \frac{n}{3} \end{pmatrix} \mu(A)^i \, M_m \, T_{n/3} \, \mu(A)^{-i},$$

and the assertion is valid if we assume

$$\kappa_i(m,n) = \kappa \begin{pmatrix} m_i \\ \frac{n_i}{3} \end{pmatrix} \kappa \begin{pmatrix} m \\ \frac{n}{3} \end{pmatrix}^{-1}.$$

So we have

$$\kappa_1(m,n) = \frac{e^{\pi i(m+n)m/3}}{e^{-\pi imn/3}} = e^{\pi i(m+2n)m/3}, \quad \kappa_2(m,n) = \frac{e^{\pi i(n(m+n)/3)}}{e^{-\pi imn/3}} = e^{\pi in(2m+n)/3}.$$

□

Matrix A from (1.6) is easily seen to be decomposable into the metaplectic representation generators as

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

Then its image under the metaplectic representation is

$$\mu(A) = \mathcal{F}\mathcal{N}_{-1/3}\mathcal{F}\mathcal{N}_{-3},$$

where $\mathcal{N}_c f(x) = e^{-2\pi icx^2} f(x)$ (cf. Ref. 7, pp. 178–179).

Let C be positive. A sequence of vectors $(x_n)_{n\in I} \subset \mathcal{H}$ for a countable set I, where $\mathcal{H}$ is a separable Hilbert space, is a *tight frame* for $\mathcal{H}$ with the bound C if for all $x \in \mathcal{H}$

$$\sum_{n\in I} |\langle x, x_n\rangle|^2 = C\|x\|^2_{\mathcal{H}}. \tag{2.3}$$

Fixed the sequence of vectors $(x_n)_{n\in I}$ such that $\sum_{n\in I} |\langle x, x_n\rangle|^2 \le C\|x\|^2$, the frame operator for this sequence is defined by

$$Sx = \sum_{n\in I} \langle x, x_n\rangle \, x_n \tag{2.4}$$

for all $x \in \mathcal{H}$. Note that the frame is tight with bound C if and only if $S = C\mathrm{Id}$. Further, if $S = \mathrm{Id}$, the underlying system $(x_n)_{n\in I}$ is an orthonormal basis if and only if all x_n's are of unit norm.

Let $Z : L^2(\mathbf{R}) \to L^2([0,1)^2)$ be the Zak transform with the parameter 1 defined as:

$$Zf\begin{pmatrix} t \\ \omega \end{pmatrix} = \sum_{n\in\mathbf{Z}} f(t-n)\, e^{2\pi in\omega}$$

with the following quasi-periodicity properties:

$$Zf\begin{pmatrix} t+1 \\ \omega \end{pmatrix} = e^{2\pi i\omega} Zf\begin{pmatrix} t \\ \omega \end{pmatrix}, \qquad Zf\begin{pmatrix} t \\ \omega+1 \end{pmatrix} = Zf\begin{pmatrix} t \\ \omega \end{pmatrix}$$

and the diagonalization properties for operators M_1 and T_1:

$$Z[M_1 f]\begin{pmatrix} t \\ \omega \end{pmatrix} = e^{2\pi i t} Zf\begin{pmatrix} t \\ \omega \end{pmatrix}, \qquad Z[T_1 f]\begin{pmatrix} t \\ \omega \end{pmatrix} = e^{-2\pi i \omega} Zf\begin{pmatrix} t \\ \omega \end{pmatrix}.$$

For the detailed discussion of properties and applications of Zak transform see, for instance, Ref. 12 - 14, 25, 26.

3. Decomposition of operators

Let $\mathbf{X} = [0, \frac{1}{3}] \times [0, 1]$ and let us introduce Piecewise Zak Transform[26 27] $\Phi : L^2(\mathbf{R}) \to L^2(\mathbf{X}, \mathbf{C}^3)$ to be the Hilbert space isomorphism such that $f \in L^2(\mathbf{R})$ is mapped to function Φf on $\mathbf{X}$ whose value at $\mathbf{t} \in \mathbf{X}$ is given by

$$\Phi f\begin{pmatrix} t \\ \omega \end{pmatrix} = \left[\ Zf\begin{pmatrix} t \\ \omega \end{pmatrix},\ Zf\begin{pmatrix} t + \frac{1}{3} \\ \omega \end{pmatrix},\ Zf\begin{pmatrix} t + \frac{2}{3} \\ \omega \end{pmatrix}\ \right].$$

The goal of this section is to gain some information about the structure of operator $\mu(A)$, studying $\Phi[\mu(A)]$, cf. (2.1).To this aim we define a transformation of $\mathbf{X}$ denoted by D as:

$$D\begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\{-3t - \omega + 1/2\} \\ \{3t\} \end{pmatrix}. \tag{3.1}$$

It is easy to see that D is of third order i.e. $D^3 = \mathrm{Id}_{\mathbf{X}}$.

Lemma 3.1. *For matrix A defined in (1.6) and transformation D defined above there exists family of unitary operators $J \in L^0(\mathbf{X}, U(3))$ and nonvanishing function $L : \mathbf{X} \to \mathbf{C}^*$ such that for any $f \in L^2(\mathbf{R})$ and for almost all $\mathbf{t} \in \mathbf{X}$*

$$\Phi[\mu(A)f]\begin{pmatrix} t \\ \omega \end{pmatrix} \quad = \quad L\begin{pmatrix} t \\ \omega \end{pmatrix} J\begin{pmatrix} t \\ \omega \end{pmatrix} \Phi f\left(D\begin{pmatrix} t \\ \omega \end{pmatrix}\right).$$

Proof. Let us prove first that the operator $J(\cdot)\Phi[\cdot](D\cdot)$ on the right hand side of the assertion establishes the same equivalence as the operator $\mu(A)$ in Lemma 2.1. We will verify it only on the generators M_1 and $T_{1/3}$. To see that it is enough we shall argue by means of the expression of representation operators in terms of ρ. Let us assume that for some operator $\mathbf{K}$ acting on $L^2(\mathbf{R})$ we have that

$$\mathbf{K}\ \rho(0,1,0)\ \mathbf{K}^{-1} \ = \ \rho(-\tfrac{1}{3}, -1, 0), \quad \mathbf{K}\ \rho(-\tfrac{1}{3}, 0, 0)\ \mathbf{K}^{-1} \ = \ \rho(0, -1, 0).$$

Now, notice that for any cyclic group generated by $(\alpha, \beta, 0) \in \mathbf{R}^2 \times \{0\}$:

$$\rho(\alpha, \beta, 0)^m = \rho(m\alpha, m\beta, 0).$$

So

$$\rho(0, m, 0) = \rho(0, 1, 0)^m, \quad \rho(-\tfrac{n}{3}, 0, 0) = \rho(-\tfrac{1}{3}, 0, 0)^n, \text{ and } \quad \rho(-\tfrac{1}{3}, -1, 0)^m = \rho(-\tfrac{m}{3}, -m, 0).$$

Thus, we have

$$\mathbf{K}\,\rho(0, m, 0)\,\mathbf{K}^{-1} \ = \ [\mathbf{K}\,\rho(0, 1, 0)\,\mathbf{K}^{-1}]^m \ = \ \rho(-\tfrac{1}{3}, -1, 0)^m \ = \ \rho(-\tfrac{m}{3}, -m, 0)$$

$$\mathbf{K}\,\rho(-\tfrac{n}{3},0,0)\,\mathbf{K}^{-1} \;=\; [\mathbf{K}\,\rho(-\tfrac{1}{3},0,0)\,\mathbf{K}^{-1}]^n \;=\; \rho(0,-1,0)^n \;=\; \rho(0,-n,0).$$

We also have for the product of such elements that

$$\mathbf{K}\,\rho(-\tfrac{n}{3},m,-\tfrac{mn}{6})\,\mathbf{K}^{-1} \;=\; [\mathbf{K}\,\rho(-\tfrac{n}{3},0,0)\,\mathbf{K}^{-1}]\,[\mathbf{K}\,\rho(0,m,0)\,\mathbf{K}^{-1}] \;=$$

$$=\; \rho(0,-n,0)\,\rho(-\tfrac{m}{3},-m,0) \;=\; \rho(-\tfrac{m}{3},-m-n,-\tfrac{mn}{6})$$

As the cyclic factors at the both sides cancel, we established

$$\mathbf{K}\,\rho(-\tfrac{n}{3},m,0)\,\mathbf{K}^{-1} \;=\; \rho(-\tfrac{m}{3},-m-n,0)\;.$$

Expressing this in terms of the operators M and T, we get

$$\mathbf{K}\,\kappa\begin{pmatrix} m \\ \frac{n}{3}\end{pmatrix}\;M_m T_{n/3}\,\mathbf{K}^{-1} \;=\; \kappa\begin{pmatrix} -m-n \\ \frac{m}{3}\end{pmatrix}\;M_{-m-n}T_{m/3}.$$

or

$$\mathbf{K}\;M_m T_{n/3}\,\mathbf{K}^{-1} \;=\; \kappa_1(m,n)\;M_{-m-n}T_{m/3}\;.$$

Hence, we demonstrated that it is enough to prove the equivalence on generators as the validity for the whole system follows.

To show that it *does* hold on the generators, we shall analyze first the image of operators M_1 and $T_{1/3}$ under mapping Φ, i.e., $\Phi[M_1]$ and $\Phi[T_{1/3}]$ The closer inspection below shows that both images belong to $L^\infty(\mathbf{X}, End(\mathbf{C}^3))$ so both original operators are *decomposable* in the direct integral $L^2(\mathbf{X},\mathbf{C}^3)$ (cf. Ref. 18, section I.6 p. 51, also Ref. 15, sec. 4.5 and Ch. 8).

For operator M_1 one finds that

$$\Phi[M_1 f]\;\begin{pmatrix} t \\ \omega\end{pmatrix} = \left[e^{2\pi i t} Zf\begin{pmatrix} t \\ \omega\end{pmatrix}, e^{2\pi i(t+\frac{1}{3})} Zf\begin{pmatrix} t+\frac{1}{3} \\ \omega\end{pmatrix}, e^{2\pi i(t+\frac{2}{3})} Zf\begin{pmatrix} t+\frac{2}{3} \\ \omega\end{pmatrix}\right],$$

and its matrix in the direct integral is diagonal:

$$\Phi[M_1]\begin{pmatrix} t \\ \omega\end{pmatrix} = \begin{bmatrix} e^{2\pi i t} & 0 & 0 \\ 0 & e^{2\pi i(t+\frac{1}{3})} & 0 \\ 0 & 0 & e^{2\pi i(t+\frac{2}{3})}\end{bmatrix}.$$

By the similar argument for $T_{1/3}$ its matrix is:

$$\Phi[T_{1/3}]\begin{pmatrix} t \\ \omega\end{pmatrix} = \begin{bmatrix} 0 & 0 & e^{-2\pi i\omega} \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}.$$

With the notation:

$$\alpha = e^{-2\pi i\omega},\quad \alpha' = e^{-2\pi i\omega'},\quad \beta = e^{2\pi i t},\quad \beta' = e^{2\pi i t'},$$

$$\gamma = \overline{\beta'\beta},\quad \lambda = e^{\pi i/3},\quad \begin{pmatrix} t' \\ \omega'\end{pmatrix} = D\begin{pmatrix} t \\ \omega\end{pmatrix}$$

let us define unitary operator $J\begin{pmatrix} t \\ \omega \end{pmatrix} \in U(3)$ by

$$J\begin{pmatrix} t \\ \omega \end{pmatrix} = 3^{-1/2}\begin{bmatrix} 1 & \overline{\beta} & \overline{\beta}^2 \\ \lambda^5\gamma & \lambda^3\gamma\overline{\beta} & \lambda\gamma\overline{\beta}^2 \\ \lambda^2\gamma^2 & \lambda^4\gamma^2\overline{\beta} & \gamma^2\overline{\beta}^2 \end{bmatrix}.$$

In the above notation

$$\Phi[M_1]\begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{bmatrix} \beta & 0 & 0 \\ 0 & \lambda^2\beta & 0 \\ 0 & 0 & \lambda^4\beta \end{bmatrix}, \quad \Phi[T_{1/3}]\begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{bmatrix} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Phi[\ \lambda\ M_{-1}T_{1/3}\ ]\begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{bmatrix} 0 & 0 & \lambda\overline{\beta}\alpha \\ \lambda^5\overline{\beta} & 0 & 0 \\ 0 & \lambda^3\overline{\beta} & 0 \end{bmatrix}.$$

By some calculations one can convince oneself that $J(\cdot)$ satisfies

$$\begin{aligned} J\begin{pmatrix} t \\ \omega \end{pmatrix}\ \Phi[\ M_1\ ]\left(D\begin{pmatrix} t \\ \omega \end{pmatrix}\right) &= \Phi[\ \lambda\ M_{-1}T_{1/3}\ ]\begin{pmatrix} t \\ \omega \end{pmatrix}\ J\begin{pmatrix} t \\ \omega \end{pmatrix}. \\ J\begin{pmatrix} t \\ \omega \end{pmatrix}\ \Phi[\ T_{1/3}\ ]\left(D\begin{pmatrix} t \\ \omega \end{pmatrix}\right) &= \Phi[\ M_{-1}\ ]\begin{pmatrix} t \\ \omega \end{pmatrix}\ J\begin{pmatrix} t \\ \omega \end{pmatrix}. \end{aligned}$$

Indeed, the left hand side of the first identity is:

$$J\begin{pmatrix} t \\ \omega \end{pmatrix}\ \Phi[M_1]\begin{pmatrix} t' \\ \omega' \end{pmatrix} = 3^{-1/2}\begin{bmatrix} \beta' & \lambda^2\beta'\overline{\beta} & \lambda^4\beta'\overline{\beta}^2 \\ \lambda^5\gamma\beta' & \lambda^5\gamma\beta'\overline{\beta} & \lambda^5\gamma\beta'\overline{\beta}^2 \\ \lambda^2\gamma^2\beta' & \gamma^2\beta'\overline{\beta} & \lambda^4\gamma^2\beta'\overline{\beta}^2 \end{bmatrix}$$

and the right hand side of the first identity is:

$$\Phi[\ \lambda\ M_{-1}T_{1/3}\ ]\begin{pmatrix} t \\ \omega \end{pmatrix}\ J\begin{pmatrix} t \\ \omega \end{pmatrix} = 3^{-1/2}\begin{bmatrix} \lambda^3\gamma^2\overline{\beta}\alpha & \lambda^5\gamma^2\overline{\beta}^2\alpha & \lambda\gamma^2\overline{\beta}^3\alpha \\ \lambda^5\overline{\beta} & \lambda^5\overline{\beta}^2 & \lambda^5\overline{\beta}^3 \\ \lambda^2\gamma\overline{\beta} & \lambda^6\gamma\overline{\beta}^2 & \lambda^4\gamma\overline{\beta}^3 \end{bmatrix}.$$

To assert their equality we use the identity $\lambda^3\gamma^3\alpha = 1$.

Then the left hand side of the second identity is:

$$J\begin{pmatrix} t \\ \omega \end{pmatrix}\ \Phi[T_{1/3}]\begin{pmatrix} t' \\ \omega' \end{pmatrix} = 3^{-1/2}\begin{bmatrix} \overline{\beta} & \overline{\beta}^2 & \alpha' \\ \lambda^3\gamma\overline{\beta} & \lambda\gamma\overline{\beta}^2 & \lambda^5\gamma\alpha' \\ \lambda^4\gamma^2\overline{\beta} & \gamma^2\overline{\beta}^2 & \lambda^2\gamma^2\alpha' \end{bmatrix}$$

and the right hand side of the second identity is:

$$\Phi[M_{-1}]\begin{pmatrix} t \\ \omega \end{pmatrix}\ J\begin{pmatrix} t \\ \omega \end{pmatrix} = 3^{-1/2}\begin{bmatrix} \overline{\beta} & \overline{\beta}^2 & \overline{\beta}^3 \\ \lambda^3\gamma\overline{\beta} & \lambda\gamma\overline{\beta}^2 & \lambda^5\gamma\overline{\beta}^3 \\ \lambda^4\gamma^2\overline{\beta} & \gamma^2\overline{\beta}^2 & \lambda^2\gamma^2\overline{\beta}^3 \end{bmatrix}.$$

To assert their equality we use the identity $\alpha' = \overline{\beta}^3$. Note that the identities $\alpha' = \overline{\beta}^3$ and $\lambda^3\gamma^3\alpha = 1$ imply that $\{\omega'\} = \{3t\}$ and $\frac{1}{3}\{3t'\} = \frac{1}{3}\{3(-t - \omega/3 + 1/6)\}$, which is just a form of D.

Currently, we have proved that the operator $J(\cdot)\Phi[\cdot](D\cdot)$ on the right hand side of the assertion establishes equivalence between $\Phi[M_m T_{n/3}](\mathbf{t})$ and $\Phi[\kappa_1(m,n)M_{-m-n}T_{m/3}](\mathbf{t})$. The same property has the operator $\Phi[\mu(A)](\cdot)$. Hence, to conclude the proof we need some type of uniqueness and this indeed holds. Observe that the representation $\rho_S|(-n/3, m)_{m,n\in\mathbf{Z}}$ is a direct integral of irreducible and non-equivalent representations. To see that, let us argue that the system of operators $(\Phi[M_m T_{n/3}](\mathbf{t}))_{m,n\in\mathbf{Z}}$ forms for each $\mathbf{t} \in \mathbf{X}$ an irreducible representation in $\mathbf{C}^3$ and that for different $\mathbf{t}$'s the central operators, namely $(\Phi[M_{3m}T_n])_{m,n\in\mathbf{Z}}$, are represented as scalar multiples of identities.

Now consider the homomorphisms $\mathbf{Z}^2 \ni (-n, 3m) \to \Phi[M_{3m}T_n](\mathbf{t})$. Formally we should consider them as homomorphisms of Heisenberg group, but the subgroup $((-n, 3m, 0))_{n,m\in\mathbf{Z}}$ is abelian and isomorphic to $\mathbf{Z}^2$. Note that the homomorphisms differ for different $\mathbf{t}$'s. It follows thence that these irreducible representations are non-equivalent.

Such representations, i.e. direct integrals of irreducible and non-equivalent representations have an important property namely the uniqueness of the equivalence operator up to the nonvanishing scalar function indexed by the domain of direct integral decomposition. For two equivalent representations π and π' of the set S let us call the operator K such that $K\pi(s) = \pi'(s)K$ for all $s \in S$ their *equivalence operator*. By Schur's lemma the equivalence operator of two finite-dimensional irreducible representations is unique up to multiplication by a constant. Note further that if any (and hence both) representation is a direct sum of non-equivalent irreducible finite-dimensional components, the equivalence operator is also unique but up to multiplication by certain nonvanishing function on the index set of direct sum. By the similar argument we get this property for the direct integrals of (almost all) non-equivalent irreducible finite-dimensional representations. *Par force* it holds for the representations $\rho_S|(-n/3, m)_{m,n\in\mathbf{Z}}$ and $\rho_S|(A(-n/3, m))_{m,n\in\mathbf{Z}}$. So their equivalence operator is unique up to multiplication by the nonvanishing scalar function L on $\mathbf{X}$ and as we found two of them: $\Phi[\mu(A)]$ and $J(\cdot)\Phi[\cdot](D\cdot)$, the assertion follows. □

One could try to prove the formula

$$\Phi[\mathcal{FN}_{-1/3}\mathcal{FN}_{-3}f]\begin{pmatrix} t \\ \omega \end{pmatrix} = L\begin{pmatrix} t \\ \omega \end{pmatrix} J\begin{pmatrix} t \\ \omega \end{pmatrix} \Phi f\left(D\begin{pmatrix} t \\ \omega \end{pmatrix}\right)$$

directly i.e. without the reference to the representation theory, but it does not seem easy.

Note that almost all points in $\mathbf{X}$ have three-element orbits under the action of D and this set can be decomposed into three disjoint subsets which are permuted by D-action. Let us denote one of these sets by $\mathbf{Y}$ and introduce isomorphism

$\Psi : L^2(\mathbf{R}) \to L^2(\mathbf{Y}, \mathbf{C}^9)$ defined by

$$\Psi \begin{pmatrix} t \\ \omega \end{pmatrix} = \left(\Phi \begin{pmatrix} t \\ \omega \end{pmatrix}, \Phi \left(D \begin{pmatrix} t \\ \omega \end{pmatrix} \right), \Phi \left(D^2 \begin{pmatrix} t \\ \omega \end{pmatrix} \right) \right)$$

and by $\Psi[Q]$ we denote an operator $\Psi Q \Psi^{-1}$. Operators decomposable in the direct integral $L^2(\mathbf{X}, \mathbf{C}^3)$ related to Φ will be also decomposable in $L^2(\mathbf{Y}, \mathbf{C}^9)$ related to Ψ, but it is also so for the operators of the form described in Lemma 3.1. From Lemma 3.1 we infer that

$$\Psi[\mu(A)] \begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{bmatrix} 0 & 0 & L_2 J_2 \\ L_0 J_0 & 0 & 0 \\ 0 & L_1 J_1 & 0 \end{bmatrix}, \tag{3.2}$$

where entries are 3×3 submatrices and $J_s \begin{pmatrix} t \\ \omega \end{pmatrix} = J \left(D^s \begin{pmatrix} t \\ \omega \end{pmatrix} \right)$, $L_s \begin{pmatrix} t \\ \omega \end{pmatrix} = L \left(D^s \begin{pmatrix} t \\ \omega \end{pmatrix} \right)$ and $D^0 := \mathrm{Id}_{\mathbf{X}}$. Analogously,

$$\Psi[\mu(A)^2] \begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{bmatrix} 0 & 0 & L_1 L_2 J_1 J_2 \\ L_2 L_0 J_2 J_0 & 0 & 0 \\ 0 & L_0 L_1 J_0 J_1 & 0 \end{bmatrix}. \tag{3.3}$$

Now we shall develop a combinatorial argument based to some extent on P. Auscher's approach.[1] Using this argument and the form of $\Psi[\mu(A)]$, we shall infer the assertion.

4. Proof of Theorem 1.2.

We shall analyze the orbit structure of system (1.4) to see how it translates into the representation structure of its frame operator.

Proof. Let m_i, n_i for $i = 1, 2$ be defined as in (2.2). Note that system (1.4) consists of f and the vectors

$$3^{-1/2} \sum_{i=0}^{2} \kappa_i(m, n) M_{m_i} T_{n_i/3} f,$$

where $\kappa_0(m, n) = 1$, $m_0 = m$, $n_0 = n$.

Frame operator W of system (1.4) multiplied by 9 equals to

$$
\begin{aligned}
9\,W = & \\
= \quad & 9\langle\,\cdot\,,f\rangle\; f + \sum_{m\geq 0,n>0} 3\left\langle\,\cdot\,,\sum_{i=0}^{2}\kappa_i(m,n)\,M_{m_i}T_{n_i/3}f\right\rangle\sum_{j=0}^{2}\kappa_j(m,n)\,M_{m_j}T_{n_j/3}f = \\
= \quad & 9\langle\,\cdot\,,f\rangle\; f + \sum_{(m,n)\neq(0,0)} \left\langle\,\cdot\,,\sum_{i=0}^{2}\kappa_i(m,n)\,M_{m_i}T_{n_i/3}f\right\rangle\sum_{j=0}^{2}\kappa_j(m,n)\,M_{m_j}T_{n_j/3}f = \qquad (4.1)\\
= \quad & \sum_{m,n\in\mathbf{Z}} \left\langle\,\cdot\,,\sum_{i=0}^{2}\kappa_i(m,n)\,M_{m_i}T_{n_i/3}f\right\rangle\sum_{j=0}^{2}\kappa_j(m,n)\,M_{m_j}T_{n_j/3}f = \qquad (4.2)\\
= \quad & \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{m,n\in\mathbf{Z}} \left\langle\,\cdot\,,\;\kappa_i(m,n)\,M_{m_i}T_{n_i/3}f\right\rangle\;\kappa_j(m,n)\,M_{m_j}T_{n_j/3}f = \qquad (4.3)\\
= \quad & \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{m,n\in\mathbf{Z}} \left\langle\,\cdot\,,\;\mu(A)^iM_mT_{n/3}\,\mu(A)^{-i}f\right\rangle\;\mu(A)^jM_mT_{n/3}\,\mu(A)^{-j}f\,.
\end{aligned}
$$

Equality (4.1) follows from the fact that the function

$$\left\langle\,\cdot\,,\sum_{i=0}^{2}\kappa_i(m,n)M_{m_i}T_{n_i/3}f\right\rangle\sum_{j=0}^{2}\kappa_j(m,n)M_{m_j}T_{n_j/3}f$$

is constant on the orbits of B action and we switch from summing its values over the orbit representatives to summing over all orbit elements. From bijectivity of B follows

$$\forall_{h\in L^2(\mathbf{R})}\quad \sum_{m,n\in\mathbf{Z}}\sum_{i=0}^{2}|\langle\,h,\;M_{m_i}T_{n_i/3}\,f\,\rangle|^2\;\leq\;9\,\|h\|^2, \tag{4.4}$$

which, combined with Cauchy–Schwarz inequality, justifies the interchange of sums in (4.2). Hence also boundedness of operator W follows. Equality (4.3) follows again from Corollary 2.1.

In the next step we shall use Janssen's Representation Theorem (Ref. 14, sec. 1.4.1), from which one has that

$$\sum_{m,n\in\mathbf{Z}}\langle\,\cdot\,,M_mT_{n/3}f_1\rangle\;M_mT_{n/3}f_2\;=\;3\sum_{m,n\in\mathbf{Z}}\langle f_2,M_{3m}T_nf_1\rangle\;M_{3m}T_n, \tag{4.5}$$

at least in the weak sense. In the original formulation condition (A) is assumed, being however possible to relax in the present argument as it holds at least for a dense subspace in $L^2(\mathbf{R})$ (or, more precisely, for the Cartesian product of such subspaces, since we consider the weak convergence) and it is all we ask for.

So from (4.3) we obtain that

$$9\ \langle Wh_1, h_2\rangle\ =$$

$$= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ h_1\ ,\ \mu(A)^i M_m T_{n/3}\,\mu(A)^{-i}\ f\ \Big\rangle\ \Big\langle\ \mu(A)^j M_m T_{n/3}\,\mu(A)^{-j} f, h_2\Big\rangle\ =$$

$$= \sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ \mu(A)^{-i} h_1\ ,\ M_m T_{n/3}\,\mu(A)^{-i}\ f\ \Big\rangle\ \Big\langle\ M_m T_{n/3}\,\mu(A)^{-j} f,\ \mu(A)^{-j} h_2\Big\rangle .$$

By (4.5) applied for h_i from the dense subspace of $L^2(\mathbf{R})$ one gets

$$9\ \langle Wh_1, h_2\rangle\ =$$

$$= 3\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ \mu(A)^{-j} f, M_{3m} T_n\ \mu(A)^{-i}\ f\Big\rangle\ \ \Big\langle\ M_{3m} T_n \mu(A)^{-i} h_1\ ,\ \mu(A)^{-j} h_2\ \Big\rangle\ =$$

$$= 3\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ f, \mu(A)^{j} M_{3m} T_n\ \mu(A)^{-i}\ f\Big\rangle\ \ \Big\langle\ \mu(A)^{j}\ M_{3m} T_n \mu(A)^{-i} h_1\ ,\ h_2\ \Big\rangle\ =$$

$$= 3\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ f\ ,\ M_{3m_j} T_{n_j}\,\mu(A)^{j-i} f\Big\rangle\ \ \Big\langle\ M_{3m_j} T_{n_j}\,\mu(A)^{j-i} h_1\ ,\ h_2\ \Big\rangle\ \ .$$

Certainly, B is a bijection of $\mathbf{Z}\times\frac{1}{3}\mathbf{Z}$. Thus,

$$9\ \langle Wh_1, h_2\rangle\ =\ 3\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ f\ ,\ M_{3m} T_n\,\mu(A)^{j-i} f\Big\rangle\ \ \Big\langle\ M_{3m} T_n\,\mu(A)^{j-i} h_1\ ,\ h_2\ \Big\rangle\ =$$

$$=\ 9\sum_{k=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ f\ ,\ M_{3m} T_n\,\mu(A)^{k} f\Big\rangle\ \ \Big\langle\ M_{3m} T_n\,\mu(A)^{k} h_1\ ,\ h_2\ \Big\rangle$$

and by density in $L^2(\mathbf{R})$ of the subspace from which h_i's come

$$W\ =\sum_{k=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ f, M_{3m} T_n\,\mu(A)^{k} f\ \Big\rangle\quad M_{3m} T_n\,\mu(A)^{k}\ \ .$$

Or, alternatively, by renumerating the terms

$$W\ =\sum_{k=0}^{2}\sum_{m,n\in\mathbf{Z}} \Big\langle\ f,\ \mu(A)^{k}\, M_{3m} T_n\, f\ \Big\rangle\quad \mu(A)^{k}\, M_{3m} T_n\ \ .$$

Having done this, we are in position to complete the proof. Indeed, from formulae (3.2) and (3.3) we find that

$$\Psi[W] = \begin{bmatrix} f_0^0\mathrm{Id} & f_1^1 L_1 J_1 & f_2^2 L_1 L_2 J_1 J_2 \\ f_0^2 L_2 L_0 J_2 J_0 & f_1^0\mathrm{Id} & f_2^1 L_2 J_2 \\ f_0^1 L_0 J_0 & f_1^2 L_0 L_1 J_0 J_1 & f_2^0\mathrm{Id} \end{bmatrix} \tag{4.6}$$

where J_s, L_s are as in (3.2) and (3.3), $\begin{pmatrix} t_s \\ \omega_s \end{pmatrix} = D^s \begin{pmatrix} t \\ \omega \end{pmatrix}$ and

$$\begin{aligned} f_s^k \begin{pmatrix} t \\ \omega \end{pmatrix} &= \Phi \left[\sum_{m,n \in \mathbf{Z}} \left\langle\, f,\, \mu(A)^k\, M_{3m} T_n\, f \,\right\rangle\;\; M_{3m} T_n \right] \begin{pmatrix} t_s \\ \omega_s \end{pmatrix} \\ &= \sum_{m,n \in \mathbf{Z}} \left\langle\, f,\, \mu(A)^k\, M_{3m} T_n\, f \,\right\rangle\;\; e^{2\pi i \cdot (3mt_s - n\omega_s)}, \end{aligned}$$

but also

$$\Psi[\mathrm{Id}_{L^2(\mathbf{R})}] = \begin{bmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{bmatrix}.$$

Since Ψ is an isomorphism, we have

$$W = \mathrm{Id}_{L^2(\mathbf{R})} \Leftrightarrow \Psi[W] \begin{pmatrix} t \\ \omega \end{pmatrix} = \Psi[\mathrm{Id}_{L^2(\mathbf{R})}] \begin{pmatrix} t \\ \omega \end{pmatrix} \quad \text{for almost all } \begin{pmatrix} t \\ \omega \end{pmatrix} \in \mathbf{Y}.$$

As L_s are non-vanishing functions of their arguments, the off-diagonal terms in (4.6) are 0 only if f_s^k themselves are 0 for $k \neq 0$. Thus, all functions f_s^k should be constant and equal identically to 0 on $\mathbf{Y}$ for $k \neq 0$ or, equivalently, f_0^k should equal 0 on $\mathbf{X}$ for $k \neq 0$. The functions $e^{2\pi i \cdot (3mt - n\omega)}$ form an orthonormal basis of characters on $\mathbf{X}$. Since the Fourier series of f_0^k's are l^2-convergent and their Fourier coefficients are $\langle f,\, \mu(A)^k\, M_{3m} T_n\, f \rangle$, we obtain that

$$\left\langle\, f,\, \mu(A)^k\, M_{3m} T_n\, f \,\right\rangle = \;\; 0 \quad \text{for } k \neq 0\,.$$

Similarly, the condition for $k = 0$ being that f_s^0 is constant and identically 1 on $\mathbf{Y}$ is equivalent to f_0^0 being identically 1 on $\mathbf{X}$ and equivalent to $\langle f, M_{3m} T_n f \rangle = \; \delta_{m0}\, \delta_{n0}$. This condition though is exactly the Wexler-Raz identity[19] and is equivalent to the condition that $\left(\, M_m\, T_{n/3}\, f \,\right)_{m,n \in \mathbf{Z}}$ is a tight frame with bound 3. Hence, (1.5) is equivalent to $W = \mathrm{Id}_{L^2(\mathbf{R})}$. □

Corollary 4.1. *Condition (1.5) holds if and only if for almost all* $\mathbf{t} \in \mathbf{X}$ *the system*

$$\left(\;\; \Phi f(\mathbf{t}),\;\; J(\mathbf{t})\, \Phi f\, (D\mathbf{t})\,,\;\; J(\mathbf{t})\, J\, (D\mathbf{t})\;\, \Phi f\, \left(D^2 \mathbf{t}\right)\;\; \right) \tag{4.7}$$

is an orthonormal basis for $\mathbf{C}^3$.

Proof. Applying Theorem 12.2.4 [27] with $p = 1$, $q = 3$, $r = 1$, we infer that because $(M_m T_{n/3} f)_{m,n \in \mathbf{Z}}$ is a tight frame and $\|f\|_{L^2(\mathbf{R})} = 1$, it follows that $\|\Phi f(\mathbf{t})\|_{\mathbf{C}^3} = 1$ and so only the orthogonality remains to be shown.

By unitarity of Z the characterization condition (1.5) is equivalent to

$$\left\langle\;\; Z[\, \mu(A)^k\, M_{3m} T_n\, f] \begin{pmatrix} t \\ \omega \end{pmatrix},\;\, Zf \begin{pmatrix} t \\ \omega \end{pmatrix} \;\;\right\rangle_{L^2(\mathbf{X})} = 0 \quad \text{for all } m, n \in \mathbf{Z}\,,$$

which, in turn, is equivalent to

$$\left\langle \ \Phi[\mu(A)^k f](\mathbf{t}),\ \Phi f(\mathbf{t}) \ \right\rangle_{\mathbf{C}^3} = 0 \quad \text{for almost all } \mathbf{t} \in \mathbf{X}\ .$$

From Lemma 3.1 we obtain that for almost all $\mathbf{t} \in \mathbf{X}$

$$L(\mathbf{t}) \left\langle \ J(\mathbf{t})\, \Phi f\,(D\mathbf{t})\,,\ \Phi f(\mathbf{t}) \ \right\rangle = 0\ ,$$

$$L(\mathbf{t})L(D\mathbf{t}) \left\langle \ J(\mathbf{t})\ J\,(D\mathbf{t})\ \Phi f\left(D^2\mathbf{t}\right),\ \Phi f(\mathbf{t}) \ \right\rangle = 0\ .$$

From unitarity of $\mu(A)$ it also follows that for almost all $\mathbf{t} \in \mathbf{X}$

$$\overline{L(\mathbf{t})}L(\mathbf{t})L(D\mathbf{t}) \left\langle \ J(\mathbf{t})\ J\,(D\mathbf{t})\ \Phi f\left(D^2\mathbf{t}\right),\ J(\mathbf{t})\ \Phi f\,(D(\mathbf{t})) \ \right\rangle = 0 \quad .$$

Since $L(\mathbf{t})$ is almost nowhere vanishing, the assertion follows. □

Remark 4.1. It is easily seen by Corollary 4.1 that if we define atom f via

$$\Psi f(\mathbf{t}) = \Big(\ V(\mathbf{t})\, e_1,\ \ J(\mathbf{t})^{-1}\, V(\mathbf{t})\, e_2,\ \ J\,(D\mathbf{t})^{-1}\ J(\mathbf{t})^{-1}\, V(\mathbf{t})\, e_3 \ \Big)$$

where (e_1, e_2, e_3) is a standard basis of $\mathbf{C}^3$ and $V \in L^0(\mathbf{Y}, U(3))$, condition (4.7) holds and vice versa - all atoms satisfying (4.7) are of this form. Hence, the set of atoms satisfying (1.5) is in $1-1$ correspondence with $L^0(\mathbf{Y}, U(3))$.

In case of the classical Wilson system also all atoms f satisfying the characterization condition are in $1-1$ correspondence with $L^0\,(\mathbf{Y}_2, U(2))$, where $\mathbf{X}_2 = \left[0, \frac{1}{2}\right] \times [0,1]$, $B_2 : \mathbf{X}_2 \to \mathbf{X}_2$ defined by $B_2(t,\omega) = (t, 1-\omega)$ and $\mathbf{Y}_2 = \mathbf{X}_2 / B_2$ is the set of representatives of orbits under B_2 action.

The appropriate characterization condition is (cf. Ref. 23 (20)):

$$Z_2 f(t,\omega)\ \overline{Z_2 f(t+\tfrac{1}{2}, 1-\omega)} \ = \ Z_2 f(t+\tfrac{1}{2},\omega)\ \overline{Z_2 f(t, 1-\omega)} \qquad (4.8)$$

$$\text{where} \quad Z_2 f(t,\omega) \ = \ 2^{1/2} \sum_{n\in\mathbf{Z}} f(2(t-n))\, e^{2\pi i n\omega}$$

and it is equivalent to $\left\langle \ \Phi f(t,\omega),\ \ J(t,\omega)\, \Phi f(B_2(t,\omega)) \ \right\rangle_{\mathbf{C}^2} \ = \ 0$ for almost all $(t,\omega) \in \mathbf{X}_2$, where $\Phi f(t,\omega) = [Z_2 f(t,\omega), Z_2 f(t+1/2,\omega)]$ and $J(t,\omega) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for all $(t,\omega) \in \mathbf{X}_2$.

Acknowledgments

I would like to thank Prof. Karlheinz Gröchenig, University of Vienna, for inspiring discussions and comments, when working on this problem. Also I would like to thank Prof. Bruno Torresani for his hospitality during my stay in 2003 at Laboratoire d'Analyse, Topologie et Probabilites, Centre de Mathematique et Informatique, Universite de Provence, Marseille under the coverage of European Commission grant "Harmonic Analysis and Statistics in Signal and Image Processing" HPRN-CT-2002-00285, where the part of the research presented here was done.

References

1. P. Auscher, Remarks on the local Fourier bases, in *Wavelets: Mathematics and Applications*, Studies in Advanced Mathematics, eds. J. Benedetto and M. Frazier (CRC Press, Boca Raton, 1994) pp. 203–218.
2. K. Bittner, Linear approximation and reproduction of polynomials by Wilson bases, *J. Fourier Anal. Appl.* **8** (2002) 85–108.
3. H. Bölcskei, K. Gröchenig, F. Hlawatsch, H.G. Feichtinger, Oversampled Wilson Expansions, *IEEE Signal Processing Letters* **4** (1997) 106–108.
4. H. Bölcskei, K. Gröchenig, F. Hlawatsch, H.G. Feichtinger, Discrete-time Wilson expansions, Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis, Paris (France), (1996) 525–528.
5. I. Daubechies, S. Jaffard, J.-L. Journé, A simple Wilson orthonormal basis with exponential decay, *SIAM J. Math. Anal.* **22** (1991) 554–573.
6. H.G. Feichtinger, K. Gröchenig, D. Walnut, Wilson bases and modulation spaces, *Math. Nachr.*, **155** (1992) 7–17.
7. G.B. Folland, *Harmonic analysis in phase space*, Ann. of Math. Studies, Vol. 122, (Princeton University Press, Princeton, 1989).
8. G.B. Folland, *A course in abstract harmonic analysis*, (CRC Press, Boca Raton, 1995).
9. H. Führ, *Abstract harmonic analysis of continuous wavelet transforms*, Springer Lecture Notes in Math., Vol. 1863, (Berlin, 2005).
10. K. Gröchenig , *Foundations of Time-Frequency Analysis*, Applied and Numerical Harmonic Analysis, (Birkhaüser, Boston, 2001).
11. K. Gröchenig, D. Walnut, A Riesz basis for Bargmann-Fock space related to sampling and interpolation, *Arkiv för Math.*, **30** (1992) 283–295.
12. A.J.E.M. Janssen, Bargmann transform, Zak transform, and coherent states, *J. Math. Phys.*, **23** (1982) 720–731.
13. A.J.E.M. Janssen, The Zak transform: a signal transform for sampled time-continuous signals, *Philips J. Res.*, **43** (1988) 23–69.
14. A.J.E.M. Janssen, The duality condition for Weyl-Heisenberg frames, in: *Gabor Analysis and Algorithms. Theory and applications.*, Applied and Numerical Harmonic Analysis, eds. H.G. Feichtinger and Th. Strohmer (Birkhäuser, 1998) pp. 33–84.
15. A.A. Kirillov, *Elements of the theory of representations*, Grundlehren der mathematischen wissenschaften, (Springer-Verlag, 1976).
16. G. Kutyniok, Th. Strohmer, Wilson bases for general time-frequency lattices, *SIAM Journal on Mathematical Analysis* **37** (2005) 685–711.
17. G.W. Mackey, *The theory of unitary group representations*, Chicago Lectures in Mathematics, (The University Chicago Press, Chicago, 1976).
18. K. Maurin, *General Eigenfunction Expansions and Unitary Representations of Topological Groups* , Mathematical Monographs, Vol. 48, (PWN, Warsaw, 1968).
19. J. Wexler, S. Raz , Discrete Gabor Expansions, *Signal Processing* **21** (1990) 207–221.
20. K.G. Wilson, Generalized Wannier Functions, Cornell University, (1987), preprint.
21. P. Wojdyłło, Characterization of Wilson Systems for General Lattices, *International Journal of Wavelets, Multiresolution and Information Processing*, **6** (2008) 305–314.
22. P. Wojdyłło, Gabor and Wavelet Frames. Geometry and Applications, Ph. D. thesis (in English), Department of Mathematics, Computer Science, and Mechanics, Warsaw University, Warsaw (2001).
23. P. Wojdyłło, Modified Orthonormal Wilson Bases, *Sampling Theory in Signal and Image Processing* **6** (2007) 223–235.
24. P. Wojdyłło, Unconditional basis in Bargmann space–new proof of Gröchenig-Walnut theorem, *Math. Nachr.* **221** (2001) 169–181.

25. J. Zak, Lattice operators in crystals for Bravais and reciprocal vectors, *Phys. Rev. B* **12** (1975) 3023–3026.
26. M. Zibulski, Y. Zeevi, Discrete multi-window Gabor type transforms, *IEEE Trans. Signal Proc.* **45** (1997) 1428–1442.
27. M. Zibulski, Y. Zeevi, M. Porat, Multi-window Gabor schemes in signal and image representations, in: *Gabor Analysis and Algorithms. Theory and applications.*, Applied and Numerical Harmonic Analysis, eds. H.G. Feichtinger and Th. Strohmer (Birkhäuser, 1998), pp. 381–407.