Minuscule Exceptional Schubert Varieties

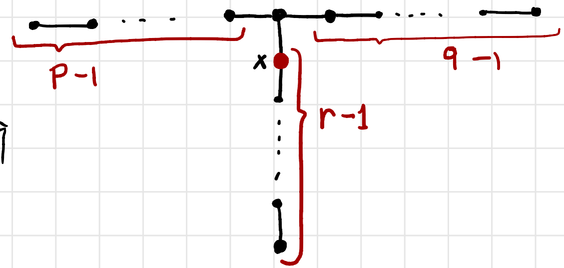
Jacinta Torres

Jagiellonnian University

joint work with Sara Angela Filipinni

and Jerzy Weyman

Motivation



- $T_{p,q,r}$ is of Dynkin type $\Longleftrightarrow$ R_{gen} is Noetherian (Weyman)

Format $T_{p,q,r}$

$$0 \to R_1 \xrightarrow{d_1} R_2 \xrightarrow{d_2} R_3 \xrightarrow{d_3} R$$

$\operatorname{rank}(d_1) + 1 = p$
$\operatorname{rank}(d_2) - 1 = q$
$\operatorname{rank}(d_3) + 1 = r$

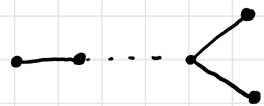
- For all Dynkin types, there exists a Schubert variety of codimension 3 in G/P_x, with an open subset whose minimal free resolution has the format $T_{p,q,r}$. (Sam-Weyman)

Dynkin types (simply laced Dynkin diagrams)

A_n

$n-1$ nodes

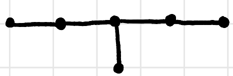
D_n



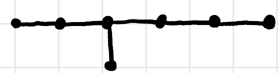
n nodes

Classical types

E_6

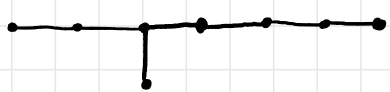


E_7



Exceptional types

E_8



$(R, X, R^\vee, X^\vee)$

root system

Reductive algebraic group G $\rightsquigarrow$ Dynkin diagram

$SL(n,\mathbb{C})$, $GL(n,\mathbb{C})$ $\rightsquigarrow$ type A_{n-1}

$SO(2n,\mathbb{C})$ $\rightsquigarrow$ type D_n

where $SO(2n,\mathbb{C}) = \{ A \in GL(2n,\mathbb{C}) \mid Q(Ax, Ay) = Q(x,y) \}$

for a non-degenerate, symmetric bilinear form $Q : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \longrightarrow \mathbb{C}$.

Explicit descriptions of E_6, E_7, E_8 (later more descriptions)

https://arxiv.org/pdf/1403.5120.pdf#page25

Weights and patterns

- Integral weights $\overset{1:1}{\longleftrightarrow}$ Labellings of the Dynkin diagram with integers $\sigma_i \in \mathbb{Z}$.

$\sigma = (\sigma_1, \dots, \sigma_n)$

- The Weyl group $W = \langle s_1, \dots, s_n \rangle$ acts on the weight σ by $\sigma_i \mapsto -\sigma_i$ and adding σ_i to all σ_j such that j is a node adjacent to i.

- The fundamental weight ω_i is defined by $\sigma_j = \delta_{ij}$.

Example (E_6)

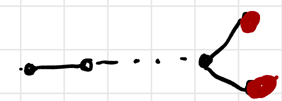
ω_2

$$\begin{array}{ccccc} {}_1 0 & 0_3 & 0_4 & 0_5 & 0_6 \\ & & 1_2 & & \end{array} \xrightarrow[s_2]{} \begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ & & -1 & & \end{array} \xrightarrow[s_4]{} \begin{array}{ccccc} 0 & 1 & -1 & 1 & 0 \\ & & 0 & & \end{array}$$

Fundamental Representations

- Nodes in Dynkin diagram $\xleftrightarrow{1:1}$ Fundamental weights
- Fundamental weight $\omega_i \rightsquigarrow$ Fundamental representation $V(\omega_i)$
- Type A_{n-1} : $V(\omega_i) = \Lambda^i \mathbb{C}^n$ for $i = 1, \dots, n-1$
- Type D_n : $V(\omega_i) = \Lambda^i \mathbb{C}^{2n}$ for $i = 1, \dots, n-2$ + two half-spin representations $V(S^+), V(S^-)$

"even" spinors "odd spinors"



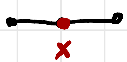
Exceptional types : later.

Parabolic subgroups and quotients

Node x in Dynkin diagram $\rightsquigarrow$ maximal parabolic subgroup $P_x \subseteq G$

$$\underbrace{G/P_x}_{\text{projective variety, smooth}} \hookrightarrow \mathbb{P}(V(\omega_x))$$

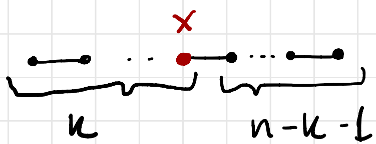
<u>Example</u> $n=4$



$$P_x = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}$$

$$GL(4,\mathbb{C})/P_x \cong Gr(2,4)$$

In general:



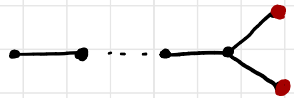
$\rightsquigarrow$

$$P_x = \left\{ \begin{array}{c|c} & n \\ k & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\ & n-k \end{array} \right\}$$

$$G/P_x \cong Gr(k,n) \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{C}^n)$$

Plücker embedding

Type D_n



Let x be one of the red vertices.

Fix $Q : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \longrightarrow \mathbb{C}$ as before. A subspace $V \subseteq \mathbb{C}^{2n}$ is <u>isotropic</u> if $Q(v,w)=0$ for all $v,w \in V$. The isotropic Grassmannian is

$$IG(n,2n) = \{ V \in Gr(n,2n) \mid V \text{ is isotropic} \}.$$

The homogeneous space $SO(2n,\mathbb{C})/P_x$ is one of the two connected components of $IG(n,2n)$.

$$\rightsquigarrow SO(2n,\mathbb{C})/P_x \hookrightarrow \mathbb{P}(V(S^+)) \text{ or } \mathbb{P}(V(S^-))$$

- Each of these connected components is called a variety of (even, odd, resp.) pure spinors.

G/P projective

Schubert varieties

B = upper-triangular matrices in $GL(n,\mathbb{C})$.

$S_n \hookrightarrow GL(n,\mathbb{C})$ (permutation matrices)

$GL(n,\mathbb{C}) = \bigcup_{w \in S_n} BwB$. In $GL(n,\mathbb{C})/B$, $X_w = \overline{BwB/B}$ are the Schubert vars.

Let $W_{P_x} = \langle S_x \rangle$. The Schubert varieties in G/P_x are those of the form $X_w = \overline{BwP_x}/P_x$, where B is the Borel subgroup contained in P_x. $w \in W/W_{P_x}$

- The codimension of X_w is the length of w.
- Each Schubert variety X_w contains an open cell $Y_w = BwP_x/P_x$ such that $X_w \setminus Y_w$ is a union of Schubert varieties of smaller dimension.
- Schubert varieties are Cohen-Macaulay, normal.

Example In $Gr(k,n)$, for $w \in W/W_{P_x}$, the open cell Y_I is given by the subspaces spanned by the rows of matrices of the form

$$\begin{array}{c} \\ \\ i_j \\ \\ \end{array} \begin{pmatrix} & \vdots & \\ & 0 & \\ 0 \cdots 0 & 1 & \\ & 0 & \end{pmatrix}$$

0's to the left and below the "diagonal" 1's

$I = i_1 < \cdots < i_k$

$\updownarrow$

$w \in W/W_{P_x}$

$W_{P_x} = \langle s_x \rangle.$

If $k=3$, $n=6$, then, for $I = (1,4,5)$ we have

$$Y_I = \text{row-span of } \left\{ \begin{pmatrix} 1 & a & b & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & d \\ 0 & 0 & 0 & 0 & 1 & e \end{pmatrix} : a,b,c,d,e \in \mathbb{C} \right\}$$

Plücker coordinates Let $\mathbb{C}^n$ with canonical basis $e_1, \dots, e_n$.

Let $W \in Gr(k,n)$ be spanned by $w_1, \dots, w_k \in \mathbb{C}^n$.

The assignment
$$Gr(k,n) \xrightarrow{\varphi} \mathbb{P}(\Lambda^k \mathbb{C}^n)$$
$$W \longmapsto [w_1 \wedge \cdots \wedge w_k]$$

is a well-defined map called the Plücker embedding. For indices $i_1, \dots, i_k$ we denote by $P_{i_1 \cdots i_k}(W)$ the projection of $\varphi(W)$ to the coordinate $[e_{i_1} \wedge \cdots \wedge e_{i_k} = 1]$.

Example For $k=2$, we have $P_{ij} = -P_{ji}$, hence the Plücker coordinates fit into a skew-symmetric matrix P. Then $Gr(2,n)$ is the projective algebraic variety defined by all the Pfaffians of the 4×4 minors of P.

Example (type D_n)

Pick the hyperbolic basis $e_1, \dots, e_n, e_{\bar{1}}, \dots, e_{\bar{n}}$ of $\mathbb{C}^{2n}$ with respect to Q, that is,

$$Q(a_1e_1+\cdots+a_ne_n+a_{\bar{1}}e_{\bar{1}}+\cdots+a_{\bar{n}}e_{\bar{n}},\, b_1e_1+\cdots+b_ne_n+b_{\bar{1}}e_{\bar{1}}+\cdots+b_{\bar{n}}e_{\bar{n}}) = \sum_{i=1}^{n} a_ib_{\bar{i}} + \sum_{i=1}^{n} a_{\bar{i}}b_i$$

Then the "big open cell" ($w=\mathrm{id}$) in $IG(n,2n)$ is spanned by the rows of matrices of the form $(I_n X)$ where I_n is the $n\times n$ identity matrix and X is a skew-symmetric matrix

Example The big open cell in $Gr(3,6)$ is given by matrices of the form:

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & x_{1,4} & x_{1,5} & x_{1,6} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} & x_{2,6} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$Q(v,w) = 0$$

for all v, w rows

- 1 with itself: $x_{1,4} = 0 = x_{2,5} = x_{3,6}$
- 1 with 2: $x_{2,4} + x_{1,5} = 0$
- 1 with 3 $x_{3,4} + x_{1,6} = 0$
- 2 with 3 $x_{3,5} + x_{2,6} = 0$

- Plücker coordinates are the Pfaffians of X of all sizes. They correspond to subsets of $\{1,\dots,n\}$ of even cardinality. $G/P_x \hookrightarrow \mathbb{P}(V_{\omega_x})$

- If $n=2m+1$ is odd, the $2m\times 2m$ Pfaffians of X are the defining equations of the intersection of some Schubert variety with our open cell.

- It is known that these Pfaffians span the generic Gorenstein ideal with resolution of format $(1,n,n,1)$.

Example

Let $M = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & x_{1,5} & x_{1,6} \\ 0 & 1 & 0 & -x_{1,5} & 0 & x_{2,6} \\ 0 & 0 & 1 & -x_{1,6} & -x_{2,6} & 0 \end{pmatrix} \\ \begin{matrix} 1 & 2 & 3 & \bar{1} & \bar{2} & \bar{3} \end{matrix} \end{array}$ and X its skew-symmetric part.

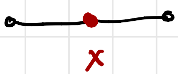
Any given subset of $\{\bar{1}, \bar{2}, \bar{3}\}$ of cardinality 2 determines a unique skew-symmetric 2×2 minor of M and X.

The subsets $\{\bar{1}, \bar{2}\}$, $\{\bar{1}, \bar{3}\}$ and $\{\bar{2}, \bar{3}\}$ correspond to $\{\bar{1}, \bar{2}, 3\}$, $\{\bar{1}, 2, \bar{3}\}$, and $\{\bar{1}, 2, \bar{3}\}$, whose corresponding determinants are the squares of the pfaffians of the appropriate 2×2 skew-symmetric minors.

In general, extremal Plücker coordinates are labelled by elements $\omega \in W/W_P$.

Example

$Gr(2,4)$



$$\begin{tikzcd}
 & & p_{14} \arrow[dr, "s_1"] & & \\
p_{12} \arrow[r, "s_2"] & p_{13} \arrow[ur, "s_3"] \arrow[dr, "s_1"'] & & p_{24} \arrow[r, "s_2"] & p_{34} \\
 & & p_{23} \arrow[ur, "s_3"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & 1\,0\,{-1} \arrow[dr, "s_1"] & & \\
0\,1\,0 \arrow[r, "s_2"] & 1\,{-1}\,1 \arrow[ur, "s_3"] \arrow[dr, "s_1"'] & & {-1}\,1\,{-1} \arrow[r, "s_2"] & 0\,{-1}\,0 \\
 & & {-1}\,0\,1 \arrow[ur, "s_3"'] & &
\end{tikzcd}$$

To parametrize the poset W/W_P, we start with the fondamental weight ω_i and act by simple reflections.

Minuscule Representations

Def. A fundamental representation is minuscule if the Weyl group acts transitively on its set of weights.

- For type A_n, all fundamental representations are minuscule.
- For type D_n, the minuscule representations are the two half-spin representations.

Example of non-minuscule representation

$G = SL(3,\mathbb{C})$, V = adjoint representation

T = diagonal matrices

$$\begin{tikzcd}
 & -1\ \ 2 \arrow[r, "2"] & 1\ -2 \arrow[dr, "1"] & \\
1\ \ 1 \arrow[ur, "1"] \arrow[dr] & & & -1\ -1 \\
 & 2\ -1 \arrow[r, "1"] & -2\ \ 1 \arrow[ur, "2"] &
\end{tikzcd}$$

- For type E_6, there are two (dual) minuscule representations of dimension 27. They are determined by the well-known configuration of 27 lines on a cubic surface.

- For type E_7, there is one minuscule rep. of dimension 56
$\rightsquigarrow$ blow-up of seven points ingeneral position in complex proj. plane.

- There are no minuscule representations in type E_8.

Theorem (S.A.F.-J.W.-T.)

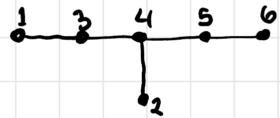
Let G be of exceptional type and $P \subseteq G$ a standard maximal parabolic subgroup stabilising a minuscule fundamental weight. Then there exists an open subset $U \subseteq G/P$ such that for $\sigma \in W_{E_6}/P_{E_6}$ and for $\sigma \in I \subseteq W_{E_7}/P_{E_7}$, the intersection $X_\sigma \cap U$ is one of the following

- complete intersection
- var. of pure spinors
- variety of complexes (minors $(2, Y)$ + ideal $(X \cdot Y)$) (Herzog '74, Kustin '93)
- Huneke-Ulrich ideal of deviation 2 ($\mathrm{Pf}(X)$ + ideal $(Y \cdot X)$) (Kustin '86)
- 2×2 minors of a 2×3 generic matrix
- 4×4 Pfaffians of a 6×6 skew-symmetric matrix

Methods

- We use descriptions of the minuscule representations by Vavilov, Luzgarev & Pevzner
- Hands-on inspection assisted by Macaulay2

Example (E_6, ω_1)



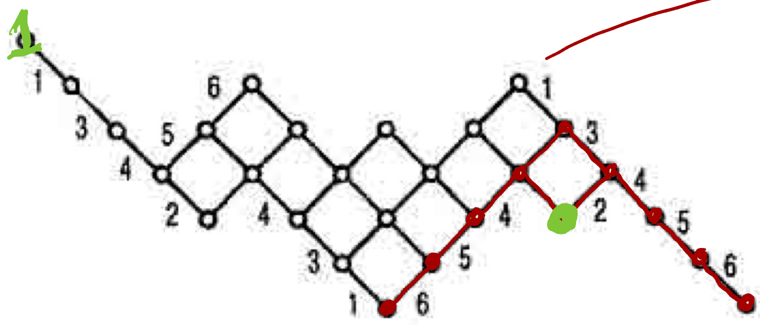


Fig. 20. $(E_6, \overline{\omega}_1)$

The equations for $X_0 \cap U$ are given by the vanishing of the ten equations shaded in in red.

$Y_0 = X_0 \cap U$ is isomorphic to the variety of pure spinors.

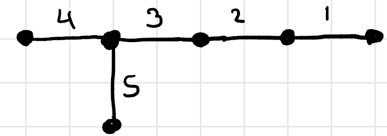
Image: Visual basic representations: an atlas (Plotkin, Semenov, Vavilov)

W/W_{P_1}

node w $\rightsquigarrow$ $I(X_w)$ is gen. by x vert. $x \not\leq w$

Bruhat order

Example (D_6, ω_6)



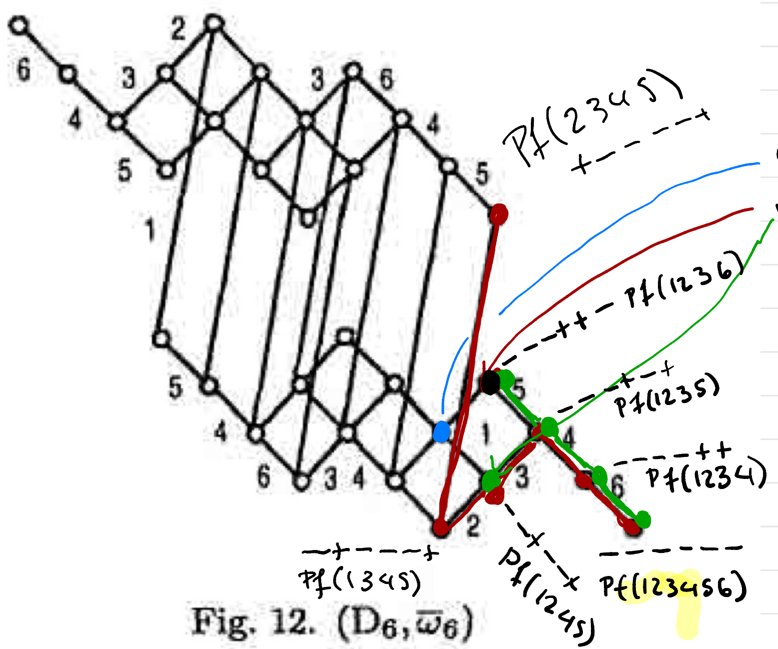


Fig. 12. $(D_6, \overline{\omega}_6)$

Schubert varieties

four equations for the green one

six equations for the red one

$\bullet \rightsquigarrow X_\bullet$ Schubert variety.

generators $\hookleftarrow$ all nodes y

$y \nleq \bullet$

Bruhat order

(D_n, ω_n)

Image: Visual basic representations: an atlas (Plotkin, Semenov, Vavilov)