The Ubiquity of Wronskians

Banach Center, October 12–18, 2011

List of Open Problems

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1 Grassmannians of subspaces of polynomials

1.1 Background

References for this section have been discussed during the talks by Kazarian and Scherbak. See e.g. [15], [16], [18], [19], [20], [21].

1.1 Consider the Grassmann variety $G_n(\mathrm{Poly}_d)$ of n-dimensional subspaces in the space of polynomials in one variable of degree at most d. Consider a plane spanned by a collection of n polynomials. The plane is called *degenerate* if the Wronski determinant of the corresponding collection of polynomials has multiple roots. Wronski degeneracy types produce a stratification of the Grassnmannian; the strata are labelled by (collections of) Young diagrams.

1.2 Recall that the Casimir operator is a central element of the universal enveloping algebra of a Lie algebra L. For example if one takes L to be the Lie algebra of skew symmetric 3×3 real matrices, the square of the modulus of the angular momentum would be the Casimir operator of $\mathfrak{so}_3$. In other words, Casimir operators are generalizations of certain kinds of motion constants. The Gaudin Hamiltonians are mutually commuting linear operators on the space of states (which is the tensor product of certain irreducible representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$): these are defined in terms of the Casimir operator of $\mathfrak{sl}_2(\mathbb{C})$ itself (Cf. [19]).

1.2 Open Problems

1.3 Study the analytic triviality of the stratification along the strata.

1.4 What is the relationship between generalized Wronskians and Schubert calculus in the polynomial Grassmannians, through ODEs having rational functions as coefficients?

1.5 Look for an interpretation of the Casimir operators (or, more generally, Gaudin Hamiltonians) in terms Grassmanians of spaces of polynomials.

1.3 Comments

M. Kazarian introduced in [15, 16] a more delicate classification (equivalence relation) of points of the Grassmannian, showing that the stratification is analytically trivial along the subvarieties of equivalent points. For certain Young diagrams the whole stratum consists of a unique equivalence class but, in general, the stratum corresponding to a given Young diagram may consist of several equivalence classes of the detailed classification, or even the equivalence classes can form continuous families. In problem 1.3 one aims either to show that the analytic type of the stratification is independent of the detailed classification or to give an example of two non-equivalent points with the Wronski degeneracy corresponding to the same Young diagrams, for which the stratification of the Grassmannian near these points are analytically non-equivalent.

2 Sections of Grassmann Bundles

2.1 Background

2.1 Let C be a smooth complex projective curve of genus $g \geq 0$, $L \in Pic^d(C)$ and $\rho : J^dL \to C$ (where J^hL is the bundle of jets of L of order $h \geq 0$): it is a vector bundle of rank $d+1$. Let $\rho_r : G \to C$, where $G := G(r+1, J^dL)$, be the associated Grassmann bundle and let

$$0 \longrightarrow \mathcal{S}_r \xrightarrow{\iota_r} \rho_* J^dL \to \mathcal{Q}_r \to 0$$

be the tautological exact sequence over G, where $\mathcal{S}_r$ is the universal vector subbundle of rank $r+1$ and $\mathcal{Q}_r$ is the universal quotient bundle of rank $d+1-r$. Let

$$\partial_h : \mathcal{S}_r \longrightarrow J^hL$$

be the composition of the universal monomorphism ι_r with the truncation morphism $J^dL \to J^hL$. In particular ∂_d can be identified with the universal monomorphism ι_r. One defines Schubert varieties associated to the truncation filtration of J^dL as follows:

$$\Omega_{(i_0 i_1 \dots i_r)}(J^\bullet L) = \{\Lambda \in G \,|\, rk_\Lambda \partial_{i_j - 1} \leq j\}$$

The Wronskian variety is:

$$\mathcal{W} := \Omega_{(01\dots,r-1,r+1)}(J^\bullet L).$$

It is a Cartier divisor as it is the zero locus of the universal Wronskian:

$$\mathbb{W}_r := \bigwedge^{r+1} \partial_r \in H^0(G, \rho_r^* \bigwedge^{r+1} J^rL \otimes (\bigwedge^{r+1} \mathcal{S}_r)^\vee)$$

Wronskian of sections of G. Let $\Gamma(\rho_r)$ be the set of all holomorphic sections of $G \to C$ (it is a quite wild object without further restrictions). If $\gamma \in \Gamma(\rho_r)$, the Wronskian of γ is

$$W_\gamma = \gamma^* \mathbb{W}_r \in H^0(C, L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$$

The zero scheme of W_γ is isomorphic to $\gamma^{-1}(\mathcal{W}_r)$, the pre-image of the Wronskian variety on C through γ. We define $\gamma^{-1}(\mathcal{W}_r)$ as the ramification locus of γ. The class in $A_*(C)$ of the ramification locus is:

$$[\gamma^{-1}(\mathcal{W})] = [\gamma^*\mathbb{W}_r] = \gamma^*[\mathbb{W}_r] = \gamma^*(c_1(\rho_r^* J^r L) - c_1(\mathcal{S}_r)) \cap [C] =$$

$$= \left[(r+1)c_1(L) + \frac{1}{2}r(r+1)c_1(K) - \gamma^* c_1(\mathcal{S}_r)\right] \cap [C].$$

If $\gamma^*\mathcal{S}_r$ is trivial, then

$$[\gamma^{-1}(\mathcal{W})] = [(r+1)c_1(L) + \frac{1}{2}r(r+1)c_1(K)] \cap [C] = (r+1)d + (g-1)r(r+1).$$

We say that $P \in C$ is a ramification point of γ if and only if $\gamma(P) \in \mathcal{W}$. For each point $P \in C$, there exists $I := (0 \le i_0 < i_1 < \ldots < i_r \le d)$ such that $\gamma(P) \in \Omega_{i_0,i_1,\ldots,i_r}(J^\bullet L)$ and $\gamma(P) \notin \Omega_J(J^\bullet L)$ for each $J \neq I := (i_0, i_1, \ldots, i_r)$ and $wt(J) \ge wt(I)$. The weight of I is $\sum_{j=0}^r (i_j - j)$. We say that $(i_0, i_1, \ldots, i_r)$ is the "vanishing" sequence of γ at P. If $\gamma(P) \in \Omega_I(J^\bullet L)$, then the vanishing sequence $(j_0, j_1, \ldots, j_r)$of γ at P is at least I, in the sense that $j_h \ge i_h$ for each $0 \le h \le d$.

2.2 Linear systems. Let

$$\Gamma_{\texttt{triv}}(\rho_r) = \{\text{sections } \gamma : C \to G(r+1, J^d L) \text{ such that } \gamma_V^* \mathcal{S}_r \text{ is trivial}\}.$$

Let $V \in G(r+1, H^0(L))$. Then $C \times V \to J^d L$ sending $(P, v) \mapsto D_d v(P) \in J^d_P L$ is a bundle monomorphism and as such it induces a holomorphic section $\gamma_V \in \Gamma_{\texttt{triv}}(\rho_r)$:

$$\begin{cases} \gamma_V & : & C & \longrightarrow & G(r+1, J^d L) \\ & & P & \longmapsto & \gamma_V(P) = (D_d V)(P) \end{cases}$$

where $(D_d V)(P) = \{D_d v(P) \,|\, v \in V\}$. In this case the Wronskian and the ramification scheme of the section γ_V coincide with the usual notion of Wronskian and ramification scheme of the linear system V.

Notice that $\Gamma_{\texttt{triv}}(\rho_r)$ can be realized as an open set of the Grassmannian $G(r+1, H^0(J^d L))$. In fact each $(r+1)$-dimensional subspace W of $H^0(J^d L)$ define a morphism

$$C \times W \to J^d L,$$

which sends (P, w) to $w(P)$. Then $\Gamma_{\texttt{triv}}(\rho_r)$ corresponds to the set of all W such that the above morphism is a monomorphism.

2.2 Open Problems

2.3 Prove or disprove the following claim:

For each $P \in C$ let $\mathrm{ev}_P : \Gamma_{\mathtt{triv}}(\rho_r) \to G(r+1, J^d L)$ sends $\gamma \mapsto \gamma(P)$. Let $P_1, \ldots, P_h$ be arbitrary distinct points on C and let $I_1, \ldots, I_h$ be strictly increasing sequences of $r+1$ non negative integers not bigger than d:

$$I_j = (0 \leq i_{j,0} < i_{j,1} < \ldots < i_{j,r} \leq d)$$

such that $wt(I_1) + \ldots + wt(I_h) = w \leq (r+1)d + (g-1)r(r+1)$. Then the subvariety

$$\mathrm{ev}_{P_1}^{-1}(\Omega_{I_1}(J^\bullet L)) \cap \ldots \cap \mathrm{ev}_{P_1}^{-1}(\Omega_{I_h}(J^\bullet L)) \tag{1}$$

which set theoretically is given by:

$$\{\gamma \in \Gamma_{\mathtt{triv}}(\rho_r) \,|\, \gamma \text{ ramifies at each } P_j \text{ with vanishing sequence at least } I_j\}$$

has codimension w in $\Gamma_{\mathtt{triv}}(\rho_r)$. In other words the intersection of the pre-images of the Schubert varieties through the evaluation maps is proper.

2.4 Compute the class in $A_*(G(r+1, H^0(J^d L))$ of the locus (1).

2.5 If $V = H^0(K)$, one gets the generalization of Weierstrass point for a section of a Grassmann bundle. Let $\rho_{g-1} : G(g, J^{2g-2}K) \to C$. Then $\Gamma_{\mathtt{triv}}(\rho_{g-1})$ can be identified with an open set of the Grassmannian $G(g, H^0(J^{2g-2}K))$. Let $\pi_{g-1} : H^0(J^{2g-2}K) \to H^0(J^{g-1}K)$. Then $\gamma \in \Gamma_{\mathtt{triv}}(\rho_{g-1})$ can be identified with a g-dimensional subspace of $H^0(H^0(J^{2g-2}K))$. If $(v_1, \ldots, v_g)$ is a basis of such a subspace, the Wronskian of γ is, up to a non zero constant,

$$p_{g-1}(v_1) \wedge \ldots \wedge p_{g-1}(v_g) \in H^0(\bigwedge^g J^{g-1}K).$$

This leads to define a generalized Wronski map:

$$G(g, H^0(J^{g-1}K)) \longrightarrow \mathbb{P}H^0(K^{\otimes \frac{g(+1)}{2}})$$

Have you any explanation of the fact that

$$\dim G(g, H^0(J^{g-1}K)) = (g-1) \cdot (g-1)g(g+1)$$

i.e. a multiple of the total weight $(= (g-1)g(g+1))$ of Weierstrass points on C?

2.3 Comments

2.6 (Related with Inna Scherbak's first talk). If $C = \mathbb{P}^1$ and $L_d \in Pic^d(\mathbb{P}^1)$, then $L_d = O_{\mathbb{P}^1}(d)$. It turns out that $J^d L_d$ is trivial as there exists a bundle isomorphism

$$C \times G(r+1, H^0(L)) \to G(r+1, J^d L_d)$$

In this case $\Gamma_{\mathtt{triv}}(\rho_r)$ coincides with the space of the linear systems $G(r+1, H^0(L))$. Further, the Schubert varieties $\Omega(F_P)$ (F_P the osculating flag) defined by I. Scherbak in her lectures coincide precisely with $\mathrm{ev}_P^{-1}(\Omega_I(J^\bullet L))$. In this case claim 2.3 is true and has been proven by Eisenbud and Harris in [3]. Thus the picture exposed in section 2.1 is the generalization of the situation that Eisenbud and Harris describe in [3] to the case of curves of positive genus.

3 Families of Curves

3.1 Background

Let $\mathfrak{X} \to S$ be a germ of a stable curve over S, i.e. $S = \mathrm{Spec}(\mathbb{C}[[t]])$, $\mathfrak{X}_\eta$ is a geometrically irreducible smooth curve (the generic fiber) and $\mathfrak{X}_0$ (the special fiber) is a stable curve of genus g (connected, nodal such that each rational component has at least three marked points). Thanks to e.g. [4, 5, 6] and [14], one can define locally free substitutes of the principal parts (Jets) in the case of the relative dualizing sheaf (because of the singularities of the special fibers, the classical principal parts of ω_π *are not* locally free). Construct

$$\rho_{g-1} : G(g, J^{2g-2}\omega_\pi) \to \mathfrak{X}.$$

Let $\mathbb{E}_\pi := \pi_*\omega_\pi$ be the Hodge bundle of the family. The vector bundle map over $\mathfrak{X}$

$$\pi^*\mathbb{E}_\pi \longrightarrow J^{2g-2}\omega_\pi$$

fails to be injective only along the special fiber. Hence it defines a rational section

$$\gamma_\omega : \mathfrak{X} - - \to G(g, J^{2g-2}K). \tag{2}$$

3.2 Open Problems

3.1 Does the rational section (2) extend to a regular section on all of $\mathfrak{X}$? If yes, this would provide a definition of Weierstrass points on a stable uninodal curve (just by intersecting the extended section with the Wronskian subvariety of $G(g, J^{2g-2}\omega_\pi)$, which can be defined precisely as in the previous section. This question is already interesting if the special fiber is a uninodal reducible curve (e.g. the union of X and Y intersecting transversally at $\{P\} = X \cap Y$): if the fiber is irreducible the map (2) is a morphism.

3.2 If the map (2) can be extended, is the extension unique?

3.3 If the map cannot be extended, does there exist a holomorphic section γ of ρ_{g-1} such that $\gamma^*\mathcal{S}_{g-1} = \pi^*\mathbb{E}_\pi$?

3.4 There is a theory of limits of Weierstrass points due to several authors (Eisenbud, Harris, Diaz, Cukierman, Esteves...). The indeterminacy of the rational section (2) can be resolved by blowing up the indeterminacy locus (not difficult, via the graph construction). This leads to a definition of Weierstrass point on a reducible curve. What is the relationship of such “Weierstrass points” with the limits à la Eisenbud and Harris?

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