Egzemplarz autorski

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On a Jacobi–Trudi Identity for Supersymmetric Polynomials

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0. INTRODUCTION

In the very late eighties a new identity for symmetric polynomials was discovered. In the form presented here the identity is a generalization of the Jacobi–Trudi identity. The latter identity expresses the Schur polynomials in a finite set of variables as a certain symmetrizing operator applied to monomials in the variables. The new identity involves two sets of variables. It expresses the super Schur polynomials as a certain symmetrizing operator applied to very simple polynomials in the two sets of variables.

It is a classical result that the Schur polynomials are the characters of the polynomial representations of SL_m. Hence the Jacobi–Trudi identity may be viewed as a character formula for SL_m. This approach was generalized to other algebraic groups by H. Weyl in his character formula.

The new identity was in fact discovered as a Weyl-type formula for the characters of polynomial representations of the Lie superalgebra $sl(m/n)$. From one side the formula was conjectured by J. van der Jeugt, J. W. B. Hughes, R. C. King and J. Thierry-Mieg [**J-H-K-T**, p.2291]. On the other side the identity was communicated without proof by A. Serge'ev to the first author who gave a proof of its validity in [**P**]. The proof, though elementary, rested on the characterization by J. Stembridge of super Schur polynomials via a certain cancellation property; therefore, the proof in [**P**] was not self-contained.

The aim of present note is to give a self-contained and elementary proof of the new identity. The method used gives a simple insight in the space of supersymmetric polynomials. Byproducts of the proof are the above mentioned characterization of super Schur polynomials given by J. Stembridge [**S**], the basis theorem and the factorization formula for super Schur polynomials given by A. Berele and A. Regev [**B-R**], and the duality formula.

1. Preliminaries

SETUP (1.1). We use throughout for partitions the notation described in Macdonald's book [M]. In particular, a partition

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

is assumed to be decreasing,

$$\lambda_1 \geq \lambda_2 \geq \cdots,$$

and the Ferrers' diagram of λ is the set of points $(i, j) \in \mathbf{N}^2$ such that $1 \leq j \leq \lambda_i$. The diagram of λ will be denoted D_λ. The diagram will always be pictured in a system of matrix coordinates where the first index i is a row index and the second j is a column index.

DEFINITION (1.2). Let m, n be non-negative integers. A partition λ will be said to be *contained in the* (m, n)*-hook*, if $\lambda_{m+1} \leq n$, or, equivalently, if the diagram D_λ is contained in the 'hook',

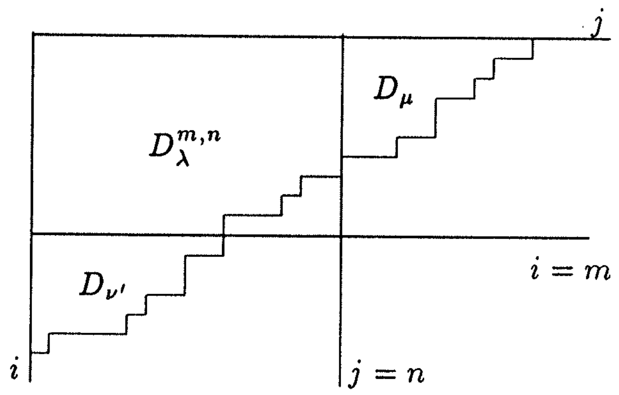
$$\{(i, j) \in \mathbf{N}^2 \mid i \leq m \text{ or } j \leq n\}.$$

Denote by $D_\lambda^{m,n}$ the following subset of D_λ:

$$D_\lambda^{m,n} := \{(i, j) \mid i \leq m,\ j \leq n,\ j \leq \lambda_i\}.$$

Assume that λ is contained in the (m, n)-hook. Then the part of D_λ outside $D_\lambda^{m,n}$ consists of two parts which – up to a translation – are diagrams of partitions: The first part consists of the points to the right of the line $j = n$, that is, of points $(i, j) \in D_\lambda$ such that $j > n$, and the second part consists of the points below the line $i = m$, that is, of points $(i, j) \in D_\lambda$ such that $i > m$. Up to a translation of the i-axis, the first part is the diagram of a partition μ; more precisely, the non-zero parts of μ are defined by $\mu_i = \lambda_i - n \quad \text{for } \lambda_i > n$. Similarly, the second part is up to a translation the transpose of the diagram of a partition ν, whose non-zero parts are defined by $\nu_j = \lambda'_j - m \quad \text{for } \lambda'_j > m$. Here λ' denotes the conjugate of λ, defined by transposing the diagram of λ, or, equivalently, by $\lambda'_j = \mathrm{Card}\{i \mid \lambda_i \geq j\}$. Note that the partitions μ and ν obtained from λ depend on m and n.

The three parts of D_λ are pictured in the figure below, where each point (i,j) is represented by a unit square with (i,j) as its lower right vertex.



DEFINITION (1.3). Let $X_m = (x_1, \ldots, x_m)$ and $Y_n = (y_1, \ldots, y_n)$ be two independent sets of indeterminates. Define polynomials $S_k(X_m/Y_n)$ for all integers k by the following equation of power series in the variable T:

$$\prod_{i=1}^{m}(1 - x_iT)^{-1}\prod_{j=1}^{n}(1 + y_jT) = \sum_k S_k(X_m/Y_n)T^k.$$

In particular, $S_k = 0$ when $k < 0$. Clearly, the polynomial S_k is homogeneous of degree equal to k.

Let λ be a partition. Denote by l the *length* of λ, that is, the number of non-zero parts of λ. Define the polynomial $S_\lambda(X_m/Y_n)$ as the following $l \times l$ determinant:

$$S_\lambda := \det[S_{\lambda_i - i + j}(X_m/Y_n)]_{1 \le i,j \le l}\,.$$

The polynomial $S_\lambda(X_m/Y_m)$ will be called the *super Schur polynomial* corresponding to the partition λ. It is homogeneous of degree equal to the weight $|\lambda| = \lambda_1 + \cdots + \lambda_l$ of λ.

Note that the polynomials $S_\lambda(X_m/Y_n)$ specialize as m and n varies: If $m \le \hat{m}$ and $n \le \hat{n}$, then the substitution of $x_i = 0$ for $m < i \le \hat{m}$ and $y_j = 0$ for $n < j \le \hat{n}$ in the polynomial $S_\lambda(X_{\hat{m}}/Y_{\hat{n}})$ yields the polynomial $S_\lambda(X_m/Y_n)$.

Note also that if $n = 0$, then the super Schur polynomial is the usual Schur polynomial $S_\lambda(X_m)$ in the single set of indeterminates X_m.

Definition (1.4). Let λ be a partition. Assume that the length of λ is less than or equal to m. Define

$$F_\lambda(X_m) := \sum_w w \left[\frac{x_1^{\lambda_1+m-1} x_2^{\lambda_2+m-2} \cdots x_m^{\lambda_m}}{\Delta(X_m)} \right]. \tag{1.4.1}$$

The sum is over all w in the group $\mathrm{Aut}(X_m)$ of all permutations of the indeterminates X_m. The denominator $\Delta = \Delta(X_m)$ is the Vandermonde determinant

$$\Delta = \prod_{1 \leq i < j \leq m} (x_i - x_j).$$

When the length of λ is greater than m, define $F_\lambda(X_m) := 0$.

The sum on the right hand side of (1.4.1) is the result of applying a symmetrizing operator. Note that the sum may be rewritten as follows:

$$\frac{1}{\Delta(X_m)} \sum_w \mathrm{sign}(w)\, w \left[x_1^{\lambda_1+m-1} x_2^{\lambda_2+m-2} \cdots x_m^{\lambda_m} \right].$$

Clearly, the sum in the latter expression is an alternating polynomial. Hence the sum is divisible by $\Delta(X_m)$. Thus F_λ is indeed a symmetric polynomial.

Lemma (1.5). (1) *For any partition λ the following equation holds:*

$$S_\lambda(X_m) = F_\lambda(X_m). \tag{1.5.1}$$

(2) *The polynomials $F_\lambda(X_m)$ for all partitions λ of length less than or equal to m form a basis of the* **Z**-*module of symmetric polynomials in X_m.*

Proof: The identity (1.5.1) is the Jacobi-Trudi identity. The reader is referred Jacobi [J] for the classical proof or to Macdonald [M] for a modern proof.

To prove (2), note that the polynomials of the form

$$\sum_w \mathrm{sign}(w) w \left[x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} \right],$$

where the sequence of exponents is strictly decreasing, form a basis of the **Z**-module of alternating polynomials. The latter statement implies the assertion in (2), because multiplication by $\Delta(X_m)$ is an isomorphism from the **Z**-module of symmetric polynomials onto the **Z**-module of alternating polynomials.

DEFINITION (1.6). Let λ be a partition. Assume that λ is contained in the (m, n)-hook. Define

$$F_\lambda(X_m/Y_n) :=$$
$$\sum_w w \left[\frac{x_1^{\mu_1+m-1} \cdots x_m^{\mu_m} \cdot y_1^{\nu_1+n-1} \cdots y_n^{\nu_n} \prod_{(i,j)\in D_\lambda^{m,n}} (x_i + y_j)}{\Delta(X_m)\Delta(Y_n)} \right],$$

where the partitions μ and ν are obtained from λ as in (1.2). The sum is over all permutations $w = u \times v$ in the product group $\mathrm{Aut}(X_m) \times \mathrm{Aut}(Y_n)$.

When λ is not contained in the (m, n)-hook, define $F_\lambda(X_m/Y_n) := 0$.

LEMMA (1.7). (1) *The polynomial $F_\lambda(X_m/Y_n)$ is homogeneous of degree equal to $|\lambda|$.*
(2) *If λ is fixed, then the polynomials $F_\lambda(X_m/Y_n)$ for varying m and n specialize, that is, if $\hat{m} \geq m$ and $\hat{n} \geq n$, then the substitution of $x_{\hat{m}} = \cdots = x_{m+1} = 0$ and $y_{\hat{n}} = \cdots = y_{n+1} = 0$ in the polynomial $F_\lambda(X_{\hat{m}}/Y_{\hat{n}})$ yields the polynomial $F_\lambda(X_m/Y_n)$.*

PROOF: The assertion in (1) is trivial.

Clearly, it suffices to prove the assertion in (2) for $\hat{m} = m + 1$ and $\hat{n} = n$. Moreover, it may be assumed that λ is contained in the $(m+1, n)$-hook. Then, by definition,

$$\text{(1.7.1)} \quad F_\lambda(X_{m+1}/Y_n) =$$
$$\sum_{\hat{w}} \hat{w} \left[\frac{x_1^{\mu_1+m} \cdots x_{m+1}^{\mu_{m+1}} y_1^{\hat{\nu}_1+n-1} \cdots y_n^{\hat{\nu}_n} \prod_{(i,j)\in D_\lambda^{m+1,n}} (x_i + y_j)}{\Delta(X_{m+1})\Delta(Y_n)} \right],$$

where the sum is over all permutations $\hat{w} = \hat{u} \times v$, where $\hat{u} \in \mathrm{Aut}(X_{m+1})$ and $v \in \mathrm{Aut}(Y_n)$. The partitions μ and $\hat{\nu}$ are those obtained from λ using the $(m + 1, n)$-hook.

Note first that the numerator of the fraction in (1.7.1) contains as a factor a power of the indeterminate x_i with an exponent that is positive for $i < m + 1$. Therefore, to evaluate the sum (1.7.1) for $x_{m+1} = 0$ it suffices to evaluate the terms corresponding to permutations $\hat{u} \times v$ such that $\hat{u}(m + 1) = m + 1$. In other words, evaluation is performed by evaluating the fraction in (1.7.1) and then forming the sum over all permutations $u \times v$ in $\mathrm{Aut}(X_m) \times \mathrm{Aut}(Y_n)$.

To prove the assertion of the Lemma, assume first that λ is not contained in the (m,n)-hook. Then, by definition, $F_\lambda(X_m/Y_n) = 0$. On the other hand, the assumption implies that $\lambda_{m+1} > n$. Hence x_{m+1} appears with the non-zero exponent μ_{m+1} in the numerator of the fraction in (1.7.1). Therefore, evaluation of the sum (1.7.1) yields zero. Hence the assertion holds under the first assumption.

Assume next that λ is contained in the (m,n)-hook. Then $F_\lambda(X_m/Y_n)$ is given by the formula in (1.6). As noted above, it suffices to prove that evaluation at $x_{m+1} = 0$ of the fraction in (1.7.1) yields the fraction in (1.6). Evaluate each factor in the fraction of (1.7.1) and divide the result by the corresponding factor of the fraction of (1.6). Clearly, the result is the following fraction:

$$\frac{x_1 \cdots x_m\, y_1^{\hat\nu_1} \cdots y_n^{\hat\nu_n} \prod_{1 \le j \le \lambda_{m+1}} y_j}{x_1 \cdots x_m \cdot 1 \cdot y_1^{\nu_1} \cdots y_n^{\nu_n}}.$$

It follows from the definition of $\hat\nu$ and ν that the latter fraction is equal to 1. Hence the assertion of the Lemma holds under the second assumption.

Thus the lemma has been proved.

2. Super Symmetric Polynomials

Setup (2.1). Fix as in Section 1 two independent sets of indeterminates $X_m = (x_1, \ldots, x_m)$ and $Y_n = (y_1, \ldots, y_n)$.

Definition (2.2). A polynomial F in the $m+n$ variables X_m and Y_n will be called *supersymmetric* if F is symmetric with respect to X_m and with respect to Y_n , and the substitution $x_m = t$, $y_n = -t$ in F yields a polynomial independent of t.

It follows immediately from the definition in (1.3) that the super Schur polynomials $S_\lambda(X_m/Y_n)$ are supersymmetric.

Note that if the polynomial F is supersymmetric, then the polynomial $F_{x_m=y_n=0}$ obtained by substitution of $x_m = 0 = y_n$ in F is supersymmetric in X_{m-1} and Y_{n-1}. Moreover, if F is supersymmetric, then the polynomial $F_{x_m=y_n=0}$ is equal to zero if and only if F is divisible by the product $P(X_m, Y_n) := \prod_i \prod_j (x_i + y_j)$.

Proposition (2.3). (1) *Every polynomial $F_\lambda(X_m/Y_n)$ is supersymmetric.*

(2) *The family of polynomials $F_\lambda(X_m/Y_n)$, where λ ranges over the partitions contained in the (m,n)-hook, forms a basis for the* $\mathbf{Z}$*-module of supersymmetric polynomials in X_m and Y_n.*

PROOF: (1) The polynomial $F_\lambda(X_m/Y_n)$ is the result of symmetrizing, and hence symmetric in X_m and Y_n.

Denote by N_λ the numerator in the fraction appearing in the definition of F_λ. Then F_λ is the quotient obtained by dividing the following sum

$$\sum_w \mathrm{sign}(w)\, w(N_\lambda) \tag{2.3.1}$$

by the product $\Delta(X_m)\Delta(Y_n)$. Clearly, the substitution $x_m = t$, $y_n = -t$ in the latter product yields a polynomial of degree $m+n-2$ in t. Therefore, to prove that F_λ is supersymmetric, it suffices to prove for each term of the sum (2.3.1) that the substitution $x_m = t$, $y_n = -t$ in it yields a polynomial of degree less than or equal to $m+n-2$ in t. Equivalently, it suffices to prove for every $i = 1,\dots,m$ and $j = 1,\dots,n$ the following assertion: The substitution $x_i = t$, $y_j = -t$ in N_λ yields a polynomial of degree less than or equal to $m+n-2$ in t.

The proof of the latter assertion is divided in two cases. Assume first that $j \le \lambda_i$, that is, (i,j) belongs to the diagram $D_\lambda^{m,n}$. Then N_λ contains the factor $x_i + y_j$. Consequently the substitution $x_i = t$, $y_j = -t$ in N_λ yields the zero polynomial.

Assume next that $j > \lambda_i$, or, equivalently, that $i > \lambda'_j$, where λ' is the conjugate partition of λ. Then, in particular, $\lambda_i \le m$ and $\lambda'_j \le n$. Hence, in the notation of Definition (1.2), $\mu_i = 0$ and $\nu_j = 0$. Therefore, the factors in N_λ that contain either x_i or y_j are the following:

$$(x_i + y_1),\dots,(x_i + y_{\lambda_i}),\ (x_1 + y_j),\dots,(x_{\lambda'_j} + y_j),\ x_i^{m-i},\ y_j^{n-j}.$$

The number of factors is the degree of their product. Hence the degree is equal to

$$\lambda_i + \lambda'_j + (m-i) + (n-j). \tag{2.3.2}$$

Consequently, the substitution $x_i = t$, $y_j = -t$ in N_λ yields a polynomial of degree equal to (2.3.2) in t. By assumption, $j > \lambda_i$ and $i > \lambda'_j$. Hence the degree (2.3.2) is less than or equal to $m+n-2$. Thus (1) has been proved.

The assertion (2) will be proved by induction. The assertion holds when $m = 0$ or $n = 0$. Indeed, if $n = 0$, then the condition on λ is that its length is less than or equal to m. Moreover, the polynomial $F_\lambda(X_m/Y_n)$ for $n = 0$ is the polynomial

$$F_\lambda(X_m) := \sum_w w\left[\frac{x_1^{m-1+\lambda_1}\cdots x_m^{\lambda_m}}{\Delta(X_m)}\right],$$

where the sum is over all permutations of X_m. By Lemma (1.5)(2), the polynomials $F_\lambda(X_m)$, where the length of λ is less then or equal to m, form a basis of the **Z**-module of symmetric polynomials in X_m. Similarly, if $n = 0$, then the condition on λ is that the conjugate partition λ' has length less than or equal to n, and the polynomial $F_\lambda(X_m/Y_n)$ for $n = 0$ is the polynomial $F_{\lambda'}(Y_n)$.

Assume next that $m > 0$ and $n > 0$. Denote by $\mathcal{H}_{m,n}$ the set of partitions contained in the (m,n)-hook. Divide the set $\mathcal{H}_{m,n}$ into the subset $\mathcal{H}_{m-1,n-1}$ consisting of partitions contained in the $(m-1,n-1)$-hook and its complement $\mathcal{H}^0$. Clearly, a partition λ in $\mathcal{H}_{m,n}$ belongs to $\mathcal{H}^0$ if and only if the part $D_\lambda^{m,n}$ of its diagram consists of all points in the rectangle $1 \le i \le m$, $1 \le j \le n$. It follows that partitions λ in $\mathcal{H}^0$ correspond bijectively to pairs (μ,ν), where μ and ν are partitions of lengths at most m and n respectively. Clearly, if $\lambda \in \mathcal{H}^0$ is the partition corresponding to (μ,ν), then

$$(2.3.3)\qquad F_\lambda(X_m/Y_n) = P(X_m,Y_n)F_\mu(X_m)F_\nu(Y_n),$$

where $P(X_m,Y_n)$ is the product considered in Definition (2.2). Moreover, by Lemma (1.5)(2), the products $F_\mu(X_m)F_\nu(Y_n)$, where μ and ν are partitions of lengths at most m and n respectively, form a basis of the **Z**-module of polynomials that are symmetric in X_m and in Y_n. Therefore, the polynomials $F_\lambda(X_m/Y_n)$, where $\lambda \in \mathcal{H}^0$, form a basis of the **Z**-module of polynomials that are symmetric in X_m and in Y_n and divisible by $P(X_m,Y_n)$. Clearly, the latter set of polynomials is equal to the set of supersymmetric polynomials for which the substitution $x_m = 0 = y_n$ yields the zero polynomial.

To finish the inductive proof of (2), let F be a given supersymmetric polynomial. Substitute $x_m = 0 = y_n$ in F. The resulting polynomial is supersymmetric in X_{m-1} and Y_{n-1}, and may therefore uniquely be written as a linear combination of the $F_\lambda(X_{m-1}, Y_{n-1})$, where λ belongs to $\mathcal{H}_{m-1,n-1}$. Subtract the same linear combination of the $F_\lambda(X_m/Y_n)$ from F. By construction, the substitution $x_m = 0 = y_n$ in the difference yields 0. Moreover, the difference is supersymmetric. Therefore, as proved above, the difference is uniquely a linear combination of the $F_\lambda(X_m/Y_n)$, where λ belongs to $\mathcal{H}^0$. Hence F is uniquely a linear combination of the $F_\lambda(X_m/Y_n)$, where λ belongs to $\mathcal{H}_{m,n}$. Thus (2) has been proved.

Thus the proposition is proved.

3. The Formula and its Consequences

THEOREM (3.1). *For every partition λ the following equation holds:*

$$S_\lambda(X_m/Y_n) = F_\lambda(X_m/Y_n).$$

PROOF: The two sides of the equation specialize when m and n varies. Indeed, the assertion for F_λ holds by Lemma (1.7)(2), and the assertion for S_λ follows immediately from the definition in (1.3). Therefore, it suffices to prove that the equation holds when m is large. Thus, without loss of generality, it will be assumed that the weight $|\lambda|$ of the partition λ is less than or equal to m.

It is clear from the definition that the polynomial $S_\lambda(X_m/Y_n)$ is supersymmetric. Therefore, by Proposition (2.3)(2), $S_\lambda(X_m/Y_n)$ is a finite linear combination

$$S_\lambda(X_m/Y_n) = \sum_{\nu \in \mathcal{H}_{m,n}} a_\nu F_\nu(X_m/Y_n), \tag{3.1.1}$$

where $a_\nu \in \mathbf{Z}$ and the sum is over all partitions ν contained in the (m,n)-hook. It has to be proven that $a_\lambda = 1$ and $a_\nu = 0$ for $\nu \neq \lambda$. The left hand side of (3.1.1) has degree equal to the weight $|\lambda|$. Therefore, for any term on the right hand side appearing with a non-zero coefficient a_ν the weight of ν is equal to the weight of λ. In particular, the length of a partition ν corresponding to a non-zero term in (3.1.1) is less than or equal to m. Consequently, it may be assumed that the sum in (3.1.1) is over partitions of length less than or equal to m.

Substitute $y_1 = \cdots = y_n = 0$ in Equation (3.1.1). It follows that

$$S_\lambda(X_m) = \sum_\nu a_\nu F_\nu(X_m), \tag{3.1.2}$$

where the sum is over partitions ν of length less than or equal to m. The polynomial on the left hand side of Equation (3.1.2) is the usual Schur polynomial. Therefore, by Lemma (1.5), (1) and (2), Equation (3.1.2) implies that $a_\lambda = 1$ and $a_\nu = 0$ for $\nu \neq \lambda$.

Thus the Theorem has been proved.

COROLLARY (3.2). *The polynomials $S_\lambda(X_m/Y_n)$, where λ ranges over all partitions contained in the (m,n)-hook, form a basis of the $\mathbf{Z}$-module of supersymmetric polynomials in X_m and Y_n.*

PROOF: The assertion holds, because the parallel assertion for the polynomials F_λ holds by Proposition (2.3)(2).

COROLLARY (3.3). *Let λ be a partition contained in the (m,n)-hook such that $\lambda_m \geq n$. Denote by μ and ν the partitions obtained from λ using the (m,n)-hook as in Definition (1.2). Then the following factorization formula holds:*

$$S_\lambda(X_m/Y_n) = S_\mu(X_m)\, S_\nu(Y_n) \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j).$$

In particular,

$$S_{(n,\dots,n)}(X_m/Y_n) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j),$$

where the partition $(n,\dots,n)$ has m non-zero parts.

PROOF: The first formula holds, because the parallel formula for the polynomials F_λ is immediate from their definition, cf. Equation (2.3.3). Clearly, the second equation is a special case.

COROLLARY (3.4). *For every partition λ the following equation holds:*

$$S_\lambda(X_m/Y_n) = S_{\lambda'}(Y_n/X_m).$$

PROOF: The equation holds, because the parallel equation for the polynomials F_λ is obvious from their definition.

NOTE (3.5). Corollary (3.2) is a refinement of both the main result of [S] and of Lemma 6.4 in [**B-R**]. The factorization formula was proven originally in [**B-R**, Theorem 6.20]. Recently, N. Bergeron and A. Garsia [**B-G**] and, independently, J. Van Der Jeugt and V. Fack [**J-F**] have used the formula in (3.1) to give a new derivation of the Littlewood-Richardson rule describing the coefficients in products of Schur polynomials.

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Note added in proof: Getting acquainted with the formula in (3.1) from a preliminary version of [P, Section 2], A. Lascoux gave still another proof of it based on the Schubert polynomial-technique.

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