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Impanga Seminar

"Deformations of rational curves on primitive symplectic varieties and applications"

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Plan:

I| Recall irreducible holomorphic symplectic (IHS) manifolds & their properties.

II| Why study "singular" IHS and rational curves on them.

III| Results (joint w/ Ch. Lehn and G. Mongardi)

IV| Sketch of some proofs.

Def.: Y is irreducible holomorphic symplectic (IHS) if Y is a compact, Kähler, simply connected manifold with $H^0(Y, \Omega^2_Y) = \mathbb{C}\cdot\sigma$, σ symplectic form.

E.g. A K3 surface S and all its punctual Hilbert schemes $S^{[n]}$ are IHS.

Main feature: $H^2(\mathrm{IHS}, \mathbb{Z})$ has a non-deg.^{te} integral quadratic form q, of sign. $(3, b_2(\mathrm{IHS})-3)$ called Beauville-Bogomolov-Fujiki (BBF) form

Fujiki: $\exists\, c_Y > 0 : \int_Y \alpha^{\dim Y} = c_Y\, q(\alpha)^{\frac{\dim Y}{2}}$, $\alpha \in H^2(Y, \mathbb{Z})$

NB: All 4 known deformation types of IHS manifolds arise from moduli spaces of sheaves on K3 or abelian surfaces.

Recall: If S K3 or abelian surface, then $\tilde{H}(S,\mathbb{Z}) = \mathbb{Z} \oplus H^2(S,\mathbb{Z}) \oplus \mathbb{Z}$ is endowed with a lattice structure $\rightsquigarrow \left(\tilde{H}(S,\mathbb{Z}), (\cdot,\cdot)\right)$ $\boxed{\text{Mukai lattice}}$

$\forall \mathcal{F}$ coherent sheaf on S $\rightsquigarrow$ a Mukai vector:

$$v(\mathcal{F}) := \left(rk(\mathcal{F}), c_1(\mathcal{F}), ch_2(\mathcal{F}) + \varepsilon(S)\, rk(\mathcal{F})\right)$$

$$\varepsilon(S) \begin{cases} 0 & \text{ab. case} \\ 1 & \text{K3 case} \end{cases}$$

Moduli of sheaves: For $v \in \tilde{H}(S, \mathbb{Z})$ consider

$$M_v := M_v(S,H) := \boxed{\text{Moduli space of sheaves on } S \text{ of Mukai vector } v \text{ which are Gieseker } H\text{-semistable wrt a polarization } H \text{ (which is } v\text{-generic)}}$$

(resp.^{ly} K_v = fiber of the Albanese map of M_v in the abelian case)

Write: $v = m w$, w primitive, $m \geq 1$.

II) Why "singular" sympl.?

$n=1$: (Huybrechts, Mukai, O'Grady, Yoshioka ...): M_v & K_v are IHS manifolds $\sim_{\text{def}}$ $K3^{[n]}$, resp. Kum_n.

$n=2$, $w^2=2$: (O'Grady): M_v / K_v are singular but possess an IHS res.^{n} of sing.^{s} $\leadsto$ OG10 / OG6 defo. types

All other cases: (Kaledin-Lehn-Sorger): M_v & K_v are singular and have no IHS resolution.

NB: In all these cases (Mukai) the smooth locus of M_v & K_v carries a symplectic form.

$\Rightarrow$ We enter the world of singular symplectic varieties!

Def : (Beauville): X normal variety

(i) A <u>symplectic form</u> on X is a closed, reflexive 2-form $\sigma \in H^0(X, \Omega_X^{[2]} := i_* \Omega^2_{X_{reg}})$ which is <u>symplectic</u> on $i: X_{reg} \hookrightarrow X$.

(ii) If σ is a symplectic form on X, we say that X has <u>symplectic singularities</u> if $\exists\, Y \longrightarrow X$ resolution of singularities such that $\sigma|_{X_{reg}}$ extends to a holo. 2-form on Y.

Let (X,σ) be normal cpct Kähler w/ symplectic sing.s

Def.: (i) X is a <u>primitive symplectic variety</u> (PSV)

if $h^1(X,\mathcal{O}_X)=0$ & $H^0(X,\Omega_X^{[2]})=\mathbb{C}\cdot\sigma$

(ii) X is an <u>irreducible symplectic variety</u> (ISV)

if X has canonical singularities and for any

$\gamma: X' \longrightarrow X$ finite cover, étale in codim. 1,

the algebra $H^0(X',\Omega_{X'}^{[*]})$ of reflexive forms on X'

is generated by $\gamma^{[*]}\sigma$.

Rmks: 1) In the smooth case the 2 notions coincide (Schwald '20).

2) Easy to see that X ISV $\Rightarrow$ X PSV, but the converse does not hold

e.g. $X = A/_{\pm}$ Kummer surface is PSV, but it has a finite cover, étale in codim 1, given by

$$A \longrightarrow A/_{\pm} \Rightarrow \text{not ISV}.$$

Why PSV's ? PSV's behave very well with respect to deformations and period mappings, namely Bakker-Lehn proved that $\mathrm{Def}^{lt}(\mathrm{PSV})$ is smooth of $\dim = h^{1,1}$ and local & global Torelli hold.

Why ISV's ? Because of the Singular BB decomposition

THM (GKP-D-GGK-HP-C-BGL): X compact Kähler with klt singularities & $K_X \equiv 0$. Then $\exists$ $\tilde{X} \longrightarrow X$ finite étale in codim 1 such that:

$$\tilde{X} = \begin{pmatrix} \text{Abelian} \\ \text{Variety} \end{pmatrix} \times \prod \begin{pmatrix} \text{(Singular)} \\ \text{Calabi-Yau} \end{pmatrix} \times \prod \big(\text{ISV's} \big)$$

Also: **THM** (Perego-Rapagnetta): Moduli spaces M_v & K_v are ISV.
(if $v = (0, mH, 0) \Rightarrow$ ask $\rho(S) = 1$)

Why study rat'l curves on PSV's?

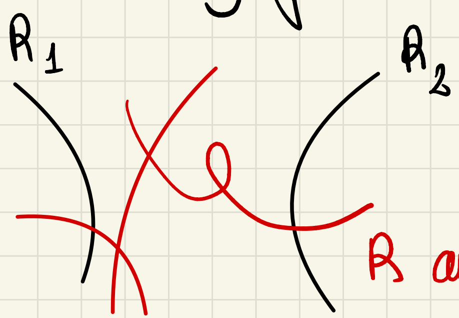
On K3 surfaces rat'l curves play a prominent rôle and arise essentially in 3 main contexts:

- degenerate fibres of elliptic fibrations (associated to $L^2=0$, L nef)
- Study of the Kähler cone

  $\Leftrightarrow R^2=-2$

  $$\mathrm{K\ddot{a}h}(S) = \{\alpha : \alpha^2>0,\ \alpha\cdot R>0,\ R \text{ smooth rat'l curve}\}$$

- Study of the CH_0 (Beauville-Voisin): despite being huge



R ample rat'l curve

$$Pic(S)\times Pic(S) \longrightarrow \mathbb{Z}\cdot p \subset CH_0(S)$$
$$(L_1,L_2) \longmapsto L_1\cdot L_2$$

p any pt on any rat'l curve

($\exists$ by Bogomolov-Mumford)

With Ch. Lehn and G. Mongardi we studied uniruled divisors on PSV's with non-zero square wrt the BBF quadratic form.

A crucial point (already for K3's) is to control their deformation theory, whence the following:

THM 1 (LMP) Let X be a projective PSV and $f: C \to X$ a genus 0 stable map. Suppose the deformations of f cover a divisor.

(1) f deforms along the Hodge locus in $\mathrm{Def}^{lt}(X)$ where $f_*[C]$ remains algebraic.

(2) For any $t \in \mathrm{Hdg}_{f_*[C]} \subset \mathrm{Def}^{lt}(X)$ the corresponding variety X_t contains a uniruled divisor (covered by the deformations of f in X_t).

We present two applications:

THM 2 (LMP): X proj. $\mathbb{Q}$-factorial PSV and $E \subset X$ a prime exceptional divisor (i.e. $q(E) < 0$). Then

(1) E is contractible on a birational $\mathbb{Q}$-factorial PSV X' locally trivial def^{n} of X. In particular E is uniruled.
Moreover the general curve in the ruling is either a smooth rat'l curve or a union of 2 smooth rat'l curves meeting transversally in a single point.

(2) $\exists$ a flat family of divisors over $\mathrm{Hdg}_{[E]}(X) \subset \mathrm{Def}^{lt}(X)$ specializing to (a multiple of) E.

THM 3 (LMP): Let $\mathcal{M} = \coprod_{d \in \mathbb{N}} \mathcal{M}_d$ the (coarse) moduli space of polarized PSV's endowed with an ample line bundle of BBF square d, and locally trivially deformation equiv.^{t} to $M_v(S,H)$ (or $K_v(S,H)$). Then $\mathcal{M}$ possesses ∞^{ly} many connected components whose pt^s correspond to polarized PSV's all containing an ample uniruled divisor proportional to the polarization.

Technical Rmk: If $v = (0, m c_1(L), 0)$, we require $\rho(S) = 1$.

IV.11: Sketch of proof of THM 1 : We know that the THM holds when X is smooth (by Charles-Magardi-P)

Claim: If X is terminal, then the gen'l $f: C \to D \subset X$ curve in the ruling avoids X_{sing}.

Assuming the Claim, if X terminal, the defo-theory of f is essentially as in the smooth case.

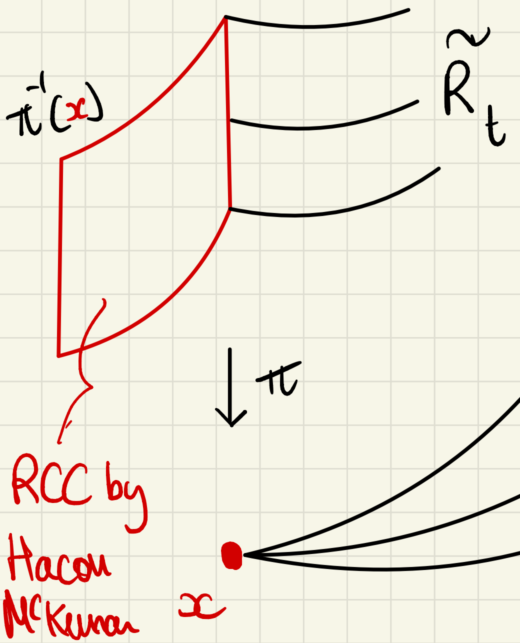
If X not terminal, by [BCHM] $\exists\, Y \xrightarrow{\pi} X$ $\mathbb{Q}$-factorial Terminalization

Y is still PSV, hence the result holds on Y and we can "descend" it to X because $\mathrm{Def}(Y, \mathrm{Exc}(\pi)) \simeq \mathrm{Def}^{lt}(X)$ (by Bakker-Lehn)

Pf of Claim:

X terminal $\overset{\text{Namikawa}}{\Longrightarrow}$ $\operatorname{codim}_X (X_{sing}) \geq 4$

$\hookrightarrow$ $\operatorname{codim}_D (D \cap X_{sing}) \geq 3$. Hence if all curves in the ruling meet X_{sing} must have a 1-dimensional family of rat'l curves $\{R_t\}$ thru the general pt $x \in D \cap X_{sing}$



R_t

$$\begin{array}{c} \tilde{D} \subset Y \text{ desing}^{n} \\ \pi \downarrow \\ D \subset X \end{array}$$

the divisor $\tilde{D} \subset Y$ is then ruled by RCC surfaces.

Consider the MRC quotient

$$\begin{array}{c} \tilde{D}^{2n-1} \\ \downarrow p \\ Q(\tilde{D})^{2n-3} \end{array}$$

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Consider $\tilde{D}^{2n-1}$ $\downarrow \rho$ the MRC quotient $Q(\tilde{D})^{2n-3}$. It's easy to see that any form on $\tilde{D}$ comes from $Q(\tilde{D})$

In particular true for $(\tilde{\sigma})^{\wedge(n-1)}_{|\tilde{D}}$, where $\tilde{\sigma}$ symplectic form

But $(2n-2)$-form on a variety of dim $2n-3$

$\hookrightarrow$ $(\tilde{\sigma})^{\wedge(n-1)}$ must vanish identically along $\tilde{D}$

and this is impossible, by the symplecticity of $\tilde{\sigma}$ (at smooth pt') and linear algebra $\blacksquare$

Sketch of proof of THM 3 :

- Perego - Rapagnetta : $\exists$ $\begin{matrix} K_u & \dashrightarrow & K_v \\ M_u & \dashrightarrow & M_v \end{matrix}$ dominant map

  from a smooth moduli space (of $K3^{[n]}$-type $\atop Kum_n$-type) onto M_v / K_v

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  $\hookrightarrow$ M_v / K_v is loc. trivial defo. equivalent to $M_{v'} / K_{v'}$
- **THM 1** : We may deform the rat'l curves on M_v / K_v to the $M_{v'} / K_{v'}$ where they stay algebraic $\blacksquare$