Pushing forward Hall-Littlewood polynomials

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SEMINAR IMPANGA, 21 May 2021

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We shall use them simultaneously by means of Hall-Littlewood classes associated with a vector bundle $E \to X$ of rank n with Chern roots $x_1, \ldots x_n$.

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For $q \leq n$, let $\pi : G^q(E) \to X$ be the Grassmann bundle parametrizing rank q quotients of E. It is endowed with the universal exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0 \, ,$$

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Then for any partitions $\lambda = (\lambda_1, \ldots, \lambda_q)$, $\mu = (\mu_1, \ldots, \mu_r)$,

$$\pi_* \big(s_\lambda(Q) \cdot s_\mu(S) \big) = s_{\lambda_1 - r, \ldots, \lambda_q - r, \mu_1, \ldots, \mu_r}(E) .$$

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For a strict partition $\lambda = (\lambda_1 > \ldots > \lambda_k > 0)$ with odd k,

$$P_\lambda = P_{\lambda_1} P_{\lambda_2,\ldots,\lambda_k} - P_{\lambda_2} P_{\lambda_1,\lambda_3,\ldots,\lambda_k} + \cdots + P_{\lambda_k} P_{\lambda_1,\ldots,\lambda_{k-1}} \,,$$

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and for positive $i > j$ we set

$$P_{i,j} = P_i P_j + 2 \sum_{d=1}^{j-1} (-1)^d P_{i+d} P_{j-d} + (-1)^j P_{i+j} .$$

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Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ be sequence of nonnegative integers. Define

$$R_\lambda(E; t) = (\tau_E)_* \Big(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} (x_i - t x_j) \Big) ,$$

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This polynomial will give rise to a Hall-Littlewood polynomial.

Proposition

If $\lambda \in \mathbb{Z}_{\geq 0}^{q}$ *and* $\mu \in \mathbb{Z}_{\geq 0}^{n-q}$ *then*

$$\pi_*\Big(R_\lambda(Q;t)R_\mu(S;t) \prod_{i \leq q < j} (x_i - tx_j)\Big) = R_{\lambda\mu}(E;t),$$

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This is seen from a commutative diagram

$$\begin{array}{ccc} Fl(Q) \times_{G^q(E)} Fl(S) & \xrightarrow{\cong} & Fl(E) \\ \downarrow{\scriptstyle \tau_Q \times \tau_S} & & \downarrow{\scriptstyle \tau_E} \\ G^q(E) & \xrightarrow[\pi]{} & X \end{array}$$

which gives

$$\pi_*(\tau_Q \times \tau_S)_* = (\tau_E)_* \,.$$

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Let S_n be the symmetric group of permutations of $\{1,\ldots,n\}$. We define the stabilizer of λ:

$$S_n^\lambda = \{w \in S_n : \lambda_{w(i)} = \lambda_i, 1 \leq i \leq n\}.$$

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Then $d = k+1$, $(m_1, \ldots, m_d) = (1^k, n-k)$, $v_\lambda(t) = v_{n-k}(t)$, $S_n^\lambda = (S_1)^k \times S_{n-k}$.

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(thinking about y_i as the Chern roots of E, we get $R_\lambda(E; t)$).

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$$R_\lambda(y_1, \ldots, y_n; t) = v_\lambda(t) \sum_{w \in S_n/S_n^\lambda} w\Big(y^\lambda \prod_{i<j, \lambda_i \neq \lambda_j} \frac{y_i - ty_j}{y_i - y_j}\Big).$$

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$$R_\lambda(y_1, \ldots, y_n; t) = v_\lambda(t) \sum_{w \in S_n/S_n^\lambda} w\Big(y^\lambda \prod_{i<j, \lambda_i \neq \lambda_j} \frac{y_i - ty_j}{y_i - y_j}\Big).$$

Proof. Any $w \in S_n$ which permutes only the digits from I_1 will fix the monomial y^λ, and by Lemma used for S_{m_1}, we can extract a factor $v_{m_1}(t)$ from R_λ.

Proposition

The polynomial $v_\lambda(t)$ divides $R_\lambda(y_1,\ldots,y_n;t)$ (i.e. all the coefficients of R_λ are divisible by $v_\lambda(t)$ in $\mathbb{Z}[t]$) and we have

$$R_\lambda(y_1,\ldots,y_n;t) = v_\lambda(t)\sum_{w\in S_n/S_n^\lambda} w\Big(y^\lambda \prod_{i<j,\lambda_i\neq\lambda_j}\frac{y_i-ty_j}{y_i-y_j}\Big).$$

Proof. Any $w\in S_n$ which permutes only the digits from I_1 will fix the monomial y^λ, and by Lemma used for S_{m_1}, we can extract a factor $v_{m_1}(t)$ from R_λ.

Repeating this procedure for I_2,...,I_d and S_{m_2}, ..., S_{m_d}, we extract succesively factors $v_{m_2}(t)$, ..., $v_{m_d}(t)$ from R_λ, i.e. a factor $v_\lambda(t)$, and get the assertion. QED

Let $\lambda \in \mathbb{Z}^n_{\geq 0}$. Extending Mcd, we set

$$P_\lambda(E;t) := \frac{1}{v_\lambda(t)} R_\lambda(E;t)$$

and call it *Hall-Littlewood polynomial*.

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As a consequence of the two Propositions and the definition of $P_\lambda(E;t)$, we get

Theorem

Let $\lambda \in \mathbb{Z}^q_{\geq 0}$ and $\mu \in \mathbb{Z}^{n-q}_{\geq 0}$. We then have

$$\pi_*\Big(\prod_{i\leq q<j}(x_i - tx_j)P_\lambda(Q;t)P_\mu(S;t)\Big) = \frac{v_{\lambda\mu}(t)}{v_\lambda(t)v_\mu(t)} P_{\lambda\mu}(E;t)\,.$$

We look at the specialization $t = 0$.

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We invoke the Jacobi-Trudi formula for $s_\lambda(E)$ with the help of the Gysin map associated to $\tau_E : Fl(E) \to X$:

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We see that $P_\lambda(E;t) = s_\lambda(E)$ for $t = 0$. Under this specialization, Theorem becomes

$$\begin{aligned}\pi_*\big((x_1 \cdots x_q)^r s_\lambda(Q) s_\mu(S)\big) &= \pi_*\big(s_{\lambda_1+r,\dots,\lambda_q+r}(Q) s_\mu(S)\big) \\ &= s_{\lambda\mu}(E)\,.\end{aligned}$$

(Józefiak-Lascoux-P)

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Either one arrives at a sequence of the form $(\ldots, i, i+1, \ldots)$, in which case $s_\lambda(E) = 0$, or one arrives in d steps at a partition μ, and then $s_\lambda(E) = (-1)^d s_\mu(E)$.

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Here e is the number of common parts of ν and σ.

We have

$$\frac{v_{\lambda\mu}}{v_\lambda v_\mu} = \frac{(1-t)\cdots(1-t^{n-k-h})}{(1-t)\cdots(1-t^{q-k})(1-t)\cdots(1-t^{n-q-h})}(1+t)^e$$

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So by Theorem we have

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- some zeros at the end of λ possible

We look at the specialization $t=-1$. Most interesting is the specialization of Gaussian polynomials.

Lemma

At $t = -1$, the Gaussian polynomial

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t)$$

specializes to zero if ab is odd and to the binomial coefficient

$$\binom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor}$$

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(with Witold Kraśkiewicz)

Indeed, we have

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t) = \frac{(1-t)(1-t^2)\cdots(1-t^{a+b})}{(1-t)\cdots(1-t^a)(1-t)\cdots(1-t^b)}\,.$$

Indeed, we have

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Since $t=-1$ is a zero with multiplicity 1 of the factor $(1-t^d)$ for even d, and a zero with multiplicity 0 for odd d,

the order of the rational function $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right](t)$ at $t=-1$ is equal to

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In the former case, we get the claimed vanishing, and

in the latter one, the product of the factors with even exponents is equal to

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The value of this function at $t=-1$ is equal to $\left[\begin{smallmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{smallmatrix}\right](1)$

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This is the requested value since the remaining factors with odd exponents give 2 in the numerator and the same number in the denominator. QED

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Note that for $i + j > 0$, we have $\mathcal{P}_{\dots,i,j,\dots} = -\mathcal{P}_{\dots,j,i,\dots}$ (follows from Mcd pp. 213-214).

Thus $\mathcal{P}_{\nu 0^{q-k}\sigma 0^{n-q-h}} = (-1)^{(q-k)h}\mathcal{P}_{\nu\sigma 0^{n-k-h}} = (-1)^{(q-k)h}\mathcal{P}_{\nu\sigma}$.

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For strict partitions ν, σ with $l(\nu) = k \leq q$ and $l(\sigma) = h \leq n - q$,

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If not, then $P_{\nu\sigma}(E) = (-I)^l P_\kappa(E)$, where l is the lenghth of the permutation which rearranges $\nu\sigma$ into the corresponding strict partition κ.

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Hall algebra : λ, μ, ν three partitions. Let M be of type λ.

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H is a commutative ring, and is generated as a $\mathbb{Z}$-algebra by $\{u_{(1^r)}\}$ (algebraically independent).

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The $\mathbb{Q}$-linear map $\psi : H \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q}$ (symmetric functions) such that

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J.A. Green, D.E. Littlewood: Representation theory of GL_n over finite fields.

A flag bundle associated to λ

Let $\lambda \in \mathbb{Z}^n_{\geq 0}$. We associate to λ a $(d-1)$-step flag bundle (with steps of lengths m_i)

$$\eta_\lambda : Fl_\lambda(E) \to X \,,$$

parametrizing flags of quotients of E of ranks

$$n - m_d, n - m_d - m_{d-1}, \ldots, n - m_d - m_{d-1} - \cdots - m_2 \,.$$

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Recall that $A(Fl_\lambda(E))$ as an $A(X)$-module is generated by S_n^λ-invariant polynomials f in the Chern roots of E.

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$$\partial_\lambda(f) = \sum_{w \in S_n/S_n^\lambda} w\left(\frac{f(x_1,\ldots,x_n)}{\prod_{i<j,\lambda_i\neq\lambda_j}(x_i - x_j)}\right).$$

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Proposition

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It follows from the two Propositions, that

$$R_\lambda(E;t) = v_\lambda(t)(\eta_\lambda)_*\Big(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\prod_{i<j,\lambda_i\neq\lambda_j}(x_i-tx_j)\Big).$$

End