Order of tangency between manifolds

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Introduction

Two plane curves, both nonsingular at a point x^0, are said to have the order of tangency (or contact) at least k at x^0 if, in properly chosen regular parametrisations, those two curves have identical Taylor polynomials of degree k about x^0.

Contact problems have been of both classical and modern interest, particularly in light of Hilbert's 15th problem to make rigorous the classical calculations of enumerative geometry, especially those undertaken by Schubert.

Why it is important to study the **"order of tangency"**? Let us discuss this notion for **Thom polynomials of singularities** (real or complex). Thom polynomials measure complexity of singularities and were studied by René Thom and many others.

An important property of Thom polynomials is their **positivity** closely related to Schubert calculus. Namely, the order of tangency allows one to define for example the **jets** of Lagrangian submanifolds. The space of these jets is a fibration over the Lagrangian Grassmannian and leads to a positive decomposition of a Lagrangian Thom polynomial in the basis of Lagrangian Schubert cycles.

Two manifolds M and $\widetilde{M}$ in $\mathbb{R}^m$, both of class C^r, $r \geq 1$, and the same dimension p, intersecting at $x^0 \in M \cap \widetilde{M}$, *have at* x^0 *the order of tangency at least* k $(1 \leq k \leq r)$, when there exist neighbourhoods $U \ni u^0$ and $\widetilde{U} \ni \tilde{u}^0$ in $\mathbb{R}^p$, parametrisations

$$q\colon (U, u^0) \to (M, x^0), \qquad \tilde{q}\colon (\widetilde{U}, \tilde{u}^0) \to (\widetilde{M}, x^0)$$

and a diffeomorphism $\Phi\colon U \to \widetilde{U}$ (all, naturally, of class C^r) such that

$$\Big(\tilde{q} \circ \Phi - q\Big)(u) \;=\; o\Big(\left|u - u^0\right|^k\Big) \tag{1}$$

when $U \ni u \to u^0$. This definition does not depend on the choice of local parametrisations q and $\tilde{q}$.

$$f(x) = o(h(x)) \quad \text{when } x \to x_0$$

means

$\lim_{x \to x_0} \frac{f(x)}{h(x)} = 0.$

"$f(x)$ is much smaller than $h(x)$ for x near x_0."

Proposition

The condition (1) is equivalent to

$$T^k_{u^0} q \;=\; T^k_{u^0}\big(\tilde{q}\circ\Phi\big)$$

where $T^k_{u^0}(\cdot)$ *means the Taylor polynomial about* u^0 *of order* k*.*

We shall refer to this fact "Taylor Prop.".

Indeed, we have

$$\tilde{q}\circ\Phi(u) - q(u) \;=\; \big(\tilde{q}\circ\Phi(u) - T^k_{u^0}\big(\tilde{q}\circ\Phi\big)(u-u^0)\big)$$
$$+\,\big(T^k_{u^0}\big(\tilde{q}\circ\Phi\big)(u-u^0) - T^k_{u^0}\,q(u-u^0)\big) + \big(T^k_{u^0}\,q(u-u^0) - q(u)\big)\,,$$

where the first and last summands are $o\Big(\big|u-u^0\big|^k\Big)$ by Taylor.

Under (1), so is the middle summand

$$T^k_{u^0}\big(\tilde{q}\circ\Phi\big)(u-u^0) - T^k_{u^0}\,q(u-u^0) \;=\; o\Big(\big|u-u^0\big|^k\Big)$$

and our assertion follows from the following general result.

Lemma 1

Let $w\in\mathbb{R}[u_1,\,u_2,\ldots,\,u_p]$, $\deg w\le k$, $w(\mathbf{u})=o\big(|\mathbf{u}|^k\big)$ *when* $\mathbf{u}\to 0$ *in* $\mathbb{R}^p$. *Then* w *is identically zero.*

The implication: Taylor Prop. $\Rightarrow$ (1) is easy.

Assume that

$$s: \ = \max\{k\colon \text{the order of tangency } \geq k\} < r\,. \qquad (2)$$

Here r is the assumed class of smoothness of manifolds, finite or infinite when a category is real. (When $r = \infty$, the condition (2) simply says that s is finite.)

Our second approach uses *pairs of curves* lying, respectively, in M and $\widetilde{M}$. We naturally assume that $T_{x^0}M = T_{x^0}\widetilde{M}$.

We now present the following **mini-max procedure**.

Theorem 1

Under (2),

$$s = \min_{v}\Big(\max_{\gamma,\tilde{\gamma}}\big(\max\big\{l\colon\ |\gamma(t)-\tilde{\gamma}(t)| = o\big(|t|^{l}\big)\ \text{when}\ t\to 0\,\big\}\big)\Big). \tag{3}$$

The **minimum** *is taken over all $0\neq v\in T_{x^0}M = T_{x^0}\widetilde{M}$. The* **outer maximum** *is taken over all pairs of curves $\gamma\subset M$, $\tilde{\gamma}\subset\widetilde{M}$ such that $\gamma(0) = x^0 = \tilde{\gamma}(0)$, and – both non-zero! – velocities $\dot{\gamma}(0)$, $\dot{\tilde{\gamma}}(0)$ are both parallel to v. The* **inner maximum** *is taken over admissible positive integers only.*

To begin the proof, take the integer s defined in (2). Then it is quick to show that the integer on the left hand side of equality (3) is at least s.

Indeed, for every fixed vector v as above, $v = dq(u^0)\mathbf{u}$ (without loss of generality, $\mathbf{u} \in \mathbb{R}^p$, $|\mathbf{u}| = 1$). We now take $\delta(t) = q(u^0 + t\mathbf{u})$ and $\tilde{\delta}(t) = \tilde{q}\big(\Phi(u^0 + t\mathbf{u})\big)$. Then

$$|\delta(t) - \tilde{\delta}(t)| \;=\; o\big(|t\mathbf{u}|^s\big) \;=\; o\big(|t|^s\big)$$

and so, in that equality,

$$\max_{\gamma,\tilde{\gamma}}\big(\max\big\{\, l\colon |\gamma(t) - \tilde{\gamma}(t)| = o\big(|t|^l\big) \text{ when } t \to 0\big\}\big) \;\geq\; s\,.$$

In view of the arbitrariness in our choice of v, the same remains true after taking the minimum over all admissible v's on equality's LHS.

The opposite inequality is more involved. It is here where a delicate assumption $s \leq r - 1$ is needed. We skip the details.

Our third approach is based on a **tower of consecutive Grassmannians** attached to a local C^r parametrisation q.

To every C^1 *immersion* $H : N \to N'$, N – an n-dimensional manifold, N' – an n'-dimensional manifold, we attach the so-called image map $\mathcal{G}H \colon N \to G_n(N')$ of the tangent map $d\,H$: for $s \in N$,

$$\mathcal{G}H(s) \;=\; dH(s)(T_sN)\,,$$

where $G_n(N')$ is the Grassmann *bundle*, with base N', of all n-planes tangent to N'.

Recall that $M, \widetilde{M} \subset \mathbb{R}^m$.

We use as previously the pair of parametrisations q and $\tilde{q}$, but now dispense with a local diffeomorphism Φ. So we are now given the mappings

$$\mathcal{G}\, q :\; U \longrightarrow G_p\big(\mathbb{R}^m\big)\,, \qquad \mathcal{G}\, \tilde{q} :\; U \longrightarrow G_p\big(\mathbb{R}^m\big)\,.$$

Upon putting $M^{(0)} = \mathbb{R}^m$, $\mathcal{G}^{(1)} = \mathcal{G}$, we get two sequences of recursively defined mappings. Namely, for $l \geq 1$,

$$\mathcal{G}^{(l)} q :\; U \longrightarrow G_p\big(M^{(l-1)}\big)\,, \qquad \mathcal{G}^{(l+1)} q \;=\; \mathcal{G}\Big(\mathcal{G}^{(l)} q\Big)$$

and

$$\mathcal{G}^{(l)} \tilde{q} :\; U \longrightarrow G_p\big(M^{(l-1)}\big)\,, \qquad \mathcal{G}^{(l+1)} \tilde{q} \;=\; \mathcal{G}\Big(\mathcal{G}^{(l)} \tilde{q}\Big),$$

where, naturally, $M^{(l)} = G_p\big(M^{(l-1)}\big)$.

Theorem 2

C^r manifolds M and $\widetilde{M}$ have at x^0 the order of tangency at least k ($1 \leq k \leq r$) iff for every parametrisations q and $\tilde{q}$ of the vicinities of x^0 in, respectively, M and $\widetilde{M}$, there holds

$$\mathcal{G}^{(k)} q\left(u^0\right) = \mathcal{G}^{(k)} \tilde{q}\left(u^0\right).$$

(We shall call this theorem: "Grassmannian Thm".)

Let now H be the graph of a C^1 mapping $h\colon \mathbb{R}^p \supset U \to \mathbb{R}^t$. That is, for $u \in U$, $H(u) = \big(u,\, h(u)\big) \in \mathbb{R}^{p+t} = \mathbb{R}^p \times \mathbb{R}^t$. Then $\mathcal{G}H(u)$ equals

$$\Big(u,\, h(u)\,;\; d\big(u,\, h(u)\big)(u)\Big) = \Big(u,\, h(u);\; \mathrm{span}\{\partial_j + h_j(u)\}\Big)$$

where $j = 1, \ldots, p$ and the symbol h_j means the partial derivative of a vector mapping h with respect to the indeterminate u_j. Moreover, $\partial_j + h_j(u)$ denotes the partial derivative of the vector mapping

$$(\iota,\, h)\colon\, U \to \mathbb{R}^p\big(u_1, \ldots, u_p\big) \times \mathbb{R}^t$$

with respect to u_j. ($\iota\colon U \hookrightarrow \mathbb{R}^p$ is the inclusion.)

But the above expression for $\mathcal{G}H(u)$ is not handy. Yet there are charts in each Grassmannian.

The chart in a typical fibre G_p over a point in the base $\mathbb{R}^{p+t}$, good for $\mathcal{G}H(u)$ consists of all the entries in the bottommost rows (indexed by numbers $p+1$, $p+2$, ..., $p+t$) in the $(p+t) \times p$ matrices

$$\left[\begin{array}{c|c|c|c} v_1 & v_2 & \ldots & v_p \end{array} \right]$$

with non-zero upper $p \times p$ minor, after multiplying the matrix on the right by the inverse of that upper $p \times p$ submatrix

That is to say, taking as the local coordinates all the entries in the rows $p+1, \ldots, p+t$ of the following matrix

$$\begin{bmatrix} v_1^1 & v_2^1 & \cdots & v_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^p & v_2^p & \cdots & v_p^p \\ \multicolumn{4}{|c|}{\begin{matrix} \mathbf{v_j^i}, & i = p+1, \dots, p+t \\ & j = 1, \dots, p \end{matrix}} \end{bmatrix} \begin{bmatrix} v_1^1 & v_2^1 & \cdots & v_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^p & v_2^p & \cdots & v_p^p \end{bmatrix}^{-1} .$$

We get a handy expression ("Jac. Form."):

$$\mathcal{G}H(u) = \Big(u,\ h(u)\,;\ \frac{\partial h}{\partial u}(u)\Big),$$

where under the symbol $\frac{\partial h}{\partial u}(u)$ are understood all the entries of this *Jacobian*.

We come back to Grass. Thm. We assume without loss of generality that both M and $\widetilde{M}$ are, in the vicinities of x^0, just graphs of C^r mappings, and the parametrisations q and $\tilde{q}$ are the graphs of those mappings. That is,

$q(u) = (u, f(u)),\ f\colon\ U \to \mathbb{R}^{m-p}\big(y_{p+1}, \ldots,\ y_m\big)$ and similarly
$\tilde{q}(u) = (u, \tilde{f}(u)),\ \tilde{f}\colon\ U \to \mathbb{R}^{m-p}\big(y_{p+1}, \ldots,\ y_m\big).$

We shall show that Taylor Prop. implies Grass. Thm.

Lemma 2

For $1 \leq l \leq k$ there exists such a local chart on the Grassmannian $G_p\big(M^{(l-1)}\big)$ in which the mapping $\mathcal{G}^{(l)}q$ evaluated at u has the form

$$\left(u,\ f(u);\ \binom{l}{1} \times f_{[1]}(u),\ \binom{l}{2} \times f_{[2]}(u),\ \ldots,\ \binom{l}{l} \times f_{[l]}(u)\right),$$

where $f_{[\nu]}(u)$ is the aggregate of all the partials of the ν-th order at u, of all the components of f, which are in the number $p^{\nu}(m-p)$.

"Chart Lemma" (we ignore the symmetricity of partials).

Proof. $l = 1$. We aready know that $\mathcal{G}^{(1)}q$ is

$$\left(u,\ f(u);\ f_{[1]}(u)\right) = \left(u,\ f(u);\ \binom{l}{1} \times f_{[1]}(u)\right).$$

Sketch of $l \Rightarrow l+1$, $l < k$. We work with $\mathcal{G}^{(l+1)}q = \mathcal{G}\left(\mathcal{G}^{(l)}q\right)$.

We use the induction assumption and use Jac. Form. We put together the groups of **same** partials.

Using $\binom{l}{\nu-1} + \binom{l}{\nu} = \binom{l+1}{\nu}$, we get that $\mathcal{G}^{(l+1)}q$ equals

$$\left(u,\ f(u), \binom{l+1}{1} \times f_{[1]}(u), \binom{l+1}{2} \times f_{[2]}(u), \ldots, \binom{l+1}{l+1} \times f_{[l+1]}(u)\right)$$

Chart Lemma is now proved by induction.

We now take $l = k$ in Chart Lemma and get, for arbitrary $u \in U$, two similar expressions for $\mathcal{G}^{(k)}q\,(u)$ and $\mathcal{G}^{(k)}\tilde{q}\,(u)$.

Suppose that Taylor Prop. – without Φ – holds for $u = u^0$. As a consequence, Grass. Thm now follows.

Conversely, assuming Grass. Thm, we get that the partial derivatives of q and $\tilde{q}$ at u^0 are mutually equal. This gives the Taylor Prop.

A natural question arises: What about branches of algebraic sets which often happen to be tangent one to another with various degrees of closeness?

Let M be a finite-dimensional real analytic manifold, d be a distance function on M induced by a Riemannian metric on M, and let $X, Y \subset M$ be closed subanalytic sets. The following important fact says that X and Y are regularly separated at any x_0:

Theorem (Łojasiewicz) For any $x_0 \in X \cap Y$ there exist $\nu > 0$ and $C > 0$ such for some neighbourhood $\Omega \subset M$ of x_0

$$d(x, X) + d(x, Y) \geq C d(x, X \cap Y)^{\nu} ,$$

where $x \in \Omega$.

The exponent ν is called a *regular separation exponent* of X and Y at x_0. The infimum of such exponents is called the *Łojasiewicz exponent* and denoted $\mathcal{L}_{x_0}(X, Y)$.

Example Let $C = \{(x,y)\colon (y-x^2)^2 = x^5\}$,

The two branches of C issuing from the point $(0,0)$,

$C_- = \{y = x^2 - x^{5/2}\,,\ x \geq 0\}$ and $C_+ = \{y = x^2 + x^{5/2}\,,\ x \geq 0\}$,

could be naturally extended to one-dimensional manifolds D_- and D_+, both of class C^2 – the graphs of functions

$$y_-(x) = x^2 - |x|^{5/2} \quad \text{and} \quad y_+(x) = x^2 + |x|^{5/2}\,,$$

respectively. The Taylor polynomials of degree 2 about $x = 0$ of y_- and y_+ coincide. Hence by Taylor Prop. D_- and D_+ have at $(0,0)$ the order of tangency 2.

This example suggests that, in the real algebraic geometry category, it would be suitable to use non-integer measures of closeness. For instance, for the above sets $y_-(x)$ and $y_+(x)$, we may take

$$\sup\{\alpha > 0 : \ y_+(x) - y_-(x) = o(|x|^\alpha) \text{ when } x \to 0\}\,.$$

This generalised order of tangency would be $5/2$ in the above example. This is the minimal regular separation exponent of the semialgebraic sets C_- and C_+. That quantity is also the Łojasiewicz exponent $\mathcal{L}_{(0,0)}\big(C_-,\ C_+\big)$.

This example generalises, for $(y - x^N)^2 = x^{2N+1}$, to a pair of C^N manifolds having the order of tangency N and the minimal separation exponent $\nu = N + \frac{1}{2}$.

Example Here we shall see that the order and exponent can be both integer and different. Consider two curves N and Z in $\mathbb{R}^2(x, y)$ intersecting at $(0, 0)$:

$$N = \{y = 0\} \quad \text{and} \quad Z = \{y^d + yx^{d-1} + x^s = 0\},$$

where $1 < d < s$, and assume that d is odd. What is their minimal regular separation exponent at $(0, 0)$? We want to present Z as the graph of some function $y(x)$.

Lemma 3

There is a locally unique function

$$y(x) = x^{s-d+1}z(x) - \ x^{s-d+1}$$

whose graph is Z*, with a* C^∞ *function* $z(x)$*,* $z(0) = 0$.

This is "$-x^{s-d+1}$" which dominates the computation.

Using $y(x)$, we compute the minimal reqular separation exponent. Here is a sketch.

We discuss the inequality defining the regular separation exponent at $(0,0)$. Let $A=(x,0)$ be the points on N, $B=(x,y(x))$ be the points on Z, and let O be the point $(0,0)$. Using the function $y(x)$, the length AB is of order $|x|^{s-d+1}$. Since AO and BO are of order $|x|$, the triangle inequality:

$$AB \leq AO + BO$$

implies the inequality from the Łojasiewicz theorem. We get that the exponent is equal to $s-d+1$.

(The order of tangency is $s-d$.)

Do we have tools to compute the exponents? Consider a singular plane curve

$$x^4 - y^3 + 6x^2y + 6y^2 - 2x^2 - 9y \, = \, 0\,.$$

This curve has two cusp-like 'return' points $P_\pm = (\pm 2, \, -1)$ and a self-intersection point $P_{\mathrm{self}} = (0, \, 3)$, all of them – critical points of the polynomial on the LHS. (The fourth critical point $(0, \, 1)$ lies off the curve.) From each of $P_\pm$ there emerge a pair of branches. This curve admits a parametrisation $x(t) = t^3 - 3t$, $y(t) = t^4 - 2t^2$.

Its Taylor expansion about $t_0 = 1$ is

$$\begin{pmatrix} t^3 - 3t \\ t^4 - 2t^2 \end{pmatrix} = P_- + (t-1)^2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + (t-1)^3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + (t-1)^4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence the Euclidean distance of points of the curve for $t = 1 - \epsilon$ and $t = 1 + \epsilon$ is $2\sqrt{17}\,\epsilon^3 + O(\epsilon^4)$, while the distances of these points to the reference point P_- are asymptotically equal $5\,\epsilon^2$ when $\epsilon \to 0^+$. So the minimal regular exponent is **3/2**. The two branches of the curve (semialgebraic sets!) from P_- are characterised by an inequality $4x - 3y + 5 \leq 0$, or else $4x - 3y + 5 \geq 0$. A general theory gives a very loose upper bound

$$\nu \leq \frac{(2 \cdot 4 - 1)^{2+2} + 1}{2} = 1201.$$

Thank you!