The Łojasiewicz exponent, hyperplane sections, and order of tangency

Piotr Pragacz
(IM PAN, Warszawa)
with Christophe Eyral

Introduction

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Let me summarize our studies of the order of tangency with Wojciech Domitrz and Piotr Mormul.

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Namely, the order of tangency allows one to define for example the **jets** of Lagrangian submanifolds.

The space of these jets is a fibration over the Lagrangian Grassmannian and leads to a positive decomposition of a Lagrangian Thom polynomial in the basis of Lagrangian Schubert cycles.

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when **there exist** a neighbourhood $U \ni u^0$ in $\mathbb{R}^p$ and parametrizations (diffeomorphisms onto their images)

$$q\colon \big(U, u^0\big) \to \big(M,\, x^0\big), \qquad \tilde{q}\colon \big(U, u^0\big) \to \big(\widetilde{M},\, x^0\big)$$

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$$\left(\tilde{q} - q\right)(u) \,=\, \mathrm{o}\left(\left|u - u^0\right|^k\right) \tag{1}$$

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means

$\lim_{u \to u_0} \frac{f(u)}{h(u)} = 0.$

"$f(u)$ is much smaller than $h(u)$ for u near u_0."

Proposition

The condition (1) *is equivalent to*

$$T^k_{u^0}\big(q\big) \;=\; T^k_{u^0}\big(\tilde{q}\big)\,, \tag{2}$$

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.(1) $\Rightarrow$ (2).

$$\mathrm{o}\Big(\big|u-u^0\big|^k\Big) = \tilde q(u) - q(u) \;=\; \big(\tilde q(u) - \;T^k_{u^0}\big(\tilde q\big)(u-u^0)\big)$$
$$+\,\big(T^k_{u^0}\big(\tilde q\big)(u-u^0) - \;T_{u^0}(q)(u-u^0)\big) + \big(T_{u^0}(q)(u-u^0) - q(u)\big)\,,$$

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$$+\,\big(T^k_{u^0}(\tilde{q})(u-u^0) - \; T_{u^0}(q)(u-u^0)\big) + \big(T_{u^0}(q)(u-u^0) - q(u)\big)\,,$$

where the first and last summands are $o\Big(\big|u-u^0\big|^k\Big)$ by Taylor.

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$$T_{u^0}^{k}\big(\tilde{q}\big)\big(u - u^0\big) - T_{u^0}^{k}\big(q\big)\big(u - u^0\big) = \mathrm{o}\Big(\big|u - u^0\big|^{k}\Big)$$

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Lemma

Let $w \in \mathbb{R}[u_1,\, u_2, \ldots,\, u_p]$, $\deg w \leq k$, $w(u) = \mathrm{o}\big(|u|^k\big)$ when $u \to 0$ in $\mathbb{R}^p$. Then w is identically zero.

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The implication: Proposition $\Rightarrow$ (1) is easy.

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Let us assume additionally that

$$s < r\,. \tag{4}$$

When $r = \infty$, the condition (4) simply says that s is finite.

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Theorem

Under (4),

$$\min_{v}\Big(\max_{\gamma,\tilde{\gamma}}\big(\max\big\{l \in \{0\}\cup\mathbb{N} : \ |\gamma(t)-\tilde{\gamma}(t)| = \mathrm{o}\big(|t|^{l}\big) \text{ when } t \to 0\ \big\}\big)\Big) = s. \tag{5}$$

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The **minimum** *is taken over all* $0\neq v\in T_{x^0}M = T_{x^0}\widetilde{M}$*. The* **outer maximum** *is taken over all pairs of* C^r *curves* $\gamma\subset M$*,* $\tilde\gamma\subset\widetilde{M}$ *such that* $\gamma(0)=x^0=\tilde\gamma(0)$*, and – both non-zero! – velocities* $\dot\gamma(0)$*,* $\dot{\tilde\gamma}(0)$ *are both parallel to* v*.*

Attention. In this theorem the assumption (4) is essential; our proof would not work in the situation $s=r$.

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$$\mathcal{G}H(s) \,=\, dH(s)(T_s N)\,, \tag{6}$$

where $G_n(N')$ is the total space of the Grassmann bundle, with base N', of all n planes tangent to N' (often denoted $G_n(T_{N'})$).

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Recall that $M, \widetilde{M} \subset \mathbb{R}^m$.

We use as previously the pair of parametrizations q and $\tilde{q}$. So we are now given the mappings

$$\mathcal{G}\, q : \; U \longrightarrow G_p\big(\mathbb{R}^m\big)\,, \qquad \mathcal{G}\, \tilde{q} : \; U \longrightarrow G_p\big(\mathbb{R}^m\big)\,.$$

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Upon putting $M^{(0)} = \mathbb{R}^m$, $\mathcal{G}^{(1)} = \mathcal{G}$, we get two sequences of recursively defined mappings. Namely, for $l \geq 1$,

$$\mathcal{G}^{(l)} q :\ U \longrightarrow G_p\big(M^{(l-1)}\big)\,, \qquad \mathcal{G}^{(l+1)} q \,=\, \mathcal{G}\Big(\mathcal{G}^{(l)} q\Big)$$

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where, naturally, $M^{(l)} = G_p\big(M^{(l-1)}\big)$.

Theorem

C^r manifolds M and $\widetilde{M}$ have at x^0 the order of tangency at least k $(1 \leq k \leq r)$ iff

$$\mathcal{G}^{(k)}q\left(u^0\right) = \mathcal{G}^{(k)}\tilde{q}\left(u^0\right)$$

for any parametrizations q and $\tilde{q}$ of M and $\widetilde{M}$ around x^0.

Lemma

For $1 \leq l \leq k$ *there exists such a local chart on the Grassmannian* $G_p\big(M^{(l-1)}\big)$ *in which the mapping* $\mathcal{G}^{(l)}q$ *evaluated at* u *has the form*

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Attention. In this lemma we distinguish mixed derivatives taken in different orders.

A natural question arises: What about branches of algebraic sets which often happen to be tangent one to another with various degrees of closeness?

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where ρ is a distance induced by any of the usual norms on $\mathbb{C}^m$.

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This exponent is an interesting metric invariant of the pointed pair $(X, Y; x^0)$.

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Consider two curves N and Z in $\mathbb{C}^2(x,y)$ intersecting at $(0,0)$:

$$N = \{y = 0\} \quad \text{and} \quad Z = \{y^d + yx^{d-1} + x^s = 0\},$$

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This is "$-x^{s-d+1}$" which dominates the computation.

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We get that the exponent is equal to $s - d + 1$. (The order of tangency is $s - d$.)

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Let X and Y be analytic subsets in $\mathbb{C}^m$, and let $x^0 \in X \cap Y$ such that $\mathcal{L}(X, Y; x^0) \geq 1$.

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$$\mathcal{L}(X \cap H_0, Y \cap H_0; x^0) \leq \mathcal{L}(X, Y; x^0).$$

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then $\nu \geq \mathcal{L}(X, Y; 0) \geq 1$, and for some $c' > 0$ we have

$$\rho(x, Y) \geq c' \rho(x, X \cap Y)^{\nu} \tag{9}$$

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Combined with (9), this gives

$$\rho(x, Y \cap H_0) \geq \rho(x, Y) \geq c'\,\rho(x, X \cap Y)^{\nu} \geq \frac{c'}{c}\,\rho(x, X \cap Y \cap H_0)^{\nu}$$

for all $x \in X \cap H_0$ near 0, so that ν is a regular separation exponent for $X \cap H_0$ and $Y \cap H_0$ at 0 as desired.

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Let $\check{\mathbb{P}}^{m-1}$ denote the set of all hyperplanes of $\mathbb{C}^m$ through 0, with its usual structure of manifold. The distance between two elements $H, K \in \check{\mathbb{P}}^{m-1}$ is the angle $\sphericalangle(H, K)$ between them, that is,

$$\sphericalangle(H, K) := \arccos \frac{\langle v, w \rangle}{|v|\,|w|},$$

where v and w are normal vectors to the hyperplanes H and K respectively, considered with their underlying real structures, and where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^{2m}$.

Let

$$\mathcal{X} := \{(H, x) \in \check{\mathbb{P}}^{m-1} \times \mathbb{C}^m \mid x \in H \cap X\}.$$

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$$\mathcal{U} := \{(H,x) \in \check{\mathbb{P}}^{m-1} \times \mathbb{C}^m \mid \sphericalangle(H_0, H) < a \text{ and } |x| < b\}$$

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of a generic $(H_0, 0)$, the set $\mathcal{X}$ is *Lipschitz equisingular* over $\check{\mathbb{P}}^{m-1} \times \{0\}$.

That is, for any $(H, 0) \in \mathcal{U} \cap (\check{\mathbb{P}}^{m-1} \times \{0\})$, there is a (germ of) Lipschitz homeomorphism

$$\varphi\colon (\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m, (H, 0)) \to (\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m, (H, 0))$$

such that $p \circ \varphi = p$ (where $p\colon \check{\mathbb{P}}^{m-1} \times \mathbb{C}^m \to \check{\mathbb{P}}^{m-1}$ is the standard projection) and $\varphi(\mathcal{X}) = \check{\mathbb{P}}^{m-1} \times (H \cap X)$ (as germs at $(H, 0)$).

Actually, if $h = (h_1, \ldots, h_{m-1})$ are coordinates in $\check{\mathbb{P}}^{m-1}$ around H_0 such that

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then locally near $(H_0, 0)$, the standard vector fields ∂_{h_j} $(1 \leq j \leq m-1)$ on $\check{\mathbb{P}}^{m-1} \times \{0\}$ can be lifted to Lipschitz vector fields v_j on $\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m$

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such that the flows of v_j preserve $\mathcal{X}$.

So, in particular, v_j is a Lipschitz vector field of the form

$$v_j(h,x) = \partial_{h_j}|_{(h,x)} + \sum_{\ell=1}^{m} w_{j\ell}(h,x)\,\partial_{x_\ell}|_{(h,x)},$$

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Using the integral curves of these Lipschitz vector fields, we prove the proposition.

Corollary

Assume that $\dim(X) = \dim(Y)$*. If* $x^0 \in \overline{X \setminus Y}$*, then the tangency order* $s(X, Y; x^0) \leq \mathcal{L}(X, Y; x^0)$*.*

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Let us first consider the special case where x^0 is an isolated point of $X \cap Y$. By the assumption, x^0 is an accumulation point of X. Then, by the inequality (11), and since the parametrization q is locally bi-Lipschitz, there exists $c > 0$ such that for all u near u^0 we have

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Thus the corollary holds true in this case.

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Thus the corollary follows from the inequality $s_n \leq \mathcal{L}_n$ (0-dimensional case).

THE END