On the automorphisms of Mukai varieties

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Prime Fano threefolds

Classification of prime Fano threefolds (Fano, Iskhovskih): smooth complex projective threefolds X such that $Pic(X) = \mathbb{Z}(-K_X)$ and $-K_X$ ample. When the anticanonical map is an embedding, codimension two linear sections are canonical curves of genus g.

g	X	g	X
2	Double sextic	7	section of Spinor variety
3	Quartic in $\mathbb{P}^4$	8	section of Grassmannian
4	Quadric $\cap$ cubic in $\mathbb{P}^5$	9	section of Lagrangian Grass.
5	Three quadrics in $\mathbb{P}^6$	10	section of adjoint variety
6	$G(2,5) \cap Q \cap L$	12	tri-isotropic Grassmannian

Mukai: vector bundle method. Classified smooth complex projective manifolds X of dimension $n \geq 4$ such that

$$Pic(X) = \mathbb{Z}H \qquad \text{and} \qquad K_X = -(n-2)H.$$

Mukai varieties

Mukai varieties are extensions of prime Fano threefolds of genus $g \leq 10$. Linear sections of Mukai varieties are Mukai varieties.

Conversely, for $g \geq 6$ there are Mukai varieties of maximal dimension. For $g \geq 7$ they are homogeneous spaces

$$M_g = G/P \hookrightarrow \mathbb{P}(V_g).$$

g	G	V_g	$\dim(V_g)$	M_g	$\dim(M_g)$
7	$Spin_{10}$	Δ_+	16	S_{10}	10
8	SL_6	$\wedge^2\mathbb{C}^6$	15	$G(2,6)$	8
9	Sp_6	$\wedge^{\langle 3\rangle}\mathbb{C}^6$	14	LG(3,6)	6
10	G_2	$\mathfrak{g}_2$	14	$X_{ad}(G_2)$	5

Automorphism groups

Main question for today:

What can be the automorphism group of a Mukai variety??

Focus on genus 7 to 10: then appears a hierarchy with respect to the dimension.

- $X = M_g$, then $Aut(X) = G/Z(G)$.
- Hyperplane section: still a big automorphism group.
- Dimension bigger than critical: positive dimensional group.
- Small dimension: trivial? Not even known for all Fano threefolds. Not even clear for general Fano threefolds!

Hyperplane sections

Start with $M_g = G/P \subset \mathbb{P}(V_g)$.
A hyperplane section is defined by a point in $\mathbb{P}(V_g^\vee)$, which is quasi-homogeneous: G acts with an open orbit.

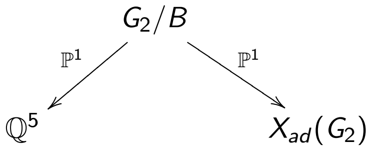
Consequence

Up to isomorphism, $\exists$ unique smooth hyperplane section X of M_g.
$Aut(X)$ is the generic stabilizer of the G-action on $\mathbb{P}(V_g^\vee)$.

How to lift an automorphism $g \in Aut(X)$ to G?
Mukai: Take more sections to reduce to K3 surfaces of genus g.
Then consider the *Mukai bundle* F, a uniquely defined stable vector bundle with special invariants. By unicity, the restrictions of F and g^*F are isomorphic. By cohomological arguments, such an isomorphism lifts to M_g.

Caveat! First part is not true in genus $g = 10$, where $V_g = \mathfrak{g}_2$ is the adjoint representation of G_2, and M_g is the *adjoint variety*. In terms of Jordan theory:

- M_g is the projectivization of the minimal nilpotent orbit.
- A general hyperplane section is defined by a regular semisimple element in $\mathfrak{g}_2$.
- So the generic stabilizer is essentially a maximal torus.
- $\rightsquigarrow$ one-dimensional family of hyperplane sections.



Adjoint varieties

More generally, there is an adjoint variety $M_{\mathfrak{g}} \subset \mathbb{P}(\mathfrak{g})$ for any simple Lie algebra $\mathfrak{g}$ (with contact structure, etc.).
Singular hyperplane sections correspond to points on the dual variety $M_{\mathfrak{g}}^{\vee} \subset \mathbb{P}(\mathfrak{g}^{\vee}) = \mathbb{P}(\mathfrak{g})$, a G-invariant hypersurface.
By Chevalley's classical theorem,

$$\mathfrak{g}//G \simeq \mathfrak{t}/W$$

and the equation of the dual is the product of the long roots.

Theorem (Prokhorov-Zaidenberg 2021)

The automorphism group of a smooth hyperplane section of $M_{\mathfrak{g}_2}$ is

$$(G_m^2) \rtimes \mathbb{Z}_2, \quad (G_m^2) \rtimes \mathbb{Z}_6, \quad (G_m \times G_a) \rtimes \mathbb{Z}_2, \quad GL_2 \rtimes \mathbb{Z}_2.$$

Critical codimension

Naive dimension count $\rightsquigarrow$ there is an integer c_g such that any linear section of M_g of codimension $< c_g$ must have positive dimensional automorphism group:

$$c_7 = 4, \quad c_8 = 3, \quad c_9 = c_{10} = 2.$$

Expectation: the general linear section of codimension $\geq c_g$ must have trivial automorphism group.

Theorem (Dedieu, M.)

The general linear section of codimension $> c_g$ has trivial automorphism group.
The general linear section of codim c_g has automorphism group

$$\mathbb{Z}_2^2, \qquad \mathbb{C}^* \times (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2), \qquad \mathbb{Z}_2^4, \qquad 1.$$

In particular, the general Fano threefold of genus $g \geq 7$ has no automorphisms ($g = 6$: Debarre-Kuznetsov , $g = 12$: Prokhorov).

General approach

Consider $X = M_g \cap L$. By Mukai's method, one shows that $Stab(L) \to Aut(X)$ is surjective. Then we observe that if $u \in Stab(L)$, then also $u_s, u_n \in Stab(L)$. So we wonder:

Can a non trivial semisimple/unipotent element in G stabilize a general subspace L of V_g of codimension c?

This can be attacked systematically:

- If u is semisimple, it acts on V_g with eigenspaces $E_1, \dots, E_m$, and the subspaces stabilized by g are parametrized by products of Grassmannians

$$G(\ell_1, E_1) \times \cdots \times G(\ell_m, E_m).$$

- If u is unipotent, we have a similar control on the dimension of the sets of stable subspaces of each dimension.

Stratifications

There are only finitely many unipotent orbits to consider. Similarly, one can stratify the semisimple orbits according to the sizes of their eigenspaces in V_g. In order to prove that the generic automorphism group is trivial in codimension c, it is then enough to prove that for each stratum S,

$$\dim(S) + d_c(S) < \dim G(c, V_g).$$

Hence a finite algorithm.

Surprise: there exist a small number of strata S for which

$$\dim(S) + d_c(S) = \dim G(c, V_g).$$

This happens in codimension $c = c_g$, only for semisimple strata, parametrizing automorphisms of order two for $g \neq 8$.

Genus 8

$M_8 = G(2,6) \subset \mathbb{P}(\wedge^2\mathbb{C}^6)$
$\rightsquigarrow$ Consider $L \subset \wedge^2\mathbb{C}^6$ general of dimension 3 to 12.

Unipotent orbits classified by partitions of 6 $\rightsquigarrow$ cannot stabilize L.

Semi-simple elements are determined by eigenvalues $t_1, \dots, t_6$.
Eigenvalues of the induced action on $\wedge^2\mathbb{C}^6$ are the $t_i t_j$, $i < j$
$\rightsquigarrow$ one has to consider the multiplicities. Beware of:

- Degenerations: some t_i's can coincide.
- Collapsings: $t_i t_j = t_p t_q$ with $(i,j) \cap (p,q) = \emptyset$.

(Same principle for the other genera.) Laborious but efficient!
Conclusion: L generic of dimension 4 to 11 has no stabilizer. Thus

$$Aut(X) = 1 \qquad \text{for} \quad X = G(2,6) \cap \mathbb{P}(L^\perp)$$

general Mukai variety of genus 8 and dimension 3, 4.

For $X = G(2,6) \cap \mathbb{P}(L^\perp)$ of dimension 5, so $\dim L = 3$, the conclusion is different.

By the previous analysis L can only be stabilized by:

- $s = z id_A + z^{-1} id_B$ for $\mathbb{C}^6 = A \oplus B$ s.t. $L \subset A \otimes B \subset \wedge^2 \mathbb{C}^6$.
- Some involutions $t = id_E - id_F$ for $\mathbb{C}^6 = E \oplus F$ such that $L = L_1 \oplus L_2$ with $L_1 \subset \wedge^2 E \oplus \wedge^2 F$ and $L_2 \subset E \otimes F$.
- Order 3 elements $u = id_P + j id_Q + j^2 id_R$ with $\mathbb{C}^6 = P \oplus Q \oplus R$, such that L is the sum of three lines contained in the three (five dimensional) eigenspaces of the induced action on $\wedge^2 \mathbb{C}^6$.

$\rightsquigarrow$ How can such transformations fit together?

The first type gives the connected component $\mathbb{C}^* \rightsquigarrow$ The pair (A, B) such that $L \subset A \otimes B$ must be unique (normalization).

Genus 8, continued

Special feature for this case:
The Pfaffian cubic cuts on $\mathbb{P}(L)$ a plane cubic C and $Stab(L)$ has to act on C. How?

- Automorphisms of the first type act trivially.
- Involutions of the second type act as symmetries w. respect to inflexion points of C.
- Order three elements act as translations by 3-torsion points.

Hence the conclusion:

$$Aut(X)/Aut^0(X) \simeq Aut_{lin}(C) \simeq (\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2.$$

Genus 7

Here M_7 is the spinor variety of dimension 10, index 8 in $\mathbb{P}^{15}$.
Small codimensional linear sections were considered before.

- In codimension 1, there is a unique smooth section; quasi-homogeneous with non reductive automorphism group.
- In codimension 2, two different types of smooth sections. The general one is quasi-homogeneous under $G_2 \times PSL_2$ (Fu-M. 2018). The special one is a compactification of $\mathbb{C}^8$ (Fu-Hwang 2018), with non reductive automorphism group. Gives counter-examples to rigidity properties for prime Fano manifolds of high index.
- In codimension 3, four different types of smooth sections. Most special one is a compactification of $\mathbb{C}^7$. General one has automorphism group PSL_2^2, not quasi-homogeneous.

Genus 7, codimension four

We focus on codimension four: Fano sixfolds X of index four, defined by a general four (co)dimensional $L \subset \Delta$, with Δ the spin representation.
The analysis of semisimple/unipotent elements yields:

L may be stabilized by finitely many involutions $s_U = id_U - id_{U^\perp}$ in SO_{10}, where U is some non-degenerate four plane in $\mathbb{C}^{10}$;
$\rightsquigarrow$ splits Δ into two eight dimensional eigenspaces Δ_+ and Δ_-, and we need $L = L_+ \oplus L_-$ for two planes $L_\pm \subset \Delta_\pm$.

To understand: do these involutions really exist in general? if yes, how many are they? which group do they generate?
The answer to the first question is YES by a dominance argument.
The answer to the last two is given by the following Theorem.

Genus 7, continued

Main Theorem for genus seven

There exist three non degenerate orthogonal planes A, B, C s.t.

$$Aut(X) = \{1, s_{A\oplus B}, s_{B\oplus C}, s_{C\oplus A}\} \simeq \mathbb{Z}_2^2.$$

Sketch of proof.

$\dim G(4,\Delta) = 4 \times (16-4) = 48$. For three orthogonal planes in $\mathbb{C}^{10}$ there are $16+12+8=36$ parameters.

When fixed, the spin representation decomposes into 4 four-dimensional spaces and L must meet each of them along a line, hence 4×3 extra parameters. Since $36+12=48$ we expect a finite non zero number of triples (A, B, C) for each L.

This is proved to be correct by computing a suitable differential.

Genus 7, conclusion

Then L is stabilized by the three involutions $s_{A\oplus B}, s_{B\oplus C}, s_{C\oplus A}$.
If t is another involution stabilizing L, the products $ts_{A\oplus B}, ts_{B\oplus C}, ts_{C\oplus A}$ must be involutions of the same type.
In particular t commutes with $s_{A\oplus B}, s_{B\oplus C}, s_{C\oplus A}$
$\rightsquigarrow$ more structure for L and contradiction by dimension count.
As a consequence the triple (A, B, C) is unique. □

Remark. For each non trivial involution s in $Aut(X)$, the fixed locus $Fix(s)$ is the union of two quartic surface scrolls in X (codimension two sections of $\mathbb{P}^1 \times \mathbb{P}^3$).
This comes from the exceptional isomorphisms

$$\mathfrak{so}(U) \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2, \qquad \mathfrak{so}(U^\perp) \simeq \mathfrak{sl}_4.$$

Genus 9

Here M_9 is the Lagrangian Grassmannian $LG(3,6)$, of dimension 6, index 4 in $\mathbb{P}^{13}$. It parametrizes three-planes in $\mathbb{C}^6$ that are isotropic w.t. to a skew-symmetric form ω.
The Plücker embedding is inside $\mathbb{P}(\wedge^{\langle 3\rangle}\mathbb{C}^6)$ where

$$\wedge^{\langle 3\rangle}\mathbb{C}^6 = Ker(\wedge^3\mathbb{C}^6 \xrightarrow{\omega} \mathbb{C}^6).$$

$\rightsquigarrow$ $LG(3,6)$ is a variety with one apparent double point.
There is a unique smooth hyperplane section of $LG(3,6)$, whose automorphism group is PSL_3.

We focus on codimension two sections, defined by a (co)dimension two subspace $L \subset \wedge^{\langle 3\rangle}\mathbb{C}^6$.

Genus 9, continued

Analysis of semisimple/unipotent elements $\rightsquigarrow$ a generic L can only be stabilized by involutions, of two possible types:

- $s_A = id_A - id_{A^\perp}$ for $A \subset \mathbb{C}^6$ a non isotropic plane,
- $t = id_E - id_F$ for $\mathbb{C}^6 = E \oplus F$ a decomposition into Lagrangian subspaces.

These possibilities can be realized in at most finitely many ways.

How many? For type I, the answer is given by the

Normalization Lemma

There exists a unique triple (A, B, C) of transverse planes in $\mathbb{C}^6$ such that $L \subset A \otimes B \otimes C$.

Genus 9, continued

Main Theorem in genus 9

The stabilizer of a general L is isomorphic with $\mathbb{Z}_2^4$, with

- 3 type I involutions s_A, s_B, s_C,
- 12 type II involutions.

Geometrically, the corresponding involutions in $Aut(X)$ can be distinguished by their fixed locus:

- a del Pezzo surface of degree four in type I,
- the union of two Veronese surfaces in type II.

QUESTION: Where does all this come from??

Relations with θ-representations

Consider the situation where a simple Lie-algebra $\mathfrak{g}$ is endowed with an automorphism θ of finite order p. This yields a $\mathbb{Z}_p$-grading

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{p-1}.$$

Then $(G_0, \mathfrak{g}_1)$ is called a θ-representation.
Typical example: $p = 2$, θ an involution $\rightsquigarrow$ symmetric spaces.

One can define a Cartan suspace of $(\mathfrak{g}, \theta)$ as a maximal subspace $\mathfrak{h} \subset \mathfrak{g}_1$ of commuting semisimple elements $\rightsquigarrow$ generalized Weyl group $W = N(H)/H$ (a complex reflection group).

Generalized Chevalley theorem

$$\mathfrak{g}_1//G_0 \simeq \mathfrak{h}/W.$$

Examples of θ-representations

For example, we have the following gradings:

$$\mathfrak{f}_4 = \mathfrak{sl}_2 \times \mathfrak{sp}_6 \oplus (\mathbb{C}^2 \otimes \wedge^{\langle 3\rangle}\mathbb{C}^6),$$

$$\mathfrak{e}_7 = \mathfrak{sl}_3 \times \mathfrak{sl}_6 \oplus (\mathbb{C}^3 \otimes \wedge^2\mathbb{C}^6) \oplus (\mathbb{C}^3 \otimes \wedge^2\mathbb{C}^6)^\vee,$$

$$\mathfrak{e}_8 = \mathfrak{sl}_4 \times \mathfrak{so}_{10} \oplus (\mathbb{C}^4 \otimes \Delta) \oplus (\wedge^2\mathbb{C}^4 \otimes \mathbb{C}^{10}) \oplus (\mathbb{C}^4 \otimes \Delta)^\vee.$$

This means that

- codimension two sections of $LG(3,6)$ are connected to $\mathfrak{f}_4$,
- codimension three sections of $G(2,6)$ are connected to $\mathfrak{e}_7$,
- codimension four sections of Δ are connected to $\mathfrak{e}_8$!

$\rightsquigarrow$ Our generic automorphism groups are what remains of the generalized Weyl groups.