The Łojasiewicz exponent of non-degenerate surface singularities

Tadeusz Krasiński

(joint results with Szymon Brzostowski and Grzegorz Oleksik)

University of Łódź

IMPANGA Seminar

May 8, 2020,

ZOOM Conference (via Internet)

Introduction

There are many versions (variants) of the Łojasiewicz inequality and the Łojasiewicz exponent.

Introduction

There are many versions (variants) of the Łojasiewicz inequality and the Łojasiewicz exponent.

The main common idea (problem) is:

We have two mappings $F \; and \; G$ (of various domains, classes, fields, etc.) such that $V(F) \subset V(G)$.

Introduction

There are many versions (variants) of the Łojasiewicz inequality and the Łojasiewicz exponent.
The main common idea (problem) is:
We have two mappings $F \; and \; G$ (of various domains, classes, fields, etc.) such that $V(F) \subset V(G)$.
Find (or prove the existence) the best exponent $\lambda \in \boldsymbol{R}$ such that the following inequality holds (the Łojasiewicz inequality)

$$\left|\left|F\right|\right| \geq C \; ||G||^{\lambda}$$

locally or globally.

Introduction

We are interested in the following local variant over **C**:

$$F = \nabla f = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}), \qquad G = (z_1, \cdots, z_n),$$

where $f: (C^n, 0) \to (C, 0)$ is an **isolated** complex singularity.

Introduction

We are interested in the following local, complex variant:

$$F = \nabla f = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right), \qquad G = (z_1, \cdots, z_n),$$

where $f\colon (C^n, 0) \to (C, 0)$ is an **isolated** complex singularity.

Of course we have

$$V(F) = V\left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right) = \{0\} = V(z_1, \cdots, z_n) = V(G)$$

and the Łojasiewicz inequality takes the form

$$\boldsymbol{\left|\left|\nabla f(z)\right|\right| \geq C\, ||z||^{\lambda}}.$$

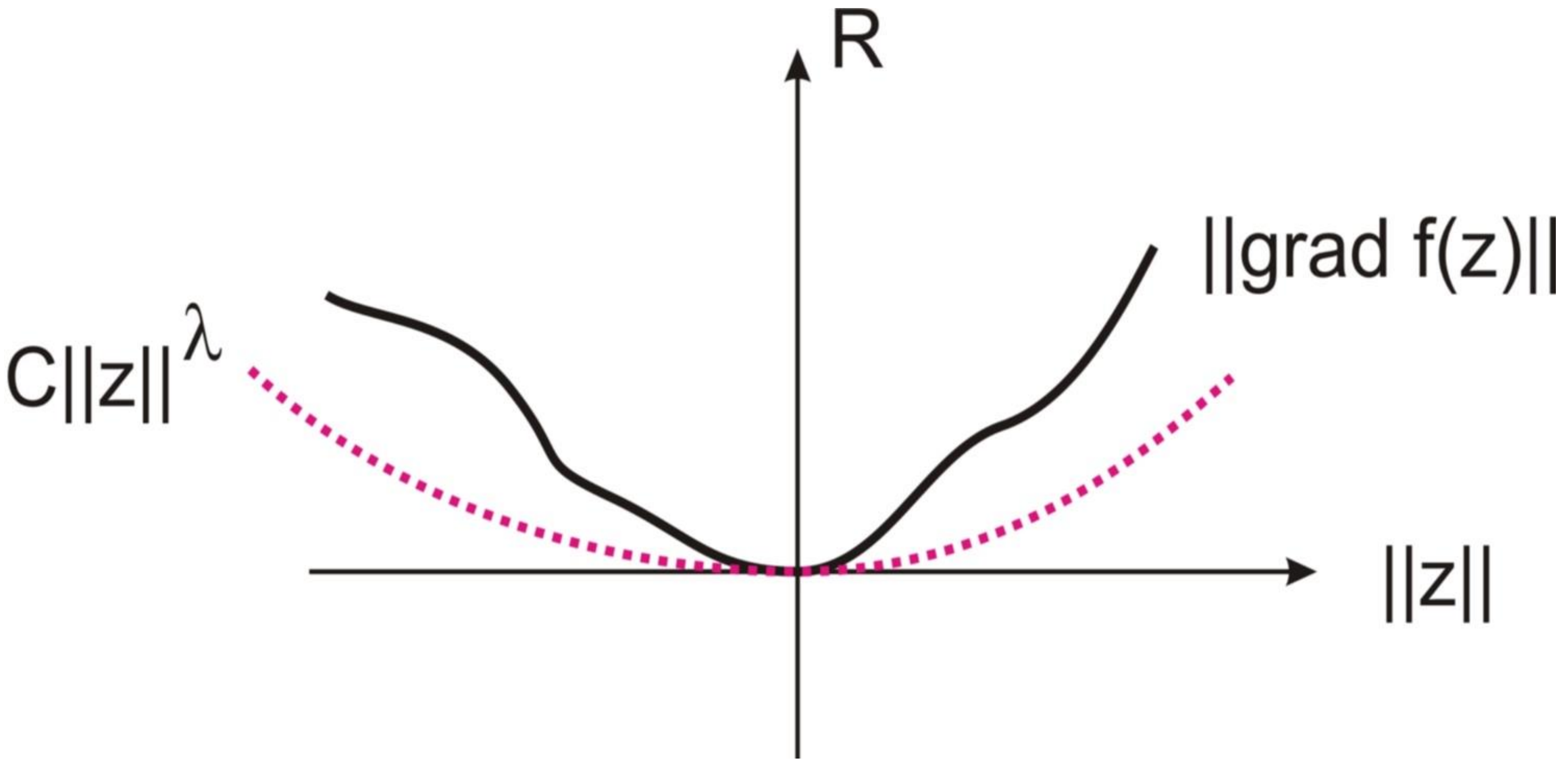
Introduction

Definition. The best exponent (the infimum) $\lambda \in \boldsymbol{R}$ such that the following inequality holds

$$\left|\left|\nabla \boldsymbol{f}(\boldsymbol{z})\right|\right| \geq \boldsymbol{C}\ ||\boldsymbol{z}||^{\lambda}$$

in a neighbourhood of the origin in $\boldsymbol{C}^n$ is the **Łojasiewicz exponent of** f and is denoted by $\mathcal{L}(f)$.

Introduction



Introduction

$\mathcal{L}(f)$ is an interesting invariant of f:

1. $[\mathcal{L}(f)] + 1$ is the C^0-sufficiency degree of f ,
2. $\mathcal{L}(f) \epsilon \boldsymbol{Q}$,
3. $\mathcal{L}(f)$ is an analytic invariant of f,
4. open problem whether $\mathcal{L}(f)$ is a topological invariant,
5. $\mathcal{L}(f)$ depends only on the ideal $\left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right) C\{z\}$

Introduction

6. $\mathcal{L}(f)$ can be calculated by means of analytic paths

$$\mathcal{L}(f) = \sup_{\Phi} \frac{\operatorname{ord}(\nabla f \circ \Phi)}{\operatorname{ord} \Phi}, \qquad \Phi(0)=0,$$

where $\Phi\colon (\boldsymbol{C}, 0) \to (\boldsymbol{C}^n, 0)$ a holomorphic curve

Introduction

6. $\mathcal{L}(f)$ can be calculated by means of analytic paths

$$\mathcal{L}(f) = \sup_{\Phi} \frac{\mathrm{ord}(\boldsymbol{\nabla f} \circ \Phi)}{\mathrm{ord}\,\Phi}, \qquad \Phi(0)=0,$$

where $\Phi: (\boldsymbol{C}, 0) \to (\boldsymbol{C}^n, 0)$ a holomorphic curve and moreover, there exists a holomorphic curve $\Phi(t)$ such that

$$||\boldsymbol{\nabla} f\left(\Phi(t)\right)|| \sim ||\Phi(t)||^{\mathcal{L}(f)}.$$

The main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.

Explanation of the main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.

Surface singularity:

$$f = f(x, y, z): (C^3, 0) \to (C, 0), \quad n = 3,$$

Explanation of the main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.

Surface singularity:

$$f = f(x, y, z): (C^3, 0) \to (C, 0), \quad n = 3,$$

$n = 1$. Trivial.
$n = 2$. Non-trivial. A. Lenarcik 1996.
$n > 3$. Open problem.

Explanation of the main result

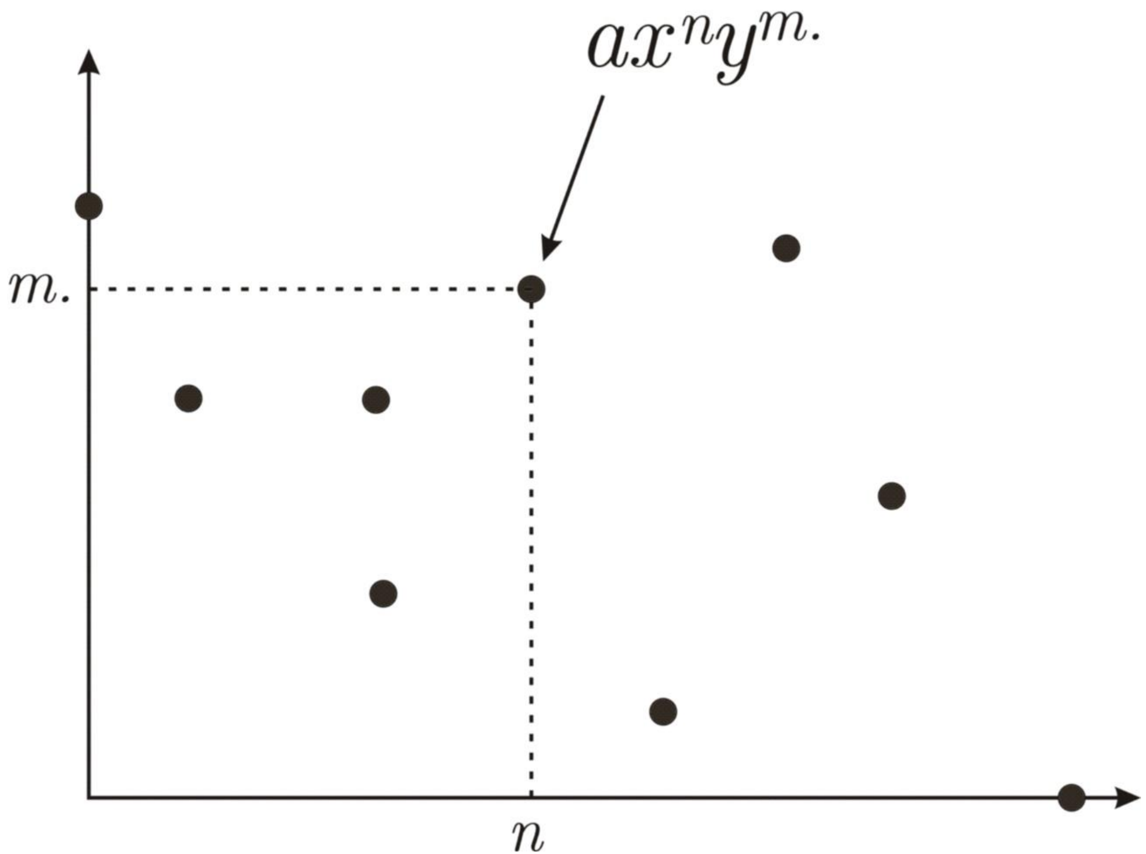
A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.

Newton polyhedron of f:

Combinatorial object in $\boldsymbol{R}^n$ associated to f

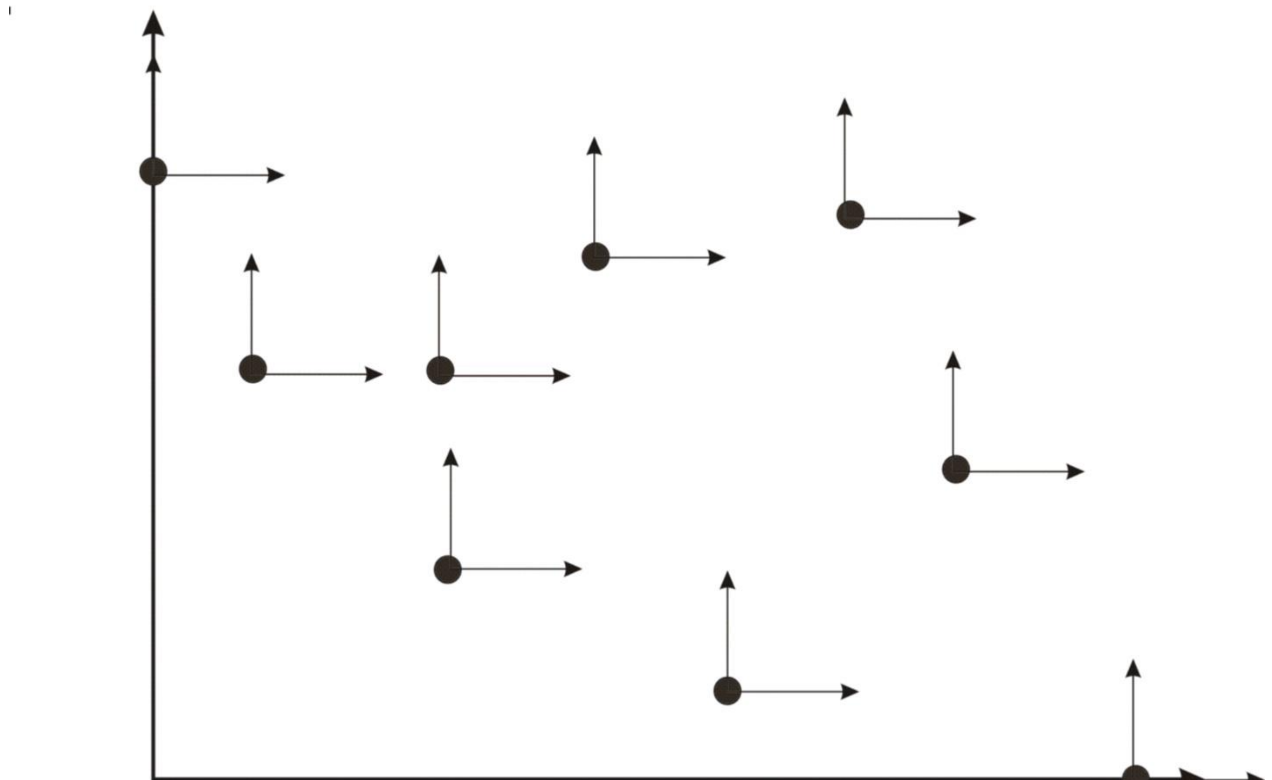
Explanation of the main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.



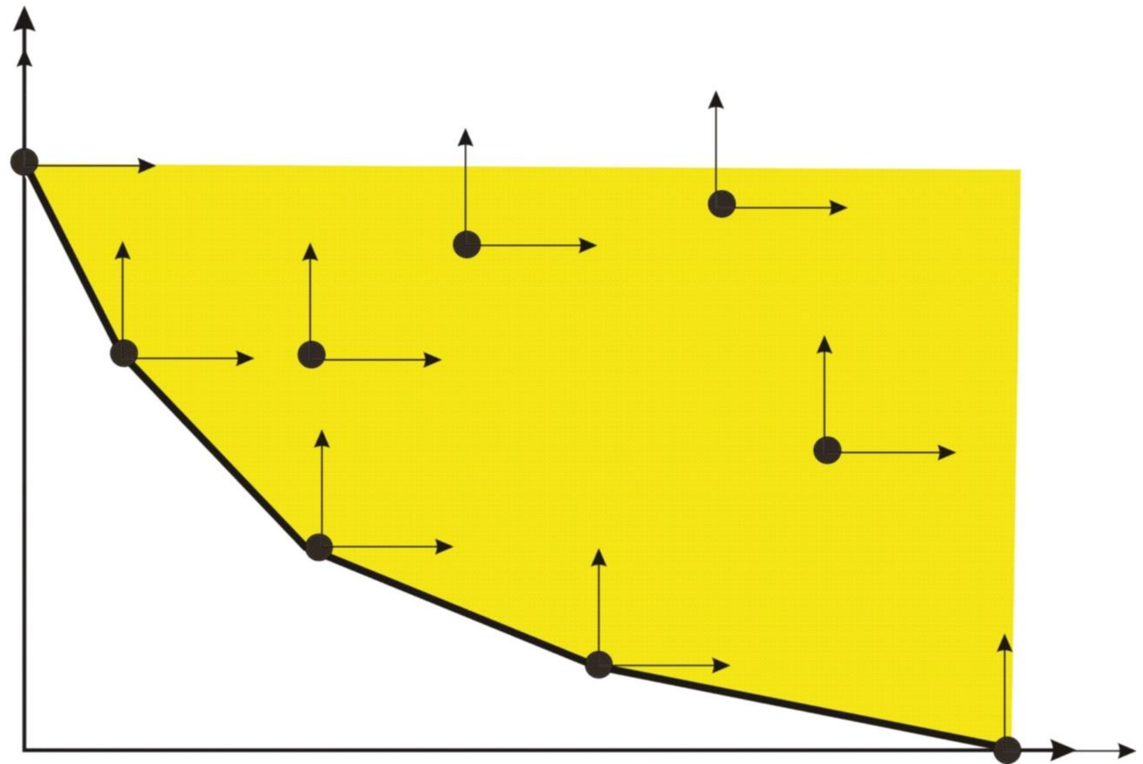
Explanation of the main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.



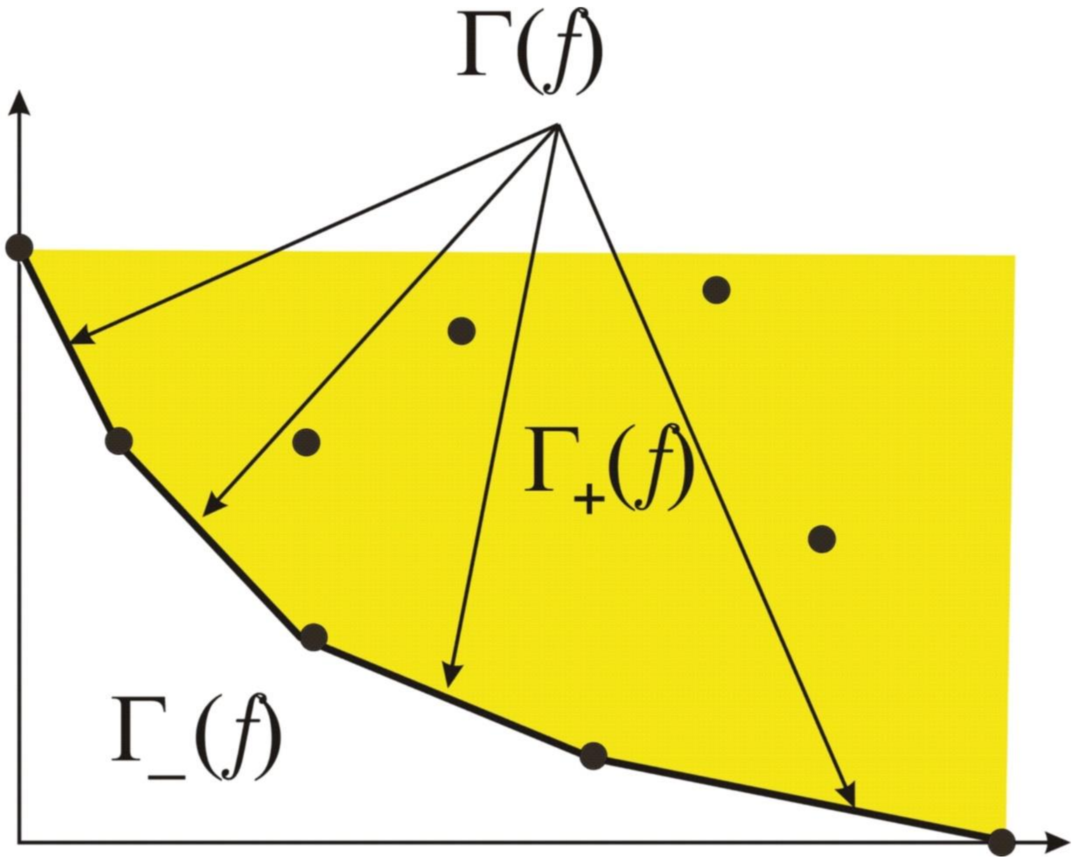
Explanation of the main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron



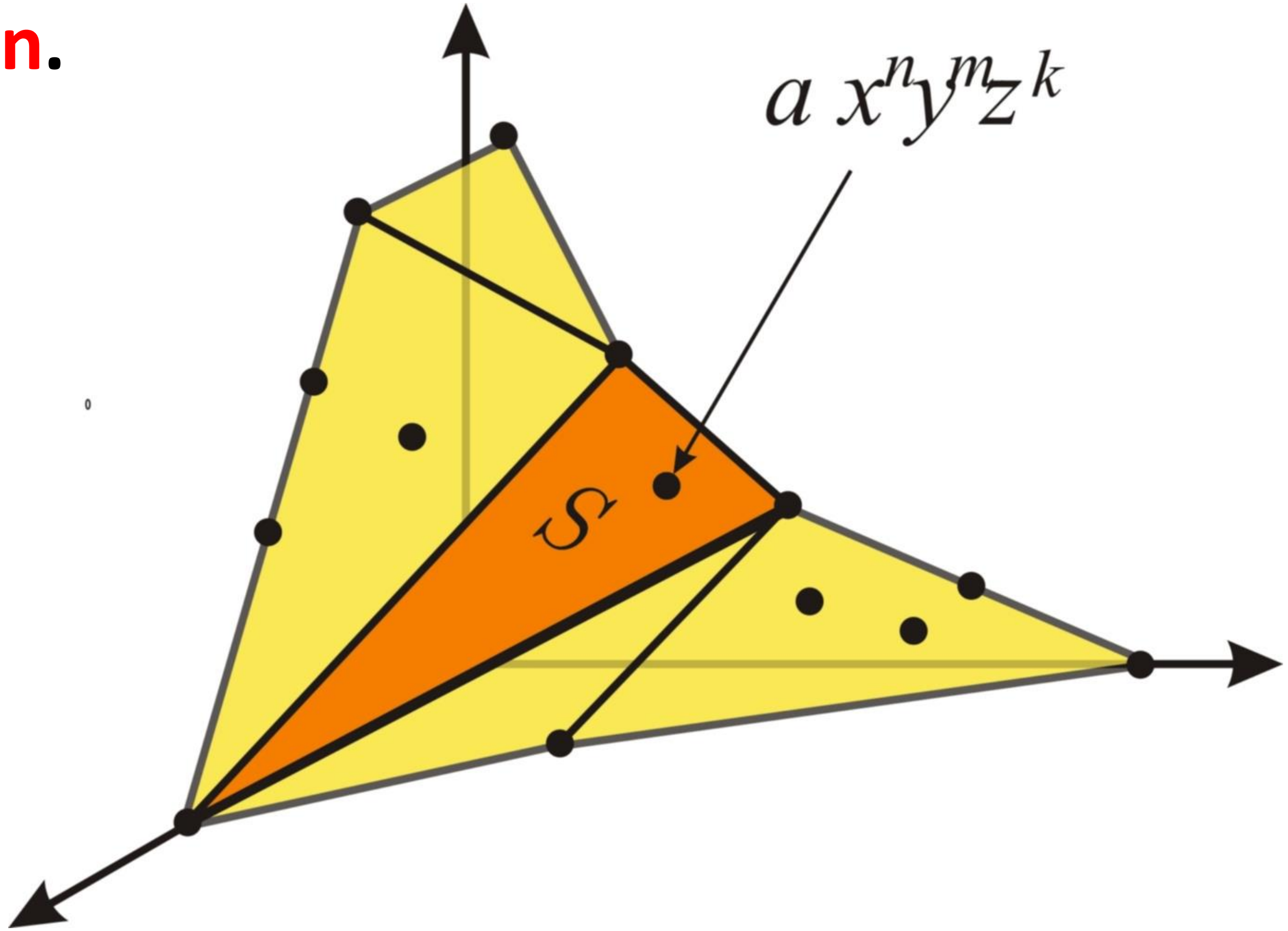
Explanation of the main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.



Explanation of the main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.



Explanation of the main result

A formula for the Łojasiewicz exponent $\mathcal{L}(f)$ of a non-degenerate surface singularity f in terms of its Newton polyhedron.

For any boundary face S of the Newton polyhedron

$$S \epsilon \Gamma(f) = \Gamma^0(f) \cup \Gamma^1(f) \cup \Gamma^2(f) \cup \dots \cup \Gamma^{n-1}(f)$$

the system of polynomial equations:

$$\frac{\partial f_S}{\partial z_1}(z) = 0,$$

$$\dots\dots\dots\dots$$

$$\frac{\partial f_S}{\partial z_n}(z) = 0$$

has no solution in $(C^*)^n$.

Explanation of the main result

Arnold's problems:

1968-2. What topological characteristics of a real (complex) polynomial are computable from the Newton diagram?

1975-1. Every interesting discrete invariant of a generic singularity with Newton polyhedron is an interesting function of the polyhedron.

1975-21. Express the main numerical invariants of a typical singularity with a given Newton diagram.

Known estimations in terms of the Newton polyhedron

Ben Lichtin (1981)
Tohizumi Fukui (1991).
Carles Bivia-Ausina (2003)
Ould Abderahmane (2005)
Pinaki Mondal (2019)
Mutsuo Oka (2018)

Lenarcik result for n=2

A. **Lenarcik** (1996) A formula for $\mathcal{L}(f)$ in 2 dimensional case (n=2). The singularity depends on two variables $f(x, y)$.

Lenarcik result for n=2

A. **Lenarcik** (1996) A formula for $\mathcal{L}(f)$ in 2 dimensional case (n=2). The singularity depends on two variables $f(x, y)$.

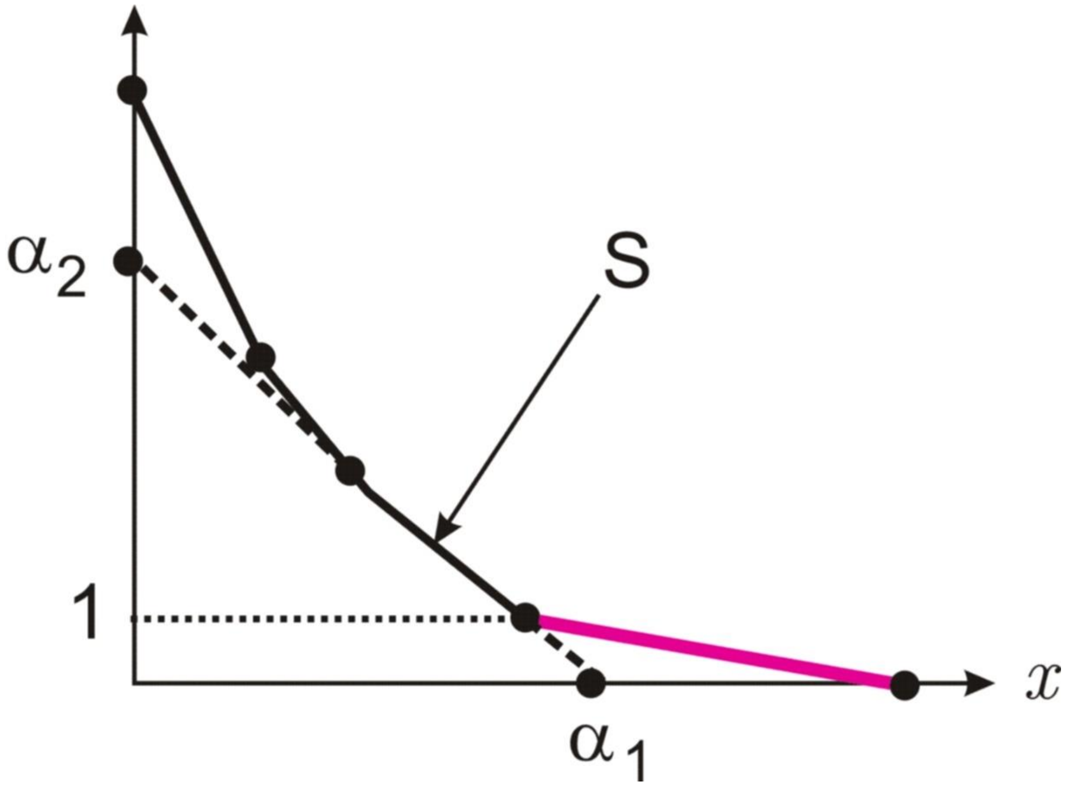
$$\mathcal{L}(f) = \begin{cases} \max\left(\alpha(S): S \in \Gamma^1(f) - E_f\right) - 1 & \text{if } \Gamma^1(f) - E_f \neq \emptyset \\ 1 & \text{if } \Gamma^1(f) - E_f = \emptyset. \end{cases}$$

E_f - exceptional segments of the Newton boundary $\Gamma(f)$.

Lenarcik result for n=2

Exceptional segment: if one of the partial derivatives of f_S is a pure power of another variables e.g. $f_S = \boldsymbol{yx^5} + x^8$ because $\frac{\partial f_S}{\partial y}(x, y) = \boldsymbol{x^5}$.

$$\alpha(S) \coloneqq \max(\alpha_1, \alpha_2)$$



The main result.

Theorem (Brzostowski, Krasiński, Oleksik). If $f:(C^3,0) \to (C,0)$ is a non-degenerate isolated singularity and $\Gamma^2(f) - E_f \neq \emptyset$ then

$$\mathcal{L}(f) = \max\left(\alpha(S): S \,\epsilon \Gamma^2(f) - E_f\right) - 1.$$

The main result.

Theorem (Brzostowski, Krasiński, Oleksik). If $f:(C^3,0)\to(C,0)$ is a non-degenerate isolated singularity and $\Gamma^2(f)-E_f\neq\emptyset$ then

$$\mathcal{L}(f)=\max\left(\alpha(S): S\,\epsilon\Gamma^2(f)-E_f\right)-1.$$

Remark. The case $\Gamma^2(f)-E_f=\emptyset$ i.e. $\Gamma^2(f)=E_f$ was solved by Oleksik in 2013. This is a very special case and relatively simpler.

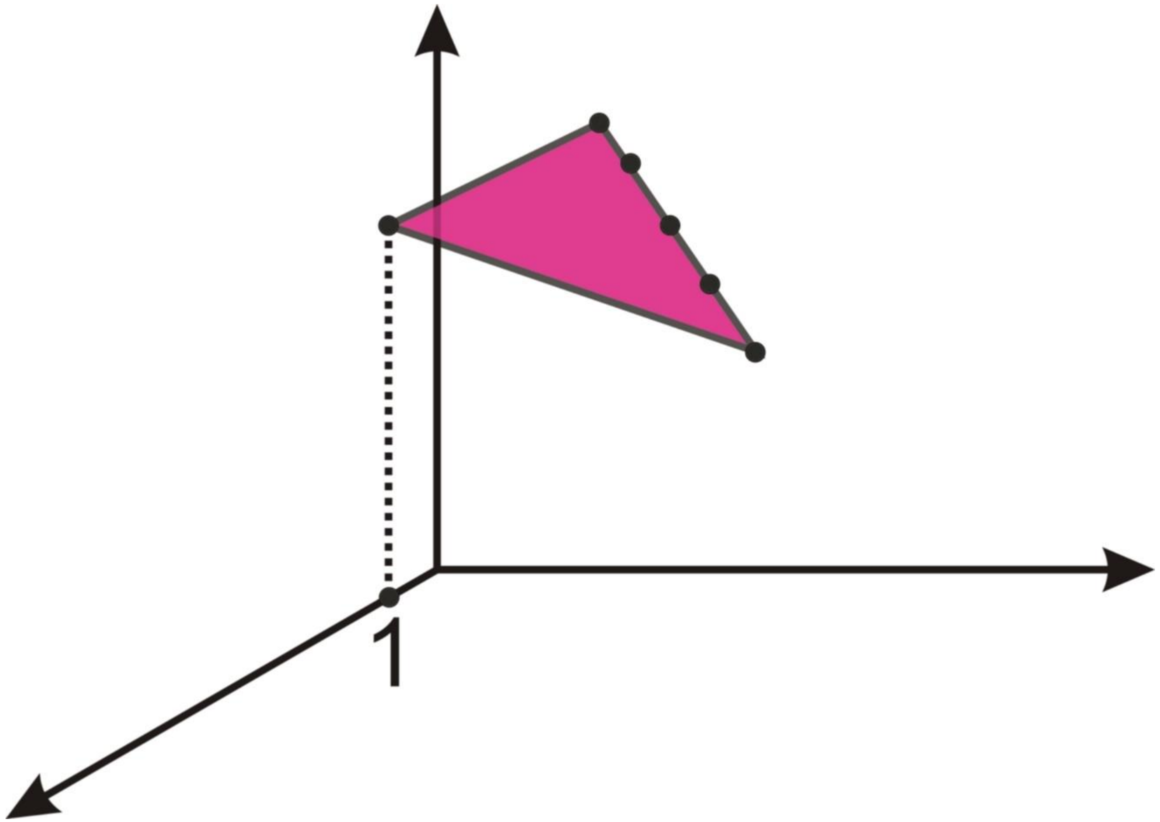
Definition of exceptional faces.

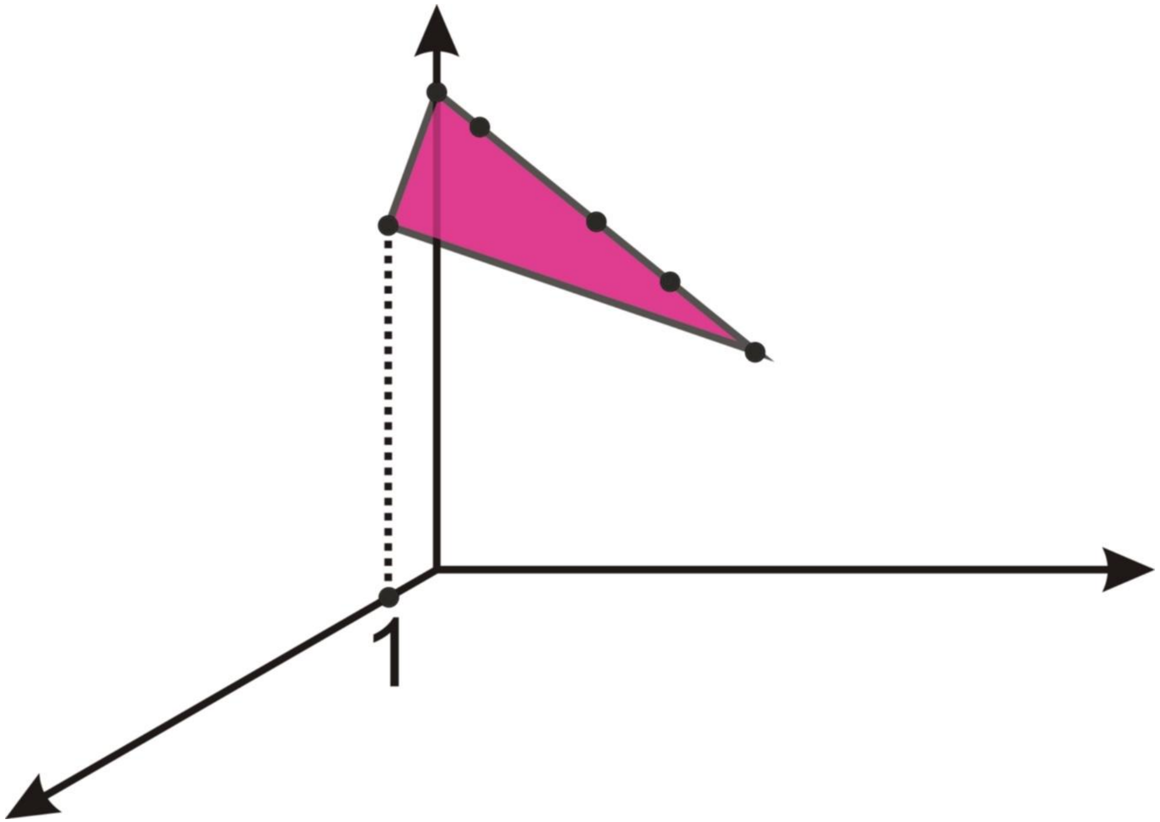
Definition:
2-dimensional face $S \epsilon \Gamma^2(f)$ is said to be exceptional if one of the partial derivatives of f_S is a pure power of another variable.

Remark. This definition may be easily transferred (generalized) to n-dimensional case.

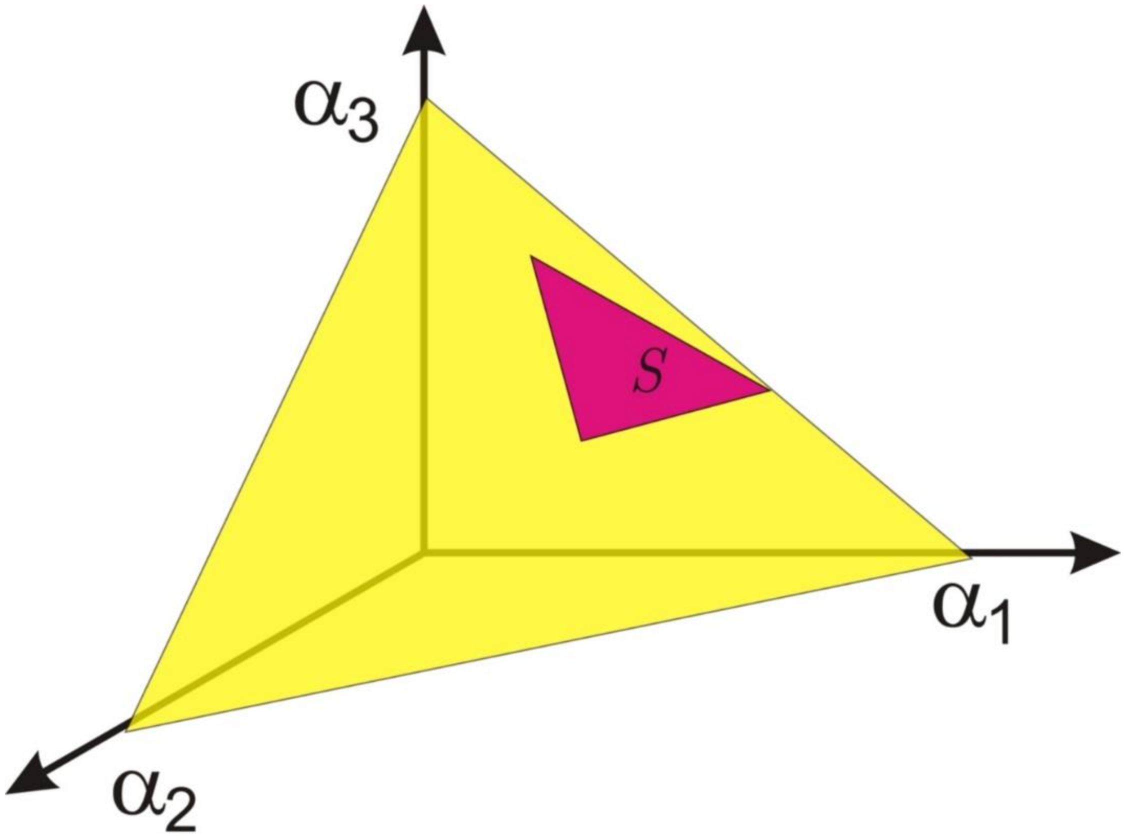
Example. $f_S = yz^5 + z^8 + x^8$

Definition of exceptional faces – geometrically.





Definition of $\alpha(S)$.



$$\alpha(S) = \max(\alpha_1, \alpha_2, \alpha_3)$$

The Oleksik result

Theorem (Oleksik 2010). If $f\colon (C^3, 0) \to (C, 0)$ is a non-degenerate isolated singularity and $\Gamma^2(f) - E_f \neq \emptyset$ then

$$\mathcal{L}(f) \leq \max\left(\alpha(S)\colon S \,\epsilon \Gamma^2(f) - E_f\right) - 1.$$

The Oleksik result

Theorem (Oleksik 2010). If $f:(C^3,0)\to(C,0)$ is a non-degenerate isolated singularity and $\Gamma^2(f)-E_f\neq\emptyset$ then

$$\mathcal{L}(f)\leq \max\left(\alpha(S): S\,\epsilon\Gamma^2(f)-E_f\right)-1.$$

We must prove the inverse inequality

$$\mathcal{L}(f)\geq \max\left(\alpha(S): S\,\epsilon\Gamma^2(f)-E_f\right)-1.$$

The idea of the proof

For inverse inequality " $\geq$ " it suffices to find a holomorphic curve $\varphi(t) = (x(t), y(t), z(t))$ such that

$$||\nabla f\left(\varphi(t)\right)|| \sim ||\varphi(t)||^{\alpha(f)},$$

where $\alpha(f) := \max\left(\alpha(S) : S \in \Gamma^2(f) - E_f\right) - 1.$

The idea of the proof

The first step to find such a curve is to „simplify” the singularity f to one which has the same Newton polyhedron, the same Łojasiewicz exponent and it is „very simple”. We apply the Brzostowski theorem.

The idea of the proof

Theorem (Brzostowski). In the class of non-degenerate isolated singularities in n-dimensional case the Łojasiewicz exponent depends only on the Newton polyhedron.

The idea of the proof

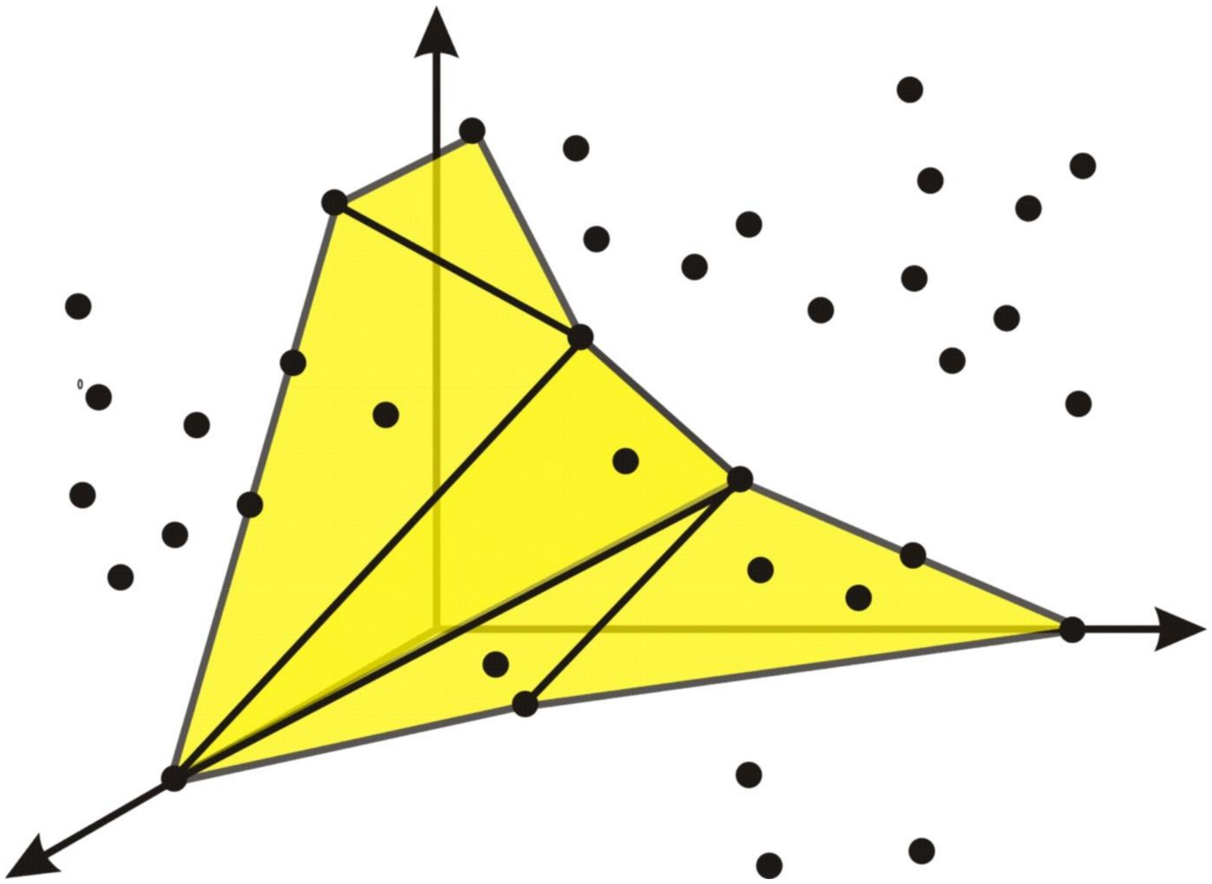
Theorem (Brzostowski). In the class of non-degenerate isolated singularities in n-dimensional case the Łojasiewicz exponent depends only on the Newton polyhedron.

Precisely
If $f, g\colon (C^n, 0) \to (C, 0)$ are isolated non-degenerate singularities and $\Gamma(f) = \Gamma(g)$ then $\mathcal{L}(f) = \mathcal{L}(g)$.

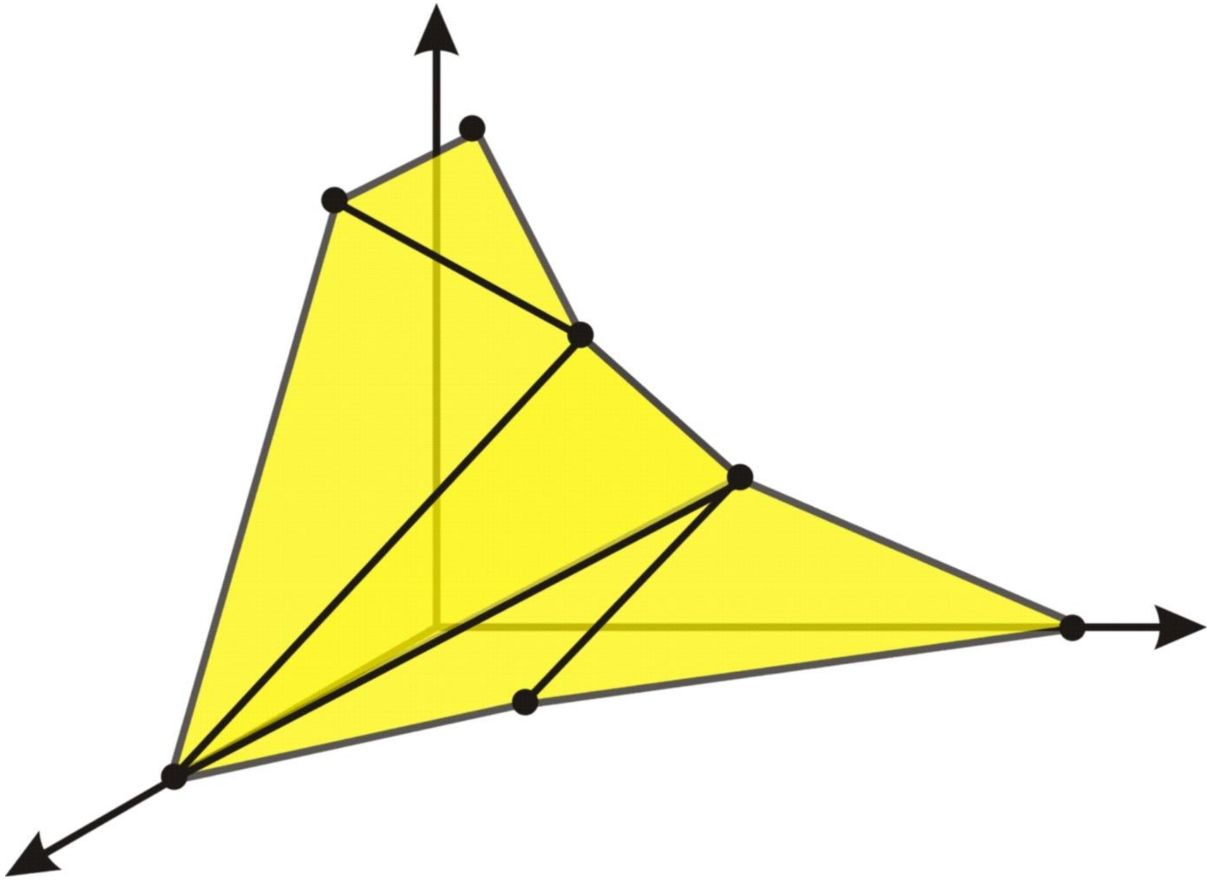
The idea of the proof

By this theorem we replace the initial singularity for another singularity which
1. has the same Newton polyhedron,
2. has no points above the Newton boundary,
3. has only vertices,
4. has generic coefficients.

The idea of the proof

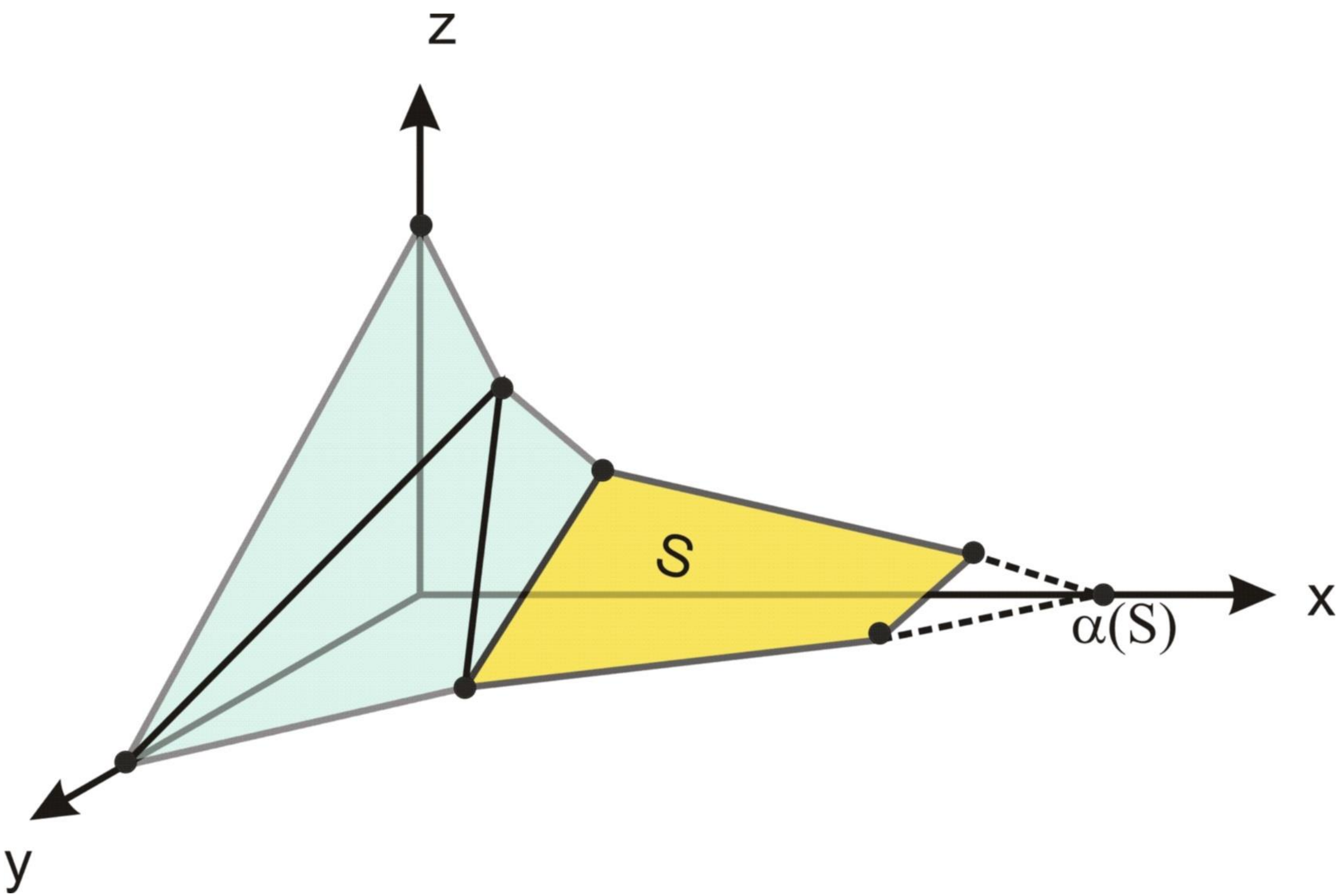


The idea of the proof



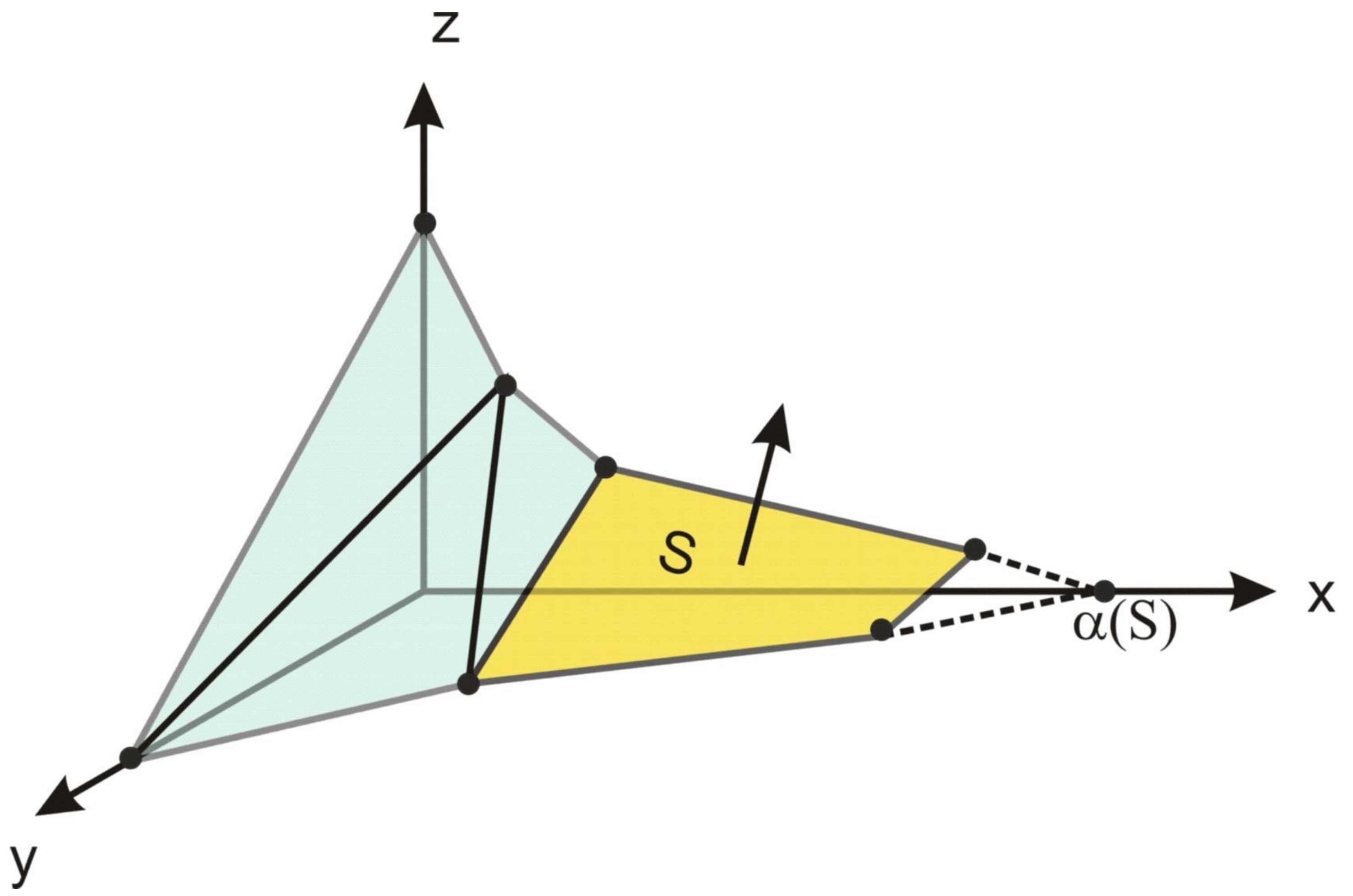
The idea of the proof

Let a **non exceptional face** $S \epsilon \Gamma^2(f)$ realize the maximum in the definition of $\alpha(f)$. Let this maximum be attained on the axis Ox.



The idea of the proof

Let $(v, u, w) \in N^3$ be a vector perpendicular to S.



The idea of the proof

For the monomial curve $\Gamma\colon \varphi(t) = (at^v, bt^u, ct^w)$ with generic coefficients $a, b, c,\quad abc \neq 0$

$$\left|\frac{\partial f}{\partial x}\big(\varphi(t)\big)\right| \sim ||\varphi(t)||^{\alpha(f)}$$

The idea of the proof

For the monomial curve Γ: $\varphi(t) = (at^v, bt^u, ct^w)$ with generic coefficients $a, b, c, \quad abc \neq 0$

$$\left|\frac{\partial f}{\partial x}(\varphi(t))\right| \sim ||\varphi(t)||^{\alpha(f)}$$

But we need

$$||\nabla f(\varphi(t))|| \sim ||\varphi(t)||^{\alpha(f)},$$

i.e.

$$||\frac{\partial f}{\partial x}(\varphi(t)), \frac{\partial f}{\partial y}(\varphi(t)), \frac{\partial f}{\partial z}(\varphi(t))|| \sim ||\varphi(t)||^{\alpha(f)}$$

The idea of the proof

The problem is to make the remaining partial derivatives $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ small enough on the monomial curve Γ or on its prolongation of the form $(at^v + \cdots, bt^u + \cdots, ct^w + \cdots)$.

The idea of the proof

The problem is to make the remaining partial derivatives $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ small enough on the monomial curve Γ or on its prolongation of the form $(at^v + \cdots, bt^u + \cdots, ct^w + \cdots)$.

It would be optimal to find such a curve for which

$$\frac{\partial f}{\partial y}\big(\varphi(t)\big) \equiv 0, \qquad \frac{\partial f}{\partial z}\big(\varphi(t)\big) \equiv 0$$

The idea of the proof

The problem is to make the remaining partial derivatives $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ small enough on the monomial curve Γ or on its prolongation of the form $(at^v + \cdots, bt^u + \cdots, ct^w + \cdots)$.

It would be optimal to find such a curve for which

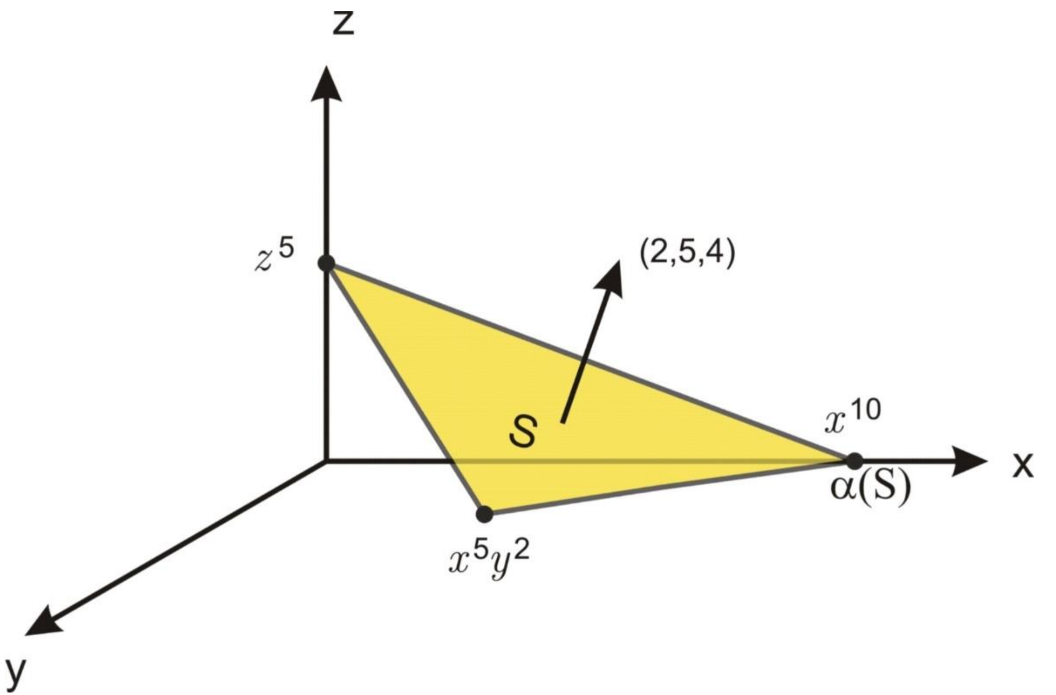
$$\frac{\partial f}{\partial y}\big(\varphi(t)\big) \equiv 0, \qquad \frac{\partial f}{\partial z}\big(\varphi(t)\big) \equiv 0$$

Unfortunately, it is not always possible.

The idea of proof

Example. For the non-exceptional face $f_S = x^{10} + x^5y^2 + z^5$ we have $(v, u, w) = (2{,}5{,}4)\ and$

$$\frac{\partial f}{\partial y} = 2x^5y + \cdots (terms\ of\ higher\ orders)$$



The idea of the proof

It is possible to find such a curve for which

$$\frac{\partial f}{\partial y}(\varphi(t)) \equiv 0, \qquad \frac{\partial f}{\partial z}(\varphi(t)) \equiv 0$$

under some assumptions on f_S.

The idea of the proof

Proposition. If the quasi-homogeneous polynomial f_S associated to the face S satisfies:

1. $\frac{\partial f_S}{\partial y}, \frac{\partial f_S}{\partial z}$ are not monomials,
2. GCD($\frac{\partial f_S}{\partial y}, \frac{\partial f_S}{\partial z}$) is at most monomial,
3. the system of 2 equations in 3 variables

(*)
$$\frac{\partial f_S}{\partial y}(x, y, z) = 0,$$
$$\frac{\partial f_S}{\partial z}(x, y, z) = 0$$

has a solution in $(C^*)^3$

The idea of the proof

then there exists a holomorphic curve $\varphi(t) = (x(t), y(t), z(t))$ such that

1. $\frac{\partial f}{\partial y} \circ \varphi \equiv 0, \frac{\partial f}{\partial z} \circ \varphi \equiv 0,$

The idea of the proof

then there exists a holomorphic curve $\varphi(t) = (x(t), y(t), z(t))$ such that

1. $\frac{\partial f}{\partial y} \circ \varphi \equiv 0, \frac{\partial f}{\partial z} \circ \varphi \equiv 0,$

2. $\varphi(t) = (at^v + \cdots, bt^u + \cdots, ct^w + \cdots)$, $abc \neq 0$, where (v, u, w) is a vector perpendicular to S and a, b, c are a solution of the above system (*),

The idea of the proof

then there exists a holomorphic curve $\varphi(t) = (x(t), y(t), z(t))$ such that

1. $\frac{\partial f}{\partial y} \circ \varphi \equiv 0, \frac{\partial f}{\partial z} \circ \varphi \equiv 0,$

2. $\varphi(t) = (at^v + \cdots, bt^u + \cdots, ct^w + \cdots)$, $abc \neq 0$, where (v, u, w) is a vector perpendicular to S and a, b, c are a solution of the above system (*),

3. $||\nabla f(\varphi(t))|| \sim ||\varphi(t)||^{\alpha(f)}$.

The idea of the proof

In the proof we use the classic:

1. The Bernstein Theorem (1975) on the existence of non-zero solutions of systems of polynomial equations with given Newton polyhedrons, and

The idea of the proof

In the proof we use classic:

1. The Bernstein Theorem (1975) on the existence of non-zero solutions of systems of polynomial equations with given Newton polyhedrons, and
2. The Maurer theorem (1980) on existence of a parametrization with „a given initial part” of an analytic space curve.

The Bernstein Theorem

Theorem. Let $f_1, \dots, f_n \in C[z_1, \dots, z_n]$ be polynomials. If the mixed volume $MV(N(f_1), \dots, N(f_n))$ of the Newton polytopes $N(f_i)$ of f_i is positive and the system $(f_1, \dots, f_n)$ is non-degenerate in the Bernstein sense then the system of equations $f_1 = 0, \dots, f_n = 0$ has exactly $MV(N(f_1), \dots, N(f_n))$ isolated solutions, counted with multiplicities.

The Bernstein Theorem

Theorem. Let $f_1, \dots, f_n \in C[z_1, \dots, z_n]$ be polynomials. If the mixed volume $MV(N(f_1), \dots, N(f_n))$ of the Newton polytopes $N(f_i)$ of f_i is positive and the system $(f_1, \dots, f_n)$ is non-degenerate in the Bernstein sense then the system of equations $f_1 = 0, \dots, f_n = 0$ has exactly $MV(N(f_1), \dots, N(f_n))$ isolated solutions, counted with multiplicities.

Mixed volume for two polynomials $MV(N(f_1), N(f_2))$, considered in the proof, has the simple form

$$MV\big(N(f_1), N(f_2)\big) = \mathrm{vol}_2(N(f_1) + N(f_2)) - \mathrm{vol}_2(N(f_1)) - \mathrm{vol}_2(N(f_2))$$

The Maurer Theorem

Theorem. Let $I \subset C\{x_1, \ldots, x_n\}$ be an ideal with one dimensional zero set and $\boldsymbol{w} = (w_1, \ldots, w_n) \in \boldsymbol{N}^n$ a weight vector of variables $\boldsymbol{x}$. If $\boldsymbol{w}$ is a tropism of I (it means w_i are non-zero positive integers and $in_{\boldsymbol{w}}F$ is not monomial for any $F \in I$) then there exists a parametrization of one irreducible component of $V(I)$ of the form

$$x_1(t) = a_1 t^{kw_1} + \cdots,$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \qquad k \in \boldsymbol{N}$$

$$x_n(t) = a_n t^{kw_n} + \cdots$$

$$a_i \neq 0.$$

The idea of the proof

To check the assumptions of the Maurer Theorem we need the following key algebraic lemma.

The idea of the proof

The key algebraic lemma:

Lemma. R – a unique factorization domain, $F, G \in R[[\boldsymbol{x}]]$ two formal series in n variables, $\boldsymbol{w}$ a weight vector of variables $\boldsymbol{x}$ and $in_{\mathbf{w}}F, in_{\mathbf{w}}G$ the initial parts of F, G with respect to these weights $\boldsymbol{w}$. If:

1. GCD($in_{\mathbf{w}}F, in_{\mathbf{w}}G$) is at most a monomial,
2. $in_{\mathbf{w}}F, in_{\mathbf{w}}G$ do not generate a monomial

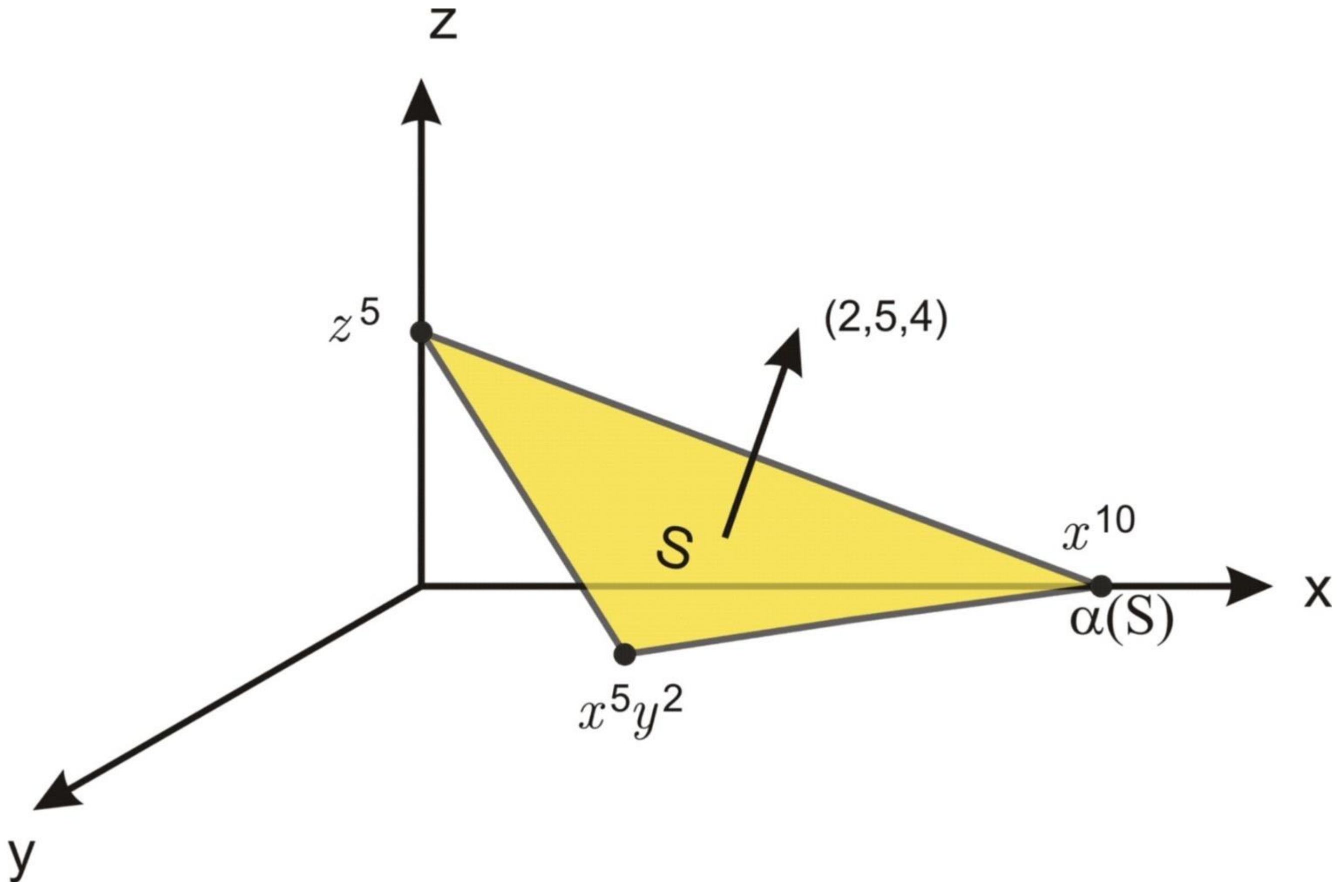
then F, G do not generate an element with $\boldsymbol{w}$-initial part being a monomial.

The idea of the proof

To complete the proof we have to consider non exceptional faces S (realizing maximum intersections with axes) which don’t satisfy conditions in the Proposition.

The idea of the proof

To complete the proof we have to consider non exceptional faces S (realizing maximum intersections with axes) which don't satisfy conditions in the Proposition.

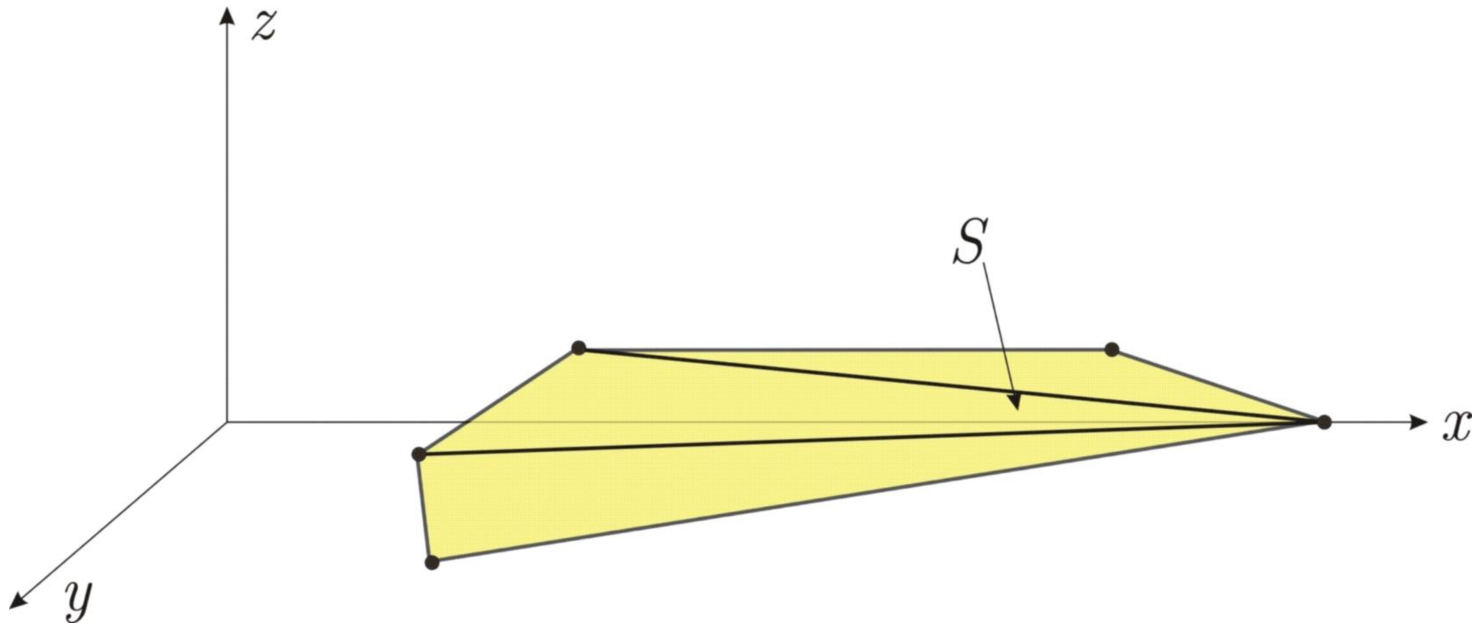


The idea of the proof

This imposes severe restrictions on the geometry of S: S must be a triangle and there is no appropriate curve with initial orders being an integer vector perpendicular to S.
In these cases we must find another appropriate face, often in some sense „hidden” one, which gives the sought curve.

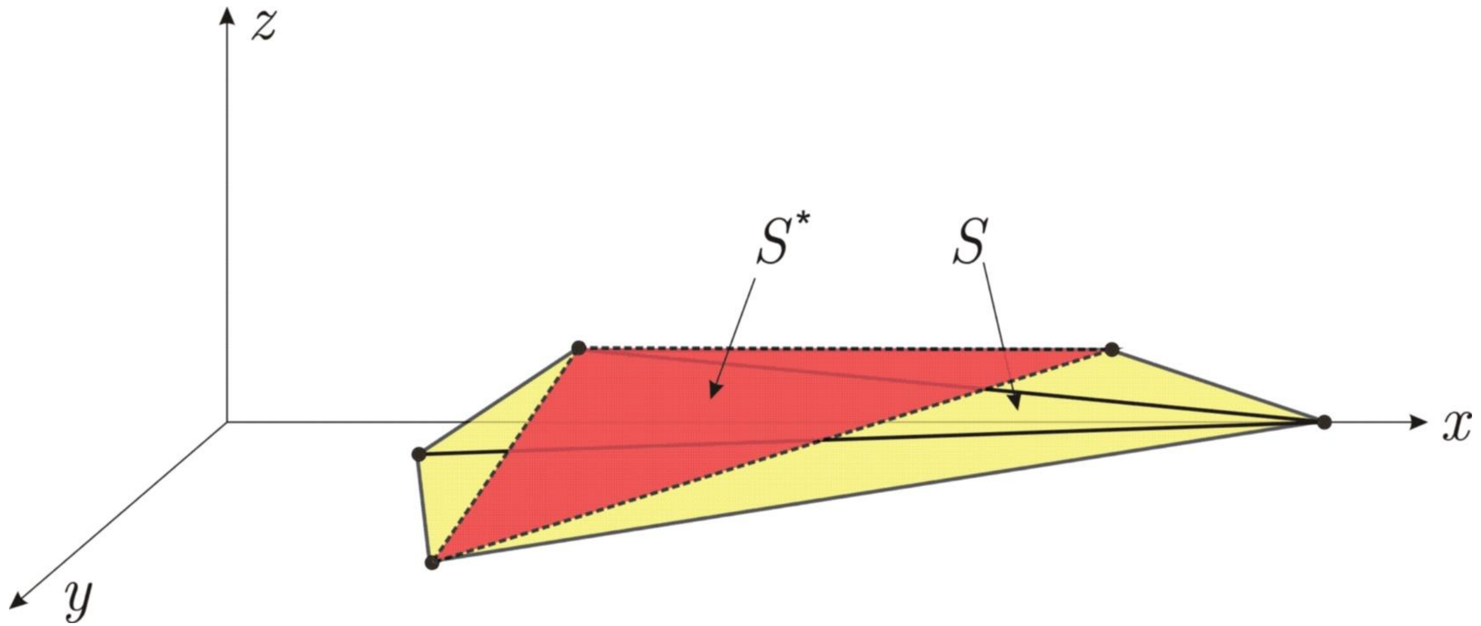
The idea of the proof

One of possible cases



The idea of the proof

One of possible cases



S^* satisfies the assumptions of the Proposition and gives an appropriate curve.

The idea of the proof

We consider all the possible cases. In each one we are able to find a curve on which the gradient of f has the required order.

Application

The solution of a Teissier's problem (question):

Whether the Łojasiewicz exponent is constant in μ-constant families of isolated singularities?

in one particular case.

Application

The solution of a Teissier's problem (question):

Whether the Łojasiewicz exponent is constant in μ-constant families of isolated singularities?

in one particular case.

Theorem. If (f_t) is a non-degenerate μ-constant deformation of an isolated surface singularity f_0 then $\mathcal{L}(f_t)$ =constant.

Problems

1. Generalize the result to n-dimensional case.

Problems

1. **Generalize the result to n-dimensional case.**

2. **Generalize the key algebraic lemma for any system of functions.**

The end

Thank you for your attention.