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CLASSES OF DETERMINANTAL VARIETIES ASSOCIATED WITH SYMMETRIC AND SKEW-SYMMETRIC MATRICES

UDC 513.6

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ABSTRACT. In this paper the authors compute the classes of subschemes of degenerations of a homomorphism of two fibrations in the Chow ring of the base.

Bibliography: 18 titles.

The starting point of enumerative problems connected with investigating singularities is the following situation: for vector bundles E and F over a scheme X and a morphism $\varphi: F \to E$ over X it is necessary to describe the set of singularities of φ; for example, for a fixed natural number q one wants to get information about a subscheme Y of the scheme X, where $\operatorname{rk}\operatorname{Coker}\varphi \geqslant q$. The same problem can be posed if some additional conditions are imposed on φ, for example if $F = E^*$ (the dual bundle) and φ is a symmetric or skew-symmetric morphism.

The class of Y in the Chow ring of the scheme X was calculated under various conditions on φ (in the case of general φ) in papers by Porteous [18], Kempf and Laksov [10], and Lascoux [14]. If X is a Grassmann variety over a field, E its tautological bundle, and F a trivial bundle, then Y is called a *special Schubert variety*. In this case the determinantal formula for its class in the Chow ring was calculated at the start of this century by the Italian mathematician Giambelli [2] long before the rise of cohomological theories that allow one to give it an appropriate interpretation.

In the case where X is a projective space, $E = \mathcal{O}(n_1) + \mathcal{O}(n_2) + \cdots$, and φ is a symmetric or skew-symmetric morphism, Giambelli [3] also got some formulas in terms of symmetric functions, which express the degree of the variety defined by all the minors of φ of a given order.

In this paper we shall prove Giambelli's formulas in complete generality, i.e. under a single assumption on the codimension of Y in X. Using the same technique we present one more (brief) proof of the formula for the class of Y in the case of general φ.

To present the main idea of our proof we consider the (relative) Grassmannian $G_q(E)$ which parametrizes the quotient-bundles of the bundle E of rank q; let $\pi: G_q(E) \to X$ be the corresponding canonical projection. In an arbitrary case (i.e. general, symmetric, and skew-symmetric) we construct a subscheme Z in $G_q(E)$ which under generic conditions is

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birationally isomorphic to Y (via π). Therefore, $\pi_*[Z] = [Y]$ in the Chow ring, where π_*: $\mathcal{A}(G_q(E)) \to \mathcal{A}(X)$ is the Gysin homomorphism. Thus, to calculate the class $[Y]$ we need to know the class $[Z]$ in $\mathcal{A}(G_q(E))$ and the description of the Gysin homomorphism. We get this by using the rich theory of Schur functions—a classical part of mathematics that plays a fundamental role in the theory of representations of symmetric and general linear groups [12], and has recently been reopened and applied in geometry [14], [15].

The general case where Y has the maximal possible codimension in X is gotten from the generic case by using the fact that Y is then a Cohen-Macaulay scheme and using a trick similar to that described by Kempf and Laksov in [10]. All this allows us to express φ as the inverse image of some ψ already defined under generic conditions.

An analogous method, applied to modules instead of Chern classes, has recently led to the construction of all syzygies of determinantal ideals (see [16], [17] and [7]).

Throughout the article, X denotes a smooth quasiprojective scheme over a field.

§1. Segre classes and Schur functions

Let E be a vector bundle over X of rank n and

$$c(E) = 1 + c_1(E) + \cdots + c_n(E)$$

the total Chern class in the Chow ring $\mathcal{A}(X)$. By the splitting principle we can represent $c(E)$ as the product of n factors

$$c(E) = (1 + a)(1 + b)\cdots, \tag{1}$$

where $a, b, \ldots$ are the first Chern classes of some one-dimensional bundles gotten from splitting E. Thus the $c_i(E)$ can be considered as the elementary symmetric functions of $a, b, \ldots$.

If F is another vector bundle over X, then $c(E + F) = c(E)c(F)$. If $c(E)$ is invertible in $\mathcal{A}(X)$, we can put $c(-E) = c(E)^{-1}$, and because of the additivity of the total Chern class on short exact sequences we can consider the c_i as mappings (not homomorphisms) of the Grothendieck ring $K(X)$ of the scheme X into $\mathcal{A}(X)$.

It turns out that the so-called Segre classes $s_i(E)$ are more convenient than the Chern classes. They are defined by the formulas

$$s_i(E) = (-1)^i c_i(-E) = c_i(-E)^*, \qquad i \geqslant 0,$$
$$s_i(E) = 0, \qquad i < 0,$$

where E^* denotes the bundle dual to E; then $s_0(E) = 1$, $s_1(E) = c_1(E)$, and $s_i(E + F) = \sum_{h+k=i} s_h(E)s_k(F)$.

By (1) the complete Segre class

$$s(E) = 1 + s_1(E) + s_2(E) + \cdots$$

can be represented as

$$s(E) = \frac{1}{(1 - a)(1 - b)\cdots}. \tag{2}$$

Thus any $s_i(E)$ is a symmetric polynomial in $a, b, \ldots$, the so-called *total homogeneous symmetric function of degree i* (the sum of all monomials of total degree i).

Let $I = (i_1, \ldots, i_p) \in \mathbf{Z}^p$ be any sequence of integers. A basic role in all our calculations will be played by determinants of the form

$$s(I; E) = \begin{vmatrix} s_{i_1}(E) & s_{i_2}(E) \cdots s_{i_p}(E) \\ s_{i_1-1}(E) & s_{i_2-1}(E) \cdots s_{i_p-1}(E) \\ \vdots & \vdots \quad\quad \vdots \end{vmatrix}.$$

In particular, $s(i; E) = s_i(E)$ and $s(1, \ldots, i; E) = c_i(E)$; we shall use both notations. According to (2), $s(I; E)$ is a symmetric function of $a, b, \ldots$, the so-called *Schur function of index* $i_1, i_2 - 1, i_3 - 2, \ldots$ (a Schur function is usually indexed by the diagonal elements of the matrix defining it; however, in the Chow ring it is more convenient to index it by the first row; for example, see Proposition 1 below).

We recall the representation of the Schur functions according to Jacobi:

$$s(I; E) = \Delta_I(a, b, \ldots)/\Delta_{0,1,2,\ldots}(a, b, \ldots),$$

where

$$\Delta_I(a, b, \ldots) = \begin{vmatrix} a^{i_1} & a^{i_2} \cdots a^{i_p} \\ b^{i_1} & b^{i_2} \cdots b^{i_p} \\ \cdots & \cdots \end{vmatrix}.$$

In particular,

$$\Delta_{0,1,2,\ldots}(a, b, \ldots) = \prod_{a \neq b} (a - b)$$

is the well-known Vandermonde determinant.

The functions $s(I; -)$, just like c_i and s_i, define a mapping of $K(X)$ into $\mathfrak{C}(X)$.

§2. The Gysin homomorphism

Let E be a vector bundle of rank n (abbreviated as rk $E = n$) over a scheme X, $q \leqslant n$, and π: $G_q(E) \to X$ the canonical projection corresponding to the relative Grassmannian parametrizing the quotient bundles of rank q of the bundle E. By definition, on $G_q(E)$ there exists a tautological short exact sequence $0 \to R \to \pi^*E \to Q \to 0$ of vector bundles, where rk $Q = q$ and rk $R = r = n - q$. For $q = 1$, $G_1(E)$ is the projective space over X, and Q is usually denoted by $\mathcal{O}(1)$.

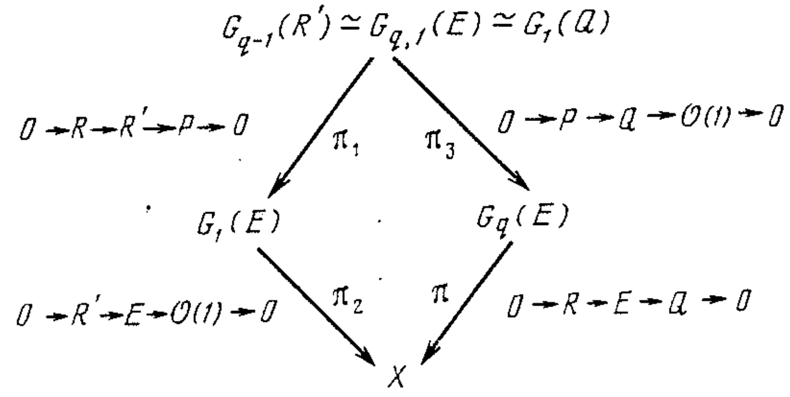
The Chow ring $\mathfrak{C}(G_q(E))$ is an extension of $\mathfrak{C}(X)$ and contains elements of the form $s(H; R)$ and $s(K; Q)$ (they play an important role in describing the structure of $\mathfrak{C}(G_q(E))$ as an $\mathfrak{C}(X)$-module).

PROPOSITION 1. *If $H \in \mathbf{N}^r$ and $K \in \mathbf{N}^q$, then*

$$\pi_*(s(H; R)s(K; Q)) = s(HK; E),$$

where $HK = (h_1, h_2, \ldots, k_1, k_2, \ldots) \in \mathbf{N}^{r+q}$.

Let $G_{q,1}(E)$ be the variety of flags which parametrizes the flags of quotient bundles $E_1 \to E_2$ of the bundle E of ranks q and 1 respectively. In our proof of the proposition, here and in the sequel, a basic role will be played by the following commutative diagram (see [14]):



(3)

It reflects the fact that the variety of flags can be considered as a Grassmannian in two different ways. The short exact sequences arising from the mappings are tautological sequences of the corresponding Grassmannians (instead of π^*E we shall just write E, and the same applies to the other mappings).

Before proving the proposition we recall a well-known lemma.

LEMMA 1. *If* $q = 1$ *and* $\xi = c_1(\mathcal{O}(1))$, *then* $\xi^k = s_k(\mathcal{O}(1))$ *and* $\pi_*(\xi^k) = s_{k-n+1}(E)$.

PROOF OF PROPOSITION 1. We induct on q. If $q = 1$, then we must show that $\pi_*(s(H;\, R)s(k;\, \mathcal{O}(1))) = s(Hk;\, E)$. Using the formula $s_i(R) = s_i(E - \mathcal{O}(1)) = s_i(E) - \xi s_{i-1}(E)$, we get

$$s(H;\, R) = \begin{vmatrix} s_{h_1}(E) & \cdots & s_{h_r}(E) & \xi^r \\ s_{h_1-1}(E) & \cdots & s_{h_r-1}(E) & \xi^{r-1} \\ \vdots & & \vdots & \vdots \\ s_{h_1-r}(E) & \cdots & s_{h_r-r}(E) & 1 \end{vmatrix}.$$

Since $r = n - 1$, by Lemma 1 we have

$$\pi_*(s(H;\, R)\xi^k) = \begin{vmatrix} s_{h_1}(E) & \cdots & s_{h_r}(E) & s_k(E) \\ s_{h_1-1}(E) & \cdots & s_{h_r-1}(E) & s_{k-1}(E) \\ \vdots & & \vdots & \vdots \end{vmatrix} = s(Hk;\, E).$$

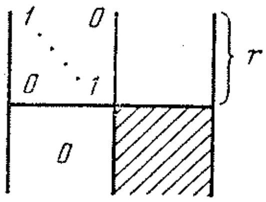
Now let $q > 1$ and $K = K'k$ by the induction hypothesis and the commutative diagram (3) we have

COROLLARY 1. *Let* D *be a vector bundle over* X. *If* $K \in \mathbf{N}^q$, *then*

$$\pi_*(s(K;\, Q - D)) = s(k_1 - r, \ldots, k_q - r;\, E - D).$$

$\pi_*(s(H;R)\, s(K;Q)) = \pi_*(\pi_{3*}(s(H;R)\, s(K';P)\, s(k;\mathcal{O}(1)))) =$
$= \pi_{1*}(\pi_{2*}(s(H;R)\, s(K';P)\, s(k;\mathcal{O}(1))) = \pi_{1*}(s(HK';R')\, s(k,\mathcal{O}(1)))) =$
$= s(HK'k;E) = s(HK;E)$.

PROOF. Since $s(H; R) = 1$ for $H = (0, 1, \ldots, r-1)$, the determinant $s(HK; E)$ has the form:



Consequently, it equals the cross-hatched determinant, with the indexing changed in this way. This proves the corollary for $D = 0$.

If $D \neq 0$, then we use the formula for $s_i(Q - D)$, we represent $s(K; Q - D)$ as a sum of determinants of the form $s(K'; Q)$ with coefficients in $\mathcal{a}(X)$, we apply the part of the corollary already proved for $D = 0$, and we return to determinantal form.

§3. Technical lemmas

In this part we present auxiliary results we shall need in §4.

LEMMA 2. *Let $I = (i_1, \ldots, i_p)$ and let F be a vector bundle of rank $m \leqslant p$. Then for any element D of the Grothendieck ring the determinant $s(I; D)$ does not change when $s_i(D)$ is replaced by $s_i(D - F)$ in the first $p - m$ rows.*

PROOF. We get the desired result by applying step-by-step to any row (from among the first $p - m$) an appropriate linear combination of the following m rows according to the equality

$$s_i(D - F) = s_i(D) - s_{i-1}(D)c_1(F) + \cdots \pm s_{i-m}(D)c_m(F).$$

PROPOSITION 2. *Let E and F be vector bundles of ranks n and m respectively, $m \leqslant n$. Then*

$$c_{mn}(E \otimes F^*) = s(m, \ldots, m+n-1; E - F).$$

PROOF. By the splitting principle we can write

$$c(E) = (1+a)(1+b)\cdots, \qquad c(F) = (1+x)(1+y)\cdots,$$

whence

$$c(E \otimes F^*) = (1+a-x)(1+b-x)\cdots(1+a-y)(1+b-y)\cdots$$

and, in particular,

$$c_{mn}(E \otimes F^*) = (a-x)(b-x)\cdots(a-y)(b-y)\cdots.$$

We first prove that $(a-x)(a-y)\cdots$ ~~lies in~~ divides $s(m, \ldots, m+n-1; E-F)$. Let $E = A + E'$ in the Grothendieck ring, where $\operatorname{rk} A = 1$ and $c(A) = 1 + a$. By Lemma 2, in the first row of the determinant $s(m, \ldots, m+n-1; E-F)$ we can replace $s_i(E-F)$ by

$$s_i(E - F - E') = s_i(A - F);$$

the first row now looks like this:

$$s_m(A-F), s_{m+1}(A-F), \ldots, s_{m+n-1}(A-F).$$

However,

$$s_{m+p}(A-F) = a^{m+p} - a^{m+p-1}c_1(F) + \cdots \pm a^p c_m(F) = a^p s_m(A-F)$$

for $p \geqslant 0$; this means that $s(m,\ldots,m+n-1;\,E-F)$ ~~lies in~~ is divided by $s_m(A-F)$, which equals $(a-x)(a-y)\cdots$.

By symmetry, $s(m,\ldots,m+n-1;\,E-F)$ ~~lies in~~ divides $c_{mn}(E\otimes F^*)$. Both elements are polynomials of the same degree in $a, b,\ldots$ and $x, y,\ldots$. Thus they are equal, since they take the same value, $(-1)^{mn}c_m(F)^n$, when $a = b = \cdots = 0$.

If $\operatorname{rk} E = n$, then $c_n(E)$ is called the *maximal Chern class* of the bundle E, and is sometimes denoted by $c_{\max}(E)$. The relative Grassmannian $\pi\colon G_q(E) \to X$ can be compared with the element $e(\pi) = c_{\max}(R\otimes Q)$. In the following lemmas we compute the images of $e(\pi)$ and other similar expressions under the homomorphism π_* in the projective case, i.e. for $q = 1$.

LEMMA 3. *If $q = 1$, then*

$$e(\pi) = \sum_{i=0}^{n-1} s(0,1,\ldots,\hat{i},\ldots,n-1;\,R)\xi^i,$$

where $\xi = c_1(\mathcal{O}(1))$ and $\operatorname{rk} E = n$.

PROOF. By Proposition 2,

$$e(\pi) = s(1,2,\ldots,n-1;\,R-\mathcal{O}(1)^*) = c_{n-1}(R-\mathcal{O}(1)^*).$$

Using the formula of linearity, we get

$$e(\pi) = \sum_{i=0}^{n-1} s(1,2,\ldots,n-i-1;\,R)\xi^i.$$

Since $s(0,1,\ldots,i-1;\,R) = 1$, the determinant $s(1,2,\ldots,n-i-1;\,R)$ equals $s(0,1,\ldots,\hat{i},\ldots,n-1;\,R)$, which proves the lemma.

COROLLARY 2. *For any $k \geqslant 0$*

$$\pi_*(e(\pi)\xi^k) = \sum_{i=0}^{n-1} s(1,2,\ldots,n-i-1,k;\,E).$$

PROOF. From Proposition 1 and Lemma 3 it follows that

$$\pi_*(e(\pi)\xi^k) = \sum_{i=0}^{n-1} \pi_*\big(s(0,1,\ldots,\hat{i},\ldots,n-1;\,R)\xi^{i+k}\big)$$

$$= \sum_{i=0}^{n-1} s(0,1,\ldots,\hat{i},\ldots,n-1,i+k;\,E).$$

Since $s(0,1,\ldots,i,\ldots,i-1;\,E) = 1$, we get the desired result.

COROLLARY 3.

$$\pi_*(e(\pi)) = \begin{cases} 1 & \text{if } n \text{ is odd},\\ 0 & \text{if } n \text{ is even}.\end{cases}$$

LEMMA 4. *If $q = 1$, then*

$$\pi_*(e(\pi)s(1,3,\ldots,2k-3;\,R)\xi^k) = s(1,3,\ldots,2k-1;\,E).$$

PROOF. We have

$$e(\pi)s(1,3,\ldots,2k-3;R)\xi^k = \left|\begin{array}{ccccc|c} s_1(R) & s_3(R) & \cdots & & s_{2k-3}(R) & 0 \\ s_0(R) & s_2(R) & \cdots & & s_{2k-4}(R) & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ & & \cdots & s_{k-3}(R) & s_{k-1}(R) & 0 \\ \hline & & \cdots & s_{k-4}(R) & s_{k-2}(R) & e(\pi)\xi^k \end{array}\right|.$$

Using the formula $s_i(E) = s_i(R) + s_{i-1}(E)\xi$, we transform this determinant into the form

$$\left|\begin{array}{cccc|c} s_1(E) & s_3(E) & \cdots & s_{2k-3}(E) & e(\pi)\xi^{2k-1} \\ s_0(E) & s_2(E) & \cdots & s_{2k-4}(E) & e(\pi)\xi^{2k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ & & \cdots & s_{k-1}(E) & e(\pi)\xi^{k+1} \\ \hline & & \cdots & s_{k-2}(E) & e(\pi)\xi^k \end{array}\right|, \tag{4}$$

by adding step-by-step to a fixed row an appropriate multiple of the preceding and then passing on to the next row. Because of Corollary 2 we can compute

$$\pi_*(e(\pi)s(1,3,\ldots,2k-3;R)\xi^k)$$

by replacing $e(\pi)\xi^j$ by $\sum_{i=0}^{n-1} s(1,2,\ldots,n-i-1,j;E)$ in the last column of (4). However, by Pieri's formula (which more generally expresses the product of any Schur function and s_p as a sum of Schur functions)

$$\begin{aligned} s_p(E)s(1,2,\ldots,2j;E) &= s(1,2,\ldots,2j-1,p+2j;E) \\ &\quad + s(1,2,\ldots,2j;p+2j;E). \end{aligned}$$

Therefore, multiplying the $(k-j)$th column of our matrix by

$$-s(1,2,\ldots,2j;E), \qquad j = 1,\ldots,k-1,$$

and by adding up to the last column, at the last step we get $s(1,3,\ldots,2k-1;E)$.

§4. The main computations

We recall that X denotes a smooth quasiprojective scheme over a field.

Let $\varphi\colon F \to E$ be a morphism of vector bundles over X, and $\pi\colon G_q(E) \to X$ the relative Grassmannian corresponding to the number q. We define a subscheme Z of $G_q(E)$ as the scheme of zeros of the composite $F \xrightarrow{\varphi} E \to Q$, where $0 \to R \to E \to Q \to 0$ is the tautological sequence on $G_q(E)$.

PROPOSITION 3. *Let $\varphi\colon F \to E$ be a morphism of vector bundles over X, $\operatorname{rk} E = n$ and $\operatorname{rk} F = m$, $m \geq n$. Moreover, let Y be a subscheme of X, where $\operatorname{rk}\operatorname{Coker}\varphi \geq q$, $q \leq n$; that is*

$$\mathcal{O}_Y = \operatorname{Coker}(\Lambda^{n-q+1}F \otimes \Lambda^{n-q+1}E^* \to \mathcal{O}_X).$$

Then the following assertions are true:

a) *If Y is nonempty, then its codimension in X is at most $q(m-n+q)$.*

b) *If π induces a birational isomorphism of Z and Y and if* $\operatorname{codim}_X Y = q(m-n+q)$, *then*

$$[Y] = s(m-n+q,\dots,m-n+2q-1;\ E-F)$$

in the Chow ring of the scheme X.

PROOF. Part a) follows from (4). Since Z is birationally isomorphic to Y, we have $\pi_*[Z] = [Y]$ in the Chow ring of X. However, Z is the scheme of zeros of a section $G_q(E) \to F^* \otimes Q$ induced by the sequence $F \to E \to Q$ and $\operatorname{rk} F^* \otimes Q = \operatorname{codim}_{G_q(E)} Z$; hence $[Z] = c_{\max}(F^* \otimes Q)$. From Proposition 2 and Corollary 1 we now get the desired result.

Let $\varphi\colon E^* \to E$ be a symmetric morphism of vector bundles over X, i.e. $\varphi^* = \varphi$. We denote by Z, just as before, the subscheme of zeros of the morphism $E^* \xrightarrow{\varphi} E \to Q$ or, equivalently, of the induced section $G_q(E) \xrightarrow{\alpha} E \otimes Q$. Since φ is symmetric, α is a section of the bundle $H = \operatorname{Ker}(E \otimes Q \to \Lambda^2(Q))$.

PROPOSITION 4. *Let $\varphi\colon E^* \to E$ be a symmetric morphism of vector bundles over X, and Y a subscheme of X, where* $\operatorname{rk}\operatorname{Coker}\varphi \geq q$, $q \leq n = \operatorname{rk} E$; *that is*

$$\mathcal{O}_Y = \operatorname{Coker}\left(\Lambda^{n-q+1}E^* \otimes \Lambda^{n-q+1}E^* \to \mathcal{O}_X\right).$$

Then the following assertions are true:

a) *If Y is nonempty, then its codimension in X is at most $q(q+1)/2$.*

b) *If π induces a birational isomorphism of Z and Y and if* $\operatorname{codim}_X Y = q(q+1)/2$, *then*

$$[Y] = 2^q s(1,3,\dots,2q-1;\ E)$$

in the Chow ring of the scheme X.

PROOF. Part a) follows from [13] (see also [5]). Just as in the proof of Proposition 3, we get $\pi_*[Z] = [Y]$ and $[Z] = c_{\max}(H)$ as long as $\operatorname{rk} H = \operatorname{codim}_{G_q(E)} Z$. In the Grothendieck ring we have $E = R + Q$, so $H = R \otimes Q + S^2(Q)$; hence

$$c_{\max}(H) = c_{\max}(R \otimes Q)\, c_{\max}(S^2(Q)).$$

By the splitting principle a formal computation enables us to find $c_{\max}(S^2(Q))$. If $c(Q) = \prod(1+a)$, then

$$c(S^2(Q)) = \prod(1+2a) \prod_{a \neq b}(1+a+b).$$

Therefore, from Jacobi's formula (see §1) we get

$$\begin{aligned} c_{\max}(S^2(Q)) &= 2^q c_q(Q) \prod_{a \neq b}(a+b) = 2^q c_q(Q) \frac{\prod_{a\neq b}(a^2-b^2)}{\prod_{a\neq b}(a-b)} \\ &= 2^q c_q(Q) \frac{\Delta_{0,2,\dots,2q-2}(a,b,\dots)}{\Delta_{0,1,\dots,q-1}(a,b,\dots)} = 2^q c_q(Q) s(0,2,\dots,2q-2;\ Q) \\ &= 2^q s(1,3,\dots,2q-1;\ Q). \end{aligned}$$

To compute $\pi_*(c_{\max}(H))$, recall that in §3 we wrote $e(\pi) = c_{\max}(R \otimes Q)$. Let $e = e(\pi)$ and $e_i = e(\pi_i)$, $i = 1,2,3$, for the other mappings in the diagram (3). It is easy to check that $e_1e_2 = ee_3$ in $\mathcal{A}(G_1(Q))$; indeed, both products equal

$$c_{\max}(P \otimes \mathcal{O}(1) + R \otimes P + R \otimes \mathcal{O}(1)).$$

We must show that

$$\pi_*(es(1,3,\ldots,2q-1;Q)) = s(1,3,\ldots,2q-1;E). \tag{5}$$

We shall show this by induction on q. For $q = 1$ formula (5) follows from Corollary 2. For $q > 1$ we use diagram (3). According to Lemma 4,

$$\pi_{3*}(e_3 s(1,3,\ldots,2q-3;P)\xi^q) = s(1,3,\ldots,2q-1;Q).$$

Therefore

$$\begin{aligned}\pi_*(es(1,3,\ldots,2q-1;Q)) &= \pi_*\big(\pi_{3*}ee_3 s(1,3,\ldots,2q-3;P)\xi^q\big)\\ &= \pi_{2_*}\big(\pi_{1_*}e_1e_2 s(1,3,\ldots,2q-3;P)\xi^q\big).\end{aligned}$$

By the induction hypothesis

$$\pi_{1_*}(e_1 s(1,3,\ldots,2q-3;P)) = s(1,3,\ldots,2q-3;R')$$

and in the last computation

$$\begin{aligned}\pi_*(es(1,3,\ldots,2q-1;Q)) &= \pi_{2_*}(e_2 s(1,3,\ldots,2q-3;R')\xi^q)\\ &= s(1,3,\ldots,2q-1;E)\end{aligned}$$

again because of Lemma 4.

To formulate the following result we shall assume that X is a scheme over a field of characteristic zero, and denote by $\varphi: E^* \to E$ a skew-symmetric morphism of vector bundles over X, i.e. $\varphi^* = -\varphi$. Let $n = \operatorname{rk} E$ and $2 \leqslant q \leqslant n$, where $n - q = 2p$ is even. We describe the subscheme Y in X defined by the Pfaffians of φ of order $2p+2$. Under the isomorphism $\operatorname{Hom}(E^*, E) \simeq E \otimes E$ the element φ corresponds to $f \in \Lambda^2 E$ (since φ is skew-symmetric). Then $f^{p+1} \in S^{p+1}(\Lambda^2 E)$; we denote by $\bar{f}^{p+1}$ the image of f^{p+1} under the natural mapping $S^{p+1}(\Lambda^2 E) \to \Lambda^{2p+2}E$ sending $(x_1 \wedge y_1)\cdots(x_{p+1} \wedge y_{p+1})$ to $x_1 \wedge y_1 \wedge \cdots \wedge x_{p+1} \wedge y_{p+1}$. For a fixed basis of E and the dual basis of E^*, the coefficients of $\bar{f}^{p+1}$ in the canonical basis of $\Lambda^{2p+2}E$ are (up to a nonzero scalar) the Pfaffians of order $2p+2$ of the matrix of φ. We define $\mathcal{O}_Y$ as $\operatorname{Coker}(\Lambda^{2p+2}E \to \mathcal{O}_X)$. From [1] it follows that Y consists of the points of the scheme X for which $\operatorname{rk}\operatorname{Coker}\varphi \geqslant q$.

If Z is defined as before, then the corresponding section $\alpha: G_q(E) \to E \otimes Q$ factors through $H' = \operatorname{Ker} E \otimes Q \to S^2(Q)$, since φ is skew-symmetric.

PROPOSITION 5. *Let X be a smooth quasiprojective scheme over a field of characteristic zero, $\varphi: E^* \to E$ a skew-symmetric morphism of vector bundles over X, Y the subscheme of X defined by the Pfaffians of order $2p+2$ of the mapping φ,* $\operatorname{rk} E = n$ *and $q = n - 2p$. Then the following assertions are true*:

a) *If Y is nonempty, then its codimension in X is at most $q(q-1)/2$.*

b) *If π induces a birational isomorphism of Z and Y and if* $\operatorname{codim}_X Y = q(q-1)/2$, *then* $[Y] = s(1,3,\ldots,2q-3;E)$ *in the Chow ring of the scheme X.*

PROOF. Part a) follows from [6]. The proof of part b) is similar to the corresponding proof of Proposition 4. By assumption we have $[Z] = c_{\max}(H')$. Since $H' = R \otimes Q + \Lambda^2 Q$

in the Grothendieck ring, we find that

$$c_{\max}(H') = c_{\max}(R \otimes Q)c_{\max}(\Lambda^2 Q).$$

Proceeding just as in the proof of Proposition 4 and using Corollary 3, we get the desired result.

§5. Passage to the generic case

We first recall one of the tricks of [10]. In the notation of Proposition 3, consider the Grassmannian $\mathbf{X} = G_n(F \oplus E)$ and its corresponding tautological sequence $0 \to T \to F \oplus E \to P \to 0$ on X. Define $\psi: F \xrightarrow{1+\varphi} F \oplus E \to P$ and $\mathbf{Y}$ as the subscheme of $\mathbf{X}$ where $\operatorname{rk}\operatorname{Coker}\psi \geqslant q$. We have a commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow\eta & & \downarrow\eta \\ \mathbf{Y} & \hookrightarrow & \mathbf{X} \end{array}$$

where η is induced by the exact sequence $0 \to F \to F \oplus E \to E \to 0$ on X. Since $\eta^*\psi = \varphi$, we find that $\eta^{-1}\mathbf{Y} = Y$. Applying Lemma 9 from [10] and noting that $\mathbf{Y}$ is a Cohen-Macaulay scheme and $\operatorname{codim}_{\mathbf{X}}\mathbf{Y} = q(m-n+q)$ (see [4]), we get the following result.

LEMMA 5. *In the notation of Proposition* 3, *if* $\operatorname{codim}_X Y = q(m-n+q)$, *then* $\eta^*[\mathbf{Y}] = [Y]$.

Now let $\varphi: E^* \to E$ be a symmetric morphism. Consider the relative Grassmannian $\mathbf{X} = G_n(E^* \oplus E)$ and its corresponding tautological sequence $0 \to T \to E^* \oplus E \xrightarrow{\rho} P \to 0$. Define a symmetric morphism

$$\psi: P^* \xrightarrow{\rho^*} E \oplus E^* \xrightarrow{\left(\begin{smallmatrix}0 & 1\\ 1 & \varphi\end{smallmatrix}\right)} E^* \oplus E \xrightarrow{\rho} P$$

and the subscheme $\mathbf{Y}$ of the points of $\mathbf{X}$ where $\operatorname{rk}\operatorname{Coker}\psi \geqslant q$. The sequence $0 \to E^* \to E^* \oplus E \to E \to 0$ on X induces an embedding $\eta: X \to \mathbf{X}$ such that the inverse image η^* transforms the exact sequence $0 \to T \to E^* \oplus E \to P \to 0$ into $0 \to E^* \to E^* \oplus E \to E \to 0$. Hence $\eta^*\psi = \varphi$ and $\eta^{-1}\mathbf{Y} = Y$. Because of [13], $\mathbf{Y}$ is a Cohen-Macaulay scheme and $\operatorname{codim}_{\mathbf{X}}\mathbf{Y} = q(q+1)/2$, so that, applying Lemma 9 from [10], we get the following result.

LEMMA 6. *In the notation of Proposition* 4, *if* $\operatorname{codim}_X Y = q(q+1)/2$, *then* $\eta^*[\mathbf{Y}] = [Y]$.

The same argument works for a skew-symmetric morphism. The fact that the corresponding scheme $\mathbf{Y}$ defined by the Pfaffians is a Cohen-Macaulay scheme and $\operatorname{codim}_{\mathbf{X}}\mathbf{Y} = q(q-1)/2$ follows from [11].

LEMMA 7. *In the notation and under the hypotheses of Proposition* 5, *if* $\operatorname{codim}_X Y = q(q-1)/2$, *then* $\eta^*[\mathbf{Y}] = [Y]$.

§6. Concluding results

THEOREM 1 (KEMPF-LAKSOV [10]). *Maintaining the notation of Proposition* 3, *let* $\operatorname{codim}_X Y = q(m-n+q)$. *Then*

$$[Y] = s(m-n+q, \ldots, m-n+2q-1; E-F)$$

in the Chow ring of the scheme X.

PROOF. We apply Proposition 3 to the scheme **X** and the morphism $\psi: F \to P$ (notation of §5). Since in the commutative diagram

$$\begin{array}{ccc} \mathbf{Z} & \hookrightarrow & G_q(P) \\ \downarrow\pi & & \downarrow\pi \\ \mathbf{Y} & \hookrightarrow & \mathbf{X} \end{array}$$

π establishes a birational isomorphism of **Z** and **Y** (because of [8] and [9], or by direct calculation) and $\operatorname{codim}_{\mathbf{X}} \mathbf{Y} = q(m-n+q)$, we find that

$$[\mathbf{Y}] = s(m-n+q,\ldots,m-n+2q-1;\, P-F)$$

in the Chow ring of **X**. The assertion of the theorem then follows from Lemma 5.

THEOREM 2. *Let X be a smooth quasiprojective scheme over a field of characteristic* $\neq 2$, $\varphi: E^* \to E$ *a symmetric morphism over X, and maintain all the other notation of Proposition* 4. *If* $\operatorname{codim}_X Y = q(q+1)/2$, *then* $[Y] = 2^q s(1,3,\ldots,2q-1;\, E)$ *in the Chow ring of the scheme X.*

PROOF. The hypotheses of Proposition 4 are fulfilled by the scheme **X** and the morphism $\psi: P^* \to P$ (see the notation of §5), since $\pi: G_q(P) \to \mathbf{X}$ establishes a birational isomorphism of the corresponding **Z** and **Y**. This follows from the arguments of [9], p. 234, which hold for fields of characteristic $\neq 2$. Thus

$$[\mathbf{Y}] = 2^q s(1,3,\ldots,2q-1;\, P),$$

and Lemma 6 completes the proof.

Similar arguments, using Proposition 5 and Lemma 7, give the result for skew-symmetric morphisms.

THEOREM 3. *Let X be a smooth quasiprojective scheme over a field of characteristic zero,* $\varphi: E^* \to E$ *a skew-symmetric morphism over X, and maintain all the other notation of Proposition* 4. *If* $\operatorname{codim}_X Y = q(q-1)/2$, *then* $[Y] = s(1,3,\ldots,2q-3;\, E)$ *in the Chow ring of the scheme X.*

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BIBLIOGRAPHY

1. David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), 447–485.

2. Giovanni Z. Giambelli, *Risoluzione del problema degli spazi secanti*, Mem. R. Accad. Sci. Torino (2) **52** (1902), 171–211.

3. ______, *Sulla varietà rappresentate coll'annullare determinanti minori contenuti in un determinante simmetrico od emisimmetrico generico di forme*, Atti. R. Accad. Sci. Torino **41** (1905/06), 102–125.

4. M. Hochster and John A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020-1058.

5. Tadeusz Józefiak, *Ideals generated by minors of a symmetric matrix*, Comment. Math. Helv. **53** (1978), 595–607.

6. Tadeusz Józefiak and Piotr Pragacz, *Ideals generated by Pfaffians*, J. Algebra **61** (1979), 189–198.

7. ______, *Syzygies de pfaffiens*, C. R. Acad. Sci. Paris Sér. A-B **287** (1978), A89-A91.

8. George R. Kempf, *On the geometry of a theorem of Riemann*, Ann. of Math. (2) **98** (1973), 178–185.

9. ______, *On the collapsing of homogeneous bundles*, Invent. Math. **37** (1976), 229–239.

10. G. R. Kempf and D. Laksov, *The determinantal formula of Schubert calculus*, Acta Math. **132** (1974), 153–162.

11.* H. Kleppe and D. Laksov, a) *The algebraic structure and deformation of Pfaffian schemes*, J. Algebra **64** (1980), 167–189.

b) *The generic perfectness of determinantal schemes*, Algebraic Geometry (Proc. Summer Meeting, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer-Verlag, 1979.

12. Donald Knutson, *λ-rings and the representation theory of the symmetric group*, Lecture Notes in Math., vol. 308, Springer-Verlag, 1973.

13. Ronald E. Kutz, *Cohen-Macaulay rings and ideal theory in rings of invariants of algebraic groups*, Trans. Amer. Math. Soc. **194** (1974), 115–129.

14. Alain Lascoux, *Puissances extérieures, déterminants et cycles de Schubert*, Bull. Soc. Math. France **102** (1974), 161–179.

15. ______, *Calcul de Schur et extension grassmanniennes des λ-anneaux*, Combinatoire et Représentation du Groupe Symétrique (Actes Table Ronde CNRS, Strasbourg, 1976), Lecture Notes in Math., vol. 579, Springer-Verlag, 1977, pp. 182–216.

16. ______, *Syzygies des variétés déterminantales*, Advances in Math. **30** (1978), 202–237.

17. ______, *Syzygies pour les mineurs de matrices symétriques*, Preprint, 1977.

18. I. R. Porteous, *Simple singularities of maps*, Proc. Liverpool Singularities Sympos. I (1969/70), Lecture Notes in Math., vol. 192, Springer-Verlag, 1971, pp. 286–307.

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* *Editor's note*. The Russian combines the title of a) with the volume reference for b). **)

** Author's note. Not Russian but Polish - French - Polish.