Effective Whitney theorem for complex polynomial mappings of the plane, IMPANGA 2021

M. Farnik, Z. Jelonek, M.A.S. Ruas

January 29, 2021

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2. To the best knowledge of the authors complex algebraic families of polynomial mappings on affine varieties have not been investigated so far.
3. Here we describe an idea of such study. We consider the family $\Omega_{\mathbb{C}^n}(d_1, \ldots, d_m)$ of polynomial mappings $F = (F_1, \ldots, F_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ of degree bounded by $(d_1, \ldots, d_m)$.
4. For a smooth affine variety $X^k \subset \mathbb{C}^n$ we also consider the family $\Omega_X(d_1, \ldots, d_m) = \{F|_X : F \in \Omega_{\mathbb{C}^n}(d_1, \ldots, d_m)\}$.

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1. If M, X, Y are affine irreducible varieties, X, Y are smooth and $\Phi\colon M \times X \to Y$ is an algebraic family of polynomial mappings such that the generic element of this family is proper then two generic members of this family are topologically equivalent (Jel 2017).
2. In particular if $X \subset \mathbb{C}^p$ is of dimension n and $m \geq n$ then any two generic members of the family $\Omega_X(d_1, \ldots, d_m)$ are topologically equivalent.
3. For example, if X is a smooth surface then the numbers $c_X(d_1, d_2)$ and $d_X(d_1, d_2)$ of cusps and double folds, respectively, of a generic member of the family $\Omega_X(d_1, d_2)$ are well-defined.

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Our aim is to describe effectively the topology of such generic mappings. We consider in this paper the simplest case, when $n = m = 2$ and $X = \mathbb{C}^2$ or X is the complex sphere $S = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 = 1\}$. In those cases we describe the topology of the set $C(F)$ of critical points of F and the topology of its discriminant $\Delta(F)$.

1. Let $\Omega_n(d_1,\ldots,d_m)$ denote the space of polynomial mappings $F:\mathbb{C}^n\to\mathbb{C}^m$ of multi-degree bounded by $d_1,\ldots,d_m$.
2. Similarly if $X\subset\mathbb{C}^n$ is a smooth affine variety we consider the family $\Omega_X(d_1,\ldots,d_m)=\{F|_X : F\in\Omega_n(d_1,\ldots,d_m)\}$. Note that $\Omega_X(d_1,\ldots,d_m)$ as algebraic variety coincides with $\Omega_n(d_1,\ldots,d_m)$.
3. By $J^q(\mathbb{C}^n,\mathbb{C}^m)$ we denote the space of q-jets of polynomial mappings $F=(f_1,\ldots,f_m):\mathbb{C}^n\to\mathbb{C}^m$.
4. If $X^n\subset\mathbb{C}^p$ is a smooth affine variety then the space $J^q(X,\mathbb{C}^m)$ has the structure of a smooth algebraic manifold and can be locally represented in the same simple way as above.

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1. By ${}_sJ^q(X,\mathbb{C}^m)$ we denote the space of multi q-jets of polynomial mappings $F=(f_1,\dots,f_n):X\to\mathbb{C}^m$.
2. We denote by Δ the set $\{(x_1,\dots x_s)\in X^s\ :\ x_i=x_j$ for some $i\neq j\}$ and for bundles $\pi_i:W_i\to X$ we denote by Δ_X the set $\{(w_1,\dots w_s)\ :\ \pi_i(w_i)=\pi_j(w_j)$ for some $i\neq j\}$.
3. We have ${}_sJ^q(X,\mathbb{C}^m)=(J^q(X,\mathbb{C}^m))^s\setminus\Delta_X$. More generally, we define the space of $(q_1,\dots,q_s)$-jets to be $J^{q_1,\dots,q_s}(X,\mathbb{C}^m):=J^{q_1}(X,\mathbb{C}^m)\times\dots\times J^{q_s}(X,\mathbb{C}^m)\setminus\Delta_X$ and call it, if there is no danger of confusion, the space of multi-jets.
4. Again, for a given polynomial mapping $F:X\to\mathbb{C}^m$ we have the mapping

$$J^{q_1,\dots,q_s}(F):X^s\backslash\Delta\mapsto(j^{q_1}(F)(x_1),\dots,j^{q_s}(F)(x_s))\in J^{q_1,\dots,q_s}(X,\mathbb{C}^m).$$

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Thom-Boardman singularities.

Let $F \in \Omega_n(d_1, ..., d_n)$ be one generic. Then $\Sigma^r(F) := \{x : \mathrm{corank}\ d_x F = r\}$ is smooth and we can consider the set $\Sigma^{r,s}(F)$ where tha map $F : \Sigma^r(F) \to \mathbb{C}^n$ drops rank s. If $\Sigma^{r,s}(F)$ is smooth we can continuing. In particular for $n = 2$ we have that $\Sigma^1(F)$ is the set of folds and $\Sigma^{1,1}$ is the set of cusps. In fact we have the following Boardman Theorem:

THEOREM. For every sequence of integers $r_1 \geq r_2 \geq ... \geq r_s \geq 0$ one can define a smooth algebraic subvariety $\Sigma^{r_1, r_2, \ldots, r_s}$ of $J^s(\mathbb{C}^n, \mathbb{C}^n)$ such that if $j^l(F)$ is transversal to all submanifolds $\Sigma^{t_1, \ldots, t_l}$ with $l < s$, then $\Sigma^{r_1, \ldots, r_s}(F)$ is well defined and

$$x \in \Sigma^{r_1, \ldots, r_s}(F) \text{ iff } j^s F(x) \in \Sigma^{r_1, \ldots, r_s}.$$

Of course this is true for arbitrary smooth manifolds. We say that the varieties $\Sigma^{t_1, \cdots t_l}$ are Thom-Boardmann strata in jet space.

1. We will also use the Thom-Boardman manifolds in the space ${}_sJ^k(X,\mathbb{C}^m)$ of multi-jets. We denote by $\delta_{\mathbb{C}^m}$ the set of all multijets $\{(w_1,\ldots,w_s) \in {}_sJ^k(X,\mathbb{C}^m)$: for all $1 \leq i,j \leq s \ : \ \pi_{\mathbb{C}^m}(w_i) = \pi_{\mathbb{C}^m}(w_j)\}$, where $\pi_{\mathbb{C}^m} : J^k(X,\mathbb{C}^m) \to \mathbb{C}^m$ is the projection.
2. We denote $(\Sigma^{I_1},\ldots,\Sigma^{I_s}) := \Sigma^{I_1} \times \ldots \times \Sigma^{I_s} \cap {}_sJ^k(X,\mathbb{C}^m)$. Moreover let $(\Sigma^{I_1},\ldots,\Sigma^{I_s})_\Delta := (\Sigma^{I_1},\ldots \times \Sigma^{I_s}) \cap \delta_{\mathbb{C}^m}$.

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1. Let $x=(x_1,...,x_s)\in X^s$, let U be a n open neighborhood of x and $f:U\to Y$ be a holomorphic mapping. Put

$$z={}_s j^k(f), y=(f(x_1),...,f(x_s)).$$

   Let ${}_sJ^k(X,Y)_x$ and ${}_sJ^k(X,Y)_{x,y}$ denote fibers of ${}_sJ^k(X,Y)$ over x and (x,y) respectively.

2. Then we have canonical identifications:

$$(*) T({}_sJ^k(X,Y)_x)_z = J^k(f^*TY)_x,$$

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Let $\mathfrak{m}_x$ denotes the ideal in $J^k(X)$ consisting of jets of functions which vanish at x. Then with respect to $(*)$ we have identification:

$$(**) T({}_sJ^k(X,Y)_{x,y})z = \mathfrak{m}_x J^k(f^*TY)_x.$$

In particular $T({}_sJ^k(X,Y)_{x,y})z$ has a structure of $J^k(X)_x$ module.

Let W be a non-void submanifold of the multi-jet bundle ${}_sJ^k(X,Y)$. We say that W is modular if:

1. W is a smooth invariant submanifold of ${}_sJ^k(X,Y)$.
2. the space $T(W_{x,y})_z$ under identification $(**)$ is a $J^k(X)_x$ submodule of $\mathfrak{m}_x J^k(f^*TY)_x$.

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Let us state the following result of Mather:

Theorem. Let $X \subset \mathbb{C}^n$ be a smooth affine algebraic subvariety and let $W \subset_s J^q(X, \mathbb{C}^m)$ be a modular submanifold. There exists a Zariski open non-empty subset U in the space of all linear mappings $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ such that for every $L \in U$ the mapping $L : X \to \mathbb{C}^m$ is transversal W.

This theorem has the following nice application (which in the real smooth case was first observed by S. Ichiki):

Corollary. Let $X \subset \mathbb{C}^n$ be an affine smooth algebraic subvariety, let $W \subset_s J^q(X, \mathbb{C}^m)$ be a modular submanifold and let $F : X \to \mathbb{C}^m$ be a polynomial mapping. There exists a Zariski open non-empty subset U in the space of all linear mappings $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ such that for every $L \in U$ the mapping $F + L : X \to \mathbb{C}^m$ is transversal to W.

Now we can state the following fundamental result (version of Mather's Theorem):

Theorem 1. Let $X^k \subset \mathbb{C}^n$ be a smooth algebraic variety of dimension k and let $W \subset {}_sJ^q(X, \mathbb{C}^m)$ be an algebraic modular submanifold. Then there is a Zariski open subset $U \subset \Omega_X(d_1, \ldots, d_m)$ such that for every $F \in U$ the mapping F is transversal to W. In particular it holds, if we take as W the Thom-Boardman manifolds $(\Sigma^{I_1}, \ldots, \Sigma^{I_s})$ and $(\Sigma^{I_1}, \ldots, \Sigma^{I_s})_\Delta$ in ${}_sJ^q(X, \mathbb{C}^m)$. Consequently, every mapping $F \in U$ satisfies the normal crossings condition, hence it is a Thom-Boardman mapping with a Normal Crossings Property.

DEFINITION. Let $F \in \Omega_2(d_1, d_2)$. We say that F is generic if F is proper, $j^1(F) \pitchfork \Sigma^1, j^2(F) \pitchfork \Sigma^{1,1}$, and additionally $j^1(F) \pitchfork \Sigma^2$.

Again by Theorem 1 the subset of generic mappings contains a Zariski open dense subset of $\Omega_2(d_1, d_2)$. Thus a general mapping is generic.

DEFINITION. Let $F : (\mathbb{C}^2, a) \to (\mathbb{C}^2, F(a))$ be a holomorphic mapping. We say that F has a simple cusp at a if F is biholomorphically equivalent to the mapping $(\mathbb{C}^2, 0) \ni (x, y) \mapsto (x, y^3 + xy) \in (\mathbb{C}^2, 0)$. It has a fold at a if F is biholomorphically equivalent to the mapping $(\mathbb{C}^2, 0) \ni (x, y) \mapsto (x, y^2) \in (\mathbb{C}^2, 0)$.

By our previous consideration we have:

THEOREM. Let $X \subset \mathbb{C}^n$ be a smooth affine surface and let $F : X \to \mathbb{C}^2$ be a generic polynomial mapping. Then F has only folds and simple cusps (and two-folds) as singularities.

THEOREM A For a general polynomial mapping $F=(f,g):\mathbb{C}^2\to\mathbb{C}^2$, $\deg f=d_1$, $\deg g=d_2$, the set $C(F)$ of critical points of F is a smooth connected curve which is transversal to the line at infinity. The curve $C(F)$ is topologically equivalent to a sphere with $\frac{(d_1+d_2-3)(d_1+d_2-4)}{2}$ handles and d_1+d_2-2 points removed.
The discriminant $\Delta(F)=F(C(F))$ of the mapping F is a curve birationally equivalent to $C(F)$ and it has only cusps and nodes as singularities. The curve $\Delta(F)$ has

$$c(F)=d_1^2+d_2^2+3d_1d_2-6d_1-6d_2+7$$

simple cusps and

$$d(F)=\frac{1}{2}\left[(d_1d_2-4)((d_1+d_2-2)^2-2)-(d-5)(d_1+d_2-2)-6\right]$$

nodes (here $d=\gcd(d_1,d_2)$).

Remark If $d_1 = d_2 = d$ then the discriminant has $2d - 2$ smooth points at infinity and at each of these points it is tangent to the line L_∞ (at infinity) with multiplicity d. If $d_1 > d_2$ then the discriminant has only one point at infinity with $d_1 + d_2 - 2$ branches $V_1, \ldots, V_{d_1+d_2-2}$ and each of these branches has delta invariant

$$\delta(V_i) = \frac{(d_1 - 1)(d_1 - d_2 - 1) + (\gcd(d_1, d_2) - 1)}{2}$$

and $V_i \cdot L_\infty = d_1$. Additionally $V_i \cdot V_j = d_1(d_1 - d_2)$. In particular the branches V_i are smooth if and only if $d_1 = d_2$ or $d_1 = d_2 + 1$.

If $S = \{(x,y,z) \in \mathbb{C}^3 : x^2+y^2+z^2 = 1\}$, then we have:

THEOREM B There is a Zariski open, dense subset $U \subset \Omega_S(d_1,d_2)$ such that for every mapping $F \in U$ the set $C(F)$ of critical points of F is a smooth connected curve, which is topologically equivalent to a sphere with $g = (d_1+d_2-2)^2$ handles and $2(d_1+d_2-1)$ points removed.
For every mapping $F \in U$ the discriminant $\Delta(F) = F(C(F))$ has only cusps and nodes as singularities. The number of cusps is equal to

$$c(F) = 2(d_1^2 + d_2^2 + 3d_1d_2 - 3d_1 - 3d_2 + 1)$$

and the number of nodes is equal to

$$d(F) = (2d_1d_2-3)D^2 - D(d_1+d_2+d-2) - 2(d_1d_2-d_1-d_2),$$

where $D = d_1+d_2-1$ and $d = \gcd(d_1,d_2)$.

Remark If $d_1 = d_2 = d$ then the discriminant has $4d-2$ smooth points at infinity and in each of these points it is tangent to the line L_∞ (at infinity) with multiplicity d. If $d_1 > d_2$ then the discriminant has only one point at infinity with $2(d_1+d_2-1)$ branches $V_1, \ldots, V_{2(d_1+d_2-1)}$ and each of these branches has delta invariant

$$\delta(V_i) = \frac{(d_1-1)(d_1-d_2-1)+(d-1)}{2}$$

and $V_i \cdot L_\infty = d_1$. Additionally $V_i \cdot V_j = d_1(d_1-d_2)$. In particular branches V_i are smooth if and only if $d_1 = d_2$ or $d_1 = d_2+1$.

How to prove Theorem A?

1. For a mapping $F=(f,g)\in\Omega_2(d_1,d_2)$, we have

$$j^1(F)=\left(x,y,f(x,y),g(x,y),\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y),\frac{\partial g}{\partial x}(x,y),\frac{\partial g}{\partial y}(x,y)\right)$$

2. The set Σ^1 is given by the equation $\phi(x,y,f,g,f_x,f_y,g_x,g_y)=f_xg_y-f_yg_x=0.$
3. Now we would like to describe the set $\Sigma^{1,1}$ effectively. In the space $J^2(\mathbb{C}^2,\mathbb{C}^2)$ we introduce coordinates

$$(x,y,f,g,f_x,f_y,g_x,g_y,f_{xx},f_{yy},f_{xy},g_{xx},g_{yy},g_{xy}).$$

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1. The set $\Sigma^{1,1}$ is given in $J^2(\mathbb{C}^2, \mathbb{C}^2)$ by three equations:

$$L_1 := f_x g_y - f_y g_x = 0,$$

$$L_2 := (f_{xx}g_y + f_x g_{xy} - f_{xy}g_x - f_y g_{xx})f_y - (f_{xy}g_y + f_x g_{yy} - f_{yy}g_x - f_y g_{xy})f_x = 0,$$

and

$$L_3 := (f_{xx}g_y + f_x g_{xy} - f_{xy}g_x - f_y g_{xx})g_y - (f_{xy}g_y + f_x g_{yy} - f_{yy}g_x - f_y g_{xy})g_x = 0.$$

2. As above by symmetry the set $\Sigma^{1,1}$ is smooth and locally is given as a complete intersection of either L_1, L_2 or L_1, L_3. We will denote by $J, J_{1,1}, J_{1,2}$ curves given by $L_1 \circ j^2(F) = 0$, $L_2 \circ j^2(F) = 0$ and $L_3 \circ j^2(F) = 0$, respectively.

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Now we show how to compute the genus of $C(F)$ and the number of cusps of a general polynomial mapping $F \in \Omega_2(d_1, d_2)$. To do this we need a series of lemmas:

LEMMA. Let L_∞ denote the line at infinity of $\mathbb{C}^2$. There is a non-empty open subset $V \subset \Omega_2(d_1, d_2)$ such that for all $(f, g) \in V$:
$\left\{\frac{\partial f}{\partial x} = 0\right\} \pitchfork \left\{\frac{\partial f}{\partial y} = 0\right\}$, $\overline{\left\{\frac{\partial f}{\partial x} = 0\right\}} \cap \overline{\left\{\frac{\partial f}{\partial y} = 0\right\}} \cap L_\infty = \emptyset$.

LEMMA. Let L_∞ denote the line at infinity of $\mathbb{C}^2$. There is a non-empty open subset $V \subset \Omega_2(d_1, d_2)$ such that for all $F = (f, g) \in V$:

1. $\overline{J(F)} \cap \overline{J_{1,1}(F)} \cap L_\infty = \emptyset$,
2. $\overline{J(F)} \pitchfork L_\infty$.

Here $\overline{J(F)}$ denotes the projective closure of the set $\{J(F) = 0\}$ etc.

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LEMMA. There is a non-empty open subset $V_1 \subset \Omega_2(d_1, d_2)$ such that for all $(f, g) \in V_1$ and every $a \in \mathbb{C}^2$: if $\frac{\partial f}{\partial x}(a) = 0$ and $\frac{\partial f}{\partial y}(a) = 0$, then $\frac{\partial g}{\partial x}(a) \neq 0$ and $\frac{\partial g}{\partial y}(a) \neq 0$.

LEMMA. There is a non-empty open subset $V_2 \subset \Omega_2(d_1, d_2)$ such that for all $(f, g) \in V_2$ we have $\{\frac{\partial f}{\partial x} = 0\} \cap \{\frac{\partial f}{\partial y} = 0\} \cap J_{1,2}(f, g) = \emptyset$.

LEMMA. There is a non-empty open subset $V_3 \subset \Omega_2(d_1, d_2)$ such that for all $(f, g) \in V_3$ the curve $J(f, g)$ is transversal to the curve $J_{1,1}(f, g)$.

THEOREM. There is a Zariski open, dense subset $U \subset \Omega_2(d_1, d_2)$ such that for every mapping $F \in U$ the mapping F has only two-folds and cusps as singularities and the number of cusps is equal to

$$d_1^2 + d_2^2 + 3d_1d_2 - 6d_1 - 6d_2 + 7.$$

Moreover, if $d_1 > 1$ or $d_2 > 1$ then the set $C(F)$ of critical points of F is a smooth connected curve, which is topologically equivalent to a sphere with $g = \frac{(d_1+d_2-3)(d_1+d_2-4)}{2}$ handles and $d_1 + d_2 - 2$ points removed.

Here we analyze the discriminant of a general mapping from $\Omega(d_1, d_2)$. Let us recall that the discriminant of the mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ is the curve $\Delta(F) := F(C(F))$, where $C(F)$ is the critical curve of F. We have:

LEMMA. There is a non-empty open subset $U \subset \Omega_2(d_1, d_2)$ such that for every mapping $F \in U$:
1) $F_{|C(F)}$ is injective outside a finite set,
2) if $p \in \Delta(F)$ then $|F^{-1}(p) \cap C(F)| \leq 2$, 3) if $|F^{-1}(p) \cap C(F)| = 2$ then the curve $\Delta(F)$ has a normal crossing at p.

Hence for a general F the only singularities of $\Delta(F)$ are cusps and nodes. We showed previously that there are exactly $c(F) = d_1^2 + d_2^2 + 3d_1d_2 - 6d_1 - 6d_2 + 7$ cusps. Now we will compute the number $d(F)$ of nodes of $\Delta(F)$. We will use the following theorem of Serre:

THEOREM. If Γ is an irreducible curve of degree d and genus g in the complex projective plane then

$$\frac{1}{2}(d-1)(d-2) = g + \sum_{z \in \mathrm{Sing}(\Gamma)} \delta_z,$$

where δ_z denotes the delta invariant of a point z.

LEMMA. Let $F=(f,g)\in\Omega(d_1,d_2)$ be a general mapping. If $d_1\geq d_2$ then $\deg\Delta(F)=d_1(d_1+d_2-2)$.

We have the following method of computing the delta invariant:

THEOREM M. (Milnor) Let $V_0 \subset \mathbb{C}^2$ be an irreducible germ of an analytic curve with the Puiseux parametrization of the form

$$z_1 = t^{a_0},\ z_2 = \sum_{i>0} \lambda_i t^{a_i},\ \text{where } \lambda_i \neq 0,\ a_1 < a_2 < a_3 < \dots$$

Let $D_j = \gcd(a_0, a_1, \dots, a_{j-1})$. Then

$$\delta_0 = \frac{1}{2} \sum_{j \geq 1} (a_j - 1)(D_j - D_{j+1}).$$

If $V = \bigcup_{i=1}^{r} V_i$ has r branches then

$$\delta(V) = \sum_{i=1}^{r} \delta(V_i) + \sum_{i<j} V_i \cdot V_j,$$

where $V \cdot W$ denotes the intersection product.

Our result follows directly from:

THEOREM. Let $F \in \Omega(d_1, d_2)$ be a general mapping. Let $d_1 \geq d_2$ and $d = \gcd(d_1, d_2)$. Denote by $\overline{\Delta}$ the projective closure of the discriminant Δ. Then

$$\sum_{z \in (\overline{\Delta} \setminus \Delta)} \delta_z = \frac{1}{2} d_1 (d_1 - d_2)(d_1 + d_2 - 2)^2 + \frac{1}{2}(-2d_1 + d_2 + d)(d_1 + d_2 - 2).$$

How to prove? It is a little bit tedious but possible!

THEOREM. There is a Zariski open, dense subset $U \subset \Omega_2(d_1, d_2)$ such that for every mapping $F \in U$ the discriminant $\Delta(F) = F(C(F))$ has only cusps and nodes as singularities. Let $d = \gcd(d_1, d_2)$. Then the number of cusps is equal to

$$c(F) = d_1^2 + d_2^2 + 3d_1d_2 - 6d_1 - 6d_2 + 7$$

and the number of nodes is equal to

$$d(F) = \frac{1}{2}\left[(d_1d_2 - 4)((d_1 + d_2 - 2)^2 - 2) - (d - 5)(d_1 + d_2 - 2) - 6\right].$$

In fact also the converse result is true (J+Farnik-submited):

THEOREM. For $d_1 d_2 > 2$, if a mapping $F \in \Omega_2(d_1, d_2)$ has $c(d_1, d_2)$ cusps and $n(d_1, d_2)$ nodes, then it has a generic topological type. In particular, mappings with the generic topological type form a Zariski open subset in $\Omega_2(d_1, d_2)$.

COROLLARY. Let $F, G \in \Omega_2(d_1, d_2)$, where $d_1 d_2 > 2$. Assume that F and G have $c(d_1, d_2)$ cusps and $n(d_1, d_2)$ nodes. Then there exist homeomorphisms $\Phi, \Psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$G = \Phi \circ F \circ \Psi.$$

DEFINITION. Let $F : (\mathbb{C}^2, a) \to (\mathbb{C}^2, F(a))$ be a holomorphic mapping. We say that F has a generalized cusp at a if F_a is proper, the curve $J(F) = 0$ is reduced near a and the discriminant of F_a is not smooth at $F(a)$.

REMARK. If F_a is proper, $J(F) = 0$ is reduced near a and $J(F)$ is singular at a then it follows from Corollary 1.11 from [Jel, 2017] that also the discriminant of F_a is singular at $F(a)$ and hence F has a generalized cusp at a.

DEFINITION. Let $F = (f, g) : (\mathbb{C}^2, a) \to (\mathbb{C}^2, F(a))$ be a holomorphic mapping. Assume that F has a generalized cusp at a point $a \in \mathbb{C}^2$. Since the curve $J(F) = 0$ is reduced near a, we have that the set $\{\nabla f = 0\} \cap \{\nabla g = 0\}$ has only isolated points near a. For a general linear mapping $T \in GL(2)$, if $F' = (f', g') = T \circ F$ then $\nabla f'$ does not vanish identically on any branch of $\{J(F) = 0\}$ near a. We say that the cusp of F at a has an index $\mu_a := \dim_{\mathbb{C}} \mathcal{O}_a/(J(F'), J_{1,1}(F')) - \dim_{\mathbb{C}} \mathcal{O}_a/(f'_x, f'_y)$.

REMARK. Using the exact sequence 1.7 from [Gaffney-Mond] we see that

$$\mu_a = \dim_{\mathbb{C}} \mathcal{O}_a/(J(F), J_{1,1}(F), J_{1,2}(F)).$$

Hence our index coincides with the classical local number of cusps defined e.g. in [Gaffny-Mond]. In particular $\mu_a \geq 1$, if F has a generalized cusp at a.

PROPOSITION. Let $F = (f, g) \in \Omega_2(d_1, d_2)$ and assume that F has a generalized cusp at $a \in \mathbb{C}^2$. If U_a is a sufficiently small ball around a then μ_a is equal to the number of simple cusps in U_a of a general mapping $F' \in \Omega_2(d_1', d_2')$, where $d_1' \geq d_1, d_2' \geq d_2$, which is sufficiently close to F in the natural topology of $\Omega_2(d_1', d_2')$.

COROLLARY 1. Let $F \in \Omega_2(d_1, d_2)$. Assume that F has generalized cusps at points $a_1, \ldots, a_r$. Then $\sum_{i=1}^{r} \mu_{a_i} \leq d_1^2 + d_2^2 + 3d_1d_2 - 6d_1 - 6d_2 + 7$.

COROLLARY 2. If $F \in \Omega(d_1, d_2)$ is a generically finite polynomial mapping with reduced critical curve, then it has not more than $d_1^2 + d_2^2 + 3d_1d_2 - 6d_1 - 6d_2 + 7$ singular points which are not folds.

Analogous theorems are true in the case of a complex sphere.

In previous sections we considered the family $\Omega_X(d_1, \ldots, d_m)$, of course we can consider also other families of polynomial mappings and try to investigate their properties. Let $\mathcal{F}$ be any algebraic family of generically-finite polynomial mappings $f_p : X \to \mathbb{C}^m;\ p \in \mathcal{F}$, where X is a smooth irreducible affine variety. We would like to know the behavior of proper mappings in a such family. In general proper mappings do not form an algebraic subset of $\mathcal{F}$ but only constructible one. However we show that there is some regular behavior in such family. We have:

Theorem.

Let P, X, Y be smooth irreducible affine algebraic varieties and let $F : P \times X \to P \times Y$ be a generically finite mapping. The mapping F induces a family $\mathcal{F} = \{f_p(\cdot) = F(p, \cdot),\ p \in P\}$. Then either there exists a Zariski open dense subset $U \subset P$ such that for every $p \in P$ the mapping f_p is proper, or there exists a Zariski open dense subset $V \subset P$ such that for every $p \in P$ the mapping f_p is not proper.

Moreover, in the first case we have:

a) for every non-proper mapping f_p in the family $\mathcal{F}$ we have $\mu(f_p) < \mu(F)$, where $\mu(f)$ denotes the geometric degree of f,

b) generic mappings in $\mathcal{F}$ are topologically equivalent, i.e., there exists a Zariski open dense subset $W \subset P$ such that for every $p, q \in W$ the mappings f_p and f_q are topologically equivalent.

Theorem.

Let $X \subset \mathbb{C}^n$ be a smooth irreducible affine variety of dimension k and let $F : X \to \mathbb{C}^m$ be a polynomial mapping. If $m \geq k$, then there exists a Zariski open dense subset U in the space of linear mappings $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ such that:

a) for every $L \in U$ the mapping $F + L$ is a finite mapping.

b) for all $L \in U$ the mappings $F + L$ are topologically equivalent.

c) for all $L \in U$ the mappings $F + L$ have only generic singularities,i.e., transversal to Thom-Boardman strata.

In particular for a given mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ we can consider the "linear" deformation $F_L = F + L;\ L \in \mathcal{L}(\mathbb{C}^2, \mathbb{C}^2)$. A general member of this deformation is locally stable and proper. If F is not "sufficiently generic", then this deformation gives a different number of cusps and folds than a "generic" deformation considered in this paper. We give here an example of a finitely $\mathcal{K}$ determined germ F which has at least two non-equivalent stable deformations.

Example. Take a finitely $\mathcal{K}$ determined germ $F(x, y) = (x, y^3)$ and consider two deformations of F: the first one linear $F_t = (x, y^3 + ty)$ and the second one given by $G_t(x, y) = (x, y^3 + txy)$. The members of the first family do not have a cusp at all and the members of the second family have exactly one cusp at 0.

This means that (contrary to the case of $\mathcal{A}$ finitely determined germs) we can not define the numbers $c(F)$ and $d(F)$ for F using stable deformations.

THANK YOU FOR ATTENTION!