Mirror symmetry of Calabi-Yau manifolds fibered by (1,8) polarized abelian surfaces

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based on a work with Hiromichi Takagi
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1. Mirror symmetry of CY 3-folds (summary)

X : Calabi-Yau 3fold $\iff$ smooth proj. var. with

$$K_X \simeq \mathcal{O}_X, H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$$

A-str. of X

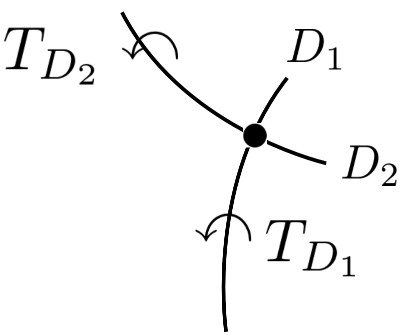
$$H^{even} = \bigoplus_{i=0}^{3} H^{i,i}(X,\mathbb{C}) \quad \curvearrowleft \; L_\kappa := \kappa \wedge -$$

$$\begin{array}{lcccccc}
H^{0,0} & \bullet & & & & \\
\oplus & \downarrow & & & & \\
H^{1,1} & \bullet & \bullet & & \bullet & \\
\oplus & \downarrow & \downarrow & \cdots & \downarrow L_\kappa & \\
H^{2,2} & \bullet & \bullet & & \bullet & \\
\oplus & \downarrow & & & & \\
H^{3,3} & \bullet & & & &
\end{array}$$

$$\left(\begin{array}{l} \text{If } \kappa \in H^{1,1}(X,\mathbb{Q}), L_\kappa \text{acts on} \\ H^{even}_{\mathbb{Q}} = H^{even} \cap H(X,\mathbb{Q}) \end{array} \right)$$

B-str. of X

$$\begin{array}{l} \mathfrak{X} \\ \downarrow^{\pi} \\ \mathcal{M}_X \end{array} \qquad R^3\pi_*\mathbb{C}_{\mathfrak{X}} \text{ over } \mathcal{M}_X$$



$$H^3(X_{b_0},\mathbb{Q}) \quad \curvearrowleft \; N_\lambda = \sum_i \lambda_i \log T_{Di}$$

$$\begin{array}{ccccccccc}
W_0 & \subset & W_2 & \subset & W_4 & \subset & W_6 & = H^3_{\mathbb{Q}} \\
\bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \\
& & \bullet & \leftarrow & \bullet & & & \\
& & \vdots & & \vdots & & & \\
& & \bullet & \leftarrow & \bullet & & &
\end{array}$$

Mirror symmetry ver 1.

For CY3 X, $\exists$ a family $\begin{array}{c}\mathfrak{X}^\vee\\ \downarrow\\ \mathcal{M}\end{array}$ of CY3 $X^\vee$ and a special boundary point $o \in \overline{\mathcal{M}}$ s.t.

$$(H^{even}(X,\mathbb{Q}), L_\kappa) \simeq (H^3(X^\vee_{b_0},\mathbb{Q}), N_\lambda)$$

Mirror symmetry ver 2. There is an isomorphism s.t.

Quantum cohomorogy of X $\simeq$ Griffiths-Yukawa coupling of the family

$$Y_{ijk} = \int_X h_i h_j h_k + (\text{quantum corrections}) \qquad C_{ijk} = -\int_{X^\vee} \Omega(X^\vee_x) \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} \Omega(X^\vee_x)$$

Gromov-Witten invariants

$\sim$ rational curves on X

Mirror symmetry ver 3. (Bershadsky,Cecotti,Ooguri,Vafa [BCOV],'93)

Higher genus Gromow-Witten theory $\leftrightarrow$ Weil-Peterson Kähler geometry on $\mathcal{M}$

Mirror symmetry ver 4. (Homological mirror symmetry, Kontsevish '94)

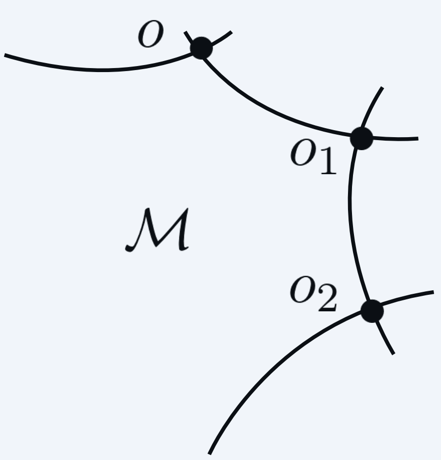
$$D^b(X) \simeq DFuk(X^\vee, \beta)$$

MS ver 1 (ver 2, ver 3) and MS ver 4 are related in an interesting way as follws:

Observation:

Studying mirror family globally over $\mathcal{M}$, we often observe $o_1, o_2, ...$ in $\mathcal{M}$ s.t.

$$X_0 := X, X_1, X_2, ... \quad \overset{ms}{\longleftrightarrow}$$



two cases:

a) $X_i \overset{\sim}{\dashrightarrow} X$

b) $X_i \not\overset{\sim}{\dashrightarrow} X$ but $D^b(X_i) \simeq D^b(X)$

(X_i is a Fourer-Mukai partner of X)

Approach to study MS and geometry of CY manifods:

1. Fix a (or select an interesting) CY3 X

2. Construct mirror family $\begin{array}{c}\mathfrak{X}^\vee\\ \downarrow \\ \mathcal{M}\end{array}$ s.t. A-str. of $X \simeq B$-str. of $X^\vee$.

3. Study $\mathcal{M}$ globally, and find birational models $X_i \overset{\sim}{\dashrightarrow} X$ and/or FM partners of X.

4. Find interesting implications.

Today, I will start with X

X : CY3 fibered by (1,8)-polarized abelian surfaces with

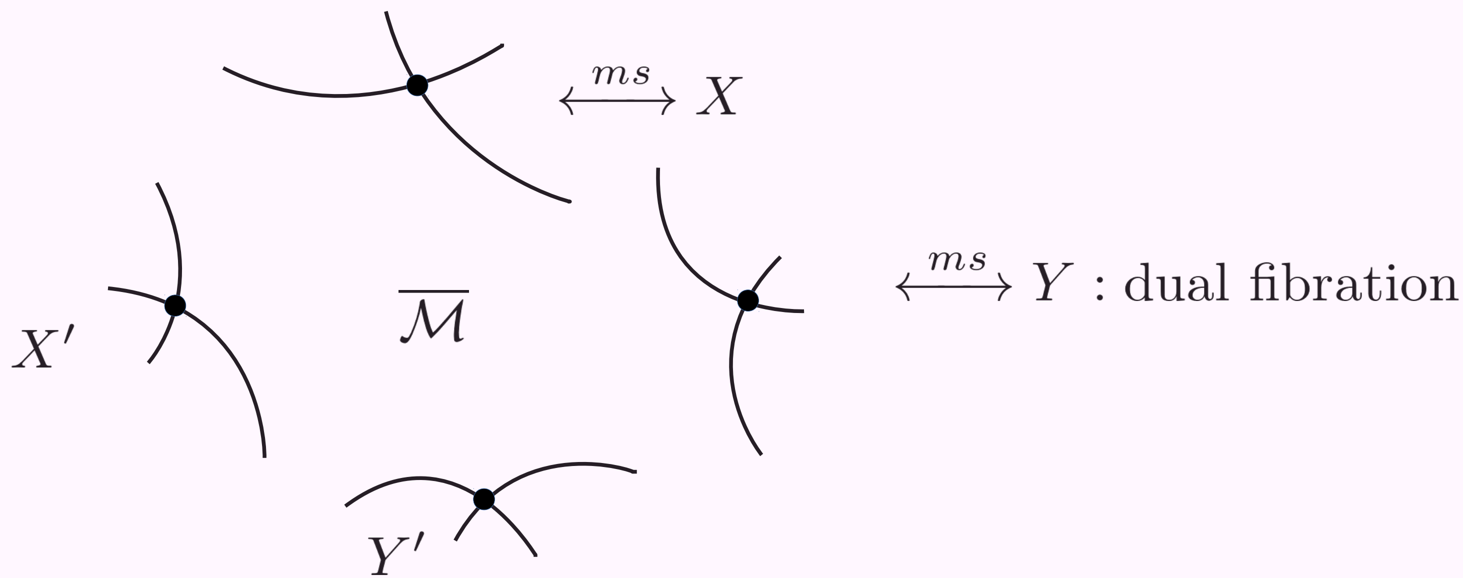
$$\begin{array}{ccccccccc} & & h^{0,0} & & & & & 1 & & & \\ & & h^{1,1} & & & & & 2 & & & \\ h^{3,0} & h^{2,1} & & h^{1,2} & h^{0,3} & = & 1 & 2 & 2 & 1 \\ & & h^{2,2} & & & & & 2 & & & \\ & & h^{3,3} & & & & & 1 & & & \end{array}$$

self-mirror?

Results:

1. We construct a mirror family $\begin{array}{c}\mathfrak{X}^\vee \\ \downarrow \\ \mathcal{M}\end{array}$ over $\mathcal{M}$, with $\overline{\mathcal{M}} = \mathbb{P}_\Delta$ of dim $= 2$.

2. We find boundary points (degeneration points) in $\overline{\mathcal{M}}$



3. Gromov-Witten invariants (g=0,1,2)

$$F_g^X = \sum_n Z_{g,n}^X(q)p^n, \qquad F_g^Y = \sum_n Z_{g,n}^Y(q)p^n$$

$Z_{g,n}^X(q)$, $Z_{g,n}^Y(q)$ are given by quasi-modular forms!

2. CY3 fibered by (1,8)-polarized abelian surfaces

(2-1) Generality of abelian surfaces

$(A, \mathcal{L})$: $(1, d)$ polarized abelian surface

$\Phi_{|\mathcal{L}|} : A \to \mathbb{P}^{d-1}$ is an embedding if $d \geq 5$

- The equations of $\mathrm{Im}(\Phi_{|\mathcal{L}|})$ were studied by Gross and Popescu ('98)

  $d \geq 10 \cdots$ generated by quadrics

  $6 \leq d < 10 \cdots$ never be generated by quadrics,

  contains equations for pencils of abelian surfaces

Brief summary and notation:

· $\Phi_{|\mathcal{L}|}$ is defined by theta functions

· $\phi_{\mathcal{L}} : A \to \hat{A}$, $x \to t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$, define $K(\mathcal{L}) := \mathrm{Ker}\phi_{\mathcal{L}}$

then, $K(\mathcal{L}) \simeq \mathbb{Z}_d \times \mathbb{Z}_d$ and $K(\mathcal{L}) \curvearrowright |\mathcal{L}| = \mathbb{P}(H^0(A, \mathcal{L}))$

not linear but proj. rep.

· $\mathcal{H}_d$: Heisenberg group, $1 \to \mu_d \to \mathcal{H}_d \to K(\mathcal{L}) \to 1$,

$$K(\mathcal{L}) \curvearrowright |\mathcal{L}| \text{ defines Schödinger rep. of } \mathcal{H}_d \begin{cases} \sigma : x_i \mapsto x_{i-1} \\ \tau : x_i \mapsto \xi^{-i} x_i \quad (\xi = e^{\frac{2\pi\sqrt{-1}}{8}}) \end{cases}$$

· additional requirement: $\mathcal{L}$ is symmetric

(2-2) Calabi-Yau 3-fold from a pencil of (1,8)-polarized abelian surfaces

- Four equations f_1, f_2, f_3, f_4 of $(2,2,2,2) \subset \mathbb{P}^7 = \mathbb{P}(H^0(A, \mathcal{L}))$

$$\frac{w_0}{2}(x_0^2 + x_4^2) + w_1(x_1x_7 + x_3x_5) + w_2x_2x_6, \quad \frac{w_0}{2}(x_2^2 + x_6^2) + w_1(x_3x_1 + x_5x_7) + w_2x_4x_0,$$
$$\frac{w_0}{2}(x_1^2 + x_5^2) + w_1(x_2x_0 + x_4x_6) + w_2x_3x_7, \quad \frac{w_0}{2}(x_3^2 + x_7^2) + w_1(x_4x_2 + x_6x_0) + w_2x_5x_1.$$

Def. $I_w := \langle f_1, f_2, f_3, f_4 \rangle$: the homogeneous ideal generated by these quadrics.

Proposition ([GP]). Let$(A, \mathcal{L})$: $(1,8)$-polarized abelian surface

(1) $I(\Phi_{|\mathcal{L}|}(A)) = I_{w(A)} + (3 \times 3$ minors of a 4×4 matrix $M_4)$

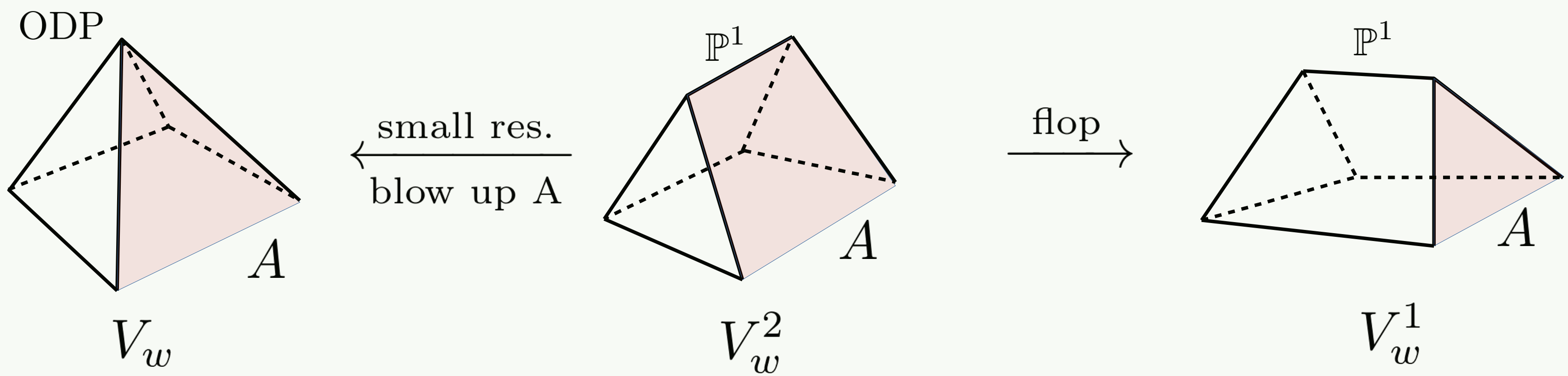
(2) $V(I_{w(A)}) =: V_w \subset \mathbb{P}^7$ defines a pencil of (1,8) abelian surfaces with

(i) 64 base points

(ii) singular exactly at these 64 points with each being ODP

Proposition ([GP,'01]).

(1)There exist small resolutions of V_w;



(2)V_w^1 has a fibration by (1,8)-polarized abelian surfaces with 64 sections.

(3)The Heisenberg group $\mathcal{H}_8$ (may be considered as $\mathbb{Z}_8 \times \mathbb{Z}_8$) acts freely on V_w^1, acting on each fiber as the group $K(\mathcal{L})$. The 64 sections make a single orbit under this action.

(4) There are 8 singular fibers, each of which is an elliptic translation scroll.

(5) $\mathrm{Pic}(V_w^1) = \mathbb{Z}[A] + \mathbb{Z}[H]$ and

$$H^3 = H^2 A = 16, \ HA^2 = A^3 = 0$$

$\cdots$ more details are known

3. Mirror family of V_w^1

- Naive expectation: V_w^1 is a self-mirror and mirror family is given by $\mathcal{V}^1 = \bigcup_w V_w^1$

  $\rightarrow$ turns out No.

- V_w^1 was studied together with free $\mathbb{Z}_8 \times \mathbb{Z}_8$ actions.

Conjecture (Gross,Pavanelli, '08).

· Mirror of V_w^1 is (a family of) $V_w^1/\mathbb{Z}_8$ with a $\mathbb{Z}_8 \subset \mathbb{Z}_8 \times \mathbb{Z}_8$.

· Mirror of $V_w^1/\mathbb{Z}_8$ is (a family of) $V_w^1/\mathbb{Z}_8 \times \mathbb{Z}_8$.

$V_w^1 \rightarrow V_w^1/\mathbb{Z}_8$

$V_w^1/\mathbb{Z}_8 \times \mathbb{Z}_8 \leftarrow V_w^1/\mathbb{Z}_8$

Theorem (Schnell, '12)

The compactified relative Picard scheme M of $V_w^1 \rightarrow \mathbb{P}^1$ is a smooth CY3.
M is isomorphic to $V_w^1/\mathbb{Z}_8 \times \mathbb{Z}_8$, hence $D^b(V_w^1/\mathbb{Z}_8 \times \mathbb{Z}_8) \simeq D^b(V_w^1)$.

Describing the group $\mathbb{Z}_8 \subset \mathbb{Z}_8 \times \mathbb{Z}_8$, we construct a family of $V_w^1/\mathbb{Z}_8$ and answer to the above conjecture affirmatively.

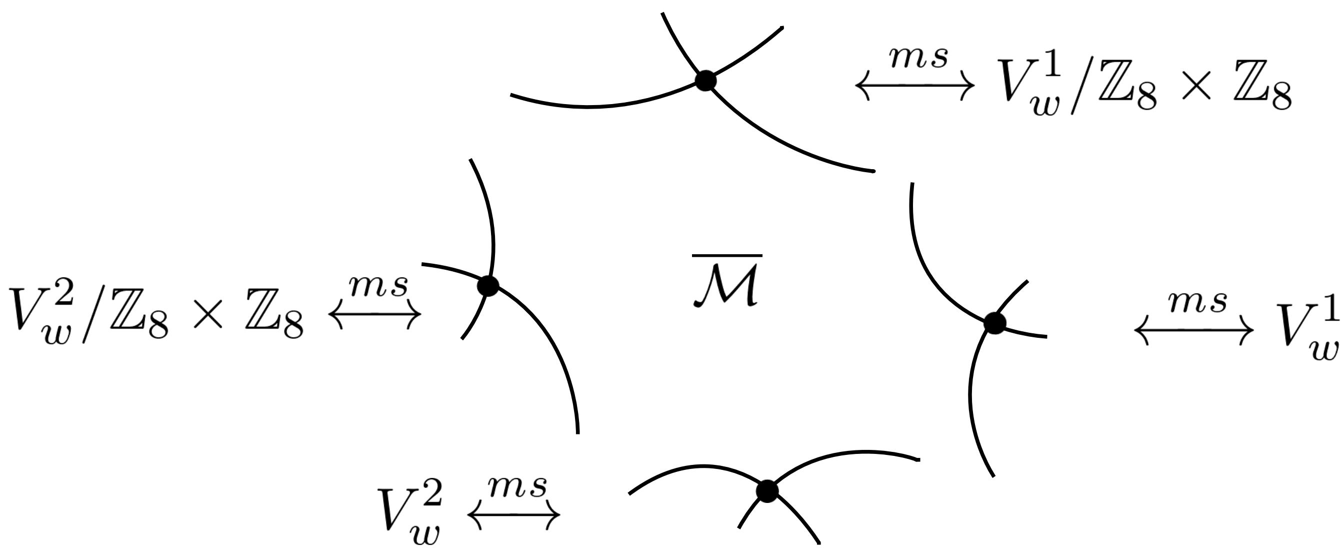
Proposition.(HT '21)

(1) Mirror of V_w^1 is a family of free quotients $V_w^1/\langle\tau\rangle$. $(\mathbb{Z}_8 \simeq \langle\tau\rangle \subset \langle\sigma,\tau\rangle)$

(2) Denote the family by $\mathcal{V}^1_{\mathbb{Z}_8} \to \mathcal{M}$. Then, mirror symmetry to

$$V_w^1,\ V_w^1/\mathbb{Z}_8\times\mathbb{Z}_8 \ \text{ and } \ V_w^2,\ V_w^2/\mathbb{Z}_8\times\mathbb{Z}_8 \ \text{(birational models)}$$

can be observed at boundary (degeneration) points in a suitably defined parameter space $\overline{\mathcal{M}}$.



I will sketch the construction of the family $\mathcal{V}^1_{\mathbb{Z}_8}$.

(3-2) Construction of the mirror family $\mathcal{V}^1_{\mathbb{Z}_8} \to \mathcal{M}$ of V^1_w

- The free action $V^1_w \curvearrowleft \mathbb{Z}_8 \times \mathbb{Z}_8 (= \mathcal{H}_8)$ is represented by

$$I_w(g.x) = I_w(x) \quad (g \in \mathcal{H}_8) \qquad (*)$$

for $I_w(x) = \langle f_1(w,x), f_2(w,x), f_3(x), f_4(x) \rangle$.

- Consider the normalizer $\mathcal{N}\mathcal{H}_8$ of $\mathcal{H}_8$ in $GL(\mathbb{C}^8)$:

$$1 \to \mathcal{H}_8 \to \mathcal{N}\mathcal{H}_8 \to SL_2(\mathbb{Z}_8) \to 1$$

then $\mathcal{N}\mathcal{H}_8 = \langle S, T, \sigma, \tau \rangle$ with $S = \frac{1}{2\sqrt{2}} \left(\xi^{-\frac{(i-j)^2}{2}}\right)_{i,j \in \mathbb{Z}_8}$, $T = \frac{1}{2\sqrt{2}} \left(\xi^{ij}\right)_{i,j \in \mathbb{Z}_8}$.

Proposition (GP,'01). The invariance $(*)$ extends to

$$I_{\rho(g).w}(g.x) = I_w(x) \quad (g \in \mathcal{N}\mathcal{H}_8)$$

for a representation $\rho : \mathcal{N}\mathcal{H}_8 \to GL(\mathbb{C}^3)$.

Can we consider a quotient of the following family $\mathcal{V}^1$ by $\mathcal{N}\mathcal{H}_8$?

$$\mathcal{V}^1 := \bigcup_{w \in \mathbb{P}^2_w} V^1_w \subset \mathbb{P}^7 \times \mathbb{P}^2_w$$

$$\downarrow \qquad \qquad \curvearrowleft \mathcal{N}\mathcal{H}_8$$

$$\mathbb{P}^2_w$$

$$
\begin{array}{ccccc}
\mathcal{V}^1 := \bigcup_{w \in \mathbb{P}^2_w} V^1_w \subset \mathbb{P}^7 \times \mathbb{P}^2_w & & & \mathcal{V}^1/\mathcal{NH}_8 & \\
\downarrow & \not{\mathcal{NH}_8} & = & \downarrow & \\
\mathbb{P}^2_w & & & \mathbb{P}^2_w/\rho(\mathcal{NH}_8) &
\end{array}
$$

fibers are only isomorphism classes. (not a univeral family)

Proposition. Define subgroups F and $\tilde{G}_0$ of $\mathcal{NH}_8$ by

$$F = \{g \in \mathcal{NH}_8 \mid \rho(g).[0,0,1]=[0,0,1]\} \text{ and } \tilde{G}_0 = \langle STS, (ST)^3 \rangle \quad (|\tilde{G}_0| = 64),$$

then we have $\rho(F) = \rho(\tilde{G}_0)$ and $1 \to \langle -1, \tau^4 \rangle \to \tilde{G}_0 \to \rho(\tilde{G}_0) \to 1$

Proposition (HT, '21). $\tilde{G}_0$ defines a good quotient of a family of $V^1_w/\langle \tau \rangle$;

$$
\begin{array}{ccccc}
\widetilde{\mathcal{V}}^1 := \bigcup_{w \in \mathbb{P}^2_w} V^1_w/\langle \tau \rangle & & \widetilde{\mathcal{V}}^1/\tilde{G}_0 & & \\
\downarrow & \rightsquigarrow & \downarrow & =: & \mathcal{V}^1_{\mathbb{Z}_8} \\
\mathbb{P}^2_w & & \mathbb{P}^2_w/\rho(\tilde{G}_0) & &
\end{array}
$$

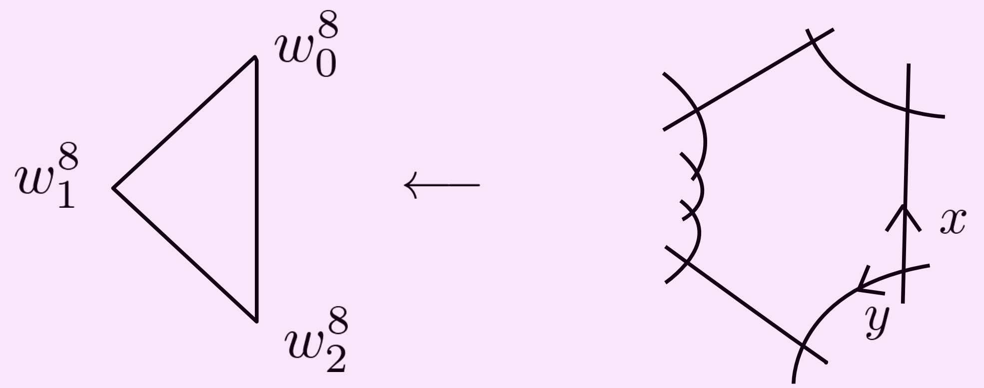
where $\rho(\tilde{G}_0) = \langle \begin{pmatrix} \xi^7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi^3 \end{pmatrix}, -1 \rangle$.

4. Mirror symmetry from the family

(4-1) The local system $R^3\pi_*\mathbb{C}_{\mathcal{V}^1_{\mathbb{Z}_8}}$ over $\mathbb{P}_\Delta$ and mirror symmetry

- Since $\rho(\tilde{G}_0)$ acts on $\mathbb{P}^2_w$ as $\mathbb{Z}_8$, we have $\mathbb{P}^2_w/\rho(\tilde{G}_0) = \mathrm{Proj}\,\mathbb{C}[w_0^8, w_1^8, w_2^8, \cdots]$.

$\mathbb{P}_\Delta :=$



Proposition (Pavanelli, '03). Consider period integrals of the family

$$w(x,y) = \int_\gamma Res\left(\frac{\sum(-1)^i x_i dx_0 \cdots \hat{dx_i} \cdots dx_7}{f_1(w)f_2(w)f_3(w)f_4(w)}\right)$$

These satisfy the following Picard-Fuchs equations:

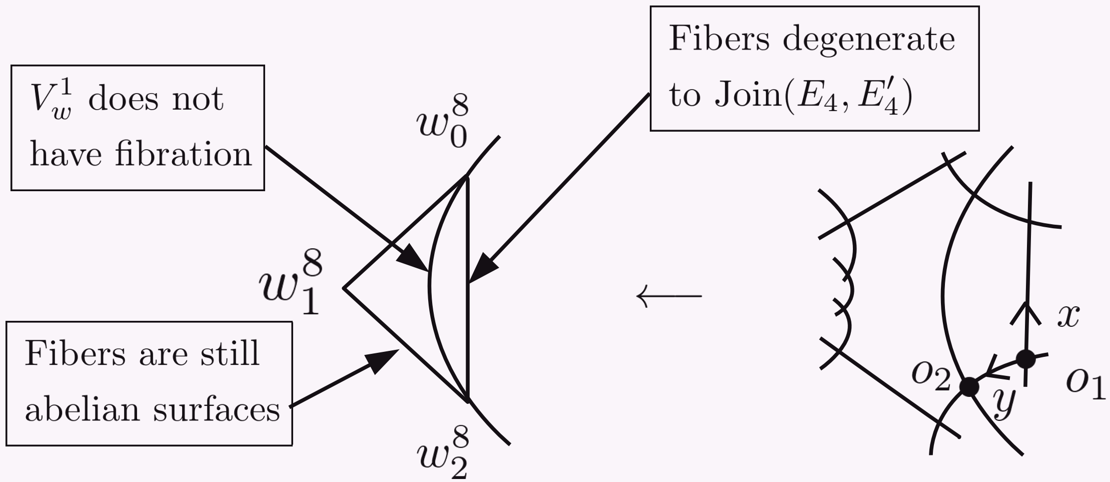
$$\mathcal{D}_1 w(x,y) = 4(1-4x)(1+4x)\theta_x^2 + \ldots.. = 0 \qquad \text{(2nd order)}$$

$$\mathcal{D}_2 w(x,y) = q_{03}(x,y)\theta_y^3 + q_{12}(x,y)\theta_x\theta_y^2\ldots.. = 0 \quad \text{(3rd order)}$$

Remark.

- Pavanelli ('03) determined the PF equations.

  He studied near $(x, y) = (0, 0)$ noticing a "strong" degeneration of V_w^1.

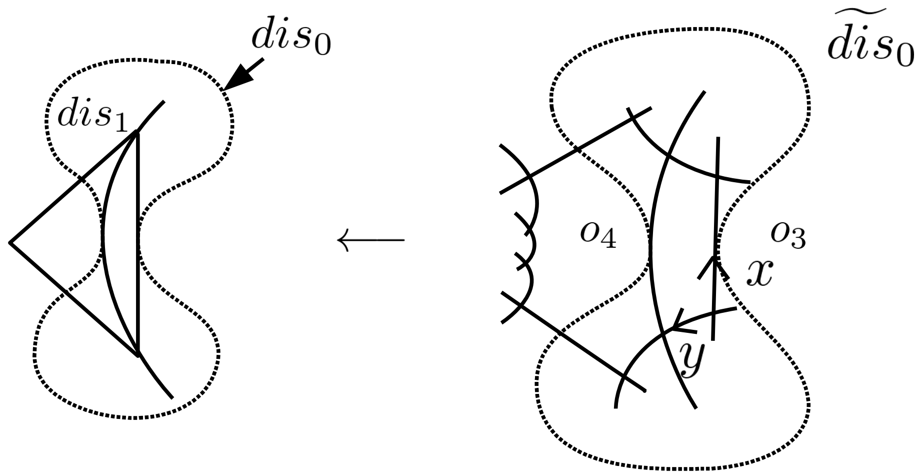


and found mirror symmetry to V_w^1 and V_w^2 (in the sense MS ver 1.)

$$V_w^1 \longleftrightarrow o_1 \qquad\qquad V_w^2 \longleftrightarrow o_2$$

- He also calculated Gromov-Witten invariants via MS ver 2.

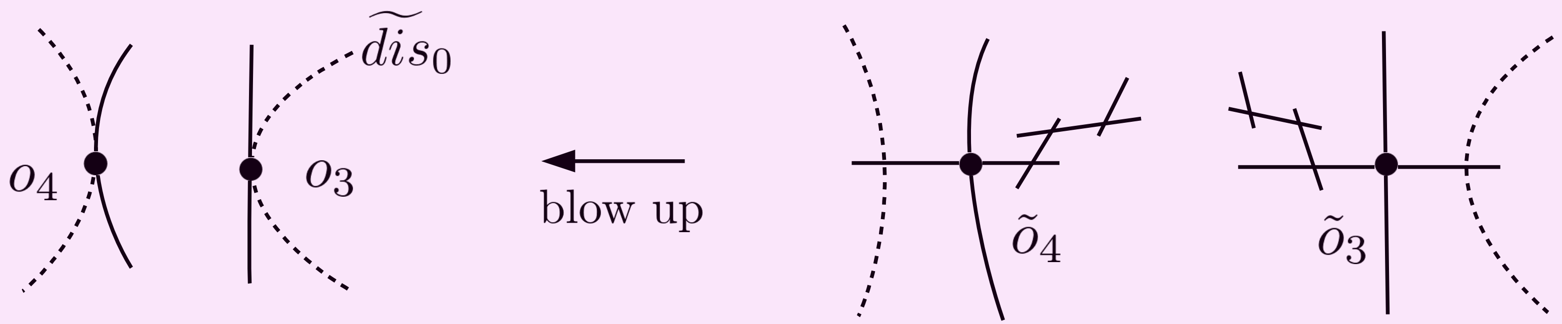
$$Y_{ijk} = \int_X h_i h_j h_k + \text{(quantum corrections)} \quad \rightsquigarrow \quad N_{g=0}(\beta) \;\; (\beta \in H_2(V, \mathbb{Z}))$$



There is another component dis_0 of the discriminant, over which $V_w^1/\mathbb{Z}_8$ has an additional ODP.

We observe 4th order tangency at o_3 and o_4.

Proposition (HT,'21). Blow up at o_3 and o_4



then

(1) at o_3 and o_4, we observe mirror correspondences

$$\text{A-str. of } V_w^1/\mathbb{Z}_8 \times \mathbb{Z}_8 \;\leftrightarrow\; \text{B-str. of } V_w^1/\mathbb{Z}_8 \text{ from } \tilde{o}_3$$
$$\text{A-str. of } V_w^2/\mathbb{Z}_8 \times \mathbb{Z}_8 \;\leftrightarrow\; \text{B-str. of } V_w^1/\mathbb{Z}_8 \text{ from } \tilde{o}_4$$

(2) We obtain Gromov-Witten invariants of $V_w^i/\mathbb{Z}_8 \times \mathbb{Z}_8$ $(i = 1, 2)$ from $\tilde{o}_3$ and $\tilde{o}_4$.

(4-2) g=0 Gromov-Witten invariants (BPS numbers)

• BPS numbers of V_w^1 from o_1

$$n_0(i,j) = n_0(\beta.H, \beta.A) \text{ for } \beta \in H_2(V_w^1, \mathbb{Z}) \text{ and } \mathrm{Pic}(V_w^1) = \mathbb{Z}H \oplus \mathbb{Z}A.$$

$j\diagdown i$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	64	512	2816	12288	46464	157696	493056	1441792	3989568
2	0	0	4096	98304	1220608	10813440	76775424	464322560	2480783360
3	0	0	2816	195072	6301056	124829696	1772620032	19764707328	183168532288
4	0	0	0	98304	10567680	478740480	13238665216	261369036800	4018366742528
5	0	0	0	12288	6301056	728901120	40797528064	1437499588608	36413468765248
6	0	0	0	0	1220608	478740480	58763759616	3812602150912	160955539341312
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$\times \frac{1}{64}$

BPS numbers $n_0(\beta) = n_0(\beta.H, \beta.A)$ with $\beta.A = n$ counts n-sections.

• BPS numbers of $V_w^1/\mathbb{Z}_8 \times \mathbb{Z}_8$ from o_3

$$n_0(i,j) = n_0(\beta.H', \beta.A') \text{ for } \beta \in H_2 \text{ and } \mathrm{Pic}(V_w^1/\mathbb{Z}_8 \times \mathbb{Z}_8) = \mathbb{Z}H' \oplus \mathbb{Z}A'.$$

$j\diagdown i$	0	..	8	..	16	..	24	..	32	..	40	..	48	..	56
0	0	..	0	..	0	..	0	..	0	..	0	..	0	..	0
1	1	..	8	..	44	..	192	..	726	..	2464	..	7704	..	22528
2	0	..	0	..	64	..	1536	..	19072	..	168960	..	1199616	..	7255040
3	0	..	0	..	44	..	3048	..	98454	..	1950464	..	27697188	..	308823552
4	0	..	0	..	0	..	1536	..	165120	..	7480320	..	206854144	..	4083891200
5	0	..	0	..	0	..	192	..	98454	..	11389080	..	637461376	..	22460931072
6	0	..	0	..	0	..	0	..	19072	..	7480320	..	918183744	..	59571908608
7	0	..	0	..	0	..	0	..	726	..	1950464	..	637461376	..	81827379400
⋮	⋮	..	⋮	..	⋮	..	⋮	..	⋮	..	⋮	..	⋮	..	⋮

$\left(\begin{array}{c} n_0(i,j) = 0 \\ \text{if } i \notin 8\mathbb{Z} \end{array} \right)$

We determined these invariants up to $g = 2$ via BCOV anomaly eqs (MS ver 3).

5. Counting functions

BPS numbers of V_w^1: $n_0(i,j) = n_0(\beta.H, \beta.A)$ for $\beta \in H_2(V_w^1, \mathbb{Z})$

$j \backslash i$	0	1	2	3	4	5	6	7	8	
0	0	0	0	0	0	0	0	0	0	
1	64	512	2816	12288	46464	157696	493056	1441792	3989568	$\leftarrow n=1$
2	0	0	4096	98304	1220608	10813440	76775424	464322560	2480783360	$\leftarrow n=2$
3	0	0	2816	195072	6301056	124829696	1772620032	19764707328	183168532288	$\leftarrow n=3$
4	0	0	0	98304	10567680	478740480	13238665216	261369036800	4018366742528	$\leftarrow n=4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Def. Let $N_0(\beta) = N_0(\beta.H, \beta.A)$ be the Gromov-Witten invariant of a class $\beta \in H_2(V_w^1, \mathbb{Z})$.
We define counting functions for n-sections by

$$Z_{0,n}(q) = \sum_{\beta.A=n} N_0(\beta) q^{\beta.H} \qquad \left(N_0(\beta) = \sum_{k|\beta} \frac{1}{k^3} n_0(\beta/k) \right)$$

Observation (HT'21). Counting functions are written by quasi-modular forms;

$$\boxed{Z_{0,n}(q) = P_{0,n}(E_2, E_4, E_6) \left(\frac{64}{\bar{\eta}(q)^8} \right)^n} \qquad \left(\bar{\eta}(q) := \prod_{n \geq 1} (1 - q^n) \right)$$

$$P_{0,1} = 1, \quad P_{0,2} = \frac{1}{4608}(8\, E_2^2 + E_4),$$

$$P_{0,3} = \frac{1}{2654208}(14 E_2^4 + 7 E_2^2 E_4 + E_4^2 + 2 E_2 E_6),$$

$$P_{0,4} = \frac{1}{2^{26} 3^7}(3008 E_2^6 + 2808 E_2^4 E_4 + 1128 E_2^2 E_4^2 + 125 E_4^3 + 1120 E_2^3 E_6 + 528 E_2 E_4 E_6 + 31 E_6^2),$$

$\cdots$.

· BPS numbers $n_0(\beta) = n_0(\beta.H, \beta.A)$ with $\beta.A = n$ counts n-sections.

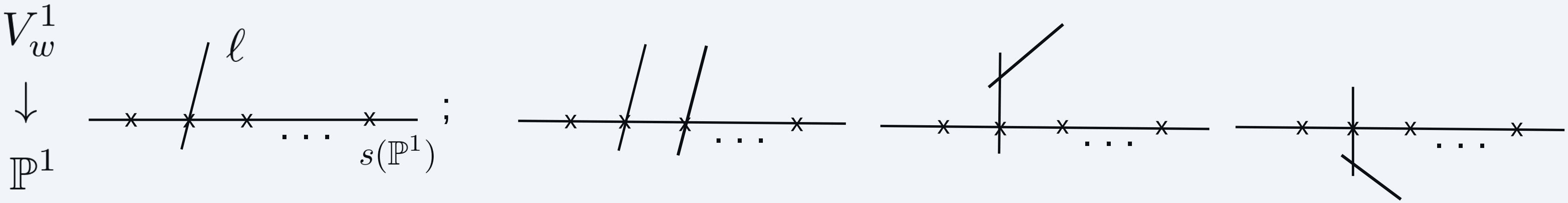
· There are 8 singular fibers (elliptic translation scrolls) of $V_w^1 \to \mathbb{P}^1$

$$T_E = \bigcup_{x \in E} \langle x, x+v \rangle$$

$n_0(0,1) = 64$ counts the sections

$n_0(1,1) = 512 = 64 \times 8$ counts the configulations (the most left)

$n_0(2,1) = 2816 = 64 \times (16 + 8 \times 2)$ counts the configurations (the right three)



Problem 1. Explain the eta functions in the denominator.

Problem 2. Under the fiberwise FM transform,

$$\text{counting } n\text{-sections} \quad \longleftrightarrow \quad \text{"counting" stable sheaves of rank } n$$
$$\text{(Euler numbers of moduli spaces } \mathfrak{M}_n \text{)}$$

Derive $Z_{0,n}(q)$ from this dual picture. This is a higer dimensional extension of a similar problem (solved) for rational elliptic surfaces.

The End

Thank You