A Gysin formula for Hall-Littlewood polynomials

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To Bill Fulton on his 75th birthday

Abstract

We give a formula for pushing forward the classes of Hall-Littlewood polynomials in Grassmann bundles, generalizing Gysin formulas for Schur S- and P-functions.

Let $E \to X$ be a vector bundle of rank n over a nonsingular variety X over an algebraically closed field. Denote by $\pi : G^q(E) \to X$ the Grassmann bundle parametrizing rank q quotients of E. Let $\pi_* : A(G^q(E)) \to A(X)$ be the homomorphism of the Chow groups of algebraic cycles modulo rational equivalence, induced by pushing-forward cycles (see [3, Chap. 1]). There exists an analogous map of cohomology groups. A goal of this note is to give a formula (see Theorem 8) for the image via π_* of Hall-Littlewood classes from the Grassmann bundle.

Hall-Littlewood polynomials appeared implicitly in Hall's study [5] of the combinatorial lattice structure of finite abelian p-groups, and explicitly in the work of Littlewood on some problems of representation theory [8]. A detailed account of the theory of Hall-Littlewood functions is given in [9].

The formula in Theorem 8 generalizes some Gysin formulas for Schur S- and P-functions. In particular, it generalizes the formula in [11, Prop. 1.3(ii)], and provides an explanation of its intriguing coefficient. We refer to [4] for general information about the appearance of Schur S- and P-functions in cohomological studies of algebraic varieties.

For some related results on push-forward, see [2, Prop. 2.1] (with the help of [1, Thm 5.5]).

Let t be an indeterminate. The main formula will be located in $A(X)[t]$, or in the extension $H^*(X, \mathbf{Z})[t]$ of the cohomology ring for a complex variety X. Let $\tau_E : Fl(E) \to X$ be the flag bundle parametrizing flags of quotients of E of ranks $n, n-1, \ldots, 1$. Suppose that $x_1, \ldots, x_n$ is a sequence of the Chern roots of E.

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Definition 1. *For a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative integers, set*

$$R_\lambda(E;t) = (\tau_E)_*\big(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j}(x_i - tx_j)\big)\,, \tag{1}$$

where $(\tau_E)_*$ acts on each coefficient of the polynomial in t separately.

The Grassmann bundle $\pi : G^q(E) \to X$ is endowed with the tautological exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0\,,$$

where $\mathrm{rank}(Q) = q$. Let $r = n - q$ be the rank of S. Suppose that $x_1 \dots, x_q$ are the Chern roots of Q and $x_{q+1}, \dots, x_n$ are the ones of S.

Proposition 2. *For sequences $\lambda = (\lambda_1, \dots, \lambda_q)$ and $\mu = (\mu_1, \dots, \mu_r)$ of nonnegative integers, we have*

$$\pi_*\big(R_\lambda(Q;t) R_\mu(S;t) \prod_{i \le q < j} (x_i - tx_j)\big) = R_{\lambda\mu}(E;t),$$

where $\lambda\mu = (\lambda_1, \dots, \lambda_q, \mu_1, \dots, \mu_r)$ is the juxtaposition of λ and μ.

Proof. Consider a commutative diagram

$$\begin{array}{ccc} Fl(Q) \times_{G^q(E)} Fl(S) & \xrightarrow{\cong} & Fl(E) \\ \downarrow{\scriptstyle \tau_Q \times \tau_S} & & \downarrow{\scriptstyle \tau = \tau_E} \\ G^q(E) & \xrightarrow[\pi]{} & X \end{array}$$

It follows that

$$\pi_*(\tau_Q \times \tau_S)_* = \tau_*\,. \tag{2}$$

Using Eq.(1) for Q and S and Eq.(2), we obtain

$$\begin{aligned} &\pi_*\big(R_\lambda(Q;t) R_\mu(S;t) \prod_{i \le q < j} (x_i - tx_j)\big) \\ &= \pi_*\big((\tau_Q)_*\big(x_1^{\lambda_1} \cdots x_q^{\lambda_q} \prod_{i<j\le q} (x_i - tx_j)\big) \cdot (\tau_S)_*\big(x_{q+1}^{\mu_1} \cdots x_n^{\mu_r} \prod_{q<i<j} (x_i - tx_j)\big) \prod_{i \le q < j} (x_i - tx_j)\big) \\ &= \pi_*(\tau_Q \times \tau_S)_*\Big(x_1^{\lambda_1} \cdots x_q^{\lambda_q} \prod_{i<j\le q} (x_i - tx_j) x_{q+1}^{\mu_1} \cdots x_n^{\mu_r} \prod_{q<i<j} (x_i - tx_j) \prod_{i \le q < j} (x_i - tx_j)\Big) \\ &= \tau_*(x_1^{\lambda_1} \cdots x_q^{\lambda_q} x_{q+1}^{\mu_1} \cdots x_n^{\mu_r} \prod_{i<j} (x_i - tx_j)) \\ &= R_{\lambda\mu}(E;t)\,. \end{aligned}$$

In the argument above, we have used the following equality:

$$\prod_{i<j\le q} (x_i - tx_j) \prod_{q<i<j} (x_i - tx_j) \prod_{i \le q < j} (x_i - tx_j) = \prod_{i<j} (x_i - tx_j)\,. \ \square$$

Definition 3. *Set*

$$v_m(t) = \prod_{i=1}^{m} \frac{1-t^i}{1-t} = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{m-1}). \tag{3}$$

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a sequence of nonnegative integers. Consider the maximal subsets I_1,...,I_d in $\{1, \ldots, n\}$, where the sequence λ is constant. Let $m_1, \ldots, m_d$ be the cardinalities of I_1,...,I_d. So we have $m_1 + \cdots + m_d = n$.

Definition 4. *Set*

$$v_\lambda(t) = \prod_{i=1}^{d} v_{m_i}(t)\,. \tag{4}$$

Example 5. Let $\nu = (\nu_1 > \ldots > \nu_k > 0)$ be a strict partition (see [9, I 1 Ex.9]) with $k \leq n$. Let $\lambda = \nu 0^{n-k}$ be the sequence ν with $n-k$ zeros added at the end. Then $d = k+1$, $(m_1, \ldots, m_d) = (1^k, n-k)$, $v_\lambda(t) = v_{n-k}(t)$.

Definition 6. *Let $\lambda = \lambda_1, \ldots, \lambda_n)$ be a sequence of nonnegative integers. Set*

$$P_\lambda(E;t) = \frac{1}{v_\lambda(t)} R_\lambda(E;t)\,. \tag{5}$$

If λ is a partition, then $P_\lambda(E;t)$ is a polynomial in t, called Hall-Littlewood polynomial (see [9, III 1,2)]).

Let $y_1, \ldots, y_n$ and t be independent indeterminates. For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, set

$$R_\lambda(y_1, \ldots, y_n; t) = \sum_{w \in S_n} w\Big(y_1^{\lambda_1} \cdots y_n^{\lambda_n} \prod_{i<j} \frac{y_i - t y_j}{y_i - y_j}\Big)\,,$$

where S_n is the symmetric group of all bijections of $\{y_1, \ldots, y_n\}$. Specializing y's to the Chern roots of E, $R_\lambda(y;t)$ becomes $R_\lambda(E;t)$.

Computing with Maple, we get the following examples.

Example 7. *For λ equal to* $(0,2,0)$, $(0,2,2,0)$, $(0,2,3,0)$, $(0,2,2,3,3)$, $(0,0,0,0,2,2)$, *$R_\lambda(y;t)$ is divisible by $v_\lambda(t)$. For λ equal to* $(0,2,0,2)$, $(0,2,0,0,2)$, $(0,2,2,0,0,0)$, $(0,2,0,2,0,2)$, *$R_\lambda(y;t)$ is not divisible by $v_\lambda(t)$.*

As a consequence of Proposition 2 we obtain the following result.

Theorem 8. *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_r)$ are sequences of nonnegative integers such that $R_\lambda(Q;t)$ is divisible by $v_\lambda(t)$ and $R_\mu(S;t)$ is divisible by $v_\mu(t)$. Then for the polynomials $P_\lambda(Q;t)$ and $P_\mu(S;t)$ we have*

$$\pi_*\Big(\prod_{i \leq q < j} (x_i - t x_j) P_\lambda(Q;t) P_\mu(S;t)\Big) = \frac{v_{\lambda\mu}(t)}{v_\lambda(t) v_\mu(t)} P_{\lambda\mu}(E;t)\,.$$

In the sequel, the sequences λ and μ will be partitions.

We first consider the specialization $t = 0$.

Example 9. We recall Schur S-functions. Let $s_i(E)$ denotes the ith complete symmetric function in the roots $x_1, \ldots, x_n$, given by

$$\sum_{i\geq 0} s_i(E) = \prod_{j=1}^{n} \frac{1}{1-x_j}\,.$$

Given a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$, we set

$$s_\lambda(E) = \left| s_{\lambda_i - i + j}(E) \right|_{1\leq i,j\leq n}.$$

(See also [9, I, 3].) Translating the Jacobi-Trudi formula (*loc.cit.*) to the Gysin map for $\tau_E : Fl(E) \to X$ (see, e.g. [11, Sect. 4]), we have

$$s_\lambda(E) = (\tau_E)_*(x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}).$$

We see that $P_\lambda(E;t) = s_\lambda(E)$ for $t = 0$. Under this specialization, the theorem becomes

$$\begin{aligned}\pi_*\big((x_1 \cdots x_q)^r s_\lambda(Q) s_\mu(S)\big) &= \pi_*\big(s_{\lambda_1+r,\ldots,\lambda_q+r}(Q) s_\mu(S)\big)\\ &= s_{\lambda\mu}(E)\,,\end{aligned}$$

a result obtained originally in [7, Prop. p. 196] and [6, Prop. 1].

If a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ is not a partition, then $s_\lambda(E)$ is either 0 or $\pm s_\mu(E)$ for some partition μ. One can rearrange λ by a sequence of operations $(\ldots, i, j, \ldots) \mapsto (\ldots, j-1, i+1, \ldots)$ applied to pairs of successive integers. Either one arrives at a sequence of the form $(\ldots, i, i+1, \ldots)$, in which case $s_\lambda(E) = 0$, or one arrives in d steps at a partition μ, and then $s_\lambda(E) = (-1)^d s_\mu(E)$.

Corollary 10. Let ν and σ be strict partitions of lengths $k \leq q$ and $h \leq r$. It follows from Eq.(3) that

$$\frac{v_{\nu 0^{q-k}\sigma 0^{r-h}}(t)}{v_{\nu 0^{q-k}}(t)\, v_{\sigma 0^{r-h}}(t)} = \begin{bmatrix} n-k-h \\ q-k \end{bmatrix}(t) \cdot (1+t)^e\,,$$

the Gaussian polynomial times $(1+t)^e$ where e is the number of common parts of ν and σ.

Thus the theorem applied to the sequences $\lambda = \nu 0^{q-k}$ and $\mu = \sigma 0^{r-h}$ yields the following equation:

$$\pi_*\Big(\prod_{i\leq q<j}(x_i - t x_j) P_\lambda(Q;t) P_\mu(S;t)\Big) = \begin{bmatrix} n-k-h \\ q-k \end{bmatrix}(t) \cdot (1+t)^e \cdot P_{\lambda\mu}(E;t)\,. \quad (6)$$

We now consider the specialization $t = -1$.

We need the following property of Gaussian polynomials, which should be known but we know no precise reference.

Lemma 11. *At $t = -1$, the Gaussian polynomial*

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t)$$

specializes to zero if ab is odd and to the binomial coefficient

$$\binom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor}$$

otherwise.

Proof. We have

$$\begin{bmatrix} a+b \\ a \end{bmatrix}(t) = \frac{(1-t)(1-t^2)\cdots(1-t^{a+b})}{(1-t)\cdots(1-t^a)(1-t)\cdots(1-t^b)}\,.$$

Since $t=-1$ is a zero with multiplicity 1 of the factor $(1-t^d)$ for even d, and a zero with multiplicity 0 for odd d, the order of the rational function $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right](t)$ at $t=-1$ is equal to

$$\lfloor (a+b)/2 \rfloor - \lfloor a/2 \rfloor - \lfloor b/2 \rfloor\,. \tag{7}$$

The order (7) is equal to 1 when a and b are odd, and 0 otherwise. In the former case, we get the claimed vanishing, and in the latter one, the product of the factors with even exponents is equal to

$$\begin{bmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{bmatrix}(t^2)\,.$$

The value of this function at $t=-1$ is equal to $\left[\begin{smallmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{smallmatrix}\right](1)$ which is the binomial coefficient

$$\binom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor}.$$

This is the requested value since the remaining factors with an odd exponent give 2 in the numerator and the same number in the denominator.

The assertions of the lemma follow. □

Example 12. Suppose that $x_1,\dots,x_n$ are independent variables. Consider Schur P-functions $P_\lambda(x_1,\dots,x_n) = P_\lambda$ defined as follows. For a strict partition $\lambda = (\lambda_1 > \ldots > \lambda_k > 0)$ with odd k,

$$P_\lambda = P_{\lambda_1} P_{\lambda_2,\dots,\lambda_k} - P_{\lambda_2} P_{\lambda_1,\lambda_3,\dots,\lambda_k} + \cdots + P_{\lambda_k} P_{\lambda_1,\dots,\lambda_{k-1}}\,,$$

and with even k,

$$P_\lambda = P_{\lambda_1,\lambda_2} P_{\lambda_3,\dots,\lambda_k} - P_{\lambda_1,\lambda_3} P_{\lambda_2,\lambda_4,\dots,\lambda_k} + \cdots + P_{\lambda_1,\lambda_k} P_{\lambda_2,\dots,\lambda_{k-1}}\,.$$

Here, $P_i = \sum s_\mu$, the sum over all hook partitions μ of i, and for positive $i > j$ we set

$$P_{i,j} = P_i P_j + 2\sum_{d=1}^{j-1} (-1)^d P_{i+d} P_{j-d} + (-1)^j P_{i+j}\,.$$

(See also [9, III 8].) It was shown in [12, p. 225] that for a strict partition λ of length k,

$$P_\lambda(x_1,\dots,x_n) = \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w\Big(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{i<j,\, i\le k} \frac{x_i + x_j}{x_i - x_j}\Big) \tag{8}$$

(see also [10, Appendix pp. 451-453], [9, Ex.1 p. 259]).

We have also, for a similar λ, the following formula for a Hall-Littlewood polynomial:

$$P_\lambda(x_1,\ldots,x_n;t)=\sum_{w\in S_n/(S_1)^k\times S_{n-k}} w\Big(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\prod_{i<j,i\le k}\frac{x_i-tx_j}{x_i-x_j}\Big) \qquad (9)$$

(see [9, (2.2) p. 208]).

We pass now to the notation from Eq.(6). It follows from comparison of (8) and (9) that for the sequence $\lambda=\nu 0^{q-k}$, we have $P_\lambda(Q;t)_{t=-1}=P_\nu(Q)$; and for $\mu=\sigma 0^{r-h}$, we have $P_\mu(S;t)_{t=-1}=P_\sigma(S)$.

For $\lambda\in\mathbf{Z}^n_{\ge 0}$ we set $\mathcal{P}:=P_\lambda(E;t)_{t=-1}$.

Note that for $i+j>0$, we have $\mathcal{P}_{\ldots,i,j,\ldots}=-\mathcal{P}_{\ldots,j,i,\ldots}$.

Thus $\mathcal{P}_{\nu 0^{q-k}\sigma 0^{r-h}}=(-1)^{(q-k)h}\mathcal{P}_{\nu\sigma 0^{n-k-h}}=(-1)^{(q-k)h}\mathcal{P}_{\nu\sigma}$.

If $e>0$, then $P_{\nu\sigma}(E)=0$; so we can assume $e=0$ without loss of generality.

We now use the notation from Corollary 10.

Specializing $t=-1$ in Eq.(6), we get by Lemma 11 the following result.

We have

$$\pi_*\big(c_{qr}(Q\otimes S)P_\nu(Q)P_\sigma(S)\big)=d_{\nu,\sigma}P_{\nu\sigma}(E)\,,$$

where $d_{\nu,\sigma}=0$ if $(q-k)(r-h)$ is odd and

$$d_{\nu,\sigma}=(-1)^{(q-k)h}\binom{\lfloor(n-k-h)/2\rfloor}{\lfloor(q-k)/2\rfloor}$$

otherwise.

This result was obtained originally in [11, Prop. 1.3(ii)] in a different way. The present approach gives an explanation of the intriguing coefficient $d_{\nu,\sigma}$.

Suppose that $\lambda=(\lambda_1,\ldots,\lambda_k)$ is not a strict partition. If there are repetitions of elements in λ, then P_λ is zero; if not then $P_\lambda=(-1)^l P_\mu$, where l is the length of the permutation which rearranges $(\lambda_1,\ldots,\lambda_k)$ into the corresponding strict partition μ.

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