# Special loci of Betti Moduli

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# Betti Moduli

$X$ sm q-proj var over $\mathbb{C}$, $\pi_1^{\mathrm{top}}(X, x)$ top fund gr based at $x \in X(\mathbb{C})$, $1 \leq r \in \mathbb{N} \rightsquigarrow \exists$ moduli space $M(X, r) =: M$ of conj cl of rk $r$ ss lin rep of $\pi_1^{\mathrm{top}}(X, x)$.

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$M$ is an aff sch$/\mathbb{Z}$. $M(\mathbb{C})$ is the set of iso cl of ss $\mathbb{C}$-loc syst of rk $r$.

# Quasi-unipotency at infinity

Fix a normal comp $j : X \hookrightarrow \bar{X}$. The conj cl of the gen $T_s$ of local fund gr around a comp $D_s$ at $\infty$ is well defined in $\pi_1^{\mathrm{top}}(X, x) \rightsquigarrow$ notion of loc mon at $\infty$ of loc syst $\mathcal{V} \in M(\mathbb{C})$.

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**Remark**

Same definition fixing the determinant $\mathcal{L}$ of $\mathcal{V}$ which is torsison $\rightsquigarrow$ $M(\mathcal{L})$ and $M(\mathcal{L})(\mathbb{C})^{\mathrm{qu}} \subset M(\mathcal{L})(\mathbb{C})$.

# Density

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**Heuristic**

- Simpson: in each irr comp of $M$, *exists* $\mathbb{C}$-VHS (viewed as a fixpoint under the $\mathbb{C}^\times$-rescaling on the Higgs field; ref in the -proj case?).
- Hope: those coming from geometry are even Zariski dense.
- Their monodromies at $\infty$ are quasi-unipotent (Brieskorn, Griffiths, Grothendieck).

# $\ell$-adic points

$\exists$ framed moduli space $M^\square(X,r) =: M^\square$ defined by the functor $\pi_1^{\rm top}(X,x) \to GL_r(A)$ on aff $\mathbb{Z}$-alg $A$. $M^\square$ fine, aff sch $/\mathbb{Z}$, $M^\square(\mathbb{C}) \xrightarrow{q} M(\mathbb{C})$ cat quotient by $GL_r$ action on frames $\rightsquigarrow$ $M^\square(\mathbb{C})^{\rm qu} = q^{-1}M(\mathbb{C})^{\rm qu}$.

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$\ell$ prime given $\rightsquigarrow$ $M^{\square}(\bar{\mathbb{Z}}_\ell) \subset M^{\square}(\bar{\mathbb{Q}}_\ell)$ = set of $\rho : \pi_1^{\mathrm{top}}(X,x) \to GL_r(\bar{\mathbb{Q}}_\ell)$ which factor through cont rep $\rho^{\text{ét}} : \pi_1^{\text{ét}}(X,x) \to GL_r(\bar{\mathbb{Z}}_\ell)$.

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**$\ell$-adic points of $M$**

$$M^{\text{ét},\ell} := q(M^{\square}(\bar{\mathbb{Z}}_\ell)) \subset M(\bar{\mathbb{Q}}_\ell)$$

is the set of étale $\bar{\mathbb{Q}}_\ell$-loc syst.

# Galois action on $\ell$-adic points

$x \in X(F)$ (if $\emptyset$ enlarge $F$) $\rightsquigarrow G \circlearrowleft \pi_1^{\text{ét}}(X_{\bar{F}}, x)$ by conj $\rightsquigarrow$

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# Special loci

**Definition**

A closed subset $S \subset M^{\square}(\bar{\mathbb{Q}}_\ell)$ is *special* if 1) $\forall$ each irred comp $S_i$ $\emptyset \neq S_i \cap M^{\square}(\bar{\mathbb{Z}}_\ell)$; 2) $U \circlearrowleft S \cap M^{\square}(\bar{\mathbb{Z}}_\ell)$ for some $U \subset G$ open.

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**Theorem B (E-Kerz '20)**

*$S$ special $\Rightarrow S \cap M^{\square}(\mathbb{C})^{\mathrm{qu}} \subset S$ Zariski dense.*

# Thm B $\Rightarrow$ Thm A

- $\bar{\mathbb{Q}}_\ell$ alg. cl. $\Rightarrow$ may replace $\mathbb{C}$ by $\bar{\mathbb{Q}}_\ell$ in Thm A (top loc syst).
- $M^\square(\bar{\mathbb{Z}}_\ell)$ $G$-invariant;
- so enough to find $\ell$ so all irr comp of $M^\square(\bar{\mathbb{Q}}_\ell)$ meet $M^\square(\bar{\mathbb{Z}}_\ell)$;
- fin many irr comp $M_i^\square/\bar{\mathbb{Q}}$, pick $\rho_i \in M_i^\square(\bar{\mathbb{Q}})$; as $\pi_1^{\rm top}(X,x)$ fin gen, all $\rho_i$ defined over $\mathcal{O}_{E,\Sigma}$, $\Sigma$ fin many places of $\mathcal{O}_E$; so $\ell \notin$ char $\Sigma$ does it.

# Proof of Thm B

- by defn, may assume $S$ irr;
- $N := \prod_s \mathbb{A}^{r-1} \times \mathbb{G}_m$;
- $$\psi : M^{\square}(\bar{\mathbb{Q}}_\ell)(\to M(\bar{\mathbb{Q}}_\ell)) \to N(\bar{\mathbb{Q}}_\ell)$$
$$\rho \mapsto \prod_s (\text{coeff of } \det(X - \rho(T_s) \cdot \mathrm{Id}))$$
- $\psi$ defined $/\mathbb{Z}_\ell$ , $G$-equivariant on $M^{\square}(\bar{\mathbb{Z}}_\ell) \to N(\bar{\mathbb{Z}}_\ell)$;
- define $N(\bar{\mathbb{Q}}_\ell)^{\mathrm{tor}} \subset N(\bar{\mathbb{Q}}_\ell)$ as the pol with torsion zeroes;
- $S \cap M^{\square}(\bar{\mathbb{Q}}_\ell)^{\mathrm{qu}} = \psi^{-1}(\psi(S) \cap N(\bar{\mathbb{Q}}_\ell)^{\mathrm{tor}})$;
- Chevalley thm $\Rightarrow \psi(S)$ constructible;
- Zariski cl $\overline{\psi(S)} =: Z$ irr and $Z \cap N(\bar{\mathbb{Z}}_\ell) \neq \emptyset$;
- so enough to prove $Z \cap N(\bar{\mathbb{Q}}_\ell)^{\mathrm{tor}} \subset Z$ Zariski dense;
- $h : T = \prod_s \mathbb{G}_m^r \to N$ separates the roots; $h$ fin surj, so enough to prove $h^{-1}Z \cap T(\bar{\mathbb{Q}}_\ell)^{\mathrm{tor}} \subset h^{-1}Z$ Zariski dense.

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Thm C more is precise: special loci are union of translates by a torsion point of motivic subtori.

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$T$ torus $/F$, then torsion points are Zariski dense in special loci.

**Remark**

Thm C more is precise: special loci are union of translates by a torsion point of motivic subtori.

**Remark**

Method of proof of Thm B comes from our proof of density thm of tame arithm $\bar{\mathbb{Q}}_\ell$-loc syst of rk 2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}/\bar{\mathbb{F}}_p$.
Generalization to higher rank would yield the Hard Lefschetz thm in char $p > 0$.