# On some properties of the Łojasiewicz exponent


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- Any pair of closed analytic subsets $X, Y \subset \mathbb{C}^m$ satisfies so-called Łojasiewicz regular separation property at any point of $X \cap Y$:

$\forall x^0 \in X \cap Y$, $\exists c, \nu > 0$ such that for some neighbourhood $U$ of $x^0$ we have

$$\rho(x, X) + \rho(x, Y) \geq c\,\rho(x, X \cap Y)^\nu \quad \text{for} \quad x \in U \tag{1}$$

where $\rho$ is the distance induced by the standard Hermitian norm on $\mathbb{C}^m$ (Łojasiewicz)

- If $x^0 \notin \mathrm{int}(X \cap Y)$, then $\nu \geq 1$

- $X$ and $Y$ satisfy (1) with a constant $\nu \geq 1$ if and only if there exist a neighbourhood $U'$ of $x^0$ and a constant $c' > 0$ such that

$$\rho(x, Y) \geq c'\rho(x, X \cap Y)^\nu \quad \text{for} \quad x \in U' \cap X$$

(Łojasiewicz, Cygan–Tworzewski, Denkowski)

- Any exponent $\nu$ satisfying (1) is called a regular separation exponent of $X$ and $Y$ at $x^0$. The infimum of such exponents is called the Łojasiewicz exponent of $X$ and $Y$ at $x^0$ and is denoted by $\mathcal{L}(X, Y; x^0)$; it is a regular separation exponent itself (Spodzieja).

**Theorem** Let $X$ and $Y$ be closed analytic subsets in $\mathbb{C}^m$ and $x^0 \in X \cap Y$ such that $\mathcal{L}(X, Y; x^0) \geq 1$. Then for a general hyperplane $H_0$ through $x^0$:

$$\mathcal{L}(X \cap H_0, Y \cap H_0; x^0) \leq \mathcal{L}(X, Y; x^0).$$

**Proposition** Let $X$ be a closed analytic subset in $\mathbb{C}^m$ and $x^0 \in X$. Then for a general hyperplane $H_0$ through $x^0$, there exist $c > 0$ and a neighbourhood $U$ of $x^0$ such that:

$$\rho(x, X \cap H_0) \leq c\, \rho(x, X) \quad \text{for } x \in U \cap H_0.$$

**Proof of the theorem** We may assume $x^0 = 0$. If $\nu$ is a regular separation exponent for $X$ and $Y$ at 0, then $\nu \geq \mathcal{L}(X, Y; 0) \geq 1$, and for some $c' > 0$ we have:

$$\rho(x, Y) \geq c' \rho(x, X \cap Y)^\nu \quad \text{for } x \in X \text{ near } 0.$$

By the proposition, for a general $H_0$, there exists $c > 0$ such that:

$$\rho(x, X \cap Y \cap H_0)^\nu \leq c\, \rho(x, X \cap Y)^\nu \quad \text{for } x \in H_0 \text{ near } 0.$$

Combining these relations gives

$$\rho(x, Y \cap H_0) \geq \rho(x, Y) \geq c'\, \rho(x, X \cap Y)^\nu \geq (c'/c)\, \rho(x, X \cap Y \cap H_0)^\nu$$

for $x \in X \cap H_0$ near 0.

**Proof of the proposition** We work in a small neighbourhood of $x^0 \equiv 0$

- $\check{\mathbb{P}}^{m-1}$ set of all hyperplanes of $\mathbb{C}^m$ through 0
- The distance between $H, K \in \check{\mathbb{P}}^{m-1}$ is the angle

$$\sphericalangle(H,K) := \arccos\Big(|\langle v,w\rangle| \Big/ (|v||w|)\Big) \in [0,\pi/2]$$

  · $v$ and $w$ normal vectors to $H$ and $K$ respectively
  · $\langle -,- \rangle$ standard Hermitian product on $\mathbb{C}^m$

► Consider the set $\mathcal{X} := \{(H,x) \in \check{\mathbb{P}}^{m-1} \times \mathbb{C}^m \mid x \in H \cap X\}$. By a theorem of Mostowski, in a neighbourhood of a generic $(H_0, 0)$, say in

$$\mathcal{U} := \{(H,x) \in \check{\mathbb{P}}^{m-1} \times \mathbb{C}^m \mid \sphericalangle(H_0,H) < a \text{ and } |x| < b\},$$

$\mathcal{X}$ is Lipschitz equisingular over $\check{\mathbb{P}}^{m-1} \times \{0\}$, i.e., for any $(H,0) \in \mathcal{U} \cap (\check{\mathbb{P}}^{m-1} \times \{0\})$, there is a (germ of) Lipschitz homeomorphism

$$\varphi\colon (\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m, (H,0)) \to (\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m, (H,0))$$

(with a Lipschitz inverse) such that $p \circ \varphi = p$ and $\varphi(\mathcal{X}) = \check{\mathbb{P}}^{m-1} \times (H \cap X)$, where $p$ is the projection on the first factor.

Actually, if $h = (h_1, \ldots, h_{m-1})$ are coordinates in $\check{\mathbb{P}}^{m-1}$ around $H_0$ such that

$$h_1(H_0) = \cdots = h_{m-1}(H_0) = 0\,,$$

and if $x = (x_1, \ldots, x_m)$ are Cartesian coordinates in $\mathbb{C}^m$, then, locally near $(H_0, 0)$, the standard "constant" vector fields $\partial_{h_j}$ on $\check{\mathbb{P}}^{m-1} \times \{0\}$ can be lifted to Lipschitz vector fields $v_j$ on $\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m$ such that the flows of $v_j$ preserve $\mathcal{X}$. So, $v_j$ is of the form

$$v_j(h,x) = \partial_{h_j}(h,x) + \sum_{\ell=1}^{m} w_{j\ell}(h,x)\,\partial_{x_\ell}(h,x),$$

so that $v_j(h,0) = \partial_{h_j}(h,0)$ and there exists a constant $c' > 0$ such that

$$|w_{j\ell}(h,x)| \leq c'\,|x| \text{ near } 0$$

for all $j, \ell$.

- $y^0 \in H_0$; we want to prove $\rho(y^0, X \cap H_0) \leq c\,\rho(y^0, X)$.

Let $y^1 \in X$ be one of the closest points to $y^0$ (i.e., $\rho(y^0, X) = |y^1 - y^0|$), and choose $H_1 \in \check{\mathbb{P}}^{m-1}$ such that $y^1 \in H_1$ and $\sphericalangle(H_0, H_1)$ is as small as possible.

**Lemma** If $(H_1, y^1) \notin \mathcal{U}$ (i.e., if $\sphericalangle(H_0, H_1) \geq a$), then $\exists a' > 0$ depending only on $a$ such that

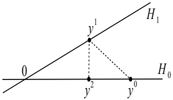
$$|y^1 - y^0| \geq a'\,|y^0|.$$

In particular, since $0 \in X \cap H_0$, we have

$$\rho(y^0, X \cap H_0) \leq |y^0| \leq (1/a')\rho(y^0, X).$$

**Proof** We may assume $H_0$: $x_m = 0$, the orthogonal projection of $y^1$ onto $H_0$ is $y^2 = (y_1^1, 0, \dots, 0)$ and $H_1$: $x_m = q_1 x_1$. Thus, if $\sphericalangle(H_0, H_1) \geq a$, we must have

$$\cos \sphericalangle(H_0, H_1) = 1/\sqrt{1 + |q_1|^2} \leq a_1 \quad \text{and} \quad |q_1| \geq a_2.$$




We may always assume $|y^0 - y^1| < (1/10)\,|y^0|$.

Thus,

$$|y^2 - y^0| \leq |y^1 - y^0| < (1/10)\,|y^0| \quad \text{and} \quad |y^2 - 0| = |y_1^1| > (9/10)\,|y^0|.$$

It follows that

$$|y^0 - y^1| \geq |y^1 - y^2| = |q_1|\,|y_1^1| \geq a_2\,(9/10)\,|y^0|.$$

Now, assume $(H_1, y^1) \in \mathcal{U}$, and let $h^1 = (h_1^1, \ldots, h_{m-1}^1)$ be the coordinates of $H_1$. Consider

$$\begin{aligned} v(h,x) &:= -\sum_{j=1}^{m-1} h_j^1 \, v_j(h,x) \\ &= -\sum_{j=1}^{m-1} h_j^1 \, \partial_{h_j}(h,x) + \sum_{\ell=1}^{m} \left( -\sum_{j=1}^{m-1} h_j^1 \, w_{j,l}(h,x) \right) \partial_{x_\ell}(h,x), \end{aligned}$$

and look at the integral curve $\gamma(t) = (h(t), x(t))$ of $v$ starting at $(H_1, y^1)$:

$$\begin{aligned} &\dot{h}_j(t) = -h_j^1, \quad \dot{x}_\ell(t) = -\sum_{j=1}^{m-1} h_j^1 \, w_{j,l}(h,x), \\ &h_j(0) = h_j^1, \qquad x_\ell(0) = y_\ell^1. \end{aligned}$$

- Flow of $v_j$ preserve $\mathcal{X}$ and $\gamma(0) \in \mathcal{X} \Rightarrow \gamma(t) \in \mathcal{X}$
- $h_j(t) = h_j^1(1-t) \Rightarrow h_j(1) = 0 \Rightarrow x(1) \in H_0$
- The length $L$ of the restriction of $x(t)$ to $[0,1]$ satisfies:

$$L := \int_0^1 |\dot{x}(t)|\, dt \leq c_1 \int_0^1 \sum_{j=1}^{m-1} \left( |h_j^1| \cdot \left( \sum_{\ell=1}^{m} |w_{j,\ell}(\gamma(t))| \right) \right) dt$$
$$\leq c_2\, |h^1| \int_0^1 |x(t)|\, dt \leq c_3\, |h^1|\, |x(0)| \leq c_4\, |y^0 - x(0)|$$

It follows that

$$\rho(y^0, X \cap H_0) \leq |y^0 - x(1)| \leq |y^0 - x(0)| + |x(0) - x(1)| \leq |y^0 - x(0)| + L$$
$$\leq (1 + c_4)\, |y^0 - x(0)| = (1 + c_4)\, \rho(y^0, X)$$

## Łojasiewicz exponent and order of tangency

$X$ and $Y$ analytic submanifolds of $\mathbb{C}^m$ of dimension $p$

- We say that the order of tangency between $X$ and $Y$ at $x^0$ is $\geq k$ if there exist parametrizations

$$q\colon (U, u^0) \to (X, x^0) \quad \text{and} \quad q'\colon (U, u^0) \to (Y, x^0),$$

($U$ open subset of $\mathbb{C}^p$) such that

$$q(u) - q'(u) = o(|u - u^0|^k)$$

as $u \to u^0$

- The order of tangency between $X$ and $Y$ at $x^0$ is the supremum of such integers $k$; it denoted by $s(X, Y; x^0)$.

**Proposition** Assume that $s(X, Y; x^0)$ is finite. If $\mathcal{L}(X, Y; x^0) \geq 1$, then

$$s(X, Y; x^0) \leq \mathcal{L}(X, Y; x^0) - 1.$$

**Proof** Write $s := s(X, Y; x^0)$, $\mathcal{L} := \mathcal{L}(X, Y; x^0)$, and $\mathbb{C}^m = \mathbb{C}^p_x \times \mathbb{C}^{m-p}_y$.

In a neighbourhood of $x^0 \equiv 0$,

$$X\colon\ y = f(x)$$

for some analytic function $f = (f_1, \dots, f_{m-p})\colon (\mathbb{C}^p_x, 0) \to (\mathbb{C}^{m-p}_y, 0)$.

Similarly, $Y\colon\ y = g(x)$; we may assume $g = 0$.

- $s'$ := smallest integer $k$ for which there exists a multi-index $\alpha$ with $|\alpha| = k$ and $D^\alpha(f - g)(0) \neq 0$

Then $s = s' - 1$.

Each $f_i$ has the Taylor expansion

$$f_i(x) = F_i(x) + o(|x|^{r_i})$$

where $F_i$ is a homogeneous polynomial of degree $r_i$. We may assume $r_1 \leq r_i$, so that $r_1 = s'$.

Let $\pi\colon \mathbb{C}^p_x \times \mathbb{C}^{m-p}_y \to \mathbb{C}^p_x$ be the standard projection, and look at

$$\pi(X \cap Y) = \{x \in \mathbb{C}^p_x;\ f(x) = 0\}.$$

**Lemma** If a line $L$ through 0 is not contained in the tangent cone $C$ of $\pi(X \cap Y)$ at 0, then $\rho(x, \pi(X \cap Y)) \sim |x|$ for $x \in L$.

So, if $F_1 \neq 0$ on $L$, then for any $x \in L$:

- $|f_1(x)| \sim |x|^{r_1} = |x|^{s'}$ and $|f_i(x)| \leq a\,|x|^{r_i} \leq a\,|x|^{s'}$
- $\rho(x, \pi(X \cap Y)) \sim |x|$ (by the lemma)

It follows that for any $(x, y) \in \pi^{-1}(L) \cap X = \{(x, y)\,;\, x \in L \text{ and } y = f(x)\}$:

- $\rho((x, y), Y) = |f(x)| \sim |x|^{s'}$
- $\rho((x, y), X \cap Y) \sim |x|$

Now the Łojasiewicz exponent $\mathcal{L}$ satisfies:

$$\rho((x, y), Y) \geq c\,\rho((x, y), X \cap Y)^{\mathcal{L}}, \text{ i.e., } |x|^{s'} \geq c|x|^{\mathcal{L}}.$$

So $s' \leq \mathcal{L}$, and hence, $s = s' - 1 \leq \mathcal{L} - 1$.

## Using the theorem, we only obtain $\mathcal{L} > s$

- Suppose $x_0$ is an isolated point of $X \cap Y$. Then $\exists c' > 0$ such that:

$$\rho(x, Y) \geq c' \, \rho(x, X \cap Y)^{\mathcal{L}} = c' \, |x - x^0|^{\mathcal{L}} \quad \text{for} \quad x \in X \text{ near } x^0,$$

or equivalently, $\rho(q(u), Y) \geq c' \, |q(u) - q(u^0)|^{\mathcal{L}}$ for $u$ near $u^0$. Since $q$ is locally bi-Lipschitz, there is a constant $c'' > 0$ such that

$$c' \, |q(u) - q(u^0)|^{\mathcal{L}} \geq c'' \, |u - u^0|^{\mathcal{L}} \quad \text{for} \quad u \text{ near } u^0.$$

Since $s$ is the order of tangency,

$$\rho(q(u), Y) \leq |q(u) - q'(u)| < c'' \, |u - u^0|^{s} \quad \text{for} \quad u \text{ near } u^0.$$

Combining these relations gives :

$$c'' \, |u - u^0|^{\mathcal{L}} \leq \rho(q(u), Y) < c'' \, |u - u^0|^{s} \quad \text{for} \quad u \text{ near } u^0.$$

- If $\dim X \cap Y = n > 0$, then take $n$ general hyperplanes $H_1, \ldots, H_n$ through $x^0$, so that

$$X \cap Y \cap H_1 \cap \cdots \cap H_n$$

is an isolated intersection. Write

$$\left.\begin{aligned} s_i &= \text{ order of tangency} \\ \mathcal{L}_i &= \text{ Łojasiewicz exponent} \end{aligned}\right| \text{ of } X \cap H_1 \cap \cdots \cap H_i \text{ and } Y \cap H_1 \cap \cdots \cap H_i \text{ at } x^0$$

Clearly, $s \leq s_1$, and by induction, $s_i \leq s_{i+1}$. By the theorem $\mathcal{L}_i \geq \mathcal{L}_{i+1}$. So altogether:

$$s \leq s_1 \leq \cdots \leq s_n < \mathcal{L}_n \leq \mathcal{L}_{n-1} \leq \cdots \leq \mathcal{L}.$$

**Thank you for your attention!**