On some properties of the Łojasiewicz exponent

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Abstract

In this note, we investigate the behaviour of the Łojasiewicz exponent under hyperplane sections and its relation to the order of tangency.

1. Introduction and statements of the results

It is well known (see [5, 6]) that any pair of closed analytic subsets $X, Y \subset \mathbf{C}^m$ ($m \geq 2$) satisfies so-called *Łojasiewicz regular separation property* at any point of $X \cap Y$. Precisely, for any $x^0 \in X \cap Y$ there are constants $c, \nu > 0$ such that for some neighbourhood $U \subset \mathbf{C}^m$ of x^0 we have

$$\rho(x, X) + \rho(x, Y) \geq c\, \rho(x, X \cap Y)^{\nu} \quad \text{for } x \in U, \tag{1}$$

where ρ is the distance induced by the standard Hermitian norm on $\mathbf{C}^m$. Note that if $x^0 \notin \operatorname{int}(X \cap Y)$, where the interior is computed in $\mathbf{C}^m$, then necessarily $\nu \geq 1$ (see [2]). Also, observe that X and Y satisfy (1) with a constant $\nu \geq 1$ if and only if there exist a neighbourhood U' of x^0 and a constant $c' > 0$ such that

$$\rho(x, Y) \geq c' \rho(x, X \cap Y)^{\nu} \quad \text{for } x \in U' \cap X \tag{2}$$

(see [5, 1, 2]). Any exponent ν satisfying the relation (1) for some U and $c > 0$ is called a *regular separation exponent* of X and Y at x^0. The infimum of such exponents is called the *Łojasiewicz exponent* of X and Y at x^0 and is denoted by $\mathcal{L}(X, Y; x^0)$; it is important to observe that the latter is a regular separation exponent itself (see [14]). The number $\mathcal{L}(X, Y; x^0)$ is an interesting metric invariant of the pointed pair $(X, Y; x^0)$ which have been the subject of vast studies in analytic geometry (see, for instance, the references in [14]).

The goal of this note is to investigate the behaviour of the Łojasiewicz exponent under hyperplane sections. Precisely we show the following theorem.

Theorem 1. *Let X and Y be closed analytic subsets in $\mathbf{C}^m$, and let $x^0 \in X \cap Y$ such that $\mathcal{L}(X, Y; x^0) \geq 1$. Then for a general hyperplane H_0 of $\mathbf{C}^m$ passing through x^0 we have*

$$\mathcal{L}(X \cap H_0, Y \cap H_0; x^0) \leq \mathcal{L}(X, Y; x^0).$$

This theorem is a consequence of the following result, which is the main part of the present work.

Theorem 2. *Let X be a closed analytic subset in $\mathbf{C}^m$, and let $x^0 \in X$. Then for a general hyperplane H_0 of $\mathbf{C}^m$ passing through x^0, there exist a constant $c > 0$ and a neighbourhood U of x^0 such that for all $x \in U \cap H_0$ we have*

$$\rho(x, X \cap H_0) \leq c\, \rho(x, X).$$

Theorems 1 and 2 are proved in Sections 2 and 3 respectively. To conclude this paper, in Section 4, we also briefly discuss the relation between the Łojasiewicz exponent and the order of tangency for pairs of closed analytic submanifolds of $\mathbf{C}^m$ with the same dimension.

2. Proof of Theorem 1

Without loss of generality, we may assume that x^0 is the origin $0 \in \mathbf{C}^m$. If ν is a regular separation exponent for X and Y at 0, then $\nu \geq \mathcal{L}(X, Y; 0) \geq 1$, and by (2), for some $c' > 0$ we have

$$\rho(x, Y) \geq c' \rho(x, X \cap Y)^\nu \tag{3}$$

for all $x \in X$ near 0. By Theorem 2, applied to $X \cap Y$, for a general hyperplane H_0 of $\mathbf{C}^m$ there is a constant $c > 0$ such that for all $x \in H_0$ near 0 we have

$$c\, \rho(x, X \cap Y)^\nu \geq \rho(x, X \cap Y \cap H_0)^\nu.$$

Combined with (3), this gives

$$\rho(x, Y \cap H_0) \geq \rho(x, Y) \geq c'\, \rho(x, X \cap Y)^\nu \geq (c'/c)\, \rho(x, X \cap Y \cap H_0)^\nu$$

for all $x \in X \cap H_0$ near 0, so that ν is a regular separation exponent for $X \cap H_0$ and $Y \cap H_0$ at 0. Applying this with $\nu = \mathcal{L}(X, Y; x^0)$ shows that

$$\mathcal{L}(X \cap H_0, Y \cap H_0; x^0) \leq \mathcal{L}(X, Y; x^0).$$

3. Proof of Theorem 2

It strongly relies on the Lipschitz equisingularity theory of complex analytic sets developed in [7] by the second named author. Throughout, we always work with Hermitian orthonormal bases $\{e_1, \ldots, e_m\}$ in $\mathbf{C}^m$, and the corresponding coordinates $x = (x_1, \ldots, x_m)$. As in Section 2, we assume $x^0 = 0$ and we work in a small neighbourhood of it.

Let $\check{\mathbf{P}}^{m-1}$ denote the set of all hyperplanes of $\mathbf{C}^m$ through 0, with its usual structure of manifold. The distance between two elements $H, K \in \check{\mathbf{P}}^{m-1}$ is the angle $\sphericalangle(H, K)$ between them, that is,

$$\sphericalangle(H, K) := \arccos \frac{|\langle v, w \rangle|}{|v|\, |w|} \in [0, \pi/2]$$

where v and w are normal vectors to the hyperplanes H and K, respectively, and $\langle \cdot, \cdot \rangle$ is the standard Hermitian product on $\mathbf{C}^m$ (see, e.g., [13]).

Step 1. Let

$$\mathcal{X} := \{(H, x) \in \check{\mathbf{P}}^{m-1} \times \mathbf{C}^m \mid x \in H \cap X\}.$$

By Proposition 1.1 of [7], in a neighbourhood

$$\mathcal{U} := \{(H,x) \in \check{\mathbf{P}}^{m-1} \times \mathbf{C}^m \mid \sphericalangle(H_0,H) < a \text{ and } |x| < b\}$$

of a generic $(H_0,0)$, we have that $\mathcal{X}$ is *Lipschitz equisingular* over $\check{\mathbf{P}}^{m-1} \times \{0\}$. That is, for any $(H,0) \in \mathcal{U} \cap (\check{\mathbf{P}}^{m-1} \times \{0\})$, there is a (germ of) Lipschitz homeomorphism

$$\varphi\colon (\check{\mathbf{P}}^{m-1} \times \mathbf{C}^m, (H,0)) \to (\check{\mathbf{P}}^{m-1} \times \mathbf{C}^m, (H,0))$$

(with a Lipschitz inverse) such that $p \circ \varphi = p$ and $\varphi(\mathcal{X}) = \check{\mathbf{P}}^{m-1} \times (H \cap X)$ (as germs at $(H,0)$). (Here, $p\colon \check{\mathbf{P}}^{m-1} \times \mathbf{C}^m \to \check{\mathbf{P}}^{m-1}$ is the standard projection.) Actually, if $h = (h_1,\ldots,h_{m-1})$ are coordinates in $\check{\mathbf{P}}^{m-1}$ around H_0 such that

$$h_1(H_0) = \cdots = h_{m-1}(H_0) = 0\,,$$

then, locally near $(H_0,0)$, the standard "constant" vector fields ∂_{h_j} $(1 \leq j \leq m-1)$ on $\check{\mathbf{P}}^{m-1} \times \{0\}$ can be lifted to Lipschitz vector fields v_j on $\check{\mathbf{P}}^{m-1} \times \mathbf{C}^m$ such that the flows of v_j preserve $\mathcal{X}$ (see the proof of Proposition 1.1 of [7], p.10). So, in particular, v_j is a Lipschitz vector field of the form

$$v_j(h,x) = \partial_{h_j}(h,x) + \sum_{\ell=1}^{m} w_{j\ell}(h,x)\, \partial_{x_\ell}(h,x),$$

so that $v_j(h,0) = \partial_{h_j}(h,0)$ and there exists a constant $c' > 0$ such that

$$|w_{j\ell}(h,x)| \leq c'\,|x| \text{ near } 0 \tag{4}$$

for all j, ℓ.

Step 2. Pick a point $y^0 \in H_0$. We want to prove that if y^0 is sufficiently close to 0, then

$$\rho(y^0, X \cap H_0) \leq c\,\rho(y^0, X) \tag{5}$$

for some constant $c > 0$ independent of y^0. Let $y^1 \in X$ be one of the closest points to y^0, that is, $\rho(y^0,X) = |y^1 - y^0|$. If $y^0 \in X$, then $\rho(y^0, X \cap H_0) = \rho(y^0,X) = 0$, and the inequality (5) is obviously true. So, hereafter, we assume that $y^0 \notin X$. Of course, without loss of generality, we may also assume that $|y^0| < b$ and $|y^1| < b$. Choose $H_1 \in \check{\mathbf{P}}^{m-1}$ such that $y^1 \in H_1$ and $\sphericalangle(H_0,H_1)$ is minimal. If $\sphericalangle(H_0,H_1) = 0$ (i.e., if $y^1 \in H_0$), then again $\rho(y^0, X \cap H_0) = \rho(y^0,X)$ and (5) is true. From now on, let us assume that $\sphericalangle(H_0,H_1) \neq 0$. Then we have the following lemma.

Lemma 3. *If $(H_1,y^1) \notin \mathcal{U}$ (i.e., if $\sphericalangle(H_0,H_1) \geq a$), then there exists $a' > 0$ depending only on a such that*

$$|y^1 - y^0| \geq a'\,|y^0|.$$

In particular, since $0 \in X \cap H_0$, if $(H_1,y^1) \notin \mathcal{U}$ then we have

$$\rho(y^0, X \cap H_0) \leq |y^0| \leq (1/a')\,\rho(y^0,X) \tag{6}$$

as desired.

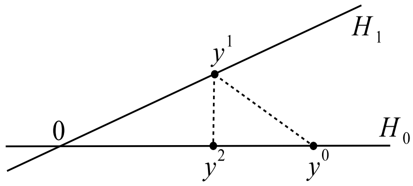

Figure 1: Hyperplanes H_0 and H_1

Proof of Lemma 3. By a proper choice of the basis $\{e_1, \dots, e_m\}$, we may assume that H_0 is defined by the equation $x_m = 0$, so that e_m is orthogonal to H_0. Now, if $x_m = \sum_{\ell=1}^{m-1} q_\ell \, x_\ell$ is an equation for H_1, then, clearly, for each $1 \leq \ell \leq m-1$, the vector $E_\ell := e_\ell + q_\ell e_m$ is in H_1. Thus, if $N = \sum_{\ell=1}^{m-1} u_\ell e_\ell + u_m e_m$ is a normal vector to H_1, then we must have $\langle N, E_\ell \rangle = 0$, and hence, $u_\ell = -u_m \bar{q}_\ell$, so that we can take $N := -\sum_{\ell=1}^{m-1} \bar{q}_\ell \, e_\ell + e_m$.

Now, saying that $\sphericalangle(H_0, H_1)$ is minimal means that

$$\cos \sphericalangle(H_0, H_1) = \frac{|\langle N, e_m \rangle|}{|N| \, |e_m|} = \frac{1}{\sqrt{1 + \sum_{\ell=1}^{m-1} |q_\ell|^2}}$$

is maximal, that is, $\sum_{\ell=1}^{m-1} |q_\ell|^2$ is minimal. By adjusting the choice of the basis, we may further assume that $y^1 = (y^1_1, 0, \dots, 0, y^1_m)$, so that its orthogonal projection onto H_0 is $y^2 := (y^1_1, 0, \dots, 0)$. As $y^1 \in H_1$, we have $q_1 = y^1_m / y^1_1 \neq 0$. Thus, $\sum_{\ell=1}^{m-1} |q_\ell|^2$ is minimal if and only if $q_2 = \cdots = q_{m-1} = 0$. So, if $\sphericalangle(H_0, H_1)$ is minimal, then H_1 is given by the equation $x_m = q_1 x_1$.

It follows that if $\sphericalangle(H_0, H_1) \geq a$ (assumption of the lemma), then we must have

$$\cos \sphericalangle(H_0, H_1) = 1/\sqrt{1 + |q_1|^2} \leq a_1,$$

and hence $|q_1| \geq a_2$, for some constants $a_1, a_2 > 0$ depending only on a. Now, clearly, we may always assume $|y^0 - y^1| < (1/10)\,|y^0|$. Thus, $|y^2 - y^0| \leq |y^1 - y^0| < (1/10)\,|y^0|$, and hence,

$$|y^2 - 0| = |y^1_1| > (9/10)\,|y^0|$$

(see Figure 1). It follows that

$$|y^0 - y^1| \geq |y^1 - y^2| = |q_1|\,|y^1_1| \geq a_2\,(9/10)\,|y^0|,$$

and this completes the proof of Lemma 3.

Step 3. Lemma 3 solves the case where $(H_1, y^1) \notin \mathcal{U}$ (see (6)). Now let us look at the case where $(H_1, y^1) \in \mathcal{U}$; here comes Lipschitz equisingularity (see Step 1). Let $h^1 = (h^1_1, \dots, h^1_{m-1})$ be the coordinates of H_1. (Note that $|h^1| \leq d \cdot \sphericalangle(H_0, H_1)$ for some constant $d > 0$ independent of H_1.) Consider the Lipschitz vector field v on $\check{\mathbf{P}}^{m-1} \times \mathbf{C}^m$ defined by

$$\begin{aligned} v(h, x) &:= -\sum_{j=1}^{m-1} h^1_j \, v_j(h, x) \\ &= -\sum_{j=1}^{m-1} h^1_j \, \partial_{h_j}(h, x) + \sum_{\ell=1}^{m} \Big(-\sum_{j=1}^{m-1} h^1_j \, w_{j,\ell}(h, x) \Big) \partial_{x_\ell}(h, x), \end{aligned}$$

and look at the integral curve $\gamma(t) = (h(t), x(t))$ of v starting at (H_1, y^1). So, in particular, we have:

$$\begin{aligned}
\dot{h}_j(t) &= -h_j^1, \quad \dot{x}_\ell(t) = -\sum_{j=1}^{m-1} h_j^1\, w_{j,\ell}(h,x), \\
h_j(0) &= h_j^1, \qquad x_\ell(0) = y_\ell^1.
\end{aligned}$$

As the flows of the vector fields v_j preserve $\mathcal{X}$ and since $\gamma(0) \in \mathcal{X}$, the curve $\gamma(t)$ lies in $\mathcal{X}$. Moreover, since $h_j(t) = h_j^1(1-t)$, we have $h_j(1) = 0$ for all j, and hence $x(1)$ lies in H_0. Finally, observe that the length $L_I(x)$ of the restriction of the curve $x(t)$ to the compact interval $I = [0,1]$ satisfies

$$\begin{aligned}
L_I(x) := \int_0^1 |\dot{x}(t)|\, dt &\le c_1 \int_0^1 \sum_{j=1}^{m-1} \Big(|h_j^1| \cdot \Big(\sum_{\ell=1}^{m} |w_{j,\ell}(\gamma(t))| \Big) \Big) dt \\
&\overset{\text{by (4)}}{\le} c_2\, |h^1| \int_0^1 |x(t)|\, dt \le c_3\, |h^1|\, |x(0)| \le c_4\, |y^0 - x(0)|
\end{aligned}$$

for some constants $c_i > 0$ independent of y^0, H_1 and y^1. The first and third inequalities are clear. The second one follows from the crucial relation (4) (i.e., from Lipschitz equisingularity). To show the last inequality, we may proceed as in the proof of Lemma 3, exchanging the roles of H_0 and H_1. Namely, for a new proper choice of the basis, we may assume that H_1 is defined by $x_m = 0$ and that $y^0 = (y_1^0, 0, \ldots, 0, y_m^0)$, so that the orthogonal projection of y^0 onto H_1 is $y^3 := (y_1^0, 0, \ldots, 0)$. As the angle $\sphericalangle(H_0, H_1)$ is minimal, we may suppose that H_0 is given by an equation of the form $x_m = q_1 x_1$. Clearly, we may also assume that $|y^0 - y^1| < (1/10)\, |y^1|$. Thus $|y^3 - y^1| \le |y^0 - y^1| < (1/10)\, |y^1|$, and hence, $|y^3 - 0| = |y_1^0| > (9/10)\, |y^1|$. It follows that

$$|y^0 - y^1| \ge |y^0 - y^3| = |q_1|\, |y_1^0| > |q_1|\, (9/10)\, |y^1|.$$

But we have

$$|h^1| \le d \cdot \sphericalangle(H_0, H_1) = d \cdot \arccos(1/\sqrt{1 + |q_1|^2}) \le d'\, |q_1|,$$

where the constants $d, d' > 0$ are independent of y^0, H_1 and y^1. It follows that

$$|y^0 - y^1| \ge \frac{9}{10\, d'}\, |h^1|\, |y^1|$$

as desired (remind that $y^1 = x(0)$). Now, by the estimate of the length $L_I(x)$ given above, we have

$$\begin{aligned}
\rho(y^0, X \cap H_0) &\le |y^0 - x(1)| \le |y^0 - x(0)| + |x(0) - x(1)| \le |y^0 - x(0)| + L_I(x) \\
&\le (1 + c_4)\, |y^0 - x(0)| = (1 + c_4)\, \rho(y^0, X),
\end{aligned}$$

and this completes the proof of Theorem 2.

Remark. Note that the proof of Theorem 2 (and hence of Theorem 1) given above only depends on the Lipschitz equisingularity theory of complex analytic sets developed in [7] by the second named author. Real versions of this theory for the semi-analytic and subanalytic categories were addressed by A. Parusiński in [9, 10, 11, 12] while the case of sets definable in a polynomially bounded o-minimal structure was obtained by Nguyen Nhan and G. Valette in [8]. Theorems 1 and 2 must then be true in these categories as well.

4. Remark on the Łojasiewicz exponent and the order of tangency

To conclude this paper, we give a lower bound for the Łojasiewicz exponent $\mathcal{L}(X,Y;x^0)$ of two p-dimensional closed analytic submanifolds X and Y of $\mathbf{C}^m$ at $x^0 \in X \cap Y$ in terms of the order of tangency of X and Y at x^0.

Following [4, 3], we say that the order of tangency between X and Y at x^0 is greater than or equal to an integer k if there exist parametrizations (i.e., biholomorphisms onto their images)

$$q\colon (U,u^0) \to (X,x^0) \quad \text{and} \quad q'\colon (U,u^0) \to (Y,x^0),$$

where $U \ni u^0$ is an open subset of $\mathbf{C}^p$, such that

$$q(u) - q'(u) = o(|u-u^0|^k) \tag{7}$$

when $U \ni u \to u^0$. The *order of tangency* between X and Y at x^0 (denoted by $s(X,Y;x^0)$) is the supremum of all such integers k.

Observation 4. *Let X and Y be p-dimensional closed analytic submanifolds of $\mathbf{C}^m$, and let $x^0 \in X \cap Y$. Suppose that $s(X,Y;x^0)$ is finite. If $\mathcal{L}(X,Y;x^0) \geq 1$, then*

$$s(X,Y;x^0) \leq \mathcal{L}(X,Y;x^0) - 1.$$

Proof. Put $s := s(X,Y;x^0)$, $\mathcal{L} := \mathcal{L}(X,Y;x^0)$, and for this proof write $\mathbf{C}^m = \mathbf{C}^p_x \times \mathbf{C}^{m-p}_y$ where $x = (x_1,\dots,x_p)$ and $y = (x_{p+1},\dots,x_m)$. As above, we assume that x^0 is the origin $0 \in \mathbf{C}^m$. In a neighbourhood of 0, the analytic submanifold X is given by $y = f(x)$ for some analytic function

$$f = (f_1,\dots,f_{m-p})\colon (\mathbf{C}^p_x,0) \to (\mathbf{C}^{m-p}_y,0).$$

Similarly, Y is also the graph of an analytic function g, and without loss of generality, we may assume that $g=0$. Now, let s' be the smallest integer k for which there exists a multi-index $\alpha = (\alpha_1,\dots,\alpha_p)$ such that $|\alpha| = \alpha_1+\dots+\alpha_p = k$ and $D^\alpha(f-g)(0) \neq 0$. Clearly, $s = s'-1$. Each component f_i has the Taylor expansion

$$f_i(x) = F_i(x) + o(|x|^{r_i})$$

where F_i is a homogeneous polynomial of degree r_i. Of course, we may assume $r_1 \leq r_i$ for all i, so that $r_1 = s'$. Consider the standard projection

$$\pi\colon \mathbf{C}^p_x \times \mathbf{C}^{m-p}_y \to \mathbf{C}^p_x,$$

and look at the hypersurface $\pi(X\cap Y) = \{x \in \mathbf{C}^p_x\,;\, f(x) = 0\}$ of $\mathbf{C}^p_x$. It is easy to see that if L is a line through 0 which is not in the tangent cone of $\pi(X \cap Y)$ at 0, then

$$\rho(x, \pi(X\cap Y)) \sim |x|$$

for $x \in L$ near 0.[1] Now, if $F_1 \neq 0$ on L, then for any $x \in L$ near 0, we also have

$$|f_1(x)| \sim |x|^{r_1} = |x|^{s'} \quad \text{and} \quad |f_i(x)| \leq a\,|x|^{r_i} \leq a\,|x|^{s'}$$

for some constant $a > 0$. It follows that for any $(x, y) \in \pi^{-1}(L) \cap X = \{(x, y)\,;\, x \in L \text{ and } y = f(x)\}$ near 0, we have

$$\rho((x, y), Y) = |f(x)| \sim |x|^{s'} \quad \text{and} \quad \rho((x, y), X \cap Y) \sim |x|.$$

Now, the Łojasiewicz exponent $\mathcal{L}$ satisfies $\rho((x, y), Y) \geq c\,\rho((x, y), X \cap Y)^{\mathcal{L}}$, that is, $|x|^{s'} \geq c\,|x|^{\mathcal{L}}$ for some constant $c > 0$. Thus $s' \leq \mathcal{L}$, and hence, $s = s' - 1 \leq \mathcal{L} - 1$.

Remark. We may also investigate the relationship between $s := s(X, Y; x^0)$ and $\mathcal{L} := \mathcal{L}(X, Y; x^0)$ using Theorem 1 but this second approach only gives the inequality $s < \mathcal{L}$. However, for completeness, let us briefly explain the argument. First, we consider the special case where x^0 is an isolated point of $X \cap Y$. In this case, there exists a constant $c' > 0$ such that

$$\rho(x, Y) \geq c'\,\rho(x, X \cap Y)^{\mathcal{L}} = c'\,|x - x^0|^{\mathcal{L}} \quad \text{for } x \in X \text{ near } x^0,$$

or equivalently, $\rho(q(u), Y) \geq c'\,|q(u) - q(u^0)|^{\mathcal{L}}$ for u near u^0. Since q is locally bi-Lipschitz, there exists a constant $c'' > 0$ such that

$$c'\,|q(u) - q(u^0)|^{\mathcal{L}} \geq c''\,|u - u^0|^{\mathcal{L}} \quad \text{for } u \text{ near } u^0.$$

Now, by (7), we have

$$\rho(q(u), Y) \leq |q(u) - q'(u)| < c''\,|u - u^0|^{s} \quad \text{for } u \text{ near } u^0.$$

Combining these relations gives

$$c''\,|u - u^0|^{\mathcal{L}} \leq \rho(q(u), Y) < c''\,|u - u^0|^{s} \quad \text{for } u \text{ near } u^0,$$

and hence $s < \mathcal{L}$.

The general case (i.e., $\dim X \cap Y = n > 0$) follows from the 0-dimensional case and Theorem 1. Indeed, take n general hyperplanes $H_1, \ldots, H_n$ in $\mathbf{C}^m$ passing through x^0, so that $X \cap Y \cap H_1 \cap \cdots \cap H_n$ is an isolated intersection. Let s_i (respectively, $\mathcal{L}_i$) denote the order of tangency (respectively, the Łojasiewicz exponent) of $X \cap H_1 \cap \cdots \cap H_i$ and $Y \cap H_1 \cap \cdots \cap H_i$ at x^0. Clearly, (7) implies $s_i \leq s_{i+1}$ while Theorem 1 shows $\mathcal{L}_i \geq \mathcal{L}_{i+1}$. (Note that since $\operatorname{int}(X \cap Y \cap H_1 \cap \cdots \cap H_i) = \emptyset$, we have $\mathcal{L}_i \geq 1$, so that Theorem 1 applies.) Now the relation $s < \mathcal{L}$ follows from the inequality $s_n < \mathcal{L}_n$ (0-dimensional case).

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