Discriminants and semi-orthogonal decompositions

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arXiv:2102.08412

Overview

Part I - Theorem

We study the derived category of coherent sheaves $D^b(X)$ on a toric variety X. The derived category has semi-orthogonal decompositions coming from wall-crossing to other birational models.

Theorem: these decompositions obey the Jordan-Hölder property.

Part II - Motivation and Conjecture

For a Calabi-Yau toric variety wall-crossing gives us many autoequivalences of $D^b(X)$. Physics/mirror symmetry predicts that these together form an action of the fundamental group of the *FI parameter space* - the complement of the discriminant in the dual toric variety.

Conjecture: the multiplicities in our decompositions agree with intersection multiplicities in the discriminant.

Semi-orthogonal decompositions

Definition

A *semi-orthogonal decomposition* of $D^b(X)$ is a sequence of full triangulated subcategories $\mathcal{C}_1, \ldots, \mathcal{C}_r \subset D^b(X)$ such that:

(i) together they generate $D^b(X)$, and

(ii) there are no morphisms from $\mathcal{C}_i$ to $\mathcal{C}_j$ if $i > j$.

- Like a semi-direct product of groups, or an algebra of block-upper-triangular matrices.
- Gives some control over $D^b(X)$ in terms of the smaller pieces, *e.g.* K-theory and homology split.
- We write

$$D^b(X) = \langle\, \mathcal{C}_1,\ \ldots,\ \mathcal{C}_r \,\rangle$$

Semi-orthogonal decompositions

Example

Let Y be a 3-fold and X be the blow-up of Y at a smooth point. Exceptional divisor is $E \cong \mathbb{P}^2$.

The sky-scraper sheaf $\mathcal{O}_E$ is an *exceptional object* in $D^b(X)$:

$$\mathrm{End}_{D^b(X)}(\mathcal{O}_E) = \mathbb{C}$$

$\implies$ The subcategory generated by $\mathcal{O}_E$ is equivalent to $D^b(pt)$.

Have

$$D^b(X) = \langle\, D^b(Y),\ D^b(pt),\ D^b(pt)\,\rangle$$

where the second and third subcategories are generated by $\mathcal{O}_E(2E)$ and $\mathcal{O}_E(E)$.

Semi-orthogonal decompositions

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Have

$$D^b(X) = \langle\, Y,\ pt,\ pt \,\rangle$$

where the second and third subcategories are generated by $\mathcal{O}_E(2E)$ and $\mathcal{O}_E(E)$.

Semi-orthogonal decompositions

Theorem (Orlov)

Let X be the blow-up of Y in a smooth subvariety Z. Then

$$D^b(X) = \langle\, Y, Z, ..., Z \,\rangle$$

where the number of copies of Z is $codim(Z) - 1$.

So semi-orthogonal decompositions appear when we do blow-ups.

What about other birational transformations?

Abelian VGIT

Example

Let $\mathbb{C}^*$ act on $\mathbb{C}^4$ with weights $(1, 1, 1, -1)$.
The two GIT quotients are:

$$X_+ \cong \mathcal{O}(-1)_{\mathbb{P}^2}$$

$$X_- \cong \mathbb{A}^3$$

We know

$$D^b(X_+) = \langle\, X_-,\ pt,\ pt \,\rangle$$

by blow-up formula.

- The *pt* here is really the fixed point (the origin) in $\mathbb{C}^4$.

Abelian VGIT

Example

Let $\mathbb{C}^*$ act on $\mathbb{C}^6$ with weights (1,1,1,1,-1,-1).
The two GIT quotients are:

$$X_+ \cong \mathcal{O}(-1)_{\mathbb{P}^2}$$

$$X_- \cong \mathbb{A}^3$$

We know

$$D^b(X_+) = \langle\, X_-,\ pt,\ pt \,\rangle$$

by blow-up formula.

- The *pt* here is really the fixed point (the origin) in $\mathbb{C}^4$.

Abelian VGIT

Example

Let $\mathbb{C}^*$ act on $\mathbb{C}^6$ with weights (1,1,1,1,-1,-1).
The two GIT quotients are:

$$X_+ \cong \mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^3}$$

$$X_- \cong \mathcal{O}(-1)^{\oplus 4}_{\mathbb{P}^1}$$

Still true that

$$D^b(X_+) = \langle\, X_-,\ pt,\ pt\,\rangle$$

- The *pt* here is really the fixed point (the origin) in $\mathbb{C}^4$.
- The number of copies of *pt* equals the sum of the weights.

Abelian VGIT

Theorem (Kawamata, Ballard–Favero–Katzarkov, Halpern-Leistner)

Let $\mathbb{C}^*$ act on U. Assume $Z = U^{\mathbb{C}^*}$ connected and

$$\kappa = \textit{weight}(\det(N_{Z/U})) \geq 0.$$

Then the two GIT quotients $X_\pm$ obey

$$D^b(X_+) = \langle\, X_-,\ Z,\ \ldots,\ Z \,\rangle$$

where κ copies of $D^b(Z)$ appear.

- Implies Orlov's blow-up formula.
- Could have $X_- = \varnothing$, *e.g.*

  $$D^b(\mathbb{P}^n) = \langle\, pt, \ldots, pt \,\rangle$$

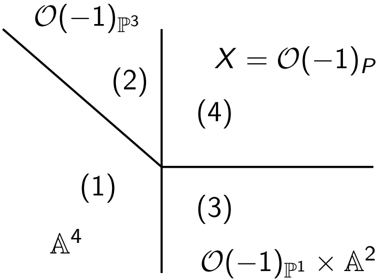
  where $\kappa = n + 1$ (Beilinson's theorem).
- If $\kappa = 0$ we have a *flop* and derived categories are equivalent.

Toric varieties

Let $(\mathbb{C}^*)^r$ act on a vector space V.

- There are many GIT quotients ("*phases*"). Each phase X_i is a toric variety.
- The space of characters has a wall-and-chamber structure, the *secondary fan*.
- A single wall crossing $X_i \rightsquigarrow X_j$ is a VGIT construction $U/\mathbb{C}^*$ where $U \subset V$ is the semi-stable locus for a character on the wall.
- We can decompose $D^b(X_i)$ by wall-crossing repeatedly and applying theorem from previous slide.
- If X_i is compact then $D^b(X_i)$ decomposes into copies of $D^b(pt)$, if not there will be bigger pieces.

Toric varieties



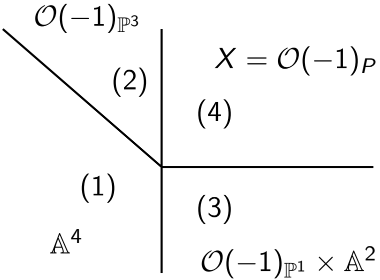
Example

Let $(\mathbb{C}^*)^2$ act on $\mathbb{C}^6$ with weights:

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}$$

Here $P = \mathbb{P}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1))_{\mathbb{P}^1}$.

Toric varieties



Example

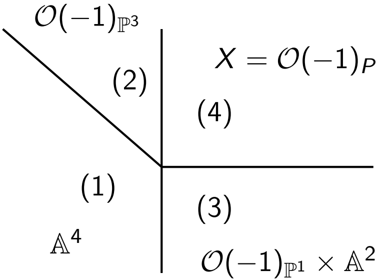
(1) $\rightsquigarrow$ (2). Blows up the origin in $\mathbb{A}^4$.

$$D^b(\mathcal{O}(-1)_{\mathbb{P}^3}) = \langle\, \mathbb{A}^4,\ pt,\ pt,\ pt \,\rangle$$

(2) $\rightsquigarrow$ (4). Blows up $\mathcal{O}(-1)_{\mathbb{P}^1}$.

$$D^b(X) = \langle\, \mathcal{O}(-1)_{\mathbb{P}^3},\ \mathcal{O}(-1)_{\mathbb{P}^1} \,\rangle = \langle\, \mathbb{A}^4,\ pt,\ pt,\ pt,\ \mathcal{O}(-1)_{\mathbb{P}^1} \,\rangle$$

Toric varieties



Example

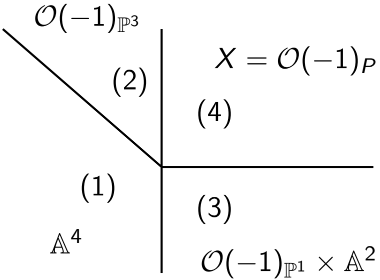
Now go a different way. (1) $\rightsquigarrow$ (3) blows up $\mathbb{A}^2$.

$$D^b(\mathcal{O}(-1)^1_{\mathbb{P}} \times \mathbb{A}^2) = \langle\, \mathbb{A}^4,\ \mathbb{A}^2 \,\rangle$$

(3) $\rightsquigarrow$ (4) blows up $\mathbb{P}^1$.

$$D^b(X) = \langle\, \mathcal{O}(-1)^1_{\mathbb{P}} \times \mathbb{A}^2,\ \mathbb{P}^1,\ \mathbb{P}^1 \,\rangle = \langle\, \mathbb{A}^4,\ \mathbb{A}^2,\ \mathbb{P}^1,\ \mathbb{P}^1 \,\rangle$$

Toric varieties



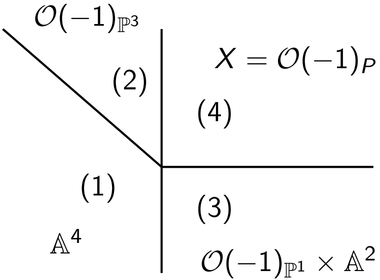
Example

(1) $\rightsquigarrow$ (2) $\rightsquigarrow$ (4) gives $D^b(X) = \langle\, \mathbb{A}^4,\ pt,\ pt,\ pt,\ \mathcal{O}(-1)_{\mathbb{P}^1}\, \rangle$.

(1) $\rightsquigarrow$ (3) $\rightsquigarrow$ (4) gives $D^b(X) = \langle\, \mathbb{A}^4,\ \mathbb{A}^2,\ \mathbb{P}^1,\ \mathbb{P}^1\, \rangle$.

But $\mathcal{O}(-1)_{\mathbb{P}^1}$ and $\mathbb{P}^1$ are toric varieties and their derived categories can also be decomposed.

Toric varieties



Example

(1) $\rightsquigarrow$ (2) $\rightsquigarrow$ (4) gives $D^b(X) = \langle\, \mathbb{A}^4,\ pt,\ pt,\ pt,\ \mathbb{A}^2,\ pt \,\rangle$.

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Toric varieties

Theorem (Kite-S.)

These semi-orthogonal decompositions of the derived categories of toric varieties satisfy the Jordan-Hölder property: the 'irreducible components' and their multiplicities are independent of choices.

- In the example we quotiented by $(\mathbb{C}^*)^2$ and the decomposition took 2 steps. For rank r it will take r steps.
- Proof not very hard.
- Jordan-Hölder property fails in general for semi-orthogonal decompositions [Bondal, Kalck, Kuznetsov, Böhning-Graf von Bothmer-Sosna].

Calabi-Yau toric varieties

Suppose $\mathbb{C}^*$ acts on U and $Z = U^{\mathbb{C}^*}$ is connected and $\kappa = 0$. Then recall $D^b(X_+) \cong D^b(X_-)$. In fact the theory gives $\mathbb{Z}$-many equivalences:

$$\Phi_k : D^b(X_+) \xrightarrow{\sim} D^b(X_-)$$

Theorem (Halpern-Leistner–Shipman)

The autoequivalence $\Phi_1^{-1}\Phi_0$ is the twist around a spherical functor:

$$F : D^b(Z) \longrightarrow D^b(X_+)$$

If $D^b(Z)$ has a semi-orthogonal decomposition then F has a corresponding factorization.

Calabi-Yau toric varieties

Let $(\mathbb{C}^*)^r$ act on a vector space V through $SL(V)$.

- All phases are Calabi-Yau.
- All phases are derived equivalent.
- Wall-crossing gives many autoequivalences of each phase.
- Physics/mirror symmetry predicts:

$$\pi_1(\text{Fayet-Iliopoulos parameter space}) \curvearrowright D^b(X_i)$$

The FI parameter space is the base of the Hori-Vafa mirror
$\approx$ complexification of space of GIT stability conditions.
$\approx$ stringy Kähler moduli space of X_i.

FI parameter space

Take the *secondary toric variety* $X^\vee$ defined by the secondary fan. Observe:

Phases $\longleftrightarrow$ toric fixed points in $X^\vee$.

Wall $\longleftrightarrow$ toric rational curve $C_{i,j}$ connecting two fixed points.

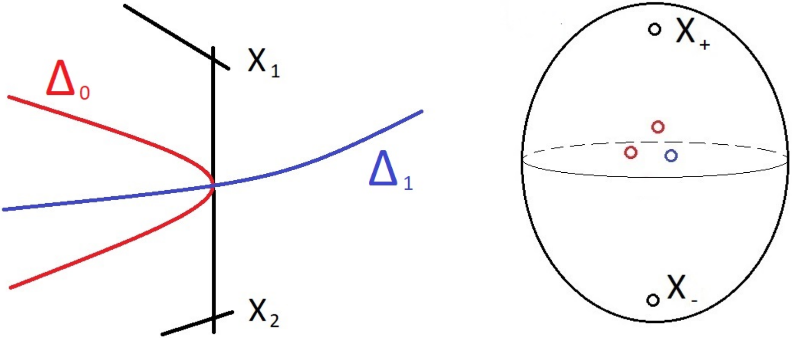
The *FI parameter space* is the open set in $X^\vee$ obtained by deleting:

1. The toric boundary.
2. The *GKZ discriminant locus*, a non-toric hypersurface

$$\Delta = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_r \quad \subset X^\vee$$

which may have several irreducible components.

FI parameter space



Loop from X_1 to X_2 and back again $\rightsquigarrow$ the wall-crossing autoequivalence of $D^b(X_1)$.

It should factor according to (i) the components of Δ, (ii) their intersection multiplicities with $C_{1,2}$.

FI parameter space

Recall that the wall-crossing autoequivalence is the twist around a spherical functor $D^b(Z) \to D^b(X_1)$. Here Z is itself a toric variety (probably not Calabi-Yau). So $D^b(Z)$ has a semi-orthogonal decomposition $\implies$ the autoequivalence factors.

Fact: the 'irreducible components' $D^b(Y_i)$ that could occur in $D^b(Z)$ biject with the components of Δ.

Conjecture (Aspinwall-Plesser-Wang, Kite-S.)

The multiplicity of a component $D^b(Y_i) \subset D^b(Z)$ agrees with the intersection multiplicity of Δ_i with $C_{1,2}$.

Theorem (Kite-S.)

This is true in the rank 2 case.

THE END.