On diagonals of flag bundles I.p. Moscow 20.11.15

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with Shizuo Kaji

Ex. $G = G_r(V)$: Grassmannian parametrizing all r-subspaces of a vector space V.

$0 \to S \to V_G \to Q \to 0$ tautological sequence of vector bundles on G

$$\begin{array}{ccc} & G \times G & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ G & & G \end{array}$$

$Hom(p_1^* S_1, p_2^* Q_2)$ of rank $=\dim G$

$$\begin{array}{c} Hom(p_1^* S_1, p_2^* Q_2) \\ \downarrow \uparrow s \\ G \times G \supset \Delta \text{ diagonal} \end{array}$$

$$p_1^* S_1 \hookrightarrow p_1^* V_{G_1} = V_{G_1 \times G_2} = p_2^* V_{G_2} \to p_2^* Q_2$$

vanishes exactly along Δ

induces section s

$$\Delta = Z(s)$$

"Diagonal property" (D): A variety X has (D) if

$\exists$ E v.b., $rk E = \dim X$ and $\exists s \in \Gamma(X \times X, E)$ s.t. $Z(s) = \Delta$

$$\begin{array}{c} E \\ \downarrow \\ X \times X \supset \Delta \end{array}$$

X has (D) $\Rightarrow$ X nonsingular

Any nonsingular curve has (D)

If X_1, X_2 have (D), then $X_1 \times X_2$ has (D)

Thm (Fulton, P) The flag varieties SL_n/P have (D).

- proof later
- related to theory of Schubert polynomials of Lascoux - Schützenberger.

Schubert polys are polynomial lifts of the classes (2
of Schubert varieties in cohomology of SL_n/B.
There is a scalar product on the polynomial ring for which the Schubert polynomials and their duals form adjont bases. The reproducing kernel of this scalar product is the top Schubert polynomial = top Chern class of the bundle realizing (D) for SL_n/B.

"Weak point property" (P) : for some $x \in X$ $\exists E$ $\mathrm{rk}\, E = \dim X$ $\begin{array}{c} \downarrow \uparrow \\ X \end{array}$

and a section s of E s.t. $Z(s) = x$.

$(D) \Rightarrow (P)$

Similar properties in topology: X sm. cpt. conn. mfd

(D_r) $\exists E \to X \times X$ smooth real of rank $\dim X$, $\exists$ smooth $s \in \Gamma(X \times X, E)$ $s \pitchfork s_E$
s.t. $Z(s) = \Delta$.

(D_c) $\dim X = 2m$, $E \to X \times X$ smooth complex of complex rk m,
$s \in \Gamma(X \times X, E)$ $s \pitchfork s_E$ s.t $Z(s) = \Delta$.

(P_r), (P_c) ...

$(D_r) \Rightarrow (P_r)$, $(D_c) \Rightarrow (P_c)$, $(D) \Rightarrow (D_c) \Rightarrow (P_r)$

An example of result in topology:

Thm (Pati-P-Srinivas) S^n has (D_r) iff
$n = 1, 2, 4, 8$.

G/B for other groups

G simple, simply connected alg. gp.
$B \subset G$ Borel, $T \subset B$ max. torus, G/B flag mld.
$\dim_{\mathbb{C}} G/B = m$; when there exists a complex vector bundle E of complex rank m on G/B s.t. $c_m(E)$ is the class of a point in $H^{2m}(G/B, \mathbb{Z})$?

<u>Thm</u> (Kaji, P) For G of type B_i $(i \geq 3)$, D_i $(i \geq 4)$ G_2, F_4 and E_i $(i = 6, 7, 8)$, the flag manifold G/B has not (P_c), and hence it has not (D_c).

<u>Pf</u> $X(T)$ group of characters of T
$K(G/B)$ Grothendieck group
Atiyah-Hirzebruch homomorphism

$$\beta_1 : S^{\bullet}(X(T)) \longrightarrow K(G/B)$$

$\lambda \in X(T)$ $\quad e^{\lambda} \longmapsto L_\lambda = G \times_B \mathbb{C}_\lambda$, line bundle on G/B

Here we regard the T-representation $\mathbb{C}_\lambda$ as a B-represent. letting the nilradical of act trivially

<u>Thm</u> (A-H, ..., Kostant-Kumar) β_1 is surjective.

In $S^1(X(T))$, every element is $\mathbb{Z}$-comb. of monomials $e^{\lambda_1} \cdot \ldots \cdot e^{\lambda_k}$, $\lambda_i \in X(T)$; $\beta_1(e^{\lambda_1} \ldots e^{\lambda_k}) = L_{\lambda_1 + \ldots + \lambda_k}$

<u>Cor.</u> In $K(G/B)$, the class of any bundle is a $\mathbb{Z}$-comb. of classes of line bundles L_μ for some $\mu \in X(T)$.

Borel characteristic homomorphism

$$c: S^{\cdot}(X(T)) \to H^*(G/B, \mathbb{Z})$$

$\lambda \in X(T)$, $e^{\lambda} \mapsto c_1(L_\lambda)$.

<u>Cor.</u> The Chern classes of any vector bundle on G/B are in the image of c.

<u>Def</u> The smallest positive integer t_G s.t. $t_G \cdot (\text{class of a point}) \in \mathrm{Im}(c)$ is called the "torsion index" of G.

<u>Thm.</u> (Borel, ...) $t_G = 1 \iff G$ is of type A_i or C_i.

This implies the theorem.

Type C_i ?

<u>Prop.</u> $Sp(2n, \mathbb{C})/B$ has $(P_{\mathbb{C}})$.

Surfaces with $(D)/k=\bar{k}$ include ruled surfaces

<u>Prop.</u> (PPS) For any smooth curve C and any rank 2 vector bundle E on C, $P(E)$ has (D).

<u>Prop. (KP)</u> Suppose that X is a projective variety with (P). Then for any vector bundle E on X, $P(E)$ has (P).

$\mathcal{O}(1)$

<u>Pf</u>
$$\begin{tikzcd} A \arrow[d, shift right] \\ x \in X \arrow[u, shift right, "s"'] \end{tikzcd} \quad Z(s) = x$$
give (P) for X

Use $G^1(E)$ instead of $P(E)$, $\mathrm{rk}\, E = r$.
There exists m s.t. $E \otimes \mathcal{O}(m)$ has $r-1$ sections $\{t_i\}$ which are independent at x.

$G^1(E(m)) = G^1(E) = F$ $\quad H = \pi^* A \oplus \mathcal{O}(1)^{\oplus r-1}$

$$\begin{array}{c} F \\ \downarrow \pi \\ X \end{array}$$

$s: F \to H$: on 1st summand we take: $\pi^* t$

on the last $r-1$ summands :

$$\underset{\pi^*(t_i)}{F \longrightarrow \pi^* E} \twoheadrightarrow \mathcal{O}(1) \quad i = 1, \dots, r-1$$

$$Z(s) = (x, (r-1)\text{-subspace of } E_x \text{ spanned by } (t_i)_x) \in G_{r-1}(E_x) = G^1(E_x)$$

$= \text{point} \implies G^1(E)$ has (P). $\square$

(D):

E vector bundle of rank n on X over a field.

$d_\cdot : 0 < d_1 < d_2 < \dots < d_{k-1} < d_k = n$ integers

$d_\cdot$-flag: $V_1 \subset V_2 \subset \dots \subset V_{k-1} \subset V_k = E_x$ $\quad \underset{x \in X}{\dim V_i = d_i}$

$\pi : Fl_{d_\cdot}(E) \to X$ flag bundle of $d_\cdot$-flags in the fibers of E.

E.g. $d_1 = d < d_2 = n$ $\quad G_d(E)$ - Grassmann bundle of d-subspaces in the fibers of E

$d = 1$ $\quad P(E)$ projective bundle of lines in the fibers of E.

Thm (K-P) (i) If X has (D_r) and $E \to X$ is a smooth real bundle, then $Fl^{\mathbb{R}}_{d_\cdot}(E)$ has (D_r).

(ii) If X has (D_c) and E is a smooth complex bundle, then $Fl^{\mathbb{C}}_{d_\cdot}(E)$ has (D_c).

X has (D), if $\exists$ v.b. $A, B \to X$, $\mathrm{rk}\, A + \mathrm{rk}\, B = \dim X$, (6

a section s of X and a section t of $B_{Z(s)}$ s.t.

$Z(t) = \Delta$.

Thm (K;P) If X has (D), then $Fl_d(E)$ has (D).

$\boxed{\text{чай-перерыв}}$

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H $\quad Fl_{d_\cdot}(E) \xrightarrow{\pi} X$

$$S_1 \subset S_2 \subset \cdots \subset S_{k-1} \subset S_k = \pi^* E \xrightarrow{q_1} Q_1 \xrightarrow{q_2} Q_2 \to \cdots \to Q_k = 0$$

$\mathrm{rk}\, S_i = d_i \;,\; Q_i = E/S_i$

Composition of Grassmann bundles:

$$Fl_{d_\cdot}(E) \to \cdots \to G_{d_3-d_2}(Q_2) \to G_{d_2-d_1}(Q_1) \to G_{d_1}(E) \to X$$

$$\dim Fl_{d_\cdot}(E) = \dim X + \sum_{i=1}^{k-1} (d_i - d_{i-1})(n - d_i)$$

$$F = Fl_{d_\cdot}(E) \;;\; F_1 = F_2 = F \;,\; p_i : F_1 \times F_2 \to F_i$$

Construct a vector bundle H on $F_1 \times F_2$

$k = 2 \qquad H = \mathrm{Hom}(p_1^* S_1, p_2^* Q_1)$

$$k \geq 3 \qquad \varphi : \bigoplus_{i=1}^{k-1} \mathrm{Hom}(p_1^* S_i, p_2^* Q_i) \to \bigoplus_{i=1}^{k-2} \mathrm{Hom}(p_1^* S_i, p_2^* Q_{i+1})$$

$$\sum_{i=1}^{k-1} h_i \longmapsto \sum_{i=1}^{k-2} \left(h_{i+1}|S_i - q_{i+1} \circ h_i \right)$$

Lemma φ is surjective.

$$H = \ker \varphi \;,\; \mathrm{rk}\, H = \sum_{i=1}^{k-1} (d_i - d_{i-1})(n - d_i).$$

Pf of thm

$$\begin{array}{c} G \\ \downarrow \uparrow s \\ X \times X \end{array} \quad Z(s) = \Delta_X$$

On $F_1 \times F_2$:

$G' = (\pi_1 \times \pi_2)^* G \qquad s' = (\pi_1 \times \pi_2)^*(s)$

$$Z = Z(s') = (\pi_1 \times \pi_2)^{-1}(\Delta_X) \subset F_1 \times F_2$$

$q_1, q_2 : X \times X \to X$ projections

On $X \times X$: $(q_1^* E)_{\Delta_X} = (q_2^* E)_{\Delta_X} \Rightarrow$

On $F_1 \times F_2$ $h_i : (p_1^* S_i)_Z \to (p_1^* E_{F_1})_Z = (p_2^* E_{F_2})_Z \to (p_2^* Q_i)_Z$

$i = 1, \dots, k-1$

These homomorphisms give rise to a section

$$h = \sum h_i' \in \Gamma\left(Z, \bigoplus_{i=1}^{k-1} Hom(p_1^* S_i, p_2^* Q_i)_Z\right)$$

Have on Z: $h_{i+1}|S_i = q_{i+1} \circ h_i$

(indeed since h_i and h_{i+1} factorize through E the two displayed homomorphisms from $(p_1^* S_i)_Z$ to $(p_2^* Q_{i+1})_Z$ are equal).

Hence $\varphi \circ h = 0$, so h induces a section $t \in \Gamma(Z, H_Z)$. Have $rk(\mathcal{E}') + rk\, H = \dim F$.

$Z(t) = \Delta_F$:

$\Delta \subset Z(t)$: taut. seq. on Grassmannians are complexes.

$f \in Z$, if $t(f) = 0$, then $f \in \Delta_F$.

Having defined s', t globally, it suffices to show the assertion locally.

$$Fl_{d_\cdot}(E) = \text{base} \times \underset{F_x}{\underset{\shortparallel}{Fl_{d_\cdot}(E_x)}}$$

for $X = pt$ this is a proof that SL_n/P has (D).

$$\pi_1(f) = \pi_2(f) = x$$

$$f = (L_1 \subset \cdots \subset L_{k-1} \subset L_k = E_x,\ M_1 \subset \cdots \subset M_{k-1} \subset M_k = E_x) \in F_x \times F_x$$

restriction h_i to $F_x \times F_x$

$$p_1^* S_i \to p_1^* V_{F_x} = V_{F_x \times F_x} = p_2^* V_{F_x} \to p_2^* Q_i$$

A $f = ((L_i), (M_i))$ it becomes

$$L_i \hookrightarrow V \twoheadrightarrow V/M_i$$

$$t(f) = 0 \Rightarrow L_i = M_i \Rightarrow Z(t) = \Delta_F. \ \square$$

Additions on (D)

Pati - P - Srinivas

(D) vs. cohomology

A line bundle $\mathcal{L} \downarrow X$ is cohomology trivial (c.t.) if $H^i(X, \mathcal{L}) = 0 \ \forall i \geq 0$

Any smooth proj. curve supports a c.t. l.b.

Any abelian variety — " —

<u>Thm</u> (i) $X \xleftarrow{p_1} X \times X \xrightarrow{p_2} X$ Suppose $Pic(X \times X) = p_1^* Pic(X) \oplus p_2^* Pic(X)$

If X has (D), then $\exists$ c.t. $\mathcal{L} \downarrow X$ s.t. $\det(E) = p_1^* \mathcal{L}^{-1} \otimes p_2^* (\mathcal{L} \otimes \omega_X^{-1})$.

(ii) If $\dim X = 2$ and $\exists$ c.t. l.b. on X, then X has (D).

Almost classified surfaces with (D)

A general K3 surface (smooth quartic in $\mathbb{P}^3$) has not (D).

If $X \sim$ ruled
abelian
K3 with 2 disjoint smooth rat'l curves (e.g. Kummer)
elliptic with section
Enriques
hyperelliptic
then X has (D).

If $\mathrm{Pic}\, X = \mathbb{Z}$, $\Gamma(X, \mathcal{O}_X(1)) \neq 0$ and X has (D)
then $X = \mathbb{P}^2$.

Prop. X sm. proj. dim $X \geq 3$ $\mathrm{Pic}\, X = \mathbb{Z}$. If X has (D) and $H^0(X, \mathcal{O}_X(1)) \neq 0$, then X is Fano and $\omega_X \cong \mathcal{O}_X(-n)$ $n \geq 2$

Cor. $X \subset \mathbb{P}^n$ sm. c.i of multidegree $(d_1, \dots, d_r)$ s.t. $r < n-3$ and $\sum d_i \geq n$, Then X has not (D).

Q_3 has not (D), Q_5 Q_7 ... have not (D).

Abelian varieties (Debarre) $(D) \sim (P)$

1. The Jacobian of a smooth proj. curve has (D).
2. $\exists$ non PPAV in dim ≥ 3 which fail to have (D).

Conj. (D) characterizes Jacobians among PPAV's with Picard number 1.