# The life and work of Hoene-Wroński

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# Una breve cronologia

**A short biography of Józef Maria Hoene-Wroński:**

| | |
|---|---|
| 1776 | — born on August 23 in Wolsztyn; |
| 1794 | — joins the Polish army; |
| 1795–1797 | — serves in the Russian army; |
| 1797–1800 | — studies in Germany; |
| 1800 | — comes to France and joins the Polish Legions in Marseilles; |
| 1803 | — publishes his first work *Critical philosophy of Kant*; |
| 1810 | — marries V. H. Sarrazin de Montferrier; |
| 1853 | — dies on August 9 in Neuilly near Paris. |

# Dagherrotipo


Józef Maria Hoene-Wroński, daguerreotype from the Kórnik Library

# Tre problemi generali

1. Discovering the relation between matter and energy (note Wroński's incredibly deep insight here);
2. the formation of celestial objects;
3. the formation of the universe from the celestial objects.

The most visible characteristic of Wroński's work is his determination to base all knowledge on philosophy, by finding the general principle, from which all other knowledge would follow.

# Parigi

## 4. Paris: solving equations, algorithms, and continued fractions.

In 1811 Wroński publishes *Philosophy of Mathematics*. Even earlier he has singled out two aspects of mathematical endeavor:

1. theories, whose aim is the study of the essence of mathematical notions;
2. algorithmic techniques, which comprise all methods leading to the computation of mathematical unknowns.

The second point above shows that Wroński was a pioneer of "algorithmic" thinking in mathematics. He gave many clever algorithms for solving important mathematical problems.

# Algoritmo di Euclide

1. Consider two normed polynomials $F(x)$ and $G(x)$. Suppose that $\deg(F) \geq \deg(G)$. Performing multiple division of $F(x)$ and $G(x)$:

$$F = *\, G + c_1 R_1, \quad G = *\, R_1 + c_2 R_2, \quad R_1 = *\, R_2 + c_3 R_3, \quad \ldots .$$

Successive coefficients " $*$ " are uniquely determined polynomials of the variable $x$ such that

$$\deg G(x) > \deg R_1(x) > \deg R_2(x) > \deg R_3(x) > \ldots .$$

Instead of the "ordinary" Euclid's algorithm, were $c_1 = c_2 = c_3 = \ldots = 1$ and where $R_i(x)$ are *rational* functions of the variable $x$ and roots of $F(x)$ and $G(x)$, one can choose $c_i$ in such a way that the successive remainders $R_i(x)$ are polynomials of the variable $x$ and of those roots. These remainders are called *subresultants*.

# Problema di fattorizzazione

Using the algorithm in 1. and passing to the limit, Wronski also solved the following important *factorization problem*:

*Suppose that we are given a normed polynomial* $W(x) \in \mathbb{C}[x]$, *which does not have roots of absolute value* 1. *Let*

$$A = \{a \in \mathbb{C} : W(a) = 0, \; |a| > 1\}, \; B = \{b \in \mathbb{C} : W(b) = 0, \; |b| < 1\}.$$

*Extract the factor* $\prod_{b \in B}(x - b)$ *from* $W(x)$ .

# Funzioni di Schur

We give – following Lascoux – Wroński's solution in terms of the *Schur functions*. The coefficients of the polynomial $W(x)$, from which we wish to extract a factor corresponding to roots of absolute value smaller than $1$, are the elementary symmetric functions of $A \cup B$ – the sum of (multi)sets $A$ and $B$. Therefore the problem boils down to expressing elementary symmetric functions of the variable $B$, in terms of the Schur functions of $A \cup B$, denoted by $S_J(A+B)$. Let the cardinality of the (multi)set $A$ be equal to $m$. For $I \in \mathbb{N}^m$ i $k, p \in \mathbb{N}$ we define

$$I(k) := (i_1{+}k, \ldots, i_m{+}k)\,, \qquad 1^p I(k) := (1, \ldots, 1, i_1{+}k, \ldots, i_m{+}k)$$

(where $1$ is present $p$ times).

# Theorema di Wroński

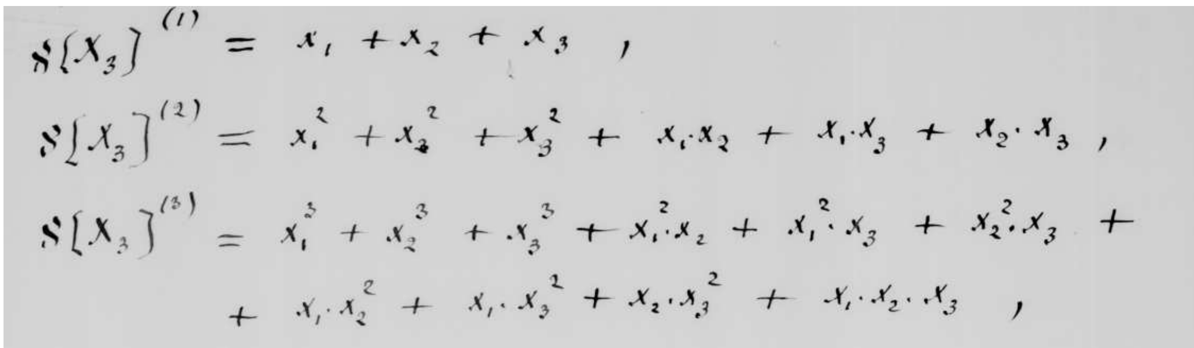
Let the cardinality of the (multi)set $B$ be equal $n$. Wroński's theorem states that

$$\prod_{b \in B}(x-b) = \lim_{k\to\infty}\left(\sum_{0\le p\le n}(-1)^p x^{n-p}\frac{S_{1^p I(k)}(A+B)}{S_{I(k)}(A+B)}\right)$$

(here $I$ is an arbitrary sequence in $\mathbb{N}^m$). Notice that the solution uses a passage to the limit; therefore besides algebraic arguments, transcendental arguments are also used. Therefore, we see that Wroński, looking for roots of algebraic equation, *did not* limit himself to using radicals.

# Funzioni di Aleph

We note that Wroński also used symmetric functions of the variables $X_n = \{x_1, \ldots, x_n\}$ and especially the aleph functions:



$$\aleph[X_3]^{(1)} = x_1 + x_2 + x_3 \,,$$
$$\aleph[X_3]^{(2)} = x_1^2 + x_2^2 + x_3^2 + x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 \,,$$
$$\aleph[X_3]^{(3)} = x_1^3 + x_2^3 + x_3^3 + x_1^2 \cdot x_2 + x_1^2 \cdot x_3 + x_2^2 \cdot x_3 + x_1 \cdot x_2^2 + x_1 \cdot x_3^2 + x_2 \cdot x_3^2 + x_1 \cdot x_2 \cdot x_3 \,,$$

A fragment of a manuscript where $n = 3$

$$\sum_{i \geq 0} \aleph[X_n]^i := \prod_{j=1}^{n} (1 - x_j)^{-1} \,,$$

i.e. $\aleph[X_n]^i$ is the sum of all monomials of degree $i$. Wroński considered these functions as "more important" than the more "popular" elementary symmetric functions.

# Frazioni continue

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}, \qquad e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\ddots}}}},$$

$$\pi = 3 + \cfrac{1}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{\ddots}}}}$$

# Frazioni continue

$$f(x) = c_0 + \cfrac{g(x)}{c_1 + \cfrac{g(x-\xi)}{c_2 + \cfrac{g(x-2\xi)}{c_3 + \cfrac{g(x-3\xi)}{\ddots}}}}$$

expressing unknown parameters
$c_0$, $c_1$, $c_2$, $\ldots$ in terms of $f(0)$, $f(\xi)$, $f(2\xi)$, $\ldots$.

This is connected with the *Thiele continued fractions*.

# Frazioni continue

A few years later Wroński gave even more general continued fractions, considering instead of one auxiliary function $g(x)$ a system of functions $g_0(x), g_1(x), \ldots$, vanishing at various points:

$$0 = g_0(\alpha_0) = g_1(\alpha_1) = g_2(\alpha_2) = \ldots.$$

Wroński gives determinantal formulas $f(\alpha_i)$, $i = 0, 1, \ldots$, for the coefficients $c_j$, $j = 0, 1, \ldots$, in the expansion

$$f(x) = c_0 + \cfrac{g_0(x)}{c_1 + \cfrac{g_1(x)}{c_2 + \cfrac{g_2(x)}{c_3 + \cfrac{g_3(x)}{\ddots}}}}.$$

These expansions are connected with the *Stieltjes continued fractions* and play a key role in interpolation theory.

# Quattro periodi più uno

1. works of the scholars of East and Egypt: concrete mathematics was practiced, without the ability to raise to abstract concepts;

2. the period from Tales and Pythagoras until the Renaissance: the human mind rose to the level of high abstraction, however the discovered mathematical truths existed as unrelated facts, not connected by a general principle ;

3. the activity of Tartaglia, Cardano, Ferrari, Cavalieri, Bombelli, Fermat, Vieta, Descartes, Kepler, . . . : mathematics rose to the study of general laws thanks to algebra, but the achievements of mathematics are still "individual" — the "general" laws of mathematics were still unknown;

# Quattro periodi più uno

4. the discovery of differential and integral calculus by Newton and Leibniz, expansion of functions into series, continued fractions popularized by Euler, generating functions of Laplace, theory of analytic functions of Lagrange. The human mind was able to raise from the consideration of quantities themselves to the consideration of their creation in the calculus of functions, i.e. differential calculus.

The fifth period should begin with the discovery of the Highest Law and algorithmic techniques by Wroński ; the development of mathematics should be based on the most general principles – “absolute ones” – encompassing all of mathematics. This is because all the methods and theories up to that time do not exhaust the essence of mathematics, as they lack a general foundation, from which everything would follow.

# Lettere ai governanti d'Europa

$a$ – degree of anarchy, $d$ – degree of despotism.

$$a = \left(\frac{m+n}{m}\cdot\frac{m+n}{n}\right)^{p-r}\cdot\left(\frac{m}{n}\right)^{p+r} = \left(\frac{m+n}{n}\right)^{2p}\cdot\left(\frac{m}{m+n}\right)^{2r},$$

$$d = \left(\frac{m+n}{m}\cdot\frac{m+n}{n}\right)^{r-p}\cdot\left(\frac{n}{m}\right)^{p+r} = \left(\frac{n}{m+n}\right)^{2p}\cdot\left(\frac{m+n}{m}\right)^{2r},$$

where $m =$ number of members of the liberal party, $p =$ the deviation of the philosophy of the liberal party from true religion, $n =$ the number of members of the religious party, $r$ $=$ the deviation of the religious party from true philosophy.

For France $p = r = 1$, $a = \left(\frac{m}{n}\right)^2$, $d = \left(\frac{n}{m}\right)^2$. Moreover, $\frac{m}{n} = 2$, and so $a = 4$, $d = \frac{1}{4}$. This means that, political freedom – in France of Wroński's time – is 4 times the normal one, and the authority of the government is 1/4 of what is essential.

# Filosofia

**8. Philosophy.** I. Kant's philosophy was the starting point of the philosophy of Wroński, who has transformed it into metaphysics in a way analogous to Hegel's approach. Wroński has not only created a philosophical system, but also its applications to politics, history, economy, law, psychology, music and education. Existence and knowledge followed from the Absolute, which he understood either as God, or as the spirit, wisdom, a thing in itself. He did not describe it, but he tried to infer from it a universal law, which he called "The Law of Creation".

# Matematica

**9. Mathematics: The Highest Law, Wronskians.**
Essentially,
Wroński worked on mathematical analysis and algebra. We have already discussed Wroński's contributions to algebra. In analysis he was especially interested in expanding functions in a *power series* and *differential equations*. Wroński's most interesting mathematical idea was his general method of expanding a function $f(x)$ of one variable $x$ into a series

$$f(x) = c_1 g_1(x) + c_2 g_2(x) + c_3 g_3(x) + \dots ,$$

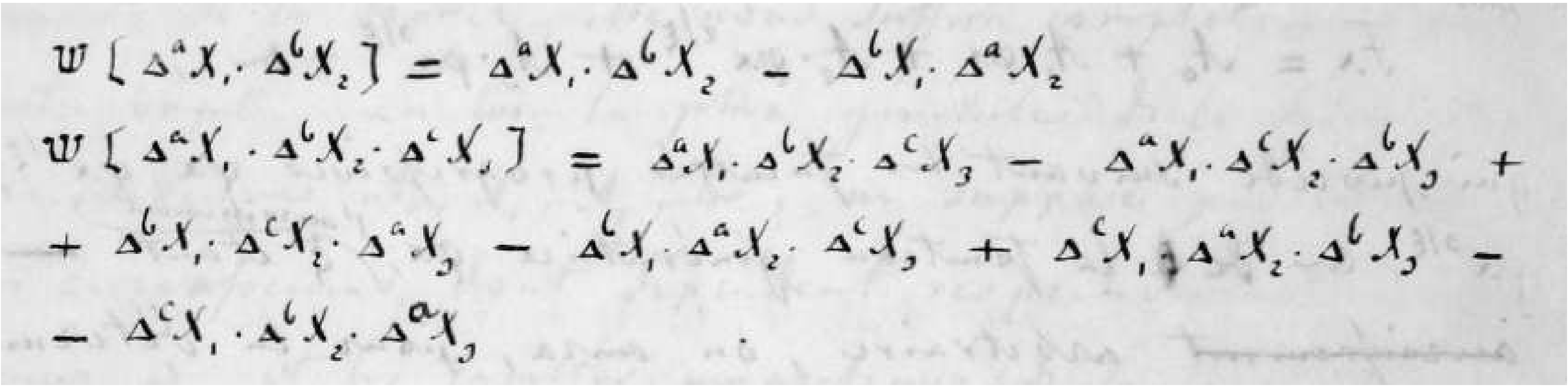
when the sequence of functions $g_1(x), g_2(x), \dots$ is given beforehand, and $c_1, c_2, \dots$ are numerical coefficients to be determined. Wroński gave his method of finding the coefficients $c_i$ the rank of *The Highest Law*.

# Wroński e Banach

These ideas were used much later by Stefan Banach, who formulated them precisely and enriched them with topological concepts, and proved that the Highest Law of Hoene-Wroński can be used in what is called today a *Banach space*, as well as in the theory of *orthogonal polynomials*. I will mention here a little known letter of Hugo Steinhaus to Zofia Pawlikowska-Brożek from 1969:

*Maybe you will find the following fact concerning two Polish mathematicians — Hoene-Wroński and Banach — interesting. In Lwow we had an edition of Wroński's work published in Paris and Banach showed me the page written by the philosopher which discussed the "Highest Law"; apparently Banach has proven to me, that Wroński is not discussing messianic philosophy — the matter concerns expanding arbitrary functions into orthogonal ones.*

# Manoscritto



$$\mathfrak{W}[\Delta^a X_1 \cdot \Delta^b X_2] = \Delta^a X_1 \cdot \Delta^b X_2 - \Delta^b X_1 \cdot \Delta^a X_2$$

$$\mathfrak{W}[\Delta^a X_1 \cdot \Delta^b X_2 \cdot \Delta^c X_3] = \Delta^a X_1 \cdot \Delta^b X_2 \cdot \Delta^c X_3 - \Delta^a X_1 \cdot \Delta^c X_2 \cdot \Delta^b X_3 + \Delta^b X_1 \cdot \Delta^c X_2 \cdot \Delta^a X_3 - \Delta^b X_1 \cdot \Delta^a X_2 \cdot \Delta^c X_3 + \Delta^c X_1 \cdot \Delta^a X_2 \cdot \Delta^b X_3 - \Delta^c X_1 \cdot \Delta^b X_2 \cdot \Delta^a X_3$$

A fragment of a manuscript of Wroński with combinatorial sums

# Wrońskiano

The Wrońskian of $n$ real functions $f_1(x), f_2(x), \ldots, f_n(x)$, which are $(n-1)$ times differentiable, is defined and denoted as follows:

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

# Palazzo a Kórnik

Palace in Kórnik near Poznań, containing a collection of Wroński's *original handwritten* manuscripts. They may contain interesting — unknown to the public yet — mathematical results.

# Sulla sua tombe in Neuilly:

THE SEARCH OF TRUTH IS A TESTIMONY TO THE POSSIBILITY OF FINDING IT.

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RICERCA DELLA VERITA E' LA TESTIMONIANZA DELLA POSSIBILITA' DI TROVARLA.