On positivity of Thom polynomials

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red herring: it was thought that $c_1^2 - 2c_2$ is positive but is not.

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Whenever we speak about the classes of algebraic cycles, we always mean their *Poincaré dual classes* in cohomology.

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If a *singularity class* Σ is "stable" (e.g. closed under the contact equivalence), then $\mathcal{T}^\Sigma$ depends on $c_i(TM - f^*TN)$.

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Giambelli's formula: The class of a *Schubert variety* in a Grassmannian is given by a Schur polynomial of the tautological bundle on it.

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Theorem. *(PP+AW, 2006) Let Σ be a nontrivial stable singularity class. Then for any partition I the coefficient α_I in*

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For any singularity class Σ, the coefficients in

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Every germ of a Lagrangian submanifold of V is the image of W via a certain germ symplectomorphism.

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A Lagrange singularity class is any closed pure dimensional algebraic subset of $\mathcal{J}^k(V)$ which is invariant w.r.t. the action of H.

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Theorem. *(P, 1986)* $\Omega_I = \widetilde{Q}_I(R^*)$, *where R is the tautological subbundle on $LG(V)$.*

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Theorem. *(MM+PP+AW, 2007) For any Lagrange singularity class Σ, the Thom polynomial $\mathcal{T}^\Sigma$ is a nonnegative combination of $\widetilde{Q}$-functions.*

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$$i^*[\Sigma] = j^*[C]\,.$$

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Additionally, we assume that Σ is stable with respect to enlarging the dimension of W.

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Since any changes of coordinates of W and ξ induce holomorphic contactomorphisms of $V \oplus \xi$, any Legendre singularity class Σ defines $\Sigma(W,\xi) \subset \mathcal{J}^k(W,\xi)$.

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The Chern classes $a_i = c_i(A)$ generate the cohomology $H^*(LG(V,\omega),\mathbf{Z}) \cong H^*(\mathcal{J}^k(W,\xi),\mathbf{Z})$ as an algebra over $H^*(X,\mathbf{Z})$.

Let us fix an approximation of $BU(1) = \bigcup_{n \in \mathbf{N}} \mathbf{P}^n$, that is we set $X = \mathbf{P}^n$, $\xi = \mathcal{O}(1)$. Let $W = \mathbf{1}^n$ be the trivial bundle of rank n.

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and is often denoted by $\mathcal{T}^{\Sigma}$. It is written in terms of the generators a_i and $s = c_1(\xi)$.

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$$F_h^+ := \bigoplus_{i=1}^{h} \alpha_i\,, \qquad F_h^- := \bigoplus_{i=1}^{h} \alpha_{n-i+1}^* \otimes \xi\,, \qquad (h = 1, 2, \ldots, n)$$

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Consider two Borel groups $B^\pm \subset Sp(V,\omega)$, preserving the flags $F_\bullet^\pm$. The orbits of $B^\pm$ in $LG(V,\omega)$ form two "opposite" cell decompositions $\{\Omega_I(F_\bullet^\pm, \xi)\}$ of $LG(V,\omega)$, indexed by

All that is functorial w.r.t. the automorphisms of the lines ξ and α_i's, (they form a torus $(\mathbf{C}^*)^{n+1}$). Thus the construction of the cell decompositions can be repeated for bundles ξ and $\{\alpha_i\}_{i=1}^n$ over any base X. We get a Lagrange Grassmann bundle

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The subsets

$$Z_{I\lambda}^- := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F_\bullet^-, \xi)$$

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Theorem. *Fix* $I \subset \rho$ *and* λ*. Suppose that the vector bundle* $\mathcal{J}$ *is globally generated. Then, in* $\mathcal{J}$*, the intersection of* $\Sigma(W,\xi)$ *with the closure of any* $\pi^{-1}(Z^{-}_{I\lambda})$ *is represented by a nonnegative cycle.*

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Consider the following three cases: the base is always $X = \mathbf{P}^n$ and

$$\xi_1 = \mathcal{O}(-2)\,, \qquad \alpha_1 = \mathcal{O}(-1)\,,$$

$$\xi_2 = \mathcal{O}(1)\,, \qquad \alpha_2 = \mathbf{1}\,,$$

$$\xi_3 = \mathcal{O}(-3)\,, \qquad \alpha_3 = \mathcal{O}(-1)\,,$$

We obtain symplectic bundles $V_i = \alpha_i^{\oplus n} \oplus (\alpha_i^* \otimes \xi_i)^{\oplus n}$ with twisted symplectic forms ω_i for $i = 1, 2, 3$.

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We have $e_{I,a,b} = e_{I,0,0}\ v_1^a v_2^b$ and $e_{I,0,0} = [\overline{\Omega_I(F^+_\bullet, \xi)}]$.

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Divide $H^*(LG(V,\omega), \mathbf{Q})$ by the relation: $q \cdot v_1 = p \cdot v_2$ that is specializing the parameters to $v_1 = p \cdot t$, $v_2 = q \cdot t$, we obtain the ring $H^*(LG(V^{(p,q)}, \omega^{(p,q)}), \mathbf{Q})$ isomorphic to the ring of Legendrian characteristic classes in degrees up to n (provided that $c_1(\xi) = v_2 - 3v_1$ is not specialized to 0.)

Theorem. *If p and q are nonnegative, $q - 3p \neq 0$, then the Thom polynomial is a nonnegative combination of the* $[\overline{\Omega_I(F_\bullet^+, \xi)}]\, t^i$*'s.*

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Theorem. *The Thom polynomial of a Legendre singularity class* Σ *is a combination:*

$$\mathcal{T}^{\Sigma} = \sum_{j \geq 0} \sum_{I} \alpha_{I,j}\ \widetilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) \cdot t^j\,.$$

Here $t = \frac{1}{2} c_1(\xi^*)$*,* $I \subset \rho$*, and* $\alpha_{I,j}$ *are nonnegative integers.*

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We know that Tp^Σ is nonzero. One shows that Tp^Σ, specialized with $f^*TC = \mathbf{1}$ i.e. $t = 0$, is also nonzero. The assertion follows from the equation.

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