

Thom polynomials and Schur functions I

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(30.08.2005; revised 15.05.2006)

To the memory of Professor Stanisław Balcerzyk (1932-2005)

Abstract

We give the Thom polynomials – via their Schur function expansions – for the singularities $I_{2,2}$, and A_3 associated with maps $(\mathbf{C}^\bullet, 0) \rightarrow (\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 0$. Moreover, for the singularities A_i (with any parameter $k \geq 0$) we analyze the “first approximation” $F^{(i)}$ to the Thom polynomial. Our computations combine the characterization of Thom polynomials via the “method of restriction equations” of Rimanyi et al. with the techniques of (super) Schur functions.

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2000 *Mathematics Subject Classification.* 05E05, 14N10, 57R45.

Key words and phrases. Thom polynomials, singularities, global singularity theory, classes of degeneracy loci, Schur functions, resultants.

*Research partially supported by KBN grant 2P03A 024 23. Part of the work was done during the author’s stay at the METU in Ankara (May 2005), supported by TÜBİTAK.

1 Introduction

The global behavior of singularities is governed by their *Thom polynomials* (cf. [27], [11], [1], [10], [24]). Knowing the Thom polynomial of a singularity η , denoted \mathcal{T}^η , one can compute the cohomology class represented by the η -points of a map. We do not attempt here to survey all activities related to computations of Thom polynomials – note that in Kleiman’s survey [11], the reader can find a summary of an early work of Thom, Porteous, Ronga, Menn, Sergeraert, Lascoux, and Roberts on Thom polynomials presented in an algebro-geometric framework.

In the present paper, following a series of papers by Rimanyi et al. [25], [23], [24], [6], [2], we study the Thom polynomials for the singularities $I_{2,2}$ and A_i associated with maps $(\mathbf{C}^\bullet, 0) \rightarrow (\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 0$.

The way of obtaining the thought Thom polynomial is through the solution of a system of linear equations, which is fine when we want to find one concrete Thom polynomial, say, for a fixed k . However, if we want to find the Thom polynomials for a series of singularities, associated with maps $(\mathbf{C}^\bullet, 0) \rightarrow (\mathbf{C}^{\bullet+k}, 0)$ with k as a parameter, we have to solve *simultaneously* a countable family of systems of linear equations. As stated by Rimanyi in [24], p. 512 :

“However, another challenge is to find Thom polynomials containing k as a parameter.”

We do it here for the restriction equations for the above mentioned singularities. In fact, the obtained functional equations in symmetric functions are of independent interest. The main novelty of the present paper over the previous articles on Thom polynomials, is an extensive use of *Schur functions*. Namely, instead of using *Chern monomial expansions* (as the authors of all previous papers constantly do), we use *Schur function expansions*. This puts a more transparent structure on computations of Thom polynomials. We hope that an expression for $\mathcal{T}^{I_{2,2}}$ (Theorem 16), as well as our expression for \mathcal{T}^{A_3} (Theorem 22), provide a support of this claim. For example, we get in this way some recursive formulas (cf., e.g., Lemma 12) that are not so easy to find using other bases, in particular the Chern monomial basis. In fact, recursions play a prominent role in the formulas of the present paper, cf. Eqs. (53) and (75).

Another feature of using the Schur function expansions for Thom polynomials is that in all known to us cases (not only those treated in the present paper), all the coefficients are nonnegative.¹

To be more precise, we use here (the specializations of) *supersymmetric* Schur functions, also called *super-S-functions* or Schur functions *in difference of alphabets* together with their three basic properties: *vanishing*, *cancell-*

¹Added 15.05.2006: Positivity of Schur function expansions of Thom polynomials has been recently proved by A. Weber and the author in [22].

tion and factorization, (cf. [3], [15], [20], [21], [16], [8], and [13]). These functions contain *resultants* among themselves. Their geometric significance was illuminated in the 80's in the author's study of *polynomials supported on degeneracy loci* (cf. [19]). In fact, in the present paper we use the point of view of that article to some extent. More precisely, given a morphism $F \rightarrow E$ of vector bundles, where $\text{rank}(E) = m$ and $\text{rank}(F) = n$, by a *j-polynomial* we understand a \mathbf{Z} -linear combination of the Schur functions $S_I(E - F)$, where partitions I are such that $(n-j)^{m-j} \subset I$ but $(n-j+1)^{m-j+1} \not\subset I$ ². In some sense, a *j-polynomial* is a “typical” polynomial supported on the j th degeneracy locus D_j of the morphism (in the sense of [19], see also [8]).

A Thom polynomial is a sum of such *j-polynomials* associated with the corresponding morphism of tangent bundles (cf. the next section). The polynomial $\mathcal{T}^{I_{2,2}}$ is a *single j-polynomial* whereas \mathcal{T}^{A_3} for $k > 0$ is the sum of two *j-polynomials* (for two consecutive *j*'s). We first determine its part related to the smaller rectangle, and then add necessary “corrections” related to the larger rectangle. For the singularities A_i (any i), we describe the *j-polynomial* part of the Thom polynomial for the largest possible j , a sort of the “first approximation” to \mathcal{T}^{A_i} (here, the corresponding rectangle is a single row of length $k + 1$).

We give here a complete proof of our formula for the Thom polynomial for the singularity $I_{2,2}$. As for the singularities A_i , we reprove the formulas of Thom and Ronga for A_1 and A_2 , and announce the Schur function expression for A_3 . We outline a proof in the Appendix. Another expression for A_3 in terms of monomials in the Chern classes was announced by Berzi, Feher, and Rimanyi in [2]. We also illustrate the methods used by reproving the result of Gaffney giving the Thom polynomial for A_4 with $k = 0$ (this was also done by Rimanyi [23] – our approach uses more extensively Schur functions). For any singularity A_i , we describe the *j-polynomial* part (denoted by $F_r^{(i)}$) of the Thom polynomial for the largest possible j .

In our calculations we use extensively the λ -ring approach to symmetric functions developed mainly in Lascoux's book [13].

2 Recollections on Thom polynomials

Our main reference for this section is [24]. We start with recalling what we shall mean by a “singularity”. Let $k \geq 0$ be a fixed integer. By *singularity* we shall mean an equivalence class of stable germs $(\mathbf{C}^\bullet, 0) \rightarrow (\mathbf{C}^{\bullet+k}, 0)$, where $\bullet \in \mathbf{N}$, under the equivalence generated by right-left equivalence (i.e. analytic reparametrizations of the source and target) and suspension (by suspension of a germ κ we mean its trivial unfolding: $(x, v) \mapsto (\kappa(x), v)$).

²We shall use the same name for (formal) alphabets \mathbb{A} and \mathbb{B} of cardinalities m and n instead of the alphabets of the Chern roots of E and F .

We recall that the *Thom polynomial* \mathcal{T}^η of a singularity η is a polynomial in the formal variables c_1, c_2, \dots that after the substitution

$$c_i = c_i(f^*TY - TX) = [c(f^*TY)/c(TX)]_i, \quad (1)$$

for a general map $f : X \rightarrow Y$ between complex analytic manifolds, evaluates the Poincaré dual $[\overline{\eta(f)}]$ of the cycle carried by the closure of

$$\eta(f) = \{x \in X : \text{the singularity of } f \text{ at } x \text{ is } \eta\}. \quad (2)$$

By *codimension of a singularity* η , $\text{codim}(\eta)$, we shall mean $\text{codim}_X(\eta(f))$ for such an f . The concept of the polynomial \mathcal{T}^η comes from Thom's fundamental paper [27]. For a detailed discussion of the *existence* of Thom polynomials, see, e.g., [1]. Thom polynomials associated with group actions were studied in [10]. In fact, the above is the "usual case" with singularities in the region where moduli (continuous families) of singularities do not occur. This will be the case of the singularities studied in the present paper. Indeed, the codimension of all these singularities does not exceed $6k+8$, the lowest codimension when moduli of singularities start.

According to Mather's classification, singularities are in one-to-one correspondence with finite dimensional \mathbf{C} -algebras. We shall use the following notation:

- A_i (of Thom-Boardman type Σ^{1_i}) will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$;
- $I_{a,b}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x, y]]/(xy, x^a + y^b)$, $b \geq a \geq 2$;
- $III_{a,b}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x, y]]/(xy, x^a, y^b)$, $b \geq a \geq 2$ (here $k \geq 1$).

Our computations of Thom polynomials for some of the above singularities, shall use the method which stems from a sequence of papers by Rimanyi et al. [25], [23], [24], [6], [2]. We sketch briefly this method, referring the interested reader for more details to these papers, the main references being the last three mentioned items.

Let $k \geq 0$ be a fixed integer, and let $\eta : (\mathbf{C}^\bullet, 0) \rightarrow (\mathbf{C}^{\bullet+k}, 0)$ be a stable singularity with a prototype $\kappa : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^{n+k}, 0)$. The *maximal compact subgroup of the right-left symmetry group*

$$\text{Aut } \kappa = \{(\varphi, \psi) \in \text{Diff}(\mathbf{C}^n, 0) \times \text{Diff}(\mathbf{C}^{n+k}, 0) : \psi \circ \kappa \circ \varphi^{-1} = \kappa\} \quad (3)$$

of κ will be denoted by G_η . Even if $\text{Aut } \kappa$ is much too large to be a finite dimensional Lie group, the concept of its maximal compact subgroup (up to conjugacy) can be defined in a sensible way (cf. [24]). It is clear that G_η can be chosen so that images of its projections to the factors $\text{Diff}(\mathbf{C}^n, 0)$ and

$\text{Diff}(\mathbf{C}^{n+k}, 0)$ are linear. Its representations via the projections on the source \mathbf{C}^n and the target \mathbf{C}^{n+k} will be denoted by $\lambda_1(\eta)$ and $\lambda_2(\eta)$. The vector bundles associated with the universal principal G_η -bundle $EG_\eta \rightarrow BG_\eta$ using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ will be called E'_η and E_η . The *total Chern class of the singularity η* is defined in $H^\bullet(BG_\eta; \mathbf{Z})$ by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)}. \quad (4)$$

The *Euler class* of η is defined in $H^{2\text{codim}(\eta)}(BG_\eta; \mathbf{Z})$ by

$$e(\eta) := e(E'_\eta). \quad (5)$$

In the following theorem we collect the information from [24], Theorem 2.4 and [6], Theorem 3.5, needed for the calculations in the present paper.

Theorem 1 *Suppose, for a singularity η , that the Euler classes of all singularities of smaller codimension than $\text{codim}(\eta)$, are not zero-divisors³. Then we have*

- (i) *if $\xi \neq \eta$ and $\text{codim}(\xi) \leq \text{codim}(\eta)$, then $\mathcal{T}^\eta(c(\xi)) = 0$;*
- (ii) *$\mathcal{T}^\eta(c(\eta)) = e(\eta)$.*

This system of equations (taken for all such ξ 's) determines the Thom polynomial \mathcal{T}^η in a unique way.

To use this method of determining the Thom polynomials for singularities, one needs their classification, see, e.g., [4].

In the forthcoming sections, we shall use these equations to compute Thom polynomials. Sometimes it will be convenient not to work with the whole maximal compact subgroup G_η but with its suitable subgroup; this subgroup should be, however, as “close” to G_η as possible (cf. [24], p. 502).

Being challenged by [24], p. 512 and especially [2], we shall find Thom polynomials *containing k as a parameter* – this seems to be a (much) more difficult task than computing Thom polynomials for separate values of k , because one must solve *simultaneously* a countable family of systems of linear equations.

To effectively use Theorem 1 we need to study the maximal compact subgroups of singularities. We recall the following recipe from [24] pp. 505–507. Let η be a singularity whose prototype is $\kappa : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^{n+k}, 0)$. The germ κ is the miniversal unfolding of another germ $\beta : (\mathbf{C}^m, 0) \rightarrow (\mathbf{C}^{m+k}, 0)$ with $d\beta = 0$. The group G_η is a subgroup of the maximal compact subgroup of the algebraic automorphism group of the local algebra Q_η of η times the unitary group $U(k-d)$, where d is the difference between the minimal number

³This is the so-called “Euler condition” (loc.cit.). The Euler condition holds true for the singularities in the present paper.

of relations and the number of generators of Q_η . With β well chosen, G_η acts as right-left symmetry group on β with representations μ_1 and μ_2 . The representations λ_1 and λ_2 are

$$\lambda_1 = \mu_1 \oplus \mu_V \quad \text{and} \quad \lambda_2 = \mu_2 \oplus \mu_V, \quad (6)$$

where μ_V is the representation of G_η on the unfolding space $V = \mathbf{C}^{n-m}$ given, for $\alpha \in V$ and $(\varphi, \psi) \in G_\eta$, by

$$(\varphi, \psi) \alpha = \psi \circ \alpha \circ \varphi^{-1}. \quad (7)$$

For example, for the singularity of type A_i : $(\mathbf{C}^\bullet, 0) \rightarrow (\mathbf{C}^{\bullet+k}, 0)$, we have $G_{A_i} = U(1) \times U(k)$ with

$$\mu_1 = \rho_1, \quad \mu_2 = \rho_1^{i+1} \oplus \rho_k, \quad \mu_V = \bigoplus_{j=2}^i \rho_1^j \oplus \bigoplus_{j=1}^i (\rho_k \otimes \rho_1^{-1}), \quad (8)$$

where ρ_j denotes the standard representation of the unitary group $U(j)$. Hence we obtain assertion (i) of the following

Proposition 2 (i) *Let $\eta = A_i$; for any k , writing x and y_1, \dots, y_k for the Chern roots of the universal bundles on $BU(1)$ and $BU(k)$,*

$$c(A_i) = \frac{1 + (i+1)x}{1+x} \prod_{j=1}^k (1 + y_j), \quad (9)$$

$$e(A_i) = i! x^i \prod_{j=1}^k (ix - y_j) \cdots (2x - y_j)(x - y_j). \quad (10)$$

(ii) *Let $\eta = I_{2,2}$; for $k \geq 0$, $G_\eta = U(1) \times U(1) \times U(k)$, and writing x_1, x_2 for the Chern roots of the universal bundles on two copies of $BU(1)$ and on $BU(k)$,*

$$c(I_{2,2}) = \frac{(1+2x_1)(1+2x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^k (1 + y_j), \quad (11)$$

$$e(I_{2,2}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) \prod_{j=1}^k (x_1 - y_j)(x_2 - y_j)(x_1 + x_2 - y_j). \quad (12)$$

(iii) *Let $\eta = III_{2,2}$; for $k \geq 1$, $G_\eta = U(2) \times U(k-1)$, and writing x_1, x_2 and y_1, \dots, y_{k-1} for the Chern roots of the universal bundles on $BU(2)$ and $BU(k-1)$,*

$$c(III_{2,2}) = \frac{(1+2x_1)(1+2x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k-1} (1 + y_j), \quad (13)$$

$$e(III_{2,2}) = (x_1 x_2)^2 (x_1 - 2x_2)(x_2 - 2x_1) \prod_{j=1}^{k-1} (x_1 - y_j) \prod_{j=1}^{k-1} (x_2 - y_j). \quad (14)$$

These assertions are obtained, in a standard way, following the instructions of [24], Sect. 4.

Notational conventions Rather than the Chern classes

$$c_i(f^*TY - TX) = [f^*c(TY)/c(TX)]_i,$$

we shall use *Segre classes* S_i of the virtual bundle $TX^* - f^*(TY^*)$, i.e. complete symmetric functions $S_i(\mathbb{A} - \mathbb{B})$ for the alphabets of the *Chern roots* \mathbb{A}, \mathbb{B} of TX^* and TY^* . The reader will find in the next section a summary of algebraic properties of the functions $S_i(\mathbb{A} - \mathbb{B})$, or, more generally, Schur functions $S_I(\mathbb{A} - \mathbb{B})$ (I runs over sequences of integers) widely used in the present paper.

Moreover, it will be more handy to use, instead of k , a “shifted” parameter

$$r := k + 1. \tag{15}$$

Sometimes, we shall write the Thom polynomial as \mathcal{T}_r^η to emphasize its dependence on r . So, e.g., in our notation, the Thom polynomial for the singularity A_1 with $\text{codim}(A_1) = r$ for $r \geq 1$ (in general, $\text{codim}(A_i) = ri$), will be : $\mathcal{T}^{A_1} = \mathcal{T}_r^{A_1} = S_r$, instead of c_{k+1} as in the papers in References. In general, a Thom polynomial in terms of the c_i 's (in those papers) will be written here as a linear combination of Schur functions obtained by changing each c_i to S_i and expanding in the Schur function basis. Another example is, for $r = 1$, the Thom polynomial for A_2 : $c_1^2 + c_2$ rewritten in the present notation as $\mathcal{T}^{A_2} = \mathcal{T}_1^{A_2} = S_{11} + 2S_2$.

3 Recollections on Schur functions

In this section we collect needed notions related to symmetric functions and prove a useful Lemma 10. We adopt the point of view of [13] for what concerns symmetric functions. Namely, given a commutative ring, we treat symmetric functions as operators acting on the ring. (Here, these commutative rings are mostly \mathbf{Z} -algebras generated by the Chern roots of the vector bundles from Proposition 2.)

Definition 3 *By an alphabet \mathbb{A} , we understand a (finite) multi-set of elements in a commutative ring.*

For $k \in \mathbf{N}$, by “an alphabet \mathbb{A}_k ” we shall mean an alphabet $\mathbb{A} = (a_1, \dots, a_k)$ (of cardinality k); ditto for $\mathbb{B}_k = (b_1, \dots, b_k)$, $\mathbb{Y}_k = (y_1, \dots, y_k)$, and $\mathbb{X}_2 = (x_1, x_2)$.

Definition 4 *Given two alphabets \mathbb{A}, \mathbb{B} , the complete functions $S_i(\mathbb{A} - \mathbb{B})$ are defined by the generating series (with z an extra variable):*

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i = \prod_{b \in \mathbb{B}} (1-bz) / \prod_{a \in \mathbb{A}} (1-az). \quad (16)$$

So $S_i(\mathbb{A}-\mathbb{B})$ interpolates between $S_i(\mathbb{A})$ – the complete homogeneous symmetric function of degree i in \mathbb{A} and $S_i(-\mathbb{B})$ – the i th elementary function in \mathbb{B} times $(-1)^i$.

The notation $\mathbb{A}-\mathbb{B}$ is compatible with the multiplication of series:

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i \cdot \sum S_j(\mathbb{A}'-\mathbb{B}')z^j = \sum S_i((\mathbb{A}+\mathbb{A}')-(\mathbb{B}+\mathbb{B}'))z^i, \quad (17)$$

the sum $\mathbb{A}+\mathbb{A}'$ denoting the union of two alphabets \mathbb{A} and \mathbb{A}' .

Convention 5 *We shall often identify an alphabet $\mathbb{A} = \{a_1, \dots, a_m\}$ with the sum $a_1 + \dots + a_m$ and perform usual algebraic operations on such elements. For example, $\mathbb{A}b$ will denote the alphabet (a_1b, \dots, a_mb) . We will give priority to the algebraic notation over the set-theoretic one. In fact, in the following, we shall use mostly alphabets of variables.*

We have $(\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}) = \mathbb{A}-\mathbb{B}$, and this corresponds to simplification of the common factor for the rational series:

$$\sum S_i((\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}))z^i = \sum S_i(\mathbb{A}-\mathbb{B})z^i. \quad (18)$$

Definition 6 *Given a sequence $I = (i_1, i_2, \dots, i_k) \in \mathbf{Z}^k$, and alphabets \mathbb{A} and \mathbb{B} , the Schur function $S_I(\mathbb{A}-\mathbb{B})$ is*

$$S_I(\mathbb{A}-\mathbb{B}) := \left| S_{i_p+p-q}(\mathbb{A}-\mathbb{B}) \right|_{1 \leq p, q \leq k}. \quad (19)$$

We shall mostly use the case when I is a partition $I = (0 \leq i_1 \leq \dots \leq i_k)$. In fact, by permuting the columns we see that any determinant of the form (19) is either zero or is, up to sign, such a determinant indexed by a partition. These functions are often called *supersymmetric Schur functions* or *Schur functions in difference of alphabets*. Their properties were studied, among others, in [3], [15], [20], [21], [16], [8], and [13]; in the present paper, we shall use the notation and conventions from this last item).

For example,

$$S_{33344}(\mathbb{A}-\mathbb{B}) = \begin{vmatrix} S_3 & S_4 & S_5 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_5 & S_6 \\ 1 & S_1 & S_2 & S_4 & S_5 \\ 0 & 1 & S_1 & S_3 & S_4 \end{vmatrix},$$

where S_i means $S_i(\mathbb{A}-\mathbb{B})$.

By Eq. (18), we get the following *cancellation property*:

$$S_I((\mathbb{A} + \mathbb{C}) - (\mathbb{B} + \mathbb{C})) = S_I(\mathbb{A} - \mathbb{B}). \quad (20)$$

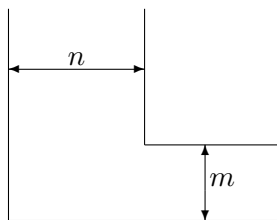
In the following, we shall identify partitions with their *Young diagrams*, as is customary.

We record the following property (*loc.cit.*), justifying the notational remark from the end of Section 2; for a partition I ,

$$S_I(\mathbb{A} - \mathbb{B}) = (-1)^{|I|} S_J(\mathbb{B} - \mathbb{A}) = S_J(\mathbb{B}^* - \mathbb{A}^*), \quad (21)$$

where J is the conjugate partition of I (i.e. the consecutive rows of J are equal to the corresponding columns of I), and \mathbb{A}^* denotes the alphabet $\{-a_1, -a_2, \dots\}$.

Fix two positive integers m and n . We shall say that a partition $I = (0 < i_1 \leq i_2 \leq \dots \leq i_k)$ is contained in the (m, n) -hook if either $k \leq m$, or $k > m$ and $i_{k-m} \leq n$. Pictorially, this means that the Young diagram of I is contained in the “tickened” hook:



We record the following *vanishing property*. Given alphabets \mathbb{A} and \mathbb{B} of cardinalities m and n , if a partition I is not contained in the (m, n) -hook, then (*loc.cit.*):

$$S_I(\mathbb{A} - \mathbb{B}) = 0 \quad (22)$$

For example,

$$S_{4569}(\mathbb{A}_2 - \mathbb{B}_4) = S_{4569}(a_1 + a_2 - b_1 - b_2 - b_3 - b_4) = 0$$

because 4569 is not contained in the $(2, 4)$ -hook.

In fact, we have the following result (*loc.cit.*).

Theorem 7 *If \mathbb{A}_m and \mathbb{B}_n are alphabets of variables, then the functions $S_I(\mathbb{A}_m - \mathbb{B}_n)$, for I running over partitions contained in the (m, n) -hook, are \mathbf{Z} -linearly independent.*

(They form a \mathbf{Z} -basis of the Abelian group of the so-called “supersymmetric functions” (*loc.cit.*).

In the present paper by a *symmetric function*, we shall mean a \mathbf{Z} -linear combination of the operators $S_I(\bullet)$. By the *degree* of such a symmetric function, we shall mean the largest weight $|I|$ of a partition I involved in its Schur function expansion.

The following useful convention stems from Lascoux’s paper [14].

Convention 8 We may need to specialize a letter to 2, but this must not be confused with taking two copies of 1. To allow one, nevertheless, specializing a letter to an (integer, or even complex) number r inside a symmetric function, without introducing intermediate variables, we write \boxed{r} for this specialization. Boxes have to be treated as single variables. For example, $S_i(2) = \binom{i+1}{2}$ but $S_i(\boxed{2}) = 2^i$. A similar remark applies to \mathbf{Z} -linear combinations of variables. We have $S_2(\mathbb{X}_2) = x_1^2 + x_1x_2 + x_2^2$ but $S_2(\boxed{x_1+x_2}) = x_1^2 + 2x_1x_2 + x_2^2$, $S_{11}(\mathbb{X}_2) = x_1x_2$ but $S_{11}(\boxed{x_1+x_2}) = 0$, $S_2(3x) = 6x^2$ but $S_2(\boxed{3x}) = 9x^2$ etc.

Definition 9 Given two alphabets \mathbb{A}, \mathbb{B} , we define their resultant:

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a \in \mathbb{A}, b \in \mathbb{B}} (a-b). \quad (23)$$

This terminology is justified by the fact that $R(\mathbb{A}, \mathbb{B})$ is the classical resultant of the polynomials $R(x, \mathbb{A})$ and $R(x, \mathbb{B})$. For example, Eq. (10) can be rewritten as

$$e(A_i) = R(x + \boxed{2x} + \cdots + \boxed{ix}, \mathbb{Y}_k + \boxed{(i+1)x})$$

and Eq. (14) is

$$e(III_{2,2}) = R(\mathbb{X}_2, \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1+x_2} + \mathbb{Y}_{k-1}).$$

We have (*loc.cit.*)

$$R(\mathbb{A}_m, \mathbb{B}_n) = S_{(n^m)}(\mathbb{A} - \mathbb{B}) = \sum_I S_I(\mathbb{A}) S_{(n^m)/I}(-\mathbb{B}), \quad (24)$$

where the sum is over all partitions $I \subset (n^m)$.

When a partition is contained in the (m, n) -hook and at the same time it contains the rectangle (n^m) , then we have the following *factorization property* (*loc.cit.*): for partitions $I = (i_1, \dots, i_m)$ and $J = (j_1, \dots, j_k)$,

$$S_{(j_1, \dots, j_k, i_1+n, \dots, i_m+n)}(\mathbb{A}_m - \mathbb{B}_n) = S_I(\mathbb{A}) R(\mathbb{A}, \mathbb{B}) S_J(-\mathbb{B}). \quad (25)$$

We now pass to the following function F . Fix positive integers m and n . For an alphabet \mathbb{A} of cardinality m , we set

$$F(\mathbb{A}, \bullet) := \sum_I S_I(\mathbb{A}) S_{n-i_m, \dots, n-i_1, n+|I|}(\bullet), \quad (26)$$

where the sum is over partitions $I = (i_1 \leq i_2 \leq \cdots \leq i_m \leq n)$.

Lemma 10 For a variable x and an alphabet \mathbb{B} of cardinality n ,

$$F(\mathbb{A}, x - \mathbb{B}) = R(x + \mathbb{A}x, \mathbb{B}). \quad (27)$$

Proof. For a fixed partition $I = (i_1 \leq i_2 \leq \dots \leq i_m \leq n)$, it follows from the factorization property (25) that

$$S_{n-i_m, \dots, n-i_1, n+|I|}(x - \mathbb{B}) = S_{(n^m)/I}(-\mathbb{B}) R(x, \mathbb{B}) x^{|I|}.$$

Hence, using $S_I(\mathbb{A}x) = S_I(\mathbb{A})x^{|I|}$, a standard factorization of a resultant, and Eq. (24), we have

$$\begin{aligned} F(\mathbb{A}, x - \mathbb{B}) &= \sum_I S_I(\mathbb{A}) S_{(n^m)/I}(-\mathbb{B}) R(x, \mathbb{B}) x^{|I|} \\ &= \sum_I S_I(\mathbb{A}x) S_{(n^m)/I}(-\mathbb{B}) R(x, \mathbb{B}) \\ &= R(\mathbb{A}x, \mathbb{B}) R(x, \mathbb{B}) = R(x + \mathbb{A}x, \mathbb{B}). \end{aligned}$$

The lemma has been proved. \square

We end this section, with an example illustrating the way we shall use symmetric functions in our computations. We consider the singularity $III_{2,2}$ (with parameter $r \geq 2$) whose codimension is $2r + 2$. We know that $\mathcal{T}^{III_{2,2}}$ is equal to the Thom polynomial for Σ^2 , and the latter polynomial was computed in [18].

But let us apply directly Theorem 1 to the singularity $III_{2,2}$. By virtue of Proposition 2, for $r \geq 3$, the equations characterizing the Thom polynomial for $III_{2,2}$ are:

$$P(-\mathbb{B}_{r-1}) = P(x - \boxed{2x} - \mathbb{B}_{r-1}) = P(x - \boxed{3x} - \mathbb{B}_{r-1}) = 0, \quad (28)$$

and additionally,

$$P(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \boxed{x_1 + x_2} - \mathbb{B}_{r-2}) = R(\mathbb{X}_2, \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1 + x_2} + \mathbb{B}_{r-2}). \quad (29)$$

Here, without loss of generality, we assume that x, x_1, x_2 , and \mathbb{B}_{r-1} are variables, and $P(\bullet)$ denotes a symmetric function. Indeed, the singularities $\neq III_{2,2}$ of codimension $\leq \text{codim}(III_{2,2})$ are: A_0, A_1, A_2 . For $r = 2$ we must add the vanishing imposed by A_3 which (similarly to $III_{2,2}$) is of codimension 6 (this is the only exception):

$$P(x - \boxed{4x} - \mathbb{B}_1) = 0. \quad (30)$$

Since the partition $(r+1, r+1)$ is not contained in the $(1, r)$ -hook, for $P = S_{r+1, r+1}(\bullet)$ we get the vanishings (28) and (30). Moreover, Eq. (29) is satisfied for this P because

$$P(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \boxed{x_1 + x_2} - \mathbb{B}_{r-2}) = R(\mathbb{X}_2, \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1 + x_2} + \mathbb{B}_{r-2}).$$

These equations characterize the Thom polynomial for $III_{2,2}$, and hence this polynomial is equal to $S_{r+1, r+1}$ in agreement with [18]. In the forthcoming computations, however, the method of restriction equations from Theorem 1 will play a principal role.

4 Thom polynomial for $I_{2,2}$

The codimension of $I_{2,2}$ (for parameter $r \geq 1$) is $3r + 1$. For $r = 1$, the Thom polynomial for $I_{2,2}$ is S_{22} (cf. [18]).

From now on, we shall assume that $r \geq 2$. For $r = 2$, the Thom polynomial for $I_{2,2}$ is (cf. [24]):

$$S_{133} + 3S_{34}.$$

By virtue of Proposition 2, the equations from Theorem 1 characterizing the Thom polynomial for $I_{2,2}$ are:

$$P(-\mathbb{B}_{r-1}) = P(x - \boxed{2x} - \mathbb{B}_{r-1}) = P(x - \boxed{3x} - \mathbb{B}_{r-1}) = 0, \quad (31)$$

and

$$P(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \mathbb{B}_{r-1}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) R(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{B}_{r-1}). \quad (32)$$

Here, without loss of generality, we assume that x , x_1 , x_2 , and \mathbb{B}_{r-1} are variables. Moreover, $P(\bullet)$ denotes a symmetric function. For the remainder of this paper, we set

$$\mathbb{D} := \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1 + x_2}. \quad (33)$$

Then, additionally, for variables x_1, x_2 and an alphabet \mathbb{B}_{r-2} , we have the vanishing imposed by $III_{2,2}$:

$$P(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0. \quad (34)$$

Indeed, the singularities $\neq I_{2,2}$ with codimension $\leq \text{codim}(I_{2,2})$ are: A_0 , A_1 , A_2 , $III_{2,2}$.

For $r \geq 1$, we set

$$P_r(\bullet) := \mathcal{T}_r^{I_{2,2}}(\bullet). \quad (35)$$

Lemma 11 (i) *A partition appearing nontrivially in the Schur function expansion of P_r contains the rectangular partition $(r + 1, r + 1)$.*

(ii) *A partition appearing nontrivially in the Schur function expansion of P_r has at most three parts.*

Proof. (i) This follows from the fact that the singularity $I_{2,2}$ belongs to the Thom-Boardman singularity Σ^2 .

(ii) We can assume that $r \geq 3$. In addition to information contained in (i), we shall use Eq. (34):

$$P_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0.$$

By virtue of (i), we can use factorization property (25) to all summands of

$$P_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = \sum_I \alpha_I S_I(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) \quad (36)$$

(we assume that $\alpha_I \neq 0$). We divide each summand of this last polynomial by the resultant

$$R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2}).$$

Suppose that the resulting factor of S_I is:

$$S_{p,q}(\mathbb{X}_2) S_J(-\mathbb{D} - \mathbb{B}_{r-2}), \quad (37)$$

cf. (25). Since $|I| = 3r + 1$, we have

$$|J| \leq r - 1. \quad (38)$$

Now, let us assume that I has more than 3 parts, that is J has more than 2 parts. This assumption (together with the inequality (38)) implies that

$$S_J(-\mathbb{B}_{r-2}) \neq 0$$

(\mathbb{B}_{r-2} is an alphabet of variables). Expanding (37), we get among summands the following one of largest possible degree $|J|$ in \mathbb{B}_{r-2} :

$$S_{p,q}(\mathbb{X}_2) S_J(-\mathbb{B}_{r-2}) \neq 0. \quad (39)$$

Take in the sum

$$\sum_I \alpha_I S_{p,q}(\mathbb{X}_2) S_J(-\mathbb{D} - \mathbb{B}_{r-2})$$

the (sub)sum of all the nonzero summands of the form (37) with the largest possible weight of J . Since Schur polynomials are independent this subsum is nonzero and moreover it is \mathbf{Z} -linearly independent of other summands both in the sum indexed by partitions with ≥ 3 parts, and as well as in that indexed by partitions with 2 parts (this last sum does not depend on \mathbb{B}_{r-2}). Hence, there is no \mathbf{Z} -linear combination of S_I 's which involve nontrivially I with more than three parts and possibly also those with 3 and 2 parts, that satisfies Eq. (34). Assertion (ii) has been proved. \square

(For example, S_{1144} cannot appear in the Schur function expansion of P_3 because $S_{1144}(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_1)$ after division by the resultant contains the summand $S_{11}(-\mathbb{B}_1) = S_2(\mathbb{B}_1)$, which does not occur in similar expressions for $S_{55}, S_{46}, S_{244}, S_{145}$.)

The following lemma gives a recursive description of P_r . Denote by τ the linear endomorphism on the \mathbf{Z} -module of Schur functions corresponding to partitions of length ≤ 3 that sends a Schur function S_{i_1, i_2, i_3} to S_{i_1+1, i_2+1, i_3+1} . Let P_r^o denote the sum of those terms in the Schur function expansion of P_r which correspond to partitions of length ≤ 2 . Note that $P_1^o = S_{22}$.

Lemma 12 *With this notation, for $r \geq 2$, we have the following recursive equation:*

$$P_r = P_r^o + \tau(P_{r-1}). \quad (40)$$

Proof. Write

$$P_r = \sum_I \alpha_I S_I = \sum_J \alpha_J S_J + \sum_K \alpha_K S_K, \quad (41)$$

where J have 2 parts and $K = (k_1, k_2, k_3)$ have 3 parts (we assume that $\alpha_I \neq 0$). We set

$$Q = \sum_K \alpha_K S_{k_1-1, k_2-1, k_3-1}, \quad (42)$$

and our goal is to show that $Q = P_{r-1}$. Since a partition I appearing nontrivially in the Schur function expansion of P_r must contain the partition $(r+1, r+1)$, then any partition K above contains the partition (r, r) . Since this last partition is not contained in the $(1, r-1)$ -hook, Eqs. (31) with r replaced by $r-1$ are automatically fulfilled by virtue of the vanishing property (22). Note that Eq. (34) is a particular case of Eq. (32). Indeed, specializing b_{r-1} to $\boxed{x_1+x_2}$ in Eq. (32), we get Eq. (34). Therefore it suffices to show that

$$Q(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) R(\mathbb{X}_2 + \boxed{x_1+x_2}, \mathbb{B}_{r-2}). \quad (43)$$

where $\mathbb{E} = \boxed{2x_1} + \boxed{2x_2}$. We apply to each summand

$$\alpha_K S_{k_1-1, k_2-1, k_3-1}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})$$

of $Q(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})$ the factorization property (25), and divide it by the resultant

$$R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-2}).$$

Suppose that the resulting factor is:

$$\alpha_K S_{a,b}(\mathbb{X}_2) S_c(-\mathbb{E} - \mathbb{B}_{r-2}), \quad (44)$$

where $(k_1 - 1, k_2 - 1, k_3 - 1) = (c, r + a, r + b)$.

Performing the same division of

$$x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) R(\mathbb{X}_2 + \boxed{x_1+x_2}, \mathbb{B}_{r-2})$$

we get $R(\boxed{x_1+x_2}, \mathbb{B}_{r-2})$. Thus the wanted equation $Q = P_{r-1}$ is equivalent to

$$\sum_{a+b+c=r-2} \alpha_K S_{a,b}(\mathbb{X}_2) S_c(-\mathbb{E} - \mathbb{B}_{r-2}) = R(\boxed{x_1+x_2}, \mathbb{B}_{r-2}). \quad (45)$$

To prove Eq. (45) we use Eqs. (32) and (41) for P_r :

$$\sum_I \alpha_I S_I(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) R(\mathbb{X}_2 + \boxed{x_1+x_2}, \mathbb{B}_{r-1}).$$

Using again the factorization property (this time w.r.t. the larger rectangle $(r+1)^2$) and dividing both sides of the last equation by the resultant

$$R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-1}).$$

we get the identity

$$\sum_{p+q+j=r-1} \alpha_I S_{p,q}(\mathbb{X}_2) S_j(-\mathbb{E} - \mathbb{B}_{r-1}) = R(\boxed{x_1+x_2}, \mathbb{B}_{r-1}). \quad (46)$$

Since

$$S_j(-\mathbb{E} - \mathbb{B}_{r-1}) = S_j(-\mathbb{E} - \mathbb{B}_{r-2}) - b_{r-1} S_{j-1}(-\mathbb{E} - \mathbb{B}_{r-2})$$

and

$$R(\boxed{x_1+x_2}, \mathbb{B}_{r-1}) = (x_1 + x_2 - b_{r-1}) R(\boxed{x_1+x_2}, \mathbb{B}_{r-2}),$$

taking the coefficients of $(-b_{r-1})$ in both sides of Eq. (46), we get the wanted Eq. (45). The lemma has been proved. \square

(For example, writing $P_3 = \alpha S_{46} + \beta S_{55} + \gamma S_{244} + \delta S_{145}$, we get that

$$\gamma S_1(-\mathbb{E} - B_1) + \delta S_1(\mathbb{X}_2) = R(\boxed{x_1+x_2}, \mathbb{B}_1)$$

by taking the coefficients of $(-b_2)$ in both sides of

$$\alpha S_2(\mathbb{X}_2) + \beta S_{11}(\mathbb{X}_2) + \gamma S_2(-\mathbb{E} - \mathbb{B}_2) + \delta S_1(-\mathbb{E} - \mathbb{B}_2) S_1(X_2) = R(\boxed{x_1+x_2}, \mathbb{B}_2).$$

Iterating Eq. (40) gives

Corollary 13 *With the above notation, we have*

$$P_r = P_r^o + \tau(P_{r-1}^o) + \tau^2(P_{r-2}^o) + \cdots + \tau^{r-1}(P_1^o). \quad (47)$$

Of course, P_r^o is uniquely determined by its value on \mathbb{X}_2 . The following result gives this value.

Proposition 14 *For any $r \geq 1$, we have*

$$P_r^o(\mathbb{X}_2) = (x_1 x_2)^{r+1} S_{r-1}(\mathbb{D}). \quad (48)$$

Proof. We use induction on r . For $r = 1, 2$, the assertion holds true. Suppose that the assertion is true for P_i^o where $i < r$. We consider the Schur function expansion of P_r :

$$P_r = \sum_I \alpha_I S_I. \quad (49)$$

Fix a partition $I = (j, r + 1 + p, r + 1 + q)$ appearing nontrivially in (49). Note that j varies from 0 to $r - 1$ because $|I| = 3r + 1$. We obtain by the factorization property (25):

$$S_I(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R \cdot S_j(-\mathbb{D} - \mathbb{B}_{r-2}) \cdot S_{p,q}(\mathbb{X}_2).$$

where $R = R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})$. Hence, using Eq. (47), we see that

$$P_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R \cdot \left(\sum_{j=0}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) \frac{P_{r-j}^o(\mathbb{X}_2)}{(x_1 x_2)^{r-j+1}} \right). \quad (50)$$

By the induction assumption, for positive $j \leq r - 1$,

$$P_{r-j}^o(\mathbb{X}_2) = (x_1 x_2)^{r-j+1} S_{r-1-j}(\mathbb{D}).$$

Substituting this to (50), and using the vanishing (34), we obtain

$$\sum_{j=1}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) S_{r-1-j}(\mathbb{D}) + \frac{P_r^o(\mathbb{X}_2)}{(x_1 x_2)^{r+1}} = 0. \quad (51)$$

But we also have, by a formula for addition of alphabets,

$$\sum_{j=1}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) S_{r-1-j}(\mathbb{D}) + S_{r-1}(\mathbb{D}) = S_{r-1}(-\mathbb{B}_{r-2}) = 0. \quad (52)$$

Combining Eqs. (51) and (52) gives

$$P_r^o(\mathbb{X}_2) = (x_1 x_2)^{r+1} S_{r-1}(\mathbb{D}),$$

that is, the assertion of the induction. The proof of the proposition is now complete. \square

This proposition allows us to write down the Schur function decomposition of P_r^0 . The coefficients of $S_{22}, S_{34}, S_{46}, S_{58}, \dots$ (in general, $S_{i,2i-2}$), are given by the coefficients 1, 3, 7, 15, \dots in the expansion of the series:

$$\begin{aligned} & \frac{1}{(1-z)(1-2z)} \\ &= 1 + 3z + 7z^2 + 15z^3 + 31z^4 + 63z^5 + 127z^6 + \dots \end{aligned}$$

Denote these coefficients by $d_{11}, d_{21}, d_{31}, d_{41}, \dots$. Moreover, we set $d_{1j} = d_{2j} = 0$ for $j \geq 1$, $d_{3j} = d_{4j} = 0$ for $j \geq 2$, $d_{5j} = d_{6j} = 0$ for $j \geq 3$ etc. Next, denoting by d_{rj} the coefficient of $S_{r+j,2r+1-j}$ in P_r^0 , where $j = 1, \dots, [(r+1)/2]$, we have the recursive formula

$$d_{i+1,j} = d_{i,j-1} + d_{ij}. \quad (53)$$

We get the following matrix:

$$\begin{array}{cccccc}
d_{11} & 0 & 0 & 0 & 0 & \dots \\
d_{21} & 0 & 0 & 0 & 0 & \dots \\
d_{31} & d_{32} & 0 & 0 & 0 & \dots \\
d_{41} & d_{42} & 0 & 0 & 0 & \dots \\
d_{51} & d_{52} & d_{53} & 0 & 0 & \dots \\
d_{61} & d_{62} & d_{63} & 0 & 0 & \dots \\
d_{71} & d_{72} & d_{73} & d_{74} & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
=
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & \dots \\
3 & 0 & 0 & 0 & 0 & \dots \\
7 & 3 & 0 & 0 & 0 & \dots \\
15 & 10 & 0 & 0 & 0 & \dots \\
31 & 25 & 10 & 0 & 0 & \dots \\
63 & 56 & 35 & 0 & 0 & \dots \\
127 & 119 & 91 & 35 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Summing up, we have

Proposition 15 For $r \geq 1$, we have

$$P_r^o = \sum_{j=1}^{[(r+1)/2]} d_{rj} S_{r+j, 2r+1-j}, \quad (54)$$

where the d_{ij} 's are defined above.

We have the following values of $P_1^o, P_2^o, \dots, P_6^o$:

$$\begin{aligned}
& S_{22}, 3S_{34}, 7S_{46} + 3S_{55}, 15S_{58} + 10S_{67}, 31S_{6,10} + 25S_{79} + 10S_{88}, \\
& 63S_{7,11} + 56S_{8,10} + 35S_{99}.
\end{aligned}$$

Combining Proposition (15) with Eq. (47), we get

Theorem 16 For $r \geq 1$ the Thom polynomial for $I_{2,2}$, with parameter r , equals

$$P_r = \sum_{k=0}^{r-1} \sum_{\{j \geq 1: k+2j \leq r+1\}} d_{r-k,j} S_{k,r+j, 2r-k-j+1}. \quad (55)$$

We have the following values of $P_1, P_2 = \tau(P_1) + P_2^o, \dots, P_6 = \tau(P_5) + P_6^o$:

$$\begin{aligned}
& S_{22} \\
& S_{133} + 3S_{34} \\
& S_{244} + 3S_{145} + 7S_{46} + 3S_{55} \\
& S_{355} + 3S_{256} + 7S_{157} + 3S_{166} + 15S_{58} + 10S_{67} \\
& S_{466} + 3S_{367} + 7S_{268} + 3S_{277} + 15S_{169} + 10S_{178} + 31S_{6,10} + 25S_{79} + 10S_{88} \\
& S_{577} + 3S_{489} + 7S_{379} + 3S_{388} + 15S_{2,7,10} + 10S_{289} + 31S_{1,7,11} + 25S_{1,8,10} + 10S_{189} + \\
& 63S_{7,11} + 56S_{8,10} + 35S_{99}.
\end{aligned}$$

5 Towards Thom polynomials for A_i

The following function $F_r^{(i)}$ will be basic for computing the Thom polynomials for A_i ($i \geq 1$). We set

$$F_r^{(i)}(\bullet) := \sum_J S_J(\boxed{2} + \boxed{3} + \cdots + \boxed{i}) S_{r-j_{i-1}, \dots, r-j_1, r+|J|}(\bullet), \quad (56)$$

where the sum is over partitions $J \subset (r^{i-1})$, and for $i = 1$ we understand $F_r^{(1)}(\bullet) = S_r(\bullet)$.

Example 17 *We have*

$$\begin{aligned} F_r^{(2)}(\bullet) &= \sum_{j \leq r} S_j(\boxed{2}) S_{r-j, r+j}(\bullet) = \sum_{j \leq r} 2^j S_{r-j, r+j}(\bullet), \\ F_r^{(3)}(\bullet) &= \sum_{j_1 \leq j_2 \leq r} S_{j_1, j_2}(\boxed{2} + \boxed{3}) S_{r-j_2, r-j_1, r+j_1+j_2}(\bullet), \\ F_r^{(4)}(\bullet) &= \sum_{j_1 \leq j_2 \leq j_3 \leq r} S_{j_1, j_2, j_3}(\boxed{2} + \boxed{3} + \boxed{4}) S_{r-j_3, r-j_2, r-j_1, r+j_1+j_2+j_3}(\bullet), \\ F_1^{(i)}(\bullet) &= \sum_{j \leq i-1} \Lambda_j(\boxed{2} + \boxed{3} + \cdots + \boxed{i}) S_{1^{i-j-1}, j+1}(\bullet), \end{aligned}$$

where $\Lambda_j(\bullet) = (-1)^j S_j(-\bullet)$.

In the following, we shall tacitly assume that x , x_1 , x_2 , and \mathbb{B}_r are variables (though many results remain valid without this assumption).

The following result gives the key algebraic property of $F_r^{(i)}$.

Proposition 18 *We have*

$$F_r^{(i)}(x - \mathbb{B}_r) = R(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}, \mathbb{B}_r). \quad (57)$$

Proof. The assertion follows from Lemma 10 with $m = i - 1$, $n = r$, and $\mathbb{A} = \boxed{2} + \boxed{3} + \cdots + \boxed{i}$. \square

Corollary 19 *Fix an integer $i \geq 1$.*

(i) *For $p \leq i$, we have*

$$F_r^{(i)}(x - \mathbb{B}_{r-1} - \boxed{px}) = 0. \quad (58)$$

(ii) *Moreover, we have*

$$F_r^{(i)}(x - \mathbb{B}_{r-1} - \boxed{(i+1)x}) = R(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}, \mathbb{B}_{r-1} + \boxed{(i+1)x}). \quad (59)$$

Proof. Substituting in Eq. (57):

$$\mathbb{B}_r = \mathbb{B}_{r-1} + \boxed{px}$$

for $p \leq i$, and, respectively,

$$\mathbb{B}_r = \mathbb{B}_{r-1} + \boxed{(i+1)x},$$

we get the assertions. \square

Theorem 20 ([27], [26]) *The polynomials S_r and $\sum_{j \leq r} 2^j S_{r-j, r+j}$ are Thom polynomials (with parameter r) for the singularities A_1 and A_2 .*

Proof. Since only A_0 has smaller codimension than A_1 , and only A_0, A_1 are of smaller codimension than A_2 , the equations from Theorem 1 characterizing these Thom polynomials are:

$$P(-\mathbb{B}_{r-1}) = 0, \quad P(x - \mathbb{B}_{r-1} - \boxed{2x}) = R(x, \mathbb{B}_{r-1} + \boxed{2x}) \quad (60)$$

for A_1 , and

$$\begin{aligned} P(-\mathbb{B}_{r-1}) = P(x - \mathbb{B}_{r-1} - \boxed{2x}) = 0, \\ P(x - \mathbb{B}_{r-1} - \boxed{3x}) = R(x + \boxed{2x}, \mathbb{B}_{r-1} + \boxed{3x}) \end{aligned} \quad (61)$$

for A_2 . Hence the claim follows from Corollary 19. \square

Since the singularities $\neq A_3$, whose codimension is $\leq \text{codim}(A_3)$ are: A_0, A_1, A_2 and, for $r \geq 2$, $III_{2,2}$ (cf. [4]), Theorem 1 yields the following equations characterizing \mathcal{T}^{A_3} :

$$P(-\mathbb{B}_{r-1}) = P(x - \mathbb{B}_{r-1} - \boxed{2x}) = P(x - \mathbb{B}_{r-1} - \boxed{3x}) = 0, \quad (62)$$

$$P(x - \mathbb{B}_{r-1} - \boxed{4x}) = R(x + \boxed{2x} + \boxed{3x}, \mathbb{B}_{r-1} + \boxed{4x}) \quad (63)$$

$$P(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0. \quad (64)$$

By Corollary 19, the first four equations are satisfied by the function $F_r^{(3)}$. For $r = 1$, this means that

$$F_1^{(3)} = S_{111} + 5S_{12} + 6S_3 \quad (65)$$

is the Thom polynomial for A_3 . However, for $r \geq 2$, $F_r^{(3)}$ does not satisfy the last vanishing, imposed by $III_{2,2}$. In the following we shall “modify” $F_r^{(3)}$ in order to obtain the Thom polynomial for A_3 .

Let us discuss now A_4 for $r = 1$ (its codimension is 4). Then the singularities $\neq A_4$, whose codimension is $\leq \text{codim}(A_4)$ are: $A_0, A_1, A_2, A_3, I_{2,2}$. The Thom polynomial is

$$\mathcal{T}^{A_4} = S_{1111} + 9S_{112} + 26S_{13} + 24S_4 + 10S_{22}. \quad (66)$$

This Thom polynomial was originally computed in [9] via the desingularization method. Its alternative derivation via solving equations imposed by the above singularities was done in [23]).

It may be instructive for the reader to reprove here this result using the function $F_1^{(4)}$. In this way we show (on this relatively simple example) the method used later to more complicated singularities. A Thom polynomial is a sum of j -polynomials (cf. Introduction) associated with the bundle morphism $df : TX \rightarrow f^*TY$. In fact, $F_r^{(i)}$ is the j -polynomial part for the largest possible j (the corresponding rectangle is a row of length r). Then to get the correct Thom polynomial, the function $F_r^{(i)}$ must be modified by j -polynomials related with smaller j 's. We shall see this in the next section for the singularity A_3 and $r \geq 2$. In the present case, this works as follows. We have

$$F_1^{(4)} = S_{1111} + 9S_{112} + 26S_{13} + 24S_4. \quad (67)$$

By Corollary 19, this function satisfies the following equations imposed by A_0, A_1, A_2, A_3, A_4 :

$$F_1^{(4)}(0) = F_1^{(4)}(x - \boxed{2x}) = F_1^{(4)}(x - \boxed{3x}) = F_1^{(4)}(x - \boxed{4x}) = 0, \quad (68)$$

$$F_1^{(4)}(x - \boxed{5x}) = R(x + \boxed{2x} + \boxed{3x} + \boxed{4x}, \boxed{5x}). \quad (69)$$

However, $F_1^{(4)}$ does not satisfy the vanishing imposed by $I_{2,2}$. Namely, we have

$$F_1^{(4)}(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2}) = (-10)x_1x_2(x_1 - 2x_2)(x_2 - 2x_1). \quad (70)$$

To see this, invoke Proposition 18:

$$F_1^{(4)}(x - \mathbb{B}_1) = R(x + \boxed{2x} + \boxed{3x} + \boxed{4x}, \mathbb{B}_1). \quad (71)$$

Substituting to the LHS of Eq. (70) $x_1 = 0$, we get by this proposition

$$F_1^{(4)}(x_2 - \boxed{2x_2}) = R(x_2 + \boxed{2x_2} + \boxed{3x_2} + \boxed{4x_2}, \boxed{2x_2}) = 0,$$

and substituting $x_1 = 2x_2$,

$$\begin{aligned} F_1^{(4)}(x_2 - \boxed{2x_1}) &= R(x_2 + \boxed{2x_2} + \boxed{3x_2} + \boxed{4x_2}, \boxed{2x_1}) \\ &= R(x_2 + \boxed{2x_2} + \boxed{3x_2} + \boxed{4x_2}, \boxed{4x_2}) = 0. \end{aligned}$$

Therefore $x_1x_2(x_1 - 2x_2)(x_2 - 2x_2)$ divides this LHS. The coefficient -10 results from specialization $x_1 = x_2 = 1$. This implies Eq. (70).

On the other hand, the Schur function $S_{22}(\bullet)$ satisfies Eqs. (68), and Eq. (69) with its RHS replaced by zero:

$$S_{22}(0) = S_{22}(x - \boxed{2x}) = \cdots = S_{22}(x - \boxed{5x}) = 0$$

because the partition 22 is not contained in the $(1, 1)$ -hook. Moreover, we have

$$S_{22}(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2}) = R(\mathbb{X}_2, \boxed{2x_1} + \boxed{2x_2}) = x_1x_2(x_1 - 2x_2)(x_2 - 2x_2). \quad (72)$$

Combining Eq. (70) with Eq. (72), the desired expression (66) follows.

Remark 21 *Porteous [18] (see also [12]) gives a geometric account to the function $F_1^{(i)}$. By passing with his formulas to Schur function expansions, we should restrict ourselves to hook partitions (more precisely: to their conjugates). We have the following recursive formula (loc. cit.):*

$$F_1^{(i)} = \sum_{j=1}^i \frac{(i-1)!}{(i-j)!} \Lambda_j F_1^{(j)}. \quad (73)$$

Our goal now is to give an expression for the Thom polynomial for A_3 (any r) as a linear combination of Schur functions. The cases $r = 1, 2$ were already known in the literature (cf., e.g., [24]). In [2], the authors announced a certain expression in terms of the Chern monomial basis. Our expression is of different form (a linear combination of Schur functions), and for the moment we do not know how to pass from it to the one in [2]. (A computer check for small values of r shows the desired coincidence.) The Thom polynomial for $r = 1$ has been already discussed. For $r = 2$, the Thom polynomial is

$$S_{222} + 5S_{123} + 6S_{114} + 19S_{24} + 30S_{15} + 36S_6 + 5S_{33}, \quad (74)$$

and it differs from $F_2^{(3)}$ by $5S_{33}$ which is the ‘‘correction term’’ in this case. In the following, we shall find such a correction term for any r .

Define integers e_{ij} , for $i \geq 2$ and $j \geq 0$ in the following way. First, $e_{20}, e_{30}, e_{40}, \dots$ are the coefficients $5, 24, 89, \dots$ in the development of the series:

$$\begin{aligned} & \frac{5 - 6z}{(1-z)(1-2z)(1-3z)} \\ &= 5 + 24z + 89z^2 + 300z^3 + 965z^4 + 3024z^5 + 9329z^6 + \dots \end{aligned}$$

Moreover, we set $e_{2j} = e_{3j} = 0$ for $j \geq 1$, $e_{4j} = e_{5j} = 0$ for $j \geq 2$, $e_{6j} = e_{7j} = 0$ for $j \geq 3$ etc. To define the remaining e_{ij} ’s, we use the recursive formula

$$e_{i+1,j} = e_{i,j-1} + e_{ij}. \quad (75)$$

We now define the function $H_r(\bullet)$:

$$H_r(\bullet) := \sum_{k=0}^{r-2} \sum_{\{j \geq 0: k+2j \leq r-2\}} e_{r-k,j} S_{k,r+j+1,2r-k-j-1}(\bullet). \quad (76)$$

We state

Theorem 22 *The Thom polynomial for the singularity A_3 , with parameter r , is equal to $F_r^{(3)} + H_r$.*

We outline a proof of the theorem in the appendix. We shall now present some examples. We have the following matrix $[e_{ij}]$:

$$\begin{array}{cccccc}
e_{20} & 0 & 0 & 0 & 0 & \dots & 5 & 0 & 0 & 0 & 0 & \dots \\
e_{30} & 0 & 0 & 0 & 0 & \dots & 24 & 0 & 0 & 0 & 0 & \dots \\
e_{40} & e_{41} & 0 & 0 & 0 & \dots & 89 & 24 & 0 & 0 & 0 & \dots \\
e_{50} & e_{51} & 0 & 0 & 0 & \dots & 300 & 113 & 0 & 0 & 0 & \dots \\
e_{60} & e_{61} & e_{62} & 0 & 0 & \dots & 965 & 413 & 113 & 0 & 0 & \dots \\
e_{70} & e_{71} & e_{72} & 0 & 0 & \dots & 3024 & 1378 & 526 & 0 & 0 & \dots \\
e_{80} & e_{81} & e_{82} & e_{83} & 0 & \dots & 9329 & 4402 & 1904 & 526 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array} =$$

Consider the following matrix whose elements are two row partitions (the symbol “ \emptyset ” denotes the empty partition):

$$\begin{array}{ccccc}
33 & \emptyset & \emptyset & \emptyset & \emptyset \dots \\
45 & \emptyset & \emptyset & \emptyset & \emptyset \dots \\
57 & 66 & \emptyset & \emptyset & \emptyset \dots \\
69 & 78 & \emptyset & \emptyset & \emptyset \dots \\
7, 11 & 8, 10 & 9, 9 & \emptyset & \emptyset \dots \\
8, 13 & 9, 12 & 10, 11 & \emptyset & \emptyset \dots \\
9, 15 & 10, 14 & 11, 13 & 12, 12 & \emptyset \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

We use for this matrix the same “matrix coordinates” as for the previous one. Denote by $I(i, j)$ the partition occupying the (i, j) th place in this matrix. So, e.g., $I(i, 0) = (i + 1, 2i - 1)$ for $i \geq 2$. For $r \geq 2$, we set

$$H_r^o(\bullet) := \sum_{j \geq 0} e(r, j) S_{I(r, j)}(\bullet). \quad (77)$$

We have the following values of H_2^0, \dots, H_7^0 :

$$\begin{aligned}
& 5S_{33}, 24S_{45}, 89S_{57} + 24S_{66}, 300S_{69} + 113S_{78}, 965S_{7,11} + 413S_{8,10} + 113S_{99}, \\
& 3024S_{8,13} + 1378S_{9,12} + 526S_{10,11}.
\end{aligned}$$

Then we have, with the endomorphism τ defined before Lemma 12,

$$H_r = H_r^o + \tau(H_{r-1}), \quad (78)$$

or iterating,

$$H_r = H_r^o + \tau(H_{r-1}^o) + \tau^2(H_{r-2}^o) + \dots + \tau^{r-2}(H_2^o). \quad (79)$$

We have the following values of $H_2, H_3 = \tau(H_2) + H_3^o, \dots, H_7 = \tau(H_6) + H_7^o$:

$$\begin{aligned}
& 5S_{33} \\
& 5S_{144} + 24S_{45} \\
& 5S_{255} + 24S_{156} + 24S_{66} + 89S_{57} \\
& 5S_{366} + 24S_{267} + 24S_{177} + 89S_{168} + 113S_{78} + 300S_{69} \\
& 5S_{477} + 24S_{378} + 24S_{288} + 89S_{279} + 113S_{189} + 300S_{1,7,10} + 113S_{99} + 413S_{8,10} + 965S_{7,11} \\
& 5S_{588} + 24S_{489} + 24S_{399} + 89S_{3,8,10} + 113S_{2,9,10} + 300S_{2,8,11} + 113S_{1,10,10} + 413S_{1,9,11} \\
& + 965S_{1,8,12} + 526S_{10,11} + 1378S_{9,12} + 3024S_{8,13}.
\end{aligned}$$

6 Appendix

We present here an outline of the proof of Theorem 22. The first result says that the addition of H_r to $F_r^{(3)}$ is “irrelevant” for the conditions imposed by A_i , $i = 0, 1, 2, 3$.

Lemma 23 *The function $H_r(\bullet)$ satisfies Eqs. (62), and we have additionally*

$$H_r(x - \mathbb{B}_{r-1} - \boxed{4x}) = 0. \quad (80)$$

It follows from the lemma that it suffices to show that

$$(F_r^{(3)} + H_r)(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0, \quad (81)$$

i.e. we have the vanishing imposed by $III_{2,2}$. We look at the specialization

$$H_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}). \quad (82)$$

By the factorization property (25), each polynomial

$$S_{c,r+1+a,r+1+b}(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) \quad (83)$$

factorizes into:

$$R \cdot S_c(-\mathbb{D} - \mathbb{B}_{r-2}) \cdot S_{a,b}(\mathbb{X}_2), \quad (84)$$

where $R = R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})$. We set

$$V_r(\mathbb{X}_2; \mathbb{B}_{r-2}) := \frac{H_r}{R}, \quad (85)$$

so that Eq. (76) gives

$$V_r(\mathbb{X}_2; \mathbb{B}_{r-2}) = \sum e_{r-k,j} S_k(-\mathbb{D} - \mathbb{B}_{r-2}) S_{j,r-k-j-2}(\mathbb{X}_2), \quad (86)$$

where summation is as in Eq. (76).

Lemma 24 For $r \geq 2$, we have

$$V_r(\mathbb{X}_2; \mathbb{B}_{r-2}) = \sum_{i=0}^{r-2} V_{r-i}(\mathbb{X}_2; 0) S_i(-\mathbb{B}_{r-2}). \quad (87)$$

We look now at the specialization of $F_r^{(3)}$ imposed by $III_{2,2}$.

Lemma 25 The polynomial $F_r^{(3)}(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})$ is divisible by the resultant $R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})$.

Denote by $-U_r(\mathbb{X}_2; \mathbb{B}_{r-2})$ the factor resulting from the lemma.

Lemma 26 For $r \geq 2$, we have

$$U_r(\mathbb{X}_2; \mathbb{B}_{r-2}) = \sum_{i=0}^{r-2} U_{r-i}(\mathbb{X}_2; 0) S_i(-\mathbb{B}_{r-2}). \quad (88)$$

Proposition 27 For $r \geq 2$ we have

$$U_r(\mathbb{X}_2; 0) = 3^{r-2}(3S_{r-2}(\mathbb{X}_2) - 2S_{1,r-3}(\mathbb{X}_2)) = V_r(\mathbb{X}_2; 0). \quad (89)$$

Combining Lemmas 23, 24, 26, and Proposition 27, the assertion of Theorem 22 follows. Details will appear elsewhere.

Acknowledgments I thank Alain Lascoux for many helpful discussions on the problems treated in the present paper. Though its author is responsible for the exposition of the details, many computations described here were done together in November 2004 and March 2005. I am grateful to Richard Rimanyi for introducing me to his paper [24]. In addition, I thank Ozer Ozturk for pointing out several defects of the manuscript, as well as to Laszlo Feher and Andrzej Weber for comments.

Notes

1. Schur function expansions of some (other) Thom polynomials were studied in [5] as we have been informed by Feher. Rimanyi and Feher report that they and Komuves also observed the nonnegativity of the Schur function expansions of Thom polynomials, cf. [5], [7].
2. After completion of the first version of this paper I received the preprint [7] containing some results on Chern monomial expansions of Thom polynomials: an expression for the Thom series of $I_{2,2}$, supported by a computer evidence, and an inductive formula for Thom polynomials. Our expressions are of different form (\mathbf{Z} -linear combinations of Schur functions), and for the moment we do not know how to pass from them to the ones in [7].
3. Thom polynomials for A_4 and $r = 3, 4$ have been computed (January 2006) via their Schur function expansion by Ozturk, with the help of the techniques from the present paper, cf. [17]. (Note that Thom polynomial for A_4 and $r = 2$, was computed in [24].)

References

- [1] V. Arnold, V. Vasilev, V. Goryunov, O. Lyashko: *Singularities. Local and global theory*, Enc. Math. Sci. vol. 6 (Dynamical Systems VI), Springer, 1993.
- [2] G. Berczi, L. Feher, R. Rimanyi, *Expressions for resultants coming from the global theory of singularities*, in: “Topics in algebraic and noncommutative geometry”, (L. McEwan et al. eds.), Contemporary Math. AMS **324** (2003), 63–69.
- [3] A. Berele, A. Regev, *Hook Young diagrams with applications to combinatorics and to representation theory of Lie superalgebras*, Adv. in Math. **64** (1987), 118–175.
- [4] A. du Plessis, C.T.C. Wall, *The geometry of topological stability*, Oxford Univ. Press, 1995.
- [5] L. Feher, B. Komuves, *On second order Thom-Boardman singularities*, preprint, math.AG/0501498.
- [6] L. Feher, R. Rimanyi, *Calculation of Thom polynomials and other cohomological obstructions for group actions*, in: “Real and complex singularities (San Carlos 2002)” (T. Gaffney and M. Ruas eds.), Contemporary Math. **354**, (2004), 69–93.
- [7] L. Feher, R. Rimanyi, *On the structure of Thom polynomials of singularities*, in preparation.
- [8] W. Fulton, P. Pragacz, *Schubert varieties and degeneracy loci*, Springer LNM **1689** (1998).
- [9] T. Gaffney, *The Thom polynomial of $\overline{\Sigma^{1111}}$* , in: “Singularities”, Proc. Symposia in Pure Math. **40(1)**, 399–408, AMS, 1983.
- [10] M.E. Kazarian, *Characteristic classes of singularity*, in: “The Arnold-Gelfand mathematical seminars: Geometry and singularity theory” (1997), 325–340.
- [11] S. Kleiman, *The enumerative theory of singularities*, in: “Real and complex singularities, Oslo 1976” (P. Holm ed.) (1978), 297–396.
- [12] W.S. Kulikov, *Calculus of singularities of immersion of general algebraic surface in \mathbf{P}^3* , Functional Analysis and Appl. **17(3)** (1983), 15–27 (Russian).
- [13] A. Lascoux, *Symmetric functions and combinatorial operators on polynomials*, CBMS/AMS Lectures Notes **99**, Providence (2003).
- [14] A. Lascoux, *Addition of ± 1 : application to arithmetic*, Séminaire Lotharingien de Combinatoire, **B52a** (2004), 9 pp.
- [15] A. Lascoux, M-P. Schützenberger, *Formulaire raisonné de fonctions symétriques*, Université Paris 7, 1985.
- [16] I.G. Macdonald, *Symmetric functions and Hall-Littlewood polynomials*, Oxford Math. Monographs, Second Edition, 1995.

- [17] O. Ozturk, *On Thom polynomials for $A_4(-)$ via Schur functions*, to appear.
- [18] I. Porteous, *Simple singularities of maps*, in: “Proc. Liverpool Singularities I”, Springer LNM **192** (1971), 286–307.
- [19] P. Pragacz, *Enumerative geometry of degeneracy loci*, Ann. Sc. Ec. Norm. Sup. **21** (1988), 413–454.
- [20] P. Pragacz, *Algebraic-geometric applications of Schur S - and Q -polynomials*, in: “Topics in invariant theory” – Séminaire d’Algèbre Dubreil-Malliavin 1989-1990 (M-P. Malliavin ed.), Springer LNM **1478** (1991), 130–191.
- [21] P. Pragacz, A. Thorup, *On a Jacobi-Trudi identity for supersymmetric polynomials*, Adv. in Math. **95** (1992), 8–17.
- [22] P. Pragacz, A. Weber, *Positivity of Schur function expansions of Thom polynomials*, Preprint, math.AG/0605308.
- [23] R. Rimanyi, *Computation of the Thom polynomial of Σ^{1111} via symmetries of singularities*, in “Real and complex singularities” (J.W. Bruce, F. Tari eds.), Chapman&Hall/CRC RNM **412** (2000), 15–35.
- [24] R. Rimanyi, *Thom polynomials, symmetries and incidences of singularities*, Inv. Math. **143** (2001), 499–521.
- [25] R. Rimanyi, A. Szücs, *Generalized Pontrjagin-Thom construction for maps with singularities*, Topology **37** (1998), 1177–1191.
- [26] F. Ronga, *Le calcul des classes duales aux singularités de Boardman d’ordre 2*, Comm. Math. Helv. **47** (1972), 15–35.
- [27] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier **6** (1955–56), 43–87.