Wronski's "loi supréme" versus the Lagrange-Bürmann formula

Tomasz Maszczyk

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Lagrange-Bürmann series

```
>From:  edwa...@sunrise.Stanford.EDU (Larry Edwards)
>Date:  7 Nov 91 00:13:58 GMT
>Organization:  Stanford University
>Is there any general method for finding the inverse
>of a taylor series?
>That is given some arbitrary taylor series
>(assuming it does have an inverse)
>is there some way of constructing the taylor series
>of its inverse.
```

(...) What the guy wanted was something like
the Burmann-Lagrange formula.
On page 150|1 of Louis Comtet 'Advanced
Combinatorics'
Reidel 1974
one will find the required formulas.

J.W. Nienhuys,
Research Group Discrete Mathematics
Dept. of Mathematics and Computing Science
Eindhoven University of Technology
P.O. BOX 513, 5600 MB Eindhoven
The Netherlands

Inverting a local univalent analytic map $y = f(x)$, taking at $x = x_0$ a value y_0.

$$F(f^{-1}(y)) = F(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\left(\frac{x - x_0}{f(x) - y_0} \right)^n F^{(1)}(x) \right]_{x=x_0}^{(n-1)} (y - y_0)^n.$$

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In particular, for $F(x) \equiv x$ one gets the inverse series

$$f^{-1}(y) = x_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\left(\frac{x - x_0}{f(x) - y_0} \right)^n \right]_{x=x_0}^{(n-1)} (y - y_0)^n.$$

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e.g. for $n = 2$.

$$\left[\left(\frac{x-x_0}{f(x)-y_0}\right)^2 F^{(1)}(x)\right]^{(1)} =$$

$$= 2\left[\frac{x-x_0}{f(x)-y_0}\right]\cdot\left[\frac{f(x)-y_0-(x-x_0)f^{(1)}(x)}{(f(x)-y_0)^2}\right]\cdot F^{(1)}(x)$$

$$+\left[\frac{x-x_0}{f(x)-y_0}\right]^2\cdot F^{(2)}(x).$$

Is it a simple formula?

- Optical illusion caused by misuse of the symbol $[\cdots]_{x=x_0}$.
- After differentiating one gets expressions not defined at x_0.
- One can't substitute $x = x_0$. An additional passage to the limit necessary.

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$$+\left[\frac{x-x_0}{f(x)-y_0}\right]^2 \cdot F^{(2)}(x).$$

Using the Taylor expansion

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{1}{2} f^{(2)}(x_0)(x - x_0)^2 + o(|x - x_0|^2), \tag{1}$$

$$f^{(1)}(x) = f^{(1)}(x_0) + f^{(2)}(x_0)(x - x_0) + o(|x - x_0|), \tag{2}$$

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$$f^{(1)}(x) = f^{(1)}(x_0) + f^{(2)}(x_0)(x - x_0) + o(|x - x_0|), \tag{2}$$

one gets

$$\lim_{x \to x_0} \frac{f(x) - y_0 - (x - x_0) f^{(1)}(x)}{(f(x) - y_0)^2} = -\frac{1}{2} \frac{f^{(2)}(x_0)}{f^{(1)}(x_0)^2}, \tag{3}$$

$$\lim_{x \to x_0} \frac{f(x) - y_0 - (x - x_0) f^{(1)}(x_0)}{(f(x) - y_0)^2} = +\frac{1}{2} \frac{f^{(2)}(x_0)}{f^{(1)}(x_0)^2}. \tag{4}$$

- This shows that the substitution $x = x_0$ makes no sense.
- Derivatives of higher order appear in effect of passing to the limit

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- a compact formula is hard to imagine.

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The problem of division of power series was solved by Wronski in 1811, we will show the formula at the end of the talk.

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The problem of division of power series was solved by Wronski in 1811, we will show the formula at the end of the talk.
Using only derivatives one gets

$$
\begin{aligned}
g(x) &= \frac{x-x_0}{f(x)-y_0}, \\
g^{(1)}(x) &= \frac{f(x)-y_0-(x-x_0)f^{(1)}(x)}{(f(x)-y_0)^2}, \\
g^{(2)}(x) &= \frac{-2f^{(1)}(x)(f(x)-y_0-(x-x_0)f^{(1)}(x))-(f(x)-y_0)(x-x}{(f(x)-y_0)^3} \\
&\vdots
\end{aligned}
$$

Taylor's expansion allows to compute every particular derivative of g at x_0

$$
\begin{aligned}
g(x_0) &= \frac{1}{f^{(1)}}(x_0), \\
g^{(1)}(x_0) &= \frac{-\frac{1}{2}f^{(2)}}{f^{(1)2}}(x_0), \\
g^{(2)}(x_0) &= \frac{-\frac{1}{3}f^{(1)}f^{(3)} + \frac{1}{2}f^{(2)2}}{f^{(1)3}}(x_0), \\
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&\vdots
\end{aligned}
$$

but it is hard to see from that a general formula in terms of derivatives of f at x_0.

We can observe that well known 1-cocycles on the group of diffeomorphisms appear

- the affine cocycle

$$\frac{-2g^{(1)}}{g}(x_0) = \frac{f^{(2)}}{f^{(1)}}(x_0) \tag{5}$$

- and the projective one (the Schwartz derivative)

$$\frac{-3g^{(2)}}{g}(x_0) = \frac{f^{(1)}f^{(3)} - \frac{3}{2}f^{(2)2}}{f^{(1)2}}(x_0), \tag{6}$$

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but this relation reflecting geometry of higher jets is still not well understood.

To obtain algebraic formula equivalent to that of Lagrange-Bürmann we first express it algebraically by derivatives of $g(x) = \frac{x - x_0}{f(x) - y_0}$ at x_0.

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We need the following numbers

Definition

For every natural number $m \geq 0$ and a sequence $(i_1, i_2, i_3 \ldots)$ of integers almost all equal to zero we define numbers $a^{(m)}_{i_1,i_2,i_3,\ldots}$ by the following reccurence
$a^{(m)}_{i_1,i_2,i_3\ldots} = 0$, if at least one among the indices $(i_1, i_2, i_3 \ldots)$ is negative or $i_1 + 2i_2 + 3i_3 \ldots \neq m$, and

$$a^{(0)}_{0,0,0,\ldots} = 1, \tag{7}$$

$$a^{(m+1)}_{i_1,i_2,i_3,\ldots} = a^{(m)}_{i_1-1,i_2,i_3,\ldots} + \sum_{k\geq 1}(i_k+1)a^{(m)}_{i_1,\ldots,i_{k-1},i_k+1,i_{k+1}-1,i_{k+2},\ldots}. \tag{8}$$

Immediate: $a^{(m)}_{m,0,0,\dots} = 1$ for all m.

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Next five nonzero groups of terms in this reccurence

$$a^{(1)}_{1,\ldots} = 1,$$

$$a^{(2)}_{2,0,\ldots} = 1, \quad a^{(2)}_{0,1,\ldots} = 1,$$

$$a^{(3)}_{3,0,0,\ldots} = 1, \quad a^{(3)}_{1,1,0,\ldots} = 3, \quad a^{(3)}_{0,0,1,\ldots} = 1,$$

$$a^{(4)}_{4,0,0,0,\ldots} = 1, \quad a^{(4)}_{2,1,0,0,\ldots} = 6, \quad a^{(4)}_{0,2,0,0,\ldots} = 3, \quad a^{(4)}_{1,0,1,0,\ldots} = 4, \quad a^{(4)}_{0,0,0,1,\ldots} = 1,$$

$$a^{(5)}_{5,0,0,0,0,\ldots} \quad a^{(5)}_{3,1,0,0,0,\ldots} \quad a^{(5)}_{1,2,0,0,0,\ldots} \quad a^{(5)}_{2,0,1,0,0,\ldots} \quad a^{(5)}_{0,1,1,0,0,\ldots} \quad a^{(5)}_{1,0,0,1,0,\ldots} \quad a^{(5)}_{0,0,0,0,1,\ldots}$$

Immediate: $a^{(m)}_{m,0,0,\ldots} = 1$ for all m.
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a^{(1)}_{1,\ldots} = 1, \qquad
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$$

Solution in a compact form (Faà di Bruno (1825-1888)),

$$
a^{(m)}_{i_1,i_2,i_3\ldots} = \frac{m!}{i_1!i_2!i_3!\ldots 1!^{i_1}2!^{i_2}3!^{i_3}\ldots}. \tag{9}
$$

They appear in the higher chain rule of Faà di Bruno

$$(h \circ g)^{(m)} = \sum_{k=0}^{m} \left(h^{(k)} \circ g \right) \cdot \sum_{\substack{i_1+2i_2+3i_3+\ldots=m, \\ i_1+i_2+i_3+\ldots=k}} a^{(m)}_{i_1,i_2,i_3,\ldots} \prod_{r\geq 1} g^{(r)i_r}, \quad (10)$$

where for $i_r = 0$ we put $g^{(r)i_r} := 1$ and summation is restricted to $i_r \geq 0$.

Iterating the Leibniz rule in the higher chain rule with $h(z) := \frac{z^n}{n!}$ in the Lagrange-Bürmann formula we get

$$\frac{1}{n!}\left[g^n F^{(1)}\right]^{(n-1)} = \tag{11}$$
$$= \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{k=0}^{m} \frac{g^{n-k}}{(n-k)!} \sum_{\substack{i_1+2i_2+3i_3+\ldots=m,\\ i_1+i_2+i_3+\ldots=k}} a^{(m)}_{i_1,i_2,i_3,\ldots} \prod_{r\geq 1} g^{(r)i_r} \cdot F^{(n-m)}.$$

Iterating the Leibniz rule and applying the higher chain rule to the composition $H \circ G$ of $H(z) := \frac{z^{-n}}{n!}$ and

$$G(x) = \frac{f(x) - y_0}{x - x_0} = f^{(1)}(x_0) + \frac{1}{2}f^{(2)}(x_0)(x - x_0) + \frac{1}{6}f^{(3)}(x_0)(x - x_0)^2 + \tag{12}$$

in the Lagrange-Bürmann formula we get

$$F(f^{-1}(y)) = F(x_0) \tag{13}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{k=0}^{m} \binom{k+n-1}{k} \widetilde{\gamma}_{m,k} \cdot F^{(n-m)} \right] (x_0) \left(\frac{y -}{f^{(1)}(} \right.$$

where

$$\widetilde{\gamma}_{m,k} = (-1)^k k! \sum_{\substack{i_1+2i_2+3i_3+\ldots=m,\\ i_1+i_2+i_3+\ldots=k}} a^{(m)}_{i_1,i_2,i_3,\ldots} \prod_{r\geq 1} \left(\frac{\gamma_r}{r+1} \right)^{i_r}, \tag{14}$$

$$\gamma_r := \frac{f^{(r+1)}}{f^{(1)}}. \tag{15}$$

Intriguing cancellations

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- Really a mess because of the double summation in the square bracket with binomial coefficients.
- But, surprisingly, computing succeeding terms of the expansion one can see that all binomial coefficients cancel.
- We are to show that Wronski's formula, free of these redundant coefficients, explains this cancellation.

Wronski's solution

Wronski considers expansions

$$F(x) = F(x_0) + \sum_{n=1}^{\infty} c_n F_n(x),$$

where

$$F_n(x) = \frac{1}{n!}(f(x) - y_0)^n, \quad f(x_0) = y_0.$$

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The Bürmann-Lagrange formula, at least on the formal level, is an easy consequence of the fact that functionals L_n, where

$$L_n(F) := \left[\left(\frac{x - x_0}{f(x) - y_0} \right)^n F^{(1)}(x) \right]_{x=x_0}^{(n-1)}, \tag{16}$$

satisfy

$$L_m(F_n) = \delta_{m,n}. \tag{17}$$

Not a real solution.

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To get an effective formula Wronski uses "loi supréme"

$$c_n = \frac{\begin{vmatrix} F_1^{(1)} & \cdots & F_{n-1}^{(1)} & F^{(1)} \\ \vdots & & \vdots & \vdots \\ F_1^{(n)} & \cdots & F_{n-1}^{(n)} & F^{(n)} \end{vmatrix}}{\begin{vmatrix} F_1^{(1)} & \cdots & F_{n-1}^{(1)} & F_n^{(1)} \\ \vdots & & \vdots & \vdots \\ F_1^{(n)} & \cdots & F_{n-1}^{(n)} & F_n^{(n)} \end{vmatrix}}(x_0). \tag{18}$$

Using the higher chain rule for $h(z) := \frac{(z-y_0)^n}{n!}$, $g(x) := f(x)$, and the identity

$$\left(\frac{d^i}{dz^i}\left(\frac{1}{k!}(z-y_0)^k\right)\right)_{z=y_0} = \delta_{i,k}$$

one gets

$$F_k^{(m)}(x_0) = \sum_{\substack{i_1+2i_2+3i_3+\ldots=m,\\ i_1+i_2+i_3+\ldots=k}} a^{(m)}_{i_1,i_2,i_3,\ldots} \prod_{r\geq 1} f^{(r)i_r}(x_0). \tag{19}$$

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in particular

$$F_k^{(m)}(x_0) = 0, \quad \text{dla } k > m, \tag{20}$$

$$F_m^{(m)}(x_0) = f^{(1)m}(x_0). \tag{21}$$

Substituting to Wronski's formula we get

$$c_n = \frac{\begin{vmatrix} F_1^{(1)} & \cdots & F_{n-1}^{(1)} & F^{(1)} \\ \vdots & & \vdots & \vdots \\ F_1^{(n)} & \cdots & F_{n-1}^{(n)} & F^{(n)} \end{vmatrix}}{f^{(1)\frac{n(n+1)}{2}}}(x_0), \qquad (22)$$

where in first $n-1$ columns of the determinant we substitute (19).

Using quantities (15), as we did for the Bürmann-Lagrange formula, it can be simplified as follows

$$F(f^{-1}(y)) = F(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{vmatrix} 1 & 0 & \cdots & 0 & F^{(1)} \\ \gamma_{2,1} & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & F^{(n-2)} \\ \vdots & & \ddots & 1 & F^{(n-1)} \\ \gamma_{n,1} & \cdots & \cdots & \gamma_{n,n-1} & F^{(n)} \end{vmatrix} (x_0) \left(\frac{y}{f^{(} \right. \tag{23}$$

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where

$$\gamma_{m,k} := \sum_{\substack{i_1+2i_2+3i_3+\ldots=m, \\ i_1+i_2+i_3+\ldots=k}} a^{(m)}_{i_1,i_2,i_3,\ldots} \prod_{r \geq 1} \gamma_r^{i_{r+1}}, \tag{24}$$

e.g.

$$
\begin{array}{rclrclrcl}
\gamma_{2,1} & = & \gamma_1 , & & & & & & \\
\gamma_{3,1} & = & \gamma_2 , & \gamma_{3,2} & = & 3\gamma_1 , & & & \\
\gamma_{4,1} & = & \gamma_3 , & \gamma_{4,2} & = & 3\gamma_1^2 + 4\gamma_2 , & \gamma_{4,3} & = & 6\gamma_1 , \\
\gamma_{5,1} & = & \gamma_4 , & \gamma_{5,2} & = & 10\gamma_1\gamma_2 + 5\gamma_3 , & \gamma_{5,3} & = & 15\gamma_1^2 + 10\gamma_2 , \\
 & \vdots & & & \vdots & & & \vdots &
\end{array}
$$

hence

$$F(f^{-1}(y)) = F(x_0) + F^{(1)}(x_0)\frac{y-y_0}{f^{(1)}(x_0)}$$

$$+\frac{1}{2}\begin{vmatrix} 1 & F^{(1)} \\ \gamma_1 & F^{(2)} \end{vmatrix}(x_0)\left(\frac{y-y_0}{f^{(1)}(x_0)}\right)^2 + \frac{1}{6}\begin{vmatrix} 1 & 0 & F^{(1)} \\ \gamma_1 & 1 & F^{(2)} \\ \gamma_2 & 3\gamma_1 & F^{(3)} \end{vmatrix}(x_0)\left(\frac{y-y_0}{f^{(1)}(x_0)}\right)$$

$$+\frac{1}{24}\begin{vmatrix} 1 & 0 & 0 & F^{(1)} \\ \gamma_1 & 1 & 0 & F^{(2)} \\ \gamma_2 & 3\gamma_1 & 1 & F^{(3)} \\ \gamma_3 & 3\gamma_1^2+4\gamma_2 & 6\gamma_1 & F^{(4)} \end{vmatrix}(x_0)\left(\frac{y-y_0}{f^{(1)}(x_0)}\right)^4$$

$$+\frac{1}{120}\begin{vmatrix} 1 & 0 & 0 & 0 & F^{(1)} \\ \gamma_1 & 1 & 0 & 0 & F^{(2)} \\ \gamma_2 & 3\gamma_1 & 1 & 0 & F^{(3)} \\ \gamma_3 & 3\gamma_1^2+4\gamma_2 & 6\gamma_1 & 1 & F^{(4)} \\ \gamma_4 & 10\gamma_1\gamma_2+5\gamma_3 & 15\gamma_1^2+10\gamma_2 & 10\gamma_1 & F^{(5)} \end{vmatrix}(x_0)\left(\frac{y-y_0}{f^{(1)}(x_0)}\right)^5$$

In particular, for $F(x) \equiv x$ we get

$$f^{-1}(y) = x_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \begin{vmatrix} \gamma_{2,1} & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ \gamma_{n,1} & \cdots & \cdots & \cdots & \gamma_{n,n-1} \end{vmatrix} (x_0) \left(\frac{y - y}{f^{(1)}(x_0} \right.$$

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i.e.

$$f^{-1}(y) = x_0 + \frac{y - y_0}{f^{(1)}(x_0)} - \frac{1}{2}\gamma_1(x_0) \left(\frac{y - y_0}{f^{(1)}(x_0)} \right)^2$$

$$+\frac{1}{6} \begin{vmatrix} \gamma_1 & 1 \\ \gamma_2 & 3\gamma_1 \end{vmatrix} (x_0) \left(\frac{y - y_0}{f^{(1)}(x_0)} \right)^3 - \frac{1}{24} \begin{vmatrix} \gamma_1 & 1 & 0 \\ \gamma_2 & 3\gamma_1 & 1 \\ \gamma_3 & 3\gamma_1^2 + 4\gamma_2 & 6\gamma_1 \end{vmatrix} (x_0) \left(\right.$$

$$+\frac{1}{120} \begin{vmatrix} \gamma_1 & 1 & 0 & 0 \\ \gamma_2 & 3\gamma_1 & 1 & 0 \\ \gamma_3 & 3\gamma_1^2 + 4\gamma_2 & 6\gamma_1 & 1 \\ \gamma_4 & 10\gamma_1\gamma_2 + 5\gamma_3 & 15\gamma_1^2 + 10\gamma_2 & 10\gamma_1 \end{vmatrix} (x_0) \left(\frac{y - y_0}{f^{(1)}(x_0)} \right)^5 + \cdots$$

- Note the systematic way in which the coefficients arise: all the determinants are obtained by truncating the same matrix.

- Note the systematic way in which the coefficients arise: all the determinants are obtained by truncating the same matrix. It appears as an iceberg emerging from the sea.

Since multiplication of power series is simple (according to Cauchy's rule) it is enough to invert the denominator. One can assume that it doesn't vanish at $x_0 = 0$.
Wronski's formula (1811)

$$\frac{1}{a_0 + a_1 x + \cdots + a_n x^n + \cdots} = s_0 + s_1 x + \cdots + s_n x^n + \cdots , \quad (25)$$

$$s_n = \frac{(-1)^{\frac{n(n+1)}{2}}}{a_0^{n+1}} \begin{vmatrix} 0 & 0 & \cdots & a_0 & a_1 \\ 0 & 0 & \cdots & a_1 & a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{vmatrix}$$

Appendix II. Zeros of analytic functions

In 1868 de Morgan used it to the following asymptotic formula for the root of an analytic function $a_0 + a_1 x + \cdots + a_n x^n + \cdots$. If that series has exactly one root x_1 of minimal absolute value and in some disc around $x_0 = 0$ containing x_1 that series converges

$$x_1 = \lim_{n \to \infty} \frac{s_{n-1}}{s_n}.$$