Thom polynomials of invariant cones, Schur functions, and positivity

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Abstract

We generalize the notion of Thom polynomials from singularities of maps between two complex manifolds to invariant cones in representations, and collections of vector bundles. We prove that the generalized Thom polynomials, expanded in the products of Schur functions of the bundles, have nonnegative coefficients. For classical Thom polynomials associated with maps of complex manifolds, this gives an extension of our former result for stable singularities to nonnecessary stable ones. We also discuss some related aspects of Thom polynomials, which makes the article expository to some extent.

1 Introduction

The present paper is both of the research and expository character. It concerns global invariants for singularities. Our main new result here is Theorem 5 (see also Corollary 6 and 7).

To start with, we recall that the global behavior of singularities is governed by their *Thom polynomials* (cf. [33], [1], [17], and [32]). By a *singularity*, we shall mean in the paper a class of germs

$$(\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)\,,$$

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where $m, n \in \mathbf{N}$, which is closed under the right-left equivalence (i.e. analytic reparametrizations of the source and target).

Suppose that $f : M \to N$ is a map between complex manifolds, where $\dim(M) = m$ and $\dim(N) = n$. Let $V^\eta(f)$ be the cycle carried by the *closure* of the set

$$\{x \in M : \text{the singularity of } f \text{ at } x \text{ is } \eta\}\,. \tag{1}$$

We recall that the *Thom polynomial* $\mathcal{T}^\eta$ of a singularity η is a polynomial in the formal variables

$$c_1, c_2, \ldots, c_m;\ c'_1, c'_2, \ldots, c'_n\,,$$

such that after the substitution

$$c_i = c_i(TM), \quad c'_j = c_j(f^*TN)\,, \tag{2}$$

$(i = 1, \ldots, m,\ j = 1, \ldots, n)$ for a general map $f : M \to N$ between complex manifolds, it evaluates the Poincaré dual[1] of $[V^\eta(f)]$. This is the content of the Thom theorem [33]. For a detailed discussion of the *existence* of Thom polynomials, see, e.g., [1]. Thom polynomials associated with group actions were studied by Kazarian in [17].

Recall that – historically – the first "Thom polynomial" appeared in the so-called "Riemann-Hurwitz formula". Let $f : M \to N$ be a general holomorphic map of compact Riemann surfaces. This means that f is a simple covering, that is, the critical points are nondegenerate and at most one appears in each fiber. Denoting by e_x the *ramification index* of f at $x \in M$ (i.e. the number of sheets of f meeting at x), the Riemann-Hurwitz formula asserts that

$$\sum_{x \in M} (e_x - 1) = 2g(M) - 2 - \deg(f)\big(2g(N) - 2\big)\,. \tag{3}$$

Denoting by A_1 the singularity of $z \mapsto z^2$ at 0, this is equivalent to saying that the fundamental class of the *ramification divisor* of f,

$$\sum_x (e_x - 1)[x] = [V^{A_1}(f)]\,,$$

where x runs over the set of critical points of f, is given by the following expression in the first Chern classes:

$$c_1(f^*TN) - c_1(TM) = c_1(f^*TN - TM)\,. \tag{4}$$

In other words, the Riemann-Hurwitz formula says:

$$\mathcal{T}^{A_1} = c'_1 - c_1\,. \tag{5}$$

[1] In the following, we shall often omit the expression: "the Poincaré dual of".

For a wider discussion of the Riemann-Hurwitz formula and early history of Thom polynomials, we refer to Kleiman's survey article [18]. The Riemann-Hurwitz formula is true also in positive characteristic for finite separable morphisms of algebraic curves (cf. [15], Chap. IV, Sect. 2).

Thom [33] generalized the Riemann-Hurwitz formula to general maps $f : M \to N$ of complex manifolds with $n - m > 0$, the singularity being always A_1:

$$[V^{A_1}(f)] = c_{n-m+1}(f^*TN - TM) = \sum_{i=0}^{n-m+1} S_{n-m+1-i}(TM^*)c_i(f^*TN)\,, \quad (6)$$

where S_j denotes the jth Segre class.

Though for the singularity A_1 the Thom polynomials are rather simple, they start to be quite complicated even for simplest singularities coming "just after A_1", say (cf. the tables in [32]). For example, the Thom polynomial for the singularities A_4 is known only for small values of k (cf. [23]). Therefore, it is important to study the *structure* of Thom polynomials. It appears that a good tool for this task is provided by *Schur functions* [3], [26], [27], [28], [29]. Let us quote some results related to Schur function expansions of Thom polynomials of stable[2] singularities.

First, it was shown in [27] that if a representative of a stable singularity

$$\eta : (\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$$

is of Thom-Boardman type Σ^i, then all summands in the Schur function expansion of $\mathcal{T}^\eta$ are indexed by partitions containing[3] the rectangle partition

$$(n - m + i, \dots, n - m + i) \qquad (i \text{ times}).$$

This is a consequence of the structure of the *$\mathcal{P}$-ideals of the singularities* Σ^i, which were introduced and investigated in [24]. Second, in [31], the authors proved that for any partition I the coefficient α_I in the Schur function expansion of the Thom polynomial

$$\mathcal{T}^\eta = \sum_I \alpha_I S_I(TM^* - f^*TN^*)$$

is nonnegative. This result was conjectured before in [3] and independently in [26]. It appears to be a consequence of the Fulton-Lazarsfeld theory of numerical positivity of cones in ample vector bundles [8] (cf. also [21, §8]),

[2] By a *stable* singularity we mean an equivalence class of stable germs $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, where $\bullet \in \mathbf{N}$, under the equivalence generated by right-left equivalence and suspension (by suspension of a germ κ we mean its trivial unfolding: $(x, v) \mapsto (\kappa(x), v)$). For a stable singularity, its Thom polynomial is of the form $\sum_I \alpha_I S_I(TM^* - f^*TN^*)$, where S_I denotes a Schur function, cf. Sections 3 and 5.

[3] We say that one partition *is contained* in another if this holds for their Young diagrams.

combined with a functorial version of the bundles of jets, appearing in the approach to Thom polynomials via classifying spaces of singularities [17].

In the present paper, we shall prove a more general result that – we believe – will better explain a reason of the positivity in the above classical case, as well as in many other situations. To this end, we extend the definition of Thom polynomials from the singularities of maps $f : M \to N$ of complex manifolds [33] to the invariant cones in representations of the product of general linear groups

$$\prod_{i=1}^{p} GL_{n_i}\,.$$

Such Thom polynomials are naturally defined on p-tuples of vector bundles of ranks n_i. It is convenient to pass to *topological homotopy category*, where each p-tuple of bundles can be pulled back from the universal p-tuple of bundles on the product of p classifying spaces

$$\prod_{i=1}^{p} BGL_{n_i}\,.$$

Suppose that the functor associated with such a representation preserves global generateness. Our main result – Theorem 5 – then asserts that the Thom polynomial for a p-tuple of vector bundles $(E_1, E_2, \ldots, E_p)$, when expanded in the basis

$$\{S_{I_1}(E_1)\cdot S_{I_2}(E_2)\ \cdots\ S_{I_p}(E_p)\}$$

of products of Schur functions applied to the successive bundles, has *nonnegative* coefficients. The key tool is *positivity of cone classes* for globally generated vector bundles combined with the *Giambelli formula*. For a polynomial representation of $\prod_{i=1}^{p} GL_{n_i}$ of positive degree, we get, in addition, that the sum of the coefficients is *positive* (cf. Corollary 6).

Theorem 5, in the classical situation of singularities of maps $f : M \to N$ between complex manifolds, implies that for a given singularity its Thom polynomial, when expanded in the basis

$$S_I(TM^*)\cdot S_J(f^*TN)\,,$$

has nonnegative coefficients (cf. Corollary 7).

We also note that Theorem 5 implies the main result of our former paper [31] for Thom polynomials of *stable* of singularities of maps between complex manifolds, where, however, the Schur functions *in difference of bundles* were used (cf. Theorem 8).

2 Thom polynomials of invariant cones

In this section, we define "generalized Thom polynomials". Our construction is modeled on that used to the construction of classical Thom polynomials with the help of the "classifying spaces of singularities" (cf., e.g., [17]).

Suppose that $(n_1, n_2, \ldots, n_p) \in \mathbf{N}^p$ and that V is a representation of

$$G = \prod_{i=1}^{p} GL_{n_i} \,. \tag{7}$$

The representation V gives rise to a *functor* ϕ defined for a collection of bundles on a variety X:

$$E_1, E_2, \ldots, E_p \mapsto \phi(E_1, E_2, \ldots, E_p)\,,$$

with $\dim E_i = n_i$, $i = 1, \ldots, p$. By passing to the dual bundles, we may assume that the functor ϕ is covariant in each variable.

Let

$$P(E_\bullet) = P(E_1, E_2, \ldots, E_p) \tag{8}$$

be the principal G-bundle associated with the bundles $E_1, E_2, \ldots, E_p$. We define a new vector bundle:

$$V(E_\bullet) = V(E_1, E_2, \ldots, E_p) := P(E_\bullet) \times_G V \,. \tag{9}$$

Suppose now that a G-invariant cone $\Sigma \subset V$ is given. We set

$$\Sigma(E_\bullet) = \Sigma(E_1, E_2, \ldots, E_p) := P(E_\bullet) \times_G \Sigma \subset V(E_\bullet)\,. \tag{10}$$

We define the "Thom polynomial" $\mathcal{T}^\Sigma$ to be the dual class[4] of

$$[\Sigma(R^{(1)}, \ldots, R^{(p)})] \in H^*\big(V(R^{(1)}, \ldots, R^{(p)}), \mathbf{Z}\big) = H^*(BG, \mathbf{Z})\,,$$

where $R^{(i)}$, $i = 1, \ldots, p$, is the pullback of the tautological vector bundle from BGL_{n_i} to

$$BG = \prod_{i=1}^{p} BGL_{n_i} \,.$$

Then, the so defined Thom polynomial

$$\mathcal{T}^\Sigma \in H^*(BG, \mathbf{Z})$$

depends on the Chern classes of the $R^{(i)}$'s.

We shall write $\mathcal{T}^\Sigma(E_1, \ldots, E_p)$ for the Thom polynomial $\mathcal{T}^\Sigma$, with $c_j(R^{(i)})$ replaced by $c_j(E_i)$ for $i = 1, \ldots, p$.

[4] Compare the footnote 4 in [31].

Lemma 1 *For any vector bundles $E_1, E_2, \ldots, E_p$ on a variety X, the dual class*[5] *of $[\Sigma(E_\bullet)]$ in*

$$H^{2\operatorname{codim}(\Sigma)}(V(E_\bullet), \mathbf{Z}) = H^{2\operatorname{codim}(\Sigma)}(X, \mathbf{Z})$$

is equal to $\mathcal{T}^\Sigma(E_1, \ldots, E_p)$.

Proof. Each p-tuple of bundles can be pulled back from the universal p-tuple $(R^{(1)}, R^{(2)}, \ldots, R^{(p)})$ of bundles on BG using a C^∞-map. It is possible to work entirely with the algebraic varieties and maps. One can use the Totaro construction and representability for affine varieties ([34, proof of Theorem 1.3]). $\square$

Remark 2 In the situation of classical Thom polynomials [33], the functor ϕ is the functor of k-jets :

$$(E, F) \mapsto \mathcal{J}^k(E, F) = \left(\bigoplus_{i=1}^{k} \operatorname{Sym}^i E^*\right) \otimes F\,,$$

where k is large enough, adapted to the investigated class of singularities – cf. [31] for details and applications. (We note that in this situation an invariant closed subset Σ, called in [31] a "class of singularities", is automatically a cone.)

3 Schur functions and the Giambelli formula

In this section, we recall the notion of *Schur functions*. We also recall a geometric interpretation of them, namely the classical *Giambelli formula*.

Given a partition $I = (i_1, i_2, \ldots, i_l) \in \mathbf{N}^l$, where

$$i_1 \geq i_2 \geq \cdots \geq i_l \geq 0 \text{ [6]},$$

and vector bundles E and F on a variety X, the *Schur function*[7] $S_I(E - F)$ is defined by the following determinant:

$$S_I(E - F) = \Big| S_{i_p - p + q}(E - F) \Big|_{1 \leq p, q \leq l}\,, \tag{11}$$

where the entries are defined by the expression

$$\sum S_i(E - F) = \prod_b (1 - b) / \prod_a (1 - a)\,. \tag{12}$$

[5] The meaning of the "dual class" for singular X is explained in [31], Note 6.

[6] Since the most common references to Schubert Calculus use weakly decreasing partitions, we follow this convention in the present paper.

[7] Usually this family of functions is called "super Schur functions" or "Schur functions in difference of bundles".

Here, the a's and b's are the Chern roots of E and F and the LHS of Eq. (12) is the *Segre class* of the virtual bundle $E - F$. So the Schur functions $S_I(E - F)$ lie in a ring containing the Chern classes of E and F; e.g., we can take the cohomology ring $H^*(X, \mathbf{Z})$ or the Chow ring $A^*(X)$.

Given a vector bundle E and a partition I, we shall write $S_I(E)$ for $S_I(E - 0)$, where 0 is the zero vector bundle.

We refer to [20], [22], and [30] for the theory of Schur functions $S_I(E)$ and $S_I(E - F)$.

Given a smooth variety X, we shall identify its cohomology $H^*(X, \mathbf{Z})$ with its homology $H_*(X, \mathbf{Z})$, as is customary. More precisely, this identification is realized via capping the cohomology classes with the fundamental class $[X]$ of X, using the standard map:

$$\cap : H^*(X, \mathbf{Z}) \otimes H_*(X, \mathbf{Z}) \to H_*(X, \mathbf{Z}) .$$

Let V be a complex vector space of dimension N, and let $G_m(V)$ be the *Grassmannian parametrizing* m-dimensional subspaces of V. On knows that $G_m(V)$ is a smooth projective variety of dimension mn, where $n = N - m$. We shall also use the notation $G^n(V)$ for this Grassmannian. The Grassmannian $G_m(V)$ is stratified by Schubert cells; the closures of these cells are Schubert varieties $\Omega_I(V_\bullet)$, where

$$I = (n \geq i_1 \geq i_2 \geq \cdots \geq i_m \geq 0)$$

is a partition, and

$$V_\bullet : 0 = V_0 \subset V_1 \subset \cdots \subset V_N = V$$

is a complete flag of subspaces of V, with $\dim V_j = j$ for $j = 0, 1, \ldots, N$.

The precise definition of $\Omega_I(V_\bullet)$ is

$$\Omega_I(V_\bullet) = \{\Lambda \in G_m(V) : \dim(\Lambda \cap V_{n+j-i_j}) \geq j, \ j = 1, \ldots, m\} . \qquad (13)$$

This is a subvariety of codimension $|I| = i_1 + i_2 + \cdots + i_m$ in $G_m(V)$. The cohomology class $[\Omega_I(V_\bullet)]$ does not depend on a flag $V_\bullet$. We denote it by σ_I and call a *Schubert class.*

Let Q denote the tautological quotient bundle on $G_m(V)$. Then

$$\sigma_{(i)} = c_i(Q) = S_{(1,\ldots,1)}(Q) ,$$

where 1 appears i times, and – more generally – the following *Giambelli formula* [10] holds:

Proposition 3 *In the cohomology ring of $G_m(V)$, we have*

$$\sigma_I = \Big| c_{i_p - p + q}(Q) \Big|_{1 \leq p,q \leq m} = S_{I^\sim}(Q) , \qquad (14)$$

where $I^\sim$ is the conjugate partition of I (i.e. the consecutive rows of the diagram of $I^\sim$ are the transposed consecutive columns of the diagram of I).

(Cf. [12, Chap. 1, Sect. 5], [6, §9.4]).

Given a partition I, consider the partition

$$J = (n - i_m, n - i_{m-1}, \ldots, n - i_1) \,.$$

Then (*loc.cit.*) σ_J is the unique Schubert class of complementary codimension to σ_I whose intersection with σ_I is nonzero, and in fact

$$\int_{G_m(V)} \sigma_I \cdot \sigma_J = 1 \,. \tag{15}$$

The class σ_J is called the *complementary class* to σ_I.

4 Cone classes for globally generated and ample vector bundles

In the proof of our main result, we shall use the following results of Fulton and Lazarsfeld from [7], [8] (cf. also [5, Chap. 12], [21, §8]). Recall first some classical definitions and facts from [5] (we shall also follow the notation from this book). Let E be a vector bundle of rank e on X. By a *cone* in E we mean a subvariety of E which is stable under the natural $\mathbb{G}_{\mathrm{m}}$-action on E. If $C \subset E$ is a cone of pure dimension d, then one may intersect its cycle $[C]$ with the zero-section of the vector bundle:

$$z(C, E) := s_E^*([C]) \in A_{d-e}(X) \,, \tag{16}$$

where $s_E^* : A_d(E) \to A_{d-e}(X)$ is the Gysin map determined by the zero-section $s_E : X \to E$. For a projective variety X, there is a well defined *degree* map $\int_X : A_0(X) \to \mathbf{Z}$.

The following results stem from [7, Theorem 1 (A)] and [8, Theorem 2.1].

Theorem 4 *Suppose that E is a vector bundle of rank e on a projective variety X, and let $C \subset E$ be a cone of pure dimension e.*
(1) If some symmetric power of E is globally generated, then

$$\int_X z(C, E) \geq 0.$$

(2) If E is ample, then

$$\int_X z(C, E) > 0.$$

Under the assumptions of the theorem, we also have in $H_0(X, \mathbf{Z})$ the homology analog of $z(C, E)$, denoted by the same symbol, and the homology degree map $\deg_X : H_0(X, \mathbf{Z}) \to \mathbf{Z}$. They are compatible with their Chow group counterparts via the cycle map: $A_0(X) \to H_0(X, \mathbf{Z})$ (cf. [5, Chap. 19]). We thus have the same inequalities for $\deg_X\big(z(C, E)\big)$.

5 Schur function expansions of Thom polynomials

We follow the setting from Section 2. Since the Schur functions form an additive basis of the ring of symmetric functions, the Thom polynomial $\mathcal{T}^\Sigma$ is uniquely written in the following form:

$$\mathcal{T}^\Sigma = \sum \alpha_{I_1,\dots,I_p}\ S_{I_1}(R^{(1)})\ S_{I_2}(R^{(2)})\ \cdots\ S_{I_p}(R^{(p)})\,, \tag{17}$$

where $\alpha_{I_1,\dots,I_p}$ are integer coefficients.

We say that the functor ϕ, associated with a representation V, *preserves global generateness* if for a collection of globally generated vector bundles $E_1, E_2, \dots, E_p$, the bundle

$$\phi(E_1, E_2, \dots, E_p)$$

is globally generated.

Examples of functors preserving global generateness over fields of characteristic zero are *polynomial functors*. They are, at the same time, quotient functors and subfunctors of the tensor power functors (cf. [14]). On the other hand, the functors: $\mathrm{Hom}(-, E)$ with fixed E, or $\mathrm{Hom}(-,-)$, do not preserve global generateness.

The main result of the present paper is

Theorem 5 *Suppose that the functor ϕ preserves global generateness. Then the coefficients $\alpha_{I_1,\dots,I_p}$ in Eq. (17) are nonnegative. Assume additionally that there exists a projective variety X [8] of dimension greater than or equal to* $\mathrm{codim}(\Sigma)$*, and there exist vector bundles $E_1, \dots, E_p$ on X such that the bundle $\phi(E_1, E_2, \dots, E_p)$ is ample. Then at least one of the coefficients $\alpha_{I_1,\dots,I_p}$ is positive.*

Proof. We assume for simplicity that $p = 2$ (the reasoning in general case goes in the same way). We want to estimate the coefficients α_{IJ} in the universal expansion into products of Schur functions:

$$\mathcal{T}^\Sigma(E_1, E_2) = \sum_{I,J} \alpha_{IJ}\ S_I(E_1) \cdot S_J(E_2) \tag{18}$$

Let E_1 and E_2 be the pullbacks of the tautological quotient bundles from the Grassmannians $G^{n_1}(\mathbf{C}^{N_1})$ and $G^{n_2}(\mathbf{C}^{N_2})$ to

$$G^{n_1}(\mathbf{C}^{N_1}) \times G^{n_2}(\mathbf{C}^{N_2})\,,$$

where N_1 and N_2 are sufficiently large. It is enough to estimate the coefficients α_{IJ} for such E_1 and E_2. Let $\sigma_K \in H^*(G^{n_1}(\mathbf{C}^{N_1}), \mathbf{Z})$ be the complementary class to $\sigma_{I^\sim}$ and $\sigma_L \in H^*(G^{n_2}(\mathbf{C}^{N_2}), \mathbf{Z})$ be the complementary

[8] The variety X can be singular.

class to $\sigma_{J^\sim}$. By the Giambelli formula (Proposition 3) and properties of complementary Schubert classes (15), we have

$$\alpha_{IJ} = \int_{G^{n_1}(\mathbf{C}^{N_1})\times G^{n_2}(\mathbf{C}^{N_2})} \mathcal{T}^{\Sigma}(E_1, E_2)\cdot(\sigma_K \times \sigma_L)\,.$$

The vector bundles E_1 and E_2 are globally generated. Hence, by the assumption, the bundle $\phi(E_1, E_2)$ is globally generated. By Theorem 4(1), we thus have $\alpha_{IJ} \geq 0$.

Now, suppose that there exists a projective variety of dimension greater than or equal to $\mathrm{codim}(\Sigma)$, and there exist vector bundles E_1, E_2 on X such that the bundle $\phi(E_1, E_2)$ is ample. Let Y be a subvariety of X of dimension equal to $\mathrm{codim}(\Sigma)$. Then, by Theorem 4(2), we have

$$\int_Y \mathcal{T}^{\Sigma}(E_1, E_2) > 0\,.$$

Therefore, $\mathcal{T}^{\Sigma} \neq 0$, which implies that at least one of the coefficients α_{IJ} is positive. $\square$

Consider now the projective variety

$$X = \prod_{i=1}^{p} G^{n_i}(\mathbf{C}^N)\,,$$

where N is sufficiently large. We denote by Q_i the pullback to X of the tautological quotient bundle on $G^{n_i}(\mathbf{C}^N)$, $i = 1, \dots, p$. The bundle Q_i is not ample, but it is globally generated. Let L be an ample line bundle on X. Then each bundle

$$E_i = Q_i \otimes L$$

is ample (cf. [14]).

Observe the hypotheses of the theorem are satisfied by the variety X, vector bundles $E_1, \dots, E_p$, and any polynomial functor ϕ of positive degree. We thus obtain

Corollary 6 *If ϕ is a polynomial functor of positive degree, then the coefficients $\alpha_{I_1,\dots,I_p}$ in Eq. (17) are nonnegative, and their sum is positive.*

In the next corollary, we use the concept of a classical Thom polynomial associated with a map $f : M \to N$ of complex manifolds and a nontrivial class of singularities Σ (cf. [31]). We *do not*, however, assume now that Σ is stable.

By the theory of Schur functions, there exist universal coefficients $\beta_{IJ} \in \mathbf{Z}$ such that

$$\mathcal{T}^{\Sigma} = \sum_{I,J} \beta_{IJ} S_I(TM^*)\cdot S_J(f^*TN)\,. \tag{19}$$

The following result follows from Theorem 5.

Corollary 7 *For any pair of partitions I, J, we have $\beta_{IJ} \geq 0$.*

We also give an alternative proof of the main result from [31]. Let Σ be a *stable* singularity. Then by the Thom-Damon theorem ([33], [2]),

$$\mathcal{T}^{\Sigma}(c_1(M), \ldots, c_m(M), c_1(N), \ldots, c_n(N))$$

is an universal polynomial in

$$c_i(TM - f^*TN) \quad \text{where} \quad i = 1, 2, \ldots$$

(Cf. also [17, Theorem 2].)

Using the theory of supersymmetric functions (cf. [20], [22], [30]), the Thom-Damon theorem can be rephrased by saying that there exist coefficients $\alpha_I \in \mathbf{Z}$ such that

$$\mathcal{T}^{\Sigma} = \sum_I \alpha_I S_I(TM^* - f^*TN^*)\,, \tag{20}$$

the sum is over partitions I with $|I| = \operatorname{codim}(\Sigma)$. The expression in Eq. (20) is unique (*loc.cit.*).

Theorem 8 *Let Σ be a stable singularity. Then for any partition I the coefficient α_I in the Schur function expansion of the Thom polynomial $\mathcal{T}^{\Sigma}$ (cf. Eq. (20)) is nonnegative.*

Proof. By the theory of Schur functions (*loc.cit.*), we have that the coefficient of $S_I(TM^* - f^*TN^*)$ in the RHS of (20) is equal to the coefficient of $S_I(TM^*)$ in the RHS of (19), that is, $\alpha_I = \beta_{I,\emptyset}$ for any partition I. The assertion now follows from Corollary 7. $\square$

Remark 9 Note that Theorem 5 overlaps various situations already studied in the literature. Consider, e.g., a *family of quadratic forms* on the tangent bundle of an m-fold M with values in a line bundle L, i.e. a section of

$$\operatorname{Hom}(\operatorname{Sym}^2(TM), L)\,.$$

The singularities of such forms lead to Thom polynomials. The group which is relevant here is $GL_m \times GL_1$ with the natural representation in the vector space

$$\bigoplus_{i=0}^{r} \operatorname{Sym}^i(\mathbf{C}^m) \otimes \operatorname{Hom}(\operatorname{Sym}^2(\mathbf{C}^m), \mathbf{C})\,.$$

The singularity classes defined by the 0th jet are just invariant subsets of $\operatorname{Hom}(\operatorname{Sym}^2(\mathbf{C}^m), \mathbf{C})$. The corank of the quadratic form determines the singularity class.

We recover[9] the situation already described in the literature in the context of *degeneracy loci formulas* for morphisms with symmetries of rank m bundles:

$$E^* \to E \otimes L\,.$$

The degrees of projective symmetric varieties were computed in [11]. The Schur function formulas for a trivial L were given in [13], [16]. To give the formulas in full generality [25], we consider, for partitions $I = (i_1, \ldots, i_m)$ and $J = (j_1 \ldots, j_m)$, the following determinant studied in the paper [19] of Lascoux:

$$d_{I,J} = \Big| \binom{i_a + m - a}{j_b + m - b} \Big|_{1 \le a,b \le m}\,. \tag{21}$$

Then, the Thom polynomial associated with the locus of quadratic forms whose corank $\geq q$ is equal to

$$2^{-\binom{q}{2}} \sum_J 2^{|J|}\ d_{\rho_q,J}\ S_J(E) \cdot S_{\binom{q+1}{2}-|J|}(L)\,,$$

where $J = (j_1, \ldots, j_q)$ runs over partitions contained in the partition

$$\rho_q = (q, q-1, \ldots, 1)\,.$$

(Cf. [25] for details. Similar formulas are valid for antisymmetric forms (*loc.cit.*).) In particular, we obtain the positivity of the $d_{\rho_q,J}$'s – a result known before by combinatorial methods (cf. [9]).

It seems to be interesting to apply Theorem 5 to other concrete situations, where Thom polynomials of invariant cones appear.

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