Miscellany on the zero schemes of sections of vector bundles

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Notes by Ozer Ozturk

Abstract

This purely expository article is a summary of the author's lectures on topological, algebraic, and geometric properties of the zero schemes of sections of vector bundles. These lectures were delivered at the seminar *Impanga* at the Banach Center in Warsaw (2006), and at the METU in Ankara (December 11-16. 2006). A special emphasis is put on the connectedness of zero schemes of sections, and the "point" and "diagonal" properties in algebraic geometry and topology. An overview of recent results by V. Srinivas, V. Pati, and the author on these properties is given.

1 The role of global equations in algebraic geometry and topology

Algebraic objects like *polynomials* enable us to present geometric objects like varieties via equations. However, when we consider projective (compact) varieties there is a problem: every global polynomial function is constant. To overcome this problem, we can "glue" local polynomial equations with the help of some global objects: vector bundles, which are families of vector spaces over a base variety, with transition functions from full linear groups.

For a motivating example, consider the complex projective n-space $\mathbb{P}^n = \mathbb{P}^n(V)$ – the set of lines l through zero in the $(n+1)$-dimensional complex vector space $V = \bigoplus_{i=0}^{n} \mathbb{C}e_i$. Over this variety we have a subbundle of the trivial vector bundle:

$$\mathcal{O}(-1) = \{(l, x) : x \in l\}\,.$$

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This bundle is called the *tautological line bundle* or the *Hopf bundle*. The dual bundle $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}(-1)^*$ is called the *Grothendieck bundle*. Note that the space of global sections of this bundle, $\Gamma(\mathbb{P}^n(V), \mathcal{O}(1))$, is isomorphic to $S^\bullet(V^*)$. So homogeneous polynomials in the dual coordinates $x_i = e_i^*$ can be identified with the global sections of $\mathcal{O}(1)$. This – most classical example of "global equations" – admits a natural generalization to *sections* of any vector bundle.

Suppose that s is a section of a vector bundle $\mathcal{E} \to X$. Consider the *zero scheme* of s

$$Z(s) = \{x \in X : s(x) = 0\}.$$

We wish to discuss the following question:

Which properties of $Z(s)$ can be deduced from those of $\mathcal{E}$?

Apart from this question (and also in connection with it), we shall study the following two properties of varieties: We will say that a variety X has the

Weak point property if for some point $x \in X$ there exist a vector bundle $\mathcal{E}$ on X of rank $\dim X$, and a section s of $\mathcal{E}$ such that $\{x\} = Z(s)$.

Diagonal property if there exist a vector bundle $\mathcal{E}$ on $X \times X$ of rank $\dim X$, and a section s of $\mathcal{E}$ such that $\Delta = Z(s)$, where Δ denotes the diagonal in $X \times X$.

We shall write (D) for the diagonal property and (P) for the week point property. Notice that (D) implies (P) – in fact, for any point $x \in X$ – via restriction from $X \times X$ to $X \times \{x\}$.

To the best of our knowledge, (P) was a popular topic neither in algebraic geometry nor topology. It appears that a stronger variant[1] of (P) was studied in algebra and arithmetics. Let A be a finitely generated reduced algebra over an algebraically closed field k with Krull dimension d. Recall that a point x of $X = \operatorname{Spec} A$ is a *complete intersection* if the corresponding maximal ideal has height d, and is generated by d elements of A. In this case x is a regular point, but not conversely. We record the following problem:

Characterize reduced affine k-varieties such that all smooth points are complete intersections.

This problem is discussed in detail by V. Srinivas in his paper [10] in the present volume. For instance, we have the following "Affine Bloch - Belinson conjecture":

Let $k = \mathbb{Q}$. Then for any finitely generated smooth k-algebra of dimension greater than 1 every maximal ideal is a complete intersection.

[1] The word "stronger" means here that we want (P) for any point $x \in X$, and the bundle involved in (P) should be trivial.

(Note that this conjecture is confirmed yet by *no* nontrivial example (!))

It appears that also (D) was not studied systematically before. First examples of varieties with (D) are curves. When X is a smooth curve, the diagonal Δ is a Cartier divisor in $X \times X$, so (D) holds. If (D) holds for varieties X_1 and X_2 then (D) holds for $X_1 \times X_2$ too. Other known examples having (D) are $\mathbb{P}^n$, Grassmannians and, in general, flag varieties of the type SL_n/P, where P is any parabolic subbroup of SL_n. Though this is well known to experts, we sketch a simple argument, since we could not find it in the literature. The argument in the Grassmannian case, given below, is well known. Also, (D) for the variety of *complete* flags was the starting point for the theory of *Schubert polynomials* of Lascoux and Schützenberger [6]. In [2], this property was proved and used to compute the fundamental classes of flag degeneracy loci. In fact, the argument for an arbitrary flag variety, follows closely that given in [2].

Let V be an n-dimensional vector space. Fix an increasing sequence of integers

$$d_\bullet : 0 < d_1 < d_2 < \ldots < d_{k-1} < d_k = n\,.$$

Then by a *$d_\bullet$-flag* we mean an increasing sequence of linear subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = V$$

of V such that $\dim V_i = d_i$ for $i = 1, \ldots, k$. The set of all $d_\bullet$-flags forms the flag variety $F\ell^{d_\bullet}$. For example, the sequence

$$d_1 = r < d_2 = n$$

gives rise to the Grassmannian $G_r(V)$ parametrizing r-dimensional linear subspaces of V.

In analogy to the Hopf bundle $\mathcal{O}(-1)$ on the projective space, we have a rank r vector bundle on $G_r(V)$. Consider the subbundle $\mathcal{S}$ of the trivial vector bundle $\mathcal{V}_{G_r(V)}$ of rank r for which the fiber $\mathcal{S}_g$ over $g \in G_r(V)$ is just the r-dimensional subspace corresponding to g. This bundle is called the rank r *tautological subbundle* and is denoted by $\mathcal{S}$. We have the tautological vector bundle sequence over $G_r(V)$:

$$0 \to \mathcal{S} \to \mathcal{V}_{G_r(V)} \to \mathcal{Q} \to 0\,,$$

where $\mathcal{Q}$ is the rank $(n-r)$ *tautological quotient*. Let G_1 and G_2 be two copies of $G_r(V)$. Let $\mathcal{S}_1$ be the tautological subbundle on G_1, and let $\mathcal{Q}_2$ be the tautological quotient bundle on G_2. Moreover, let

$$p_i : G_1 \times G_2 \to G_i$$

denote the projection for $i = 1, 2$. Then the composition

$$p_1^*\mathcal{S}_1 \hookrightarrow p_1^*(\mathcal{V}_{G_1}) = p_2^*(\mathcal{V}_{G_2}) \to p_2^*\mathcal{Q}_2$$

gives rise to the section of the vector bundle

$$\underline{\mathrm{Hom}}(p_1^*\mathcal{S}_1, p_2^*\mathcal{Q}_2)$$

over $G_1 \times G_2$ which vanishes precisely on the diagonal. We conclude that $G_r(V)$ has (D).

In the case of arbitary $d_\bullet$-flags, the tautological sequence takes the form:

$$\mathcal{S}_1 \hookrightarrow \mathcal{S}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{S}_{k-1} \hookrightarrow \mathcal{S}_k = \mathcal{V} \overset{q_1}{\twoheadrightarrow} \mathcal{Q}_1 \overset{q_2}{\twoheadrightarrow} \mathcal{Q}_2 \overset{q_3}{\twoheadrightarrow} \cdots \overset{q_k}{\twoheadrightarrow} \mathcal{Q}_k\,,$$

where $\mathrm{rank}(\mathcal{S}_i) = d_i$ for $i = 1, \ldots, k$, and $\mathcal{Q}_i$ is the quotient of $\mathcal{V}$ by $\mathcal{S}_i$, so that $\mathrm{rank}(\mathcal{Q}_i) = n - d_i$.

Let F_1 and F_2 be two copies of $F\ell^{d_\bullet}$ and

$$p_i : F_1 \times F_2 \to F_i$$

denote the projection for $i = 1, 2$. Consider the map

$$\varphi : \bigoplus_{i=1}^{k-1} \underline{\mathrm{Hom}}(p_1^*\mathcal{S}_i, p_2^*\mathcal{Q}_i) \to \bigoplus_{i=1}^{k-2} \underline{\mathrm{Hom}}(p_1^*\mathcal{S}_i, p_2^*\mathcal{Q}_{i+1})$$

defined by

$$\varphi(\sum_{i=1}^{k-1} f_i) = \sum_{i=1}^{k-2} (f_{i+1}|_{\mathcal{S}_i} - q_{i+1} \circ f_i)\,.$$

One checks that φ is surjective. Set $\mathcal{K} = \mathrm{Ker}\,\varphi$. Then the compositions

$$p_1^*\mathcal{S}_i \hookrightarrow p_1^*(\mathcal{V}_{F_1}) = p_2^*(\mathcal{V}_{F_2}) \to p_2^*\mathcal{Q}_i$$

for $i = 1, \ldots, k$, give rise to a section s of $\mathcal{K}$ such that $\Delta = Z(s)$.

Note that

$$\mathrm{rank}\,\mathcal{K} = \sum_{i=1}^{k-1} (n - d_i)(d_i - d_{i-1}) = \dim F\ell^{d_\bullet}\,.$$

Summing up, we conclude that the flag variety $F\ell^{d_\bullet}$ has (D).

2 Connectedness of the zero schemes of sections of vector bundles

In this section, we shall discuss the connectedness properties of the zero schemes. A prototype of all results here is

Theorem 1 (Lefschetz) *The hypersurface defined by a single homogeneous polynomial equation in $\mathbb{P}^n$ is connected provided $n \geq 2$.*

One should be carefull with generalizations of this simple result: one cannot – in general – replace $\mathbb{P}^n$ by $\mathbb{C}^n$, a hypersurface in $\mathbb{P}^n$ by a hypersurface in another smooth projective variety, and single equation by several equations (these issues are discussed in detail in [11]).

Recall that a line bundle $\mathcal{L} \to X$ is called *very ample* if $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^n}(1)|_X$ for some embedding of X into $\mathbb{P}^n$. A line bundle $\mathcal{L}$ is called *ample* if there exists $m \geq 0$ such that $\mathcal{L}^{\otimes m}$ is very ample.

Theorem 2 (Lefschetz) *Let X be a smooth projective irreducible variety over $\mathbb{C}$. Let $\mathcal{L}$ be an ample line bundle over X and s be a section of $\mathcal{L}$. Then $H_q(Z(s), \mathbb{Z}) \to H_q(X, \mathbb{Z})$ is an isomorphism for $q < \dim X - 1$ and is a surjection when $q = \dim X - 1$.*

This theorem is called the *Lefschetz hyperplane theorem.* Let us record its simple consequence.

Corollary 3 *Under the assumptions of the theorem, if $\dim X \geq 2$, then $Z(s)$ is connected.*

A vector bundle $\mathcal{E} \to X$ is called *ample* if the vector bundle $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E}^*)$ is ample.

Proposition 4 (Sommese) *If $\mathcal{E}$ is a rank e vector bundle on X and $s \in \Gamma(X, \mathcal{E})$, then $\mathbb{P}(\mathcal{E}^*) \setminus Z(s^*)$ is an affine-space bundle with fiber $\mathbb{C}^{e-1}$ over $X \setminus Z(s)$. So $H_0(\mathbb{P}(\mathcal{E}^*) \setminus Z(s^*), \mathbb{Z}) = H_0(X \setminus Z(s), \mathbb{Z})$.*

Indeed, if $x \in Z(s)$ then $s(x)^*$ vanishes on entire $\mathcal{E}_x^*$. If $x \notin Z(s)$ then $s(x)^*$ vanishes on a hyperplane in $\mathcal{E}_x^*$. Therefore, the fiber at x of

$$\mathbb{P}(\mathcal{E}^*) \setminus Z(s^*) \to X \setminus Z(s)$$

is

$$\mathbb{P}(\mathcal{E}^*) \setminus H = \mathbb{C}^{e-1}\,,$$

where H is a hyperplane (cf. [9]).

Theorem 5 (Griffiths, Sommese) *Let X be an irreducible smooth projective variety over $\mathbb{C}$. Let $\mathcal{E}$ be a rank e vector bundle over X, and s a section of $\mathcal{E}$. Then $H_q(Z(s), \mathbb{Z}) \to H_q(X, \mathbb{Z})$ is an isomorphism for $q < \dim X - e$, and is a surjection when $q = \dim X - e$.*

(Cf. [4], [9].)

Corollary 6 *Under the assumptions of the last theorem, if $\dim X \geq e + 1$, then $Z(s)$ is connected.*

For a more detailed account to this theory, we refer the reader to the Tu's article [11]. In this article, the author also discusses the conectedness of *degeneracy loci* which provide natural generalizations of the zero schemes of sections of vector bundles.

3 Cohomologically trivial line bundles

We start with two simple consequences of (D). Consider the ideal sheaf $\mathcal{J}_\Delta \subset \mathcal{O}_{X\times X}$. The diagonal property (D) implies that

$$\mathcal{E}^*_{|\Delta} = \mathcal{J}_\Delta/\mathcal{J}^2_\Delta \cong \Omega^1_X$$

(the last isomorphism uses the isomorphism $\Delta \cong X$). Therefore, Ω^1_X is locally free of rank equal to dimension of X. Hence X is smooth. Also, by the Grothendieck formula [5], we obtain the following expression for the fundamental class of Δ:

$$[\Delta] = c_{\dim X}(\mathcal{E})\,.$$

From now on – unless otherwise is explicitly stated – all results and conjectures surveyed here come from the paper [8], written by the author, V. Srinivas, and V. Pati.

Definition 7 *A line bundle $\mathcal{L}$ over X is called cohomologically trivial (we shall write "c.t.") if $H^i(X,\mathcal{L}) = 0$ for all $i \geq 0$.*

Example 8 Any smooth projective curve supports a c.t. line bundle (this should be well known to experts – for a written account, cf. [8]). Any abelian variety supports a c.t. line bundle [7].

Theorem 9 *(i) Let $p_1, p_2 : X \times X \to X$ be the two projections. If X has (D), and moreover, the following isomorphism holds:*

$$\operatorname{Pic}(X\times X) \cong p_1^* \operatorname{Pic}(X) \oplus p_2^* \operatorname{Pic}(X), \tag{1}$$

then there exists a c.t. line bundle $\mathcal{L}$ over X such that

$$\det(\mathcal{E}) = p_1^*\mathcal{L}^{-1} \otimes p_2^*(\mathcal{L}\otimes\omega_X^{-1})\,.$$

(ii) If $\dim X = 2$ and there exists a c.t. line bundle on X, then X has (D).

This theorem was proved in [8]. We give now a sketch of this proof.

Apply to the exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0\,.$$

and any line bundle $\mathcal{L}$ on $X\times X$, the functor $\operatorname{Hom}(-,\mathcal{L})$ (and its derived functors) to get the sequence of global $\operatorname{Ext}_{X\times X}$'s :

$$\operatorname{Ext}^{n-1}(\mathcal{I}_\Delta,\mathcal{L}) \to \operatorname{Ext}^n(\mathcal{O}_\Delta,\mathcal{L}) \xrightarrow{\alpha} H^n(X\times X,\mathcal{L})\,. \tag{2}$$

We shall need the following cohomological result (cf. [8] and the references there):

Proposition 10 *Let $\mathcal{L}$ be a line bundle on $X \times X$ whose restriction to Δ is isomorphic to ω_Δ. Assume that there exist a rank n vector bundle $\mathcal{E}$ on $X \times X$ with $\det(\mathcal{E}^*) = \mathcal{L}$, and $s \in \Gamma(X \times X, \mathcal{E})$ satisfying $\mathrm{Im}(s^*) = \mathcal{I}_\Delta$. Then α in the exact sequence (2) vanishes. The converse holds if $n = \dim(X) = 2$.*

Suppose that there exists a vector bundle $\mathcal{E}$ on $X \times X$ of rank n such that the diagonal is the zero scheme of its section s. Let $\mathcal{L} = \det(\mathcal{E}^*)$, and form the corresponding exact sequence (2). From the proposition, we have $\alpha = 0$. Consider the dual linear map to α:

$$\alpha^* : H^n(X \times X, \mathcal{L})^* \to \mathrm{Ext}^n(\mathcal{O}_\Delta, \mathcal{L})^* \cong H^n(\Delta, \omega_\Delta) = k\,.$$

Using (1) choose $\mathcal{M} \in \mathrm{Pic}(X)$ such that

$$\mathcal{L} = \det(\mathcal{E}^*) \cong p_1^*(\mathcal{M}) \otimes p_2^*(\mathcal{M}^{-1} \otimes \omega_X)\,.$$

By Serre duality on $X \times X$, we get that

$$H^n(X\times X, \mathcal{L})^* \cong H^n(X\times X, \mathcal{L}^{-1} \otimes \omega_{X\times X}) \cong H^n(X\times X, p_1^*(\mathcal{M}^{-1} \otimes \omega_X) \otimes p_2^* \mathcal{M}).$$

From the Künneth formula, we have

$$H^n(X\times X, p_1^*(\mathcal{M}^{-1}\otimes\omega_X)\otimes p_2^*(\mathcal{M})) = \bigoplus_{i=0}^{n} H^i(X, \mathcal{M}^{-1}\otimes\omega_X)\otimes H^{n-i}(X, \mathcal{M})\,. \tag{3}$$

Further, on any summand on the right, the induced map

$$H^i(X, \mathcal{M}^{-1} \otimes \omega_X) \otimes H^{n-i}(X, \mathcal{M}) \hookrightarrow$$

$$H^n(X \times X, p_1^*(\mathcal{M}^{-1} \otimes \omega_X) \otimes p_2^*(\mathcal{M})) \xrightarrow{\alpha^*} H^n(\Delta, \omega_\Delta) = k$$

coincides with the *Serre duality pairing* on cohomology of X, and is hence a non-degenerate bilinear form, for each $0 \le i \le n$. Thus, α^* vanishes if and only if all the summands on the RHS of (3) vanish, which says that $\mathcal{M}$ is c.t.

Conversely, if $\mathcal{M}$ is c.t., then in the exact sequence (2) determined by

$$\mathcal{L} = p_1^*\mathcal{M} \otimes p_2^*(\mathcal{M}^{-1} \otimes \omega_X),$$

the map α is the zero map, by reversing the above argument. Hence, if $n = 2$, we deduce that $X \times X$ supports a vector bundle $\mathcal{E}$ of rank 2 and a section s with zero scheme Δ, by the $n = 2$ case of the proposition.

This ends our sketch of the proof from [8].

Corollary 11 *Suppose that the isomorphism (1) holds for X, and X supports no c.t. bundle. Then X has not (D).*

Let X be a smooth proper variety over an algebraic closed field. The isomorphism (1) holds for X if and only if $\mathrm{Pic}\, X$ is a finitely generated abelian group. Also, if $H^1(X, \mathcal{O}_X) = 0$ then (1) holds for X.

4 When a smooth projective surface has (D)?

We remind that for a surface to have (D) is almost equivalent to the existence of a c.t. line bundle on it. In [8], the following results were proved for a smooth projective surface X over an algebraically closed field:

1. There exists a surface Y having (D), and a birational proper map $f : Y \to X$.
2. If $f : Y \to X$ is a birational map, X has (D) and $\operatorname{Pic} X$ is finitely generated then Y has (D).
3. If X is birational to one of the following: a ruled or an abelian surface or a K3 surface with 2 disjoint smooth rational curves or an elliptic surface with a section or a complex Enriques or hyperelliptic surface, then X has (D).
4. If $\operatorname{Pic} X = \mathbb{Z}$, $\Gamma(X, \mathcal{O}_X(1)) \neq 0$ and X has (D), then $X = \mathbb{P}^2$.

More precisely, the first item says that any surface – after blowing up sufficiently many points – becomes a surface having (D).

The last item implies that (D) fails for general K3 surfaces or general hypersurfaces in $\mathbb{P}^3$ of degree greater than 3.

5 When a higher dimensional variety has (D)?

In this section, we consider varieties of dimension ≥ 3. We first consider the varieties with Picard group $\mathbb{Z}$.

Proposition 12 *Let X be a smooth projective variety of dimension $d \geq 3$ over a field with* $\operatorname{Pic} X = \mathbb{Z}$*. If X has (D) and $H^0(X, \mathcal{O}_X(1)) \neq 0$ then X is a Fano variety and $\omega_X \cong \mathcal{O}_X(-n)$ for some $n \geq 2$.*

This result has two useful consequences.

Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of multidegree $(d_1, \ldots, d_r)$ such that $r < n - 3$ and $\sum d_i \geq n$. Then X has not (D). In particular, the non-Fano hypersurfaces in the projective spaces have not (D).

Also, let X be a smooth projective Fano variety such that $b_2(X) = 1$ and $\omega_X = \mathcal{O}_X(-1)$ (i.e. X is of index 1). Then X has not (D).

Let X be a scheme and $\mathcal{L}$ be a line bundle over X. We say that X has the

$\mathcal{L}$-point property if for every point $x \in X$ there exists a vector bundle $\mathcal{F}$ over X such that $d = \operatorname{rank} \mathcal{F} = \dim X$, $\det(\mathcal{F}) = \mathcal{L}$, and there exists a section s of $\mathcal{F}$ such that $\{x\} = Z(s)$.

Note that in this case $c_1(\mathcal{F}) = c_1(\mathcal{L})$ and $c_d(\mathcal{F}) = [x]$.

Theorem 13 *Let X be a smooth proper variety over an algebraically closed field. If $\operatorname{Pic} X$ is finitely generated and X has (D), then there exists a c.t. line bundle $\mathcal{L}$ on X such that*
(i) X has the $\mathcal{L}^{-1}$-point property, and
(ii) X has the $\mathcal{L} \otimes \omega_X^{-1}$-point property.

Corollary 14 *Let X be a scheme as in Theorem 13 with finitely generated $\operatorname{Pic} X$. If for any c.t. line bundle $\mathcal{L}$, either $\mathcal{L}^{-1}$-point property fails or $\mathcal{L} \otimes \omega_X^{-1}$-point property fails then X has not (D).*

For example, if X is a smooth complex projective quadric of dimension 3, then $\mathcal{O}_X(-1)$ and $\mathcal{O}_X(-2)$ are the unique c.t. line bundles on X. One checks – with the help of the corollary – that X has not (D).

Sometimes, (D) boils down to (P). For instance, we have

Proposition 15 *Let X be a group variety over an algebraically closed field. Then X has (D) if and only if X has (P).*

Indeed, assume that X has (P). Let $\mathcal{E}$ be a vector bundle over X such that $\operatorname{rank} \mathcal{E} = \dim X$, and let $s \in \Gamma(X, \mathcal{E})$ be such that $Z(s) = \{x\}$ for some $x \in X$. Since X is a group variety we have the morphisms $\mu : X \times X \to X$ of multiplication and $i : X \to X$ of inverse. Consider the morphism

$$f : X \times X \to X$$

defined by

$$f(u, v) = \mu(\mu(u, i(v)), x)\,.$$

Since $f^{-1}(x) = \Delta$, the vector bundle $f^*\mathcal{E}$, together with section f^*s, implies (D) for X.

In particular, an abelian variety has (D) if and only if it has (P). Recently O. Debarre [1] has proved that the Jacobian of a smooth projective connected curve has (P), and that there exist non-principally polarized abelian varieties in dimension greater than 2, which fail to have (P). Moreover, he suggests that (P) may characterize Jacobians among all principally polarized abelian varieties with Picard number 1.

6 Affine case

Let k be an algebraically closed field and A be a finitely generated k-algebra. Let $X = \operatorname{Spec} A$. If $\dim X = 2$, then X has (D) by Serre's construction. What about higher dimensions?

Theorem 16 *An affine algebraic group over an algebraically closed field has (D).*

Indeed, M. Kumar and M.P. Murthy proved that an affine algebraic group over an algebraically closed field has (P). It suffices then to invoke Proposition 15.

Conjecture 17 *There exists smooth complex varieties of any dimension greater than 2 for which (D) fails.*

This conjecture leads to the following question:

Let k be an algebraically closed field and A be a regular k-algebra. Let K be an extension field of k (not necessarily algebraically closed), $A_K := A \otimes_k K$ and let $M \subset A_K$ be a maximal ideal with residue field K. Does there exist a projective A_K-module P of rank $n = \dim A$ such that there is a surjection $P \twoheadrightarrow M$?

The question has a positive answer by M.P. Murthy when K is an algebraically closed extension of k. If $X = \operatorname{Spec} A$ has (D), then the question has a positive answer for any field extension K. So a negative answer to the question would produce counterexamples to (D).

7 Diagonal property in topology

In this section, we use mostly the notation used by topologists. Let M be a compact connected oriented smooth manifold of real dimension n, and Δ be the diagonal submanifold of $M \times M$. We say that M has property (D_r) if there exist a smooth *real* vector bundle $\mathcal{E}$ over $M \times M$ with $\operatorname{rank}(\mathcal{E}) = n$ and a smooth section s of $\mathcal{E}$ such that s is transverse to the zero section $0_{\mathcal{E}}$ of $\mathcal{E}$ and $s^{-1}(0_{\mathcal{E}}) = \Delta$. If the vector bundle $\mathcal{E}$ is *orientable* then we say that M has the property (D_o). If $\dim_{\mathbb{R}} M = 2m$ and the vector bundle $\mathcal{E}$ is a smooth *complex* vector bundle of $\operatorname{rank}_{\mathbb{C}}(\mathcal{E}) = m$ then we say that M has the property (D_c). We have the following relation between these properties:

$$(D) \Rightarrow (D_c) \Rightarrow (D_o) \Rightarrow (D_r).$$

Take a Riemannian metric on M. It induces a Riemannian metric on $M \times M$, on the tangent bundle τ_M, and on all its subbundles. Let U be a closed ϵ-tubular neighborhood of Δ in $M \times M$. Then, by the tubular neighborhood theorem, we have a diffeomorphism

$$\phi : (U, \partial U) \xrightarrow{\sim} (D(\nu), S(\nu)),$$

where $D(\nu)$ is the ϵ-disc bundle of the normal bundle $\rho : \nu \to \Delta$ of Δ in $M \times M$ and $S(\nu)$ is the ϵ-sphere bundle. Then $r := \rho \circ \phi : U \to \Delta$ is a strong deformation retraction of U to Δ. So we have the following bundle diagram:

$$\begin{array}{ccc} r^*(\nu) & \longrightarrow & \rho^*(\nu) \\ {\scriptstyle\rho}\Big\downarrow & & \Big\downarrow{\scriptstyle\rho} \\ U & \xrightarrow{\phi} & D(\nu). \end{array}$$

The bundle $\rho^*(\nu)_{|S(\nu)} \to S(\nu)$ has a tautological section s given by $v \mapsto v$, which satisfies $\|s(v)\| = \epsilon$ for all $v \in S(\nu)$. So

$$\rho^*(\nu)_{|S(\nu)} = \xi \oplus \mathcal{L},$$

where $\mathcal{L}$ is the trivial line subbundle spanned by s and ξ is its orthogonal complement. Under the identification $\Delta \cong M$, $\nu \to \Delta$ is isomorphic to $\tau_M \to M$. Since M is orientable, so is ξ, and it is isomorphic to the quotient bundle

$$\rho^*(\tau_M)/\mathcal{L} \to S(\tau_M).$$

Let $\mathcal{F} := \phi^*(\xi)$. $\mathcal{F}$ is a rank $(n-1)$ subbundle of $r^*(\nu)_{|\partial U}$, and is isomorphic to $\rho^*(\tau_M)/\mathcal{L} \to S(\tau_M)$. Moreover, $\mathcal{F} \to \partial U$ is orientable.

Note that the restriction of ξ to each fiber $S(\nu_x)$ of $\rho : S(\nu) \to M$ is the tangent bundle τ_{n-1} of the sphere $S(\nu_x)$. Consequently, the bundle $\mathcal{F}$ when restricted to the fiber $r^{-1}(x)$ of the bundle $r : \partial U \to \Delta$ is isomorphic to τ_{n-1}. The following result is a *key* tool in analyzing the topological diagonal properties:

Lemma 18 *Let M, Δ and U be as above. Set $X := M \times M \setminus Int(U)$. Then M has (D_r) if and only if the rank $(n-1)$ bundle $\mathcal{F} \to \partial U$ is isomorphic to the restriction to $\partial U = \partial X$ of a smooth rank $(n-1)$ bundle $\mathcal{G}$ on X. Moreover, M has (D_o) if and only if the bundle $\mathcal{G}$ can be chosen to be orientable.*

We list here some results from [8]:

1. S^n has (D_r) if and only if $n = 1, 2, 4$ or 8 (all except the first have (D_o)).

2. Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 2$. Then M has (D_c). (This is in contrast with the algebraic situation.)

3. Let M be an almost complex manifold of $\dim_{\mathbb{C}} M = 3$. Assume that $H^1(M, \mathbb{Z}) = 0$ and $H^2(M, \mathbb{Z}) = \mathbb{Z}$. Then if M satisfies (D_c), the second Stiefel-Whitney class $w_2(M)$ vanishes (i.e. M is spin).

4. Let $M \subset \mathbb{CP}^N$ be a smooth projective variety of $\dim_{\mathbb{C}} M = 3$. Assume that M is a strict complete intersection or a set-theoretically complete intersection with $H_1(M, \mathbb{Z}) = 0$. Then M has (D_c) if and only if M is spin.

5. Let M be a smooth strict complete intersection of $\dim_{\mathbb{C}} M = 3$ in $\mathbb{CP}^n$ with $M = X_1 \cap \cdots \cap X_{n-3}$ where X_i is a smooth hypersurfaces of degree d_i. Then M has (D_c) only if $\left(n+1-\sum d_i\right)$ is even. In particular, a smooth hypersurface M in $\mathbb{CP}^4$ has (D_c) only if it is of odd degree.

6. Let $m \geq 2$. Then a smooth quadric hypersurface $Q_{2m-1} \subset \mathbb{P}^{2m}$ has not (D_c).

Note that the converse of the third item is not true. For example, S^6 is an almost complex manifold which has a spin structure but fails to have (D_c) by the first item.

From the last item one can deduce that a smooth quadric hypersurface $Q_{2m-1} \subset \mathbb{P}^{2m}$ over any algebraically closed field, has not (D).

Conjecture 19 *Let Q_n be a smooth quadric hypersurface in $\mathbb{P}^{n+1}$ (over an algebraically closed field). Then Q_n has (D) if and only if $n = 1, 2$ or 4.*

8 Three other conjectures

1. From the results on surfaces, it follows that any smooth projective toric surface has (D). We conjecture that any smooth toric variety has (D).

2. The Grassmannian of Lagrangian 2-planes in $\mathbb{C}^4$ is identified with the quadric Q_3, which has not (D). We conjecture that, in general, the homogeneous spaces Sp_n/P and $SO(n)/P$ (P being any parabolic subgroup) have not (D).[2]

3. We conjecture that (D) fails for a general cubic threefold X, though X has (P) and even the $\mathcal{O}_X(1)$-point property ($\mathcal{O}_X(-1)$ is the unique c.t. line bundle on X) – cf. [8] for details.

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