Schubert functors and Schubert polynomials

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Abstract

We construct a family of functors assigning an R-module to a flag of R-modules, where R is a commutative ring. As particular instances, we get flagged Schur functors and Schubert functors, the latter family being indexed by permutations. We identify Schubert functors for vexillary permutations with some flagged Schur functors, thus establishing a functorial analogue of a theorem from [6] and [15]. Over an infinite field, we study the trace of a Schubert module, which is a cyclic module over a Borel subgroup B, restricted to the maximal torus. The main result of the paper says that this trace is equal to the corresponding Schubert polynomial of Lascoux and Schützenberger [6]. We also investigate filtrations of B-modules associated with the Monk formula [10] and transition formula from [8].

Introduction

In the present paper, we study a certain extension of the notion of polynomial functors defined on the category of R-modules, where R is a commutative ring. If $R \supset \mathbb{Q}$, the field of rationals, such polynomial functors are classified by the so-called Schur functors and have been extensively studied ([5], [14], [9], [11], and [1]). Restricted to the vector spaces over a field of characteristic 0, Schur

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functors provide the set of all irreducible polynomial representations of the general linear group, and in this context were investigated already by Schur (cf. [13]). We refer to [12] for a survey concerning Schur functors and some of their applications.

The main notion introduced in the present paper is the notion of a *Schubert functor*. Schubert functors assign to a given flag of R-modules a new R-module, where R is a commutative ring [1]. As Schur functors are indexed by partitions, Schubert functors are indexed by permutations; this reflects their relationship with the Schubert calculus on the flag manifold. The last subject, being a part of classical algebraic geometry, goes back to Ehresmann and Bruhat. In the beginning of the seventies, the cohomology ring of the flag variety was extensively studied by Bernstein-Gelfand-Gelfand [2] and Demazure [3] with the help of some differential-like operators defined on it. In [6], Lascoux and Schützenberger have invented a more combinatorial approach; the main notion of this approach is the notion of a *Schubert polynomial* (see Section 2 for a precise definition). Schubert polynomials are indexed by permutations and generalize Schur polynomials. In his Oberwolfach talk (June 1983), Lascoux suggested the existence of a family of functors generalizing Schur functors and having similar properties as Schubert polynomials. The present paper may be treated as a realization of this program.

The main aim of this paper is to relate Schubert functors and Schubert polynomials. The fundamental relationship between Schur functors and Schur polynomials says that the trace of the restricted action of the group of diagonal matrices on a Schur module is equal to the corresponding Schur polynomial (see [13], or [9] for a modern treatment). The main result of the present paper says that the trace of the restricted action of the group of diagonal matrices on a Schubert module is equal to the corresponding Schubert polynomial. This fact is proved in Section 4. The main idea is based on an analogue of the "transition formula", invented and applied by Lascoux and Schützenberger in [8], and worked out here in the setting of Schubert modules.

Schubert functors are defined and discussed in Section 1. In Section 2, we recall a definition and properties of Schubert polynomials in the setting needed for the purposes of this paper. In particular, we describe a combinatorics standing behind the transition formula. Section 3 is devoted to study of the so-called *flagged Schur functors* which are some interesting generalizations of Schur functors, and some special cases of Schubert functors. We prove here an analogue of a theorem of Lascoux-Schützenberger [6] and Wachs [15] relating Schubert polynomials for vexillary permutations and flagged Schur polynomials. In Section 5, a filtration associated with the Monk formula [10] is

[1] Following the classical route ([13], [5], and [9]), we work here over commutative $\mathbb{Q}$-algebras R. We discuss generalizations of our results to arbitrary commutative rings or, in some cases, infinite fields in Remark 5.3.

discussed.

The main results of the present paper were announced in [4], but the full text has not been published untill now. The present article, modulo some minor modifications, follows the authors' preprint distributed in 1986. We add, at the end, a section on further developments related to the main themes of this paper.

We dedicate this paper to the mathematician who

- introduced in 1977 Schur functors to commutative algebra and combinatorics in the course of his work on the syzygies of determinantal ideals [5],
- invented in 1982 (together with M.-P. Schützenberger) Schubert polynomials ([6], [7]),
- and emphasized, with the same coauthor, the role of vexillary permutations in the study of the symmetric groups ([6], [8]).

1 Definition of Schubert functors

Let R be a commutative $\mathbb{Q}$-algebra and let

$$E. : E_1 \subset E_2 \subset \cdots \subset E_n \ \subset \cdots$$

be a flag of R-modules. Assume that $\mathcal{I} = [i_{k,l}]$, $k, l = 1, 2, \ldots$, is a matrix consisting of zeros and units and satisfying the following conditions:

1.1 *$i_{k,l} = 0$ for $k \geq l$.*

1.2 *$\sum_l i_{k,l}$ is finite for every k.*

1.3 *$\mathcal{I}$ has a finite number of nonzero rows.*

A matrix $\mathcal{I}$ with properties 1.1–1.3 will be called a *shape*. We will represent a shape graphically by replacing each unit by "$\times$" and by omitting zeros on the diagonal and under the diagonal. Moreover, we will omit all the columns which are right to the last nonzero column.

Example 1.4 The matrix

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
 & & & \ldots\ldots & & & & &
\end{array}
\quad \text{will be represented as} \quad
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & & \times & 0 & \times & 0 \\
 & & & \times & 0 & \times \\
 & & & & \times & 0 \\
 & & & & & 0
\end{array}$$

Let $i_k := \sum_{l=1}^{\infty} i_{k,l}\,,\ k = 1,2,\ldots,\ \widetilde{i}_l := \sum_{k=1}^{\infty} i_{k,l}\,,\ l = 1,2,\ldots$. We define the module $S_{\mathcal{I}}(E.)$ as the image of the map

$$\Phi_{\mathcal{I}}(E.) : \bigotimes_k S_{i_k}(E_k) \xrightarrow{\Delta_S} \bigotimes_k \bigotimes_l S_{i_{k,l}}(E_k) \xrightarrow{m_\wedge} \bigotimes_l \bigwedge^{\widetilde{i}_l} E_l$$

where Δ_S is a product of symmetrizations and $m_\wedge$ is a product of multiplications in the corresponding exterior algebras. Observe that $\Phi_{\mathcal{I}}(E.)$ is well defined because of properties 1.1–1.3. It is clear that $S_{\mathcal{I}}(-)$ defines, in fact, a *functor*: if $E.$ and $F.$ are two flags of R-modules and $f : \bigcup_n E_n \to \bigcup_n F_n$ is an R-homomorphism such that $f(E_n) \subset F_n$, then f induces in a natural way a homomorphism $S_{\mathcal{I}}(E.) \to S_{\mathcal{I}}(F.)$.

For a sequence $I : i_1 \geq i_2 \geq \cdots \geq i_r$ of nonnegative integers, we define the *flagged Schur module* $S_I(E.)$ as $S_{\mathcal{I}}(E.)$ where the shape $\mathcal{I} = [i_{k,l}]$ is defined as follows. Let $c := \max\{s + i_s : s = 1,2,\ldots,r\}$, then

$$i_{k,l} := \begin{cases} 1 & \text{for } k = 1,2,\ldots,r,\ c - i_k + 1 \leq l \leq c \\ 0 & \text{otherwise.} \end{cases}$$

Let now $\mu = \mu(1), \mu(2), \mu(3), \ldots$ be a permutation, i.e. bijection of the set of natural numbers which is the identity on the complement of some finite set [2]. For every positive integer k, we define

$$I_k(\mu) := \{l : l > k,\ \mu(k) > \mu(l)\}$$

which will be called the kth *inversion set* of μ. The sequence of numbers $i_k = |I_k(\mu)|$ (the cardinality of $I_k(\mu)$), $k = 1,2,\ldots$, is called the *code* of μ. A code determines the corresponding permutation in a unique way. Note that $\sum i_k$ is the length of μ. The *index* of the permutation μ is defined to be the number $\sum (k-1) i_k$. The *shape* of μ is defined to be the matrix

$$\mathcal{I}_\mu = [i_{k,l}] := [\chi_k(l)]\,,\ \ k,l = 1,2,\ldots$$

where χ_k is the characteristic function of the set $I_k(\mu)$.

Example 1.5 For $\mu = 5,2,1,6,4,3,7,8,\ldots$, the code of μ is $(4,1,0,2,1,0,\ldots)$, and the shape $\mathcal{I}_\mu$ is

$$\begin{array}{ccccc} \times & \times & 0 & \times & \times \\ & \times & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & \times & \times \\ & & & & \times \end{array}$$

[2] We display a permutation by the sequence of its values; the composition of permutations will be denoted by "$\circ$".

We define $S_\mu(E.)$—the *Schubert module* associated with a permutation μ and a flag $E.$, as the module $S_{\mathcal{I}_\mu}(E.)$; this gives rise to the corresponding *Schubert functor* $S_\mu(-)$.

Remark 1.6 Let $E. : E_1 \subset E_2 \subset \cdots$ be a flag of vector spaces over a field K of characteristic 0, $\dim E_i = i$. Let B be the group of linear transformations f of $E := \bigcup E_i$ which preserve $E.$, i.e. $f(E_i) = E_i$. Observe that the vector spaces used in the definition of $S_{\mathcal{I}}(E.)$ are B-modules (modules over the group ring $K[B]$ will be called B-modules for brevity) and the maps $\Delta_S, m_\wedge, \Phi_{\mathcal{I}}$ are morphisms of B-modules. Moreover, $\bigotimes_k S_{i_k}(E_k)$ is a cyclic B-module. Therefore $S_{\mathcal{I}}(E.)$ is also a *cyclic* B-module. Let $\{e_i : i = 1, 2, \ldots\}$ be a basis of E such that $e_1, e_2, \ldots, e_k$ span E_k. Then $S_{\mathcal{I}}(E)$ as a B-submodule in $\bigotimes_l \bigwedge^{\widetilde{i_l}} E_l$ is generated by the element

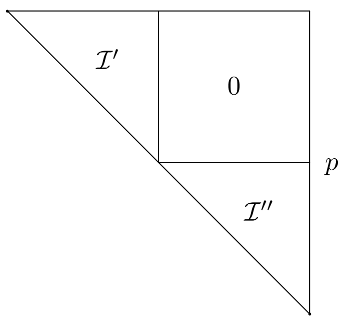
$$C_{\mathcal{I}} := \bigotimes_l e_{k_{1,l}} \wedge e_{k_{2,l}} \wedge \cdots \wedge e_{k_{\widetilde{i_l},l}}$$

where $k_{1,l} < k_{2,l} < \cdots < k_{\widetilde{i_l},l}$ are precisely those indices for which $i_{k_{r,l},l} = 1$. In particular, if in the lth column of $\mathcal{I}$ the lowest "$\times$" appears in the sth row, then we can replace $\bigwedge^{\widetilde{i_l}} E_l$ by $\bigwedge^{\widetilde{i_l}} E_s$ in the definition of $S_{\mathcal{I}}(E.)$. For a given permutation μ, we will write C_μ instead of $C_{\mathcal{I}_\mu}$.

Example 1.7 $C_{5,2,1,6,4,3,7,\ldots} = e_1 \otimes e_1 \wedge e_2 \otimes e_1 \wedge e_4 \otimes e_1 \wedge e_4 \wedge e_5$ (an element of $E_1 \otimes \bigwedge^2 E_2 \otimes \bigwedge^2 E_4 \otimes \bigwedge^3 E_5$).

Let us record the following immediate consequence of the definition of $S_{\mathcal{I}}(E.)$.

Lemma 1.8 *Assume that for a shape $\mathcal{I} = [i_{k,l}]$, $k, l = 1, 2, \ldots$, there exists p such that $i_{k,l} = 0$ for $k \le p$ and $l > p$. Let $\mathcal{I}'$ be a shape $[i'_{k,l}]$ such that $i'_{k,l} = i_{k,l}$ for $k \le p$ and $i'_{k,l} = 0$ for $k > p$. Let $\mathcal{I}''$ be a shape $[i''_{k,l}]$ such that $i''_{k,l} = 0$ for $k \le p$ and $i''_{k,l} = i_{k,l}$ for $k > p$. Pictorially*



Then $S_{\mathcal{I}}(E.) \simeq S_{\mathcal{I}'}(E.) \otimes S_{\mathcal{I}''}(E.)$.

2 Review of Schubert polynomials

In this section, we collect some facts on Schubert polynomials, as defined in [6] and [7], useful in other parts of this paper. Let $A := \mathbb{Z}[x_1, x_2, \ldots]$ be a polynomial ring in a countable set of variables. For every i, we consider an operator $\partial_i : A \to A$ defined for $f \in A$ by

$$\partial_i(f) := \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}} .$$

For the ith simple reflection $\tau_i = 1, \ldots, i-1, i+1, i, i+2, \ldots$, we set $\partial_{\tau_i} := \partial_i$.

Proposition 2.1 *Let $\mu = \sigma_k \circ \cdots \circ \sigma_1 = \pi_k \circ \cdots \circ \pi_1$ be two reduced decompositions of a permutation μ into simple reflections. Then*

$$\partial_{\sigma_k} \circ \cdots \circ \partial_{\sigma_1}(f) = \partial_{\pi_k} \circ \cdots \circ \partial_{\pi_1}(f)$$

for every $f \in A$.

Therefore, for a given permutation μ, we can define ∂_μ as $\partial_{\sigma_k} \circ \cdots \circ \partial_{\sigma_1}$ independently of a reduced decomposition of μ chosen. For a given μ, let n be a positive integer such that $\mu(k) = k$ for $k > n$. The *Schubert polynomial X_μ* is defined by

$$X_\mu := \partial_{\mu^{-1} \circ \omega_n}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0)$$

where ω_n is the permutation $n, n-1, \ldots, 2, 1, n+1, n+2, \ldots$. Observe that the above definition does not depend on a choice of n, because

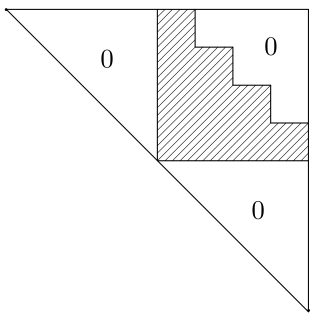
$$\partial_n \circ \cdots \circ \partial_2 \circ \partial_1(x_1^n x_2^{n-1} \cdots x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} .$$

Schubert polynomials satisfy the following properties:

2.2 *The Schubert polynomial X_μ is symmetric with respect to x_k and x_{k+1} iff $\mu(k) < \mu(k+1)$ (or equivalently $i_k \leq i_{k+1}$).*

2.3 *If $\mu(1) < \mu(2) < \cdots < \mu(k) > \mu(k+1) < \mu(k+2) < \cdots$ (or equivalently $i_1 \leq i_2 \leq \cdots \leq i_k$, $0 = i_{k+1} = i_{k+2} = \cdots$), then X_μ equals the Schur polynomial $s_{i_k, \ldots, i_2, i_1}(x_1, \ldots, x_k)$ (cf. [9] for this last notion).*

Pictorially, the shape of μ can be displayed as



2.4 *If* $i_1 \geq i_2 \geq \cdots$ *, then* $X_\mu = x_1^{i_1} x_2^{i_2} \cdots$ *is a monomial.*

If the sets $I_k(\mu)$ ordered by inclusion form a chain, we call μ a *vexillary* permutation (see [6] and [8] for more information on vexillary permutations).

Example 2.5 $3,6,5,1,4,2,7,\ldots$ is a vexillary permutation contrary to $6,3,5,1,2,4,7,\ldots$.

$$\mathcal{I}_{365142\ldots} = \begin{array}{ccccc} 0 & 0 & \times & 0 & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & 0 & 0 \\ & & & & \times \end{array} \qquad \mathcal{I}_{635124\ldots} = \begin{array}{ccccc} \times & \times & \times & \times & \times \\ & \times & 0 & \times & 0 \\ & & 0 & \times & 0 \\ & & & \times & \times \\ & & & & 0 \end{array}$$

Recall the notion of a flagged Schur function $s_{i_1,\ldots,i_k}(b_1,\ldots,b_k)$ where $i_1 \geq \cdots \geq i_k$ and $0 < b_1 \leq \cdots \leq b_k$ are two sequences of nonnegative integers. One sets

$$s_{i_1,\ldots,i_k}(b_1,\ldots,b_k) := \det\Big(s_{i_p-p+q}(x_1,\ldots,x_{b_p})\Big)_{1\leq p,q\leq k}$$

where s_l is the complete homogeneous symmetric function of degree l (for an alternative definition in terms of standard tableaux, see [15]). For a given sequence of nonnegative integers $I = (i_1,\ldots,i_n)$, we define $I^{\leq}$ (resp. $I^{\geq}$) as the increasing (resp. decreasing) reordering of I. The following result stems from [6] and [15]:

Theorem 2.6 *Let* μ *be a vexillary permutation with the code* $(i_1,i_2,\ldots)$ *and assume* $i_n \neq 0$*,* $i_r = 0$ *for* $r > n$*. Then*

$$X_\mu = s_{(i_1,\ldots,i_n)^{\geq}}\left(\min I_1(\mu) - 1,\ldots,\min I_n(\mu) - 1\right)^{\leq}.$$

We finish this section with some facts on multiplication of Schubert polynomials.

2.7 *The Monk formula for multiplication by* X_{τ_k} ([10], [7])*:*

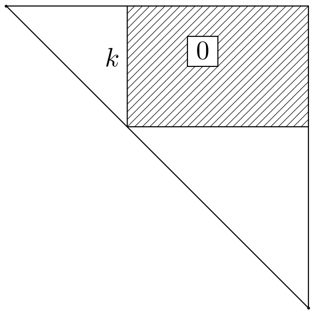
$$X_\mu \cdot (x_1 + \cdots + x_k) = \sum X_{\mu\circ\tau_{p,q}},$$

where $\tau_{p,q}$*,* $p < q$*, denotes the permutation*

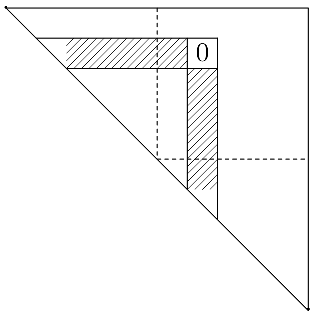
$$1,\ldots,p-1,q,p+1,\ldots,q-1,p,q+1,\ldots$$

and the summation ranges over all p,q *such that* $p \leq k$*,* $q > k$ *and* $l(\mu\circ\tau_{p,q}) = l(\mu)+1$*.*

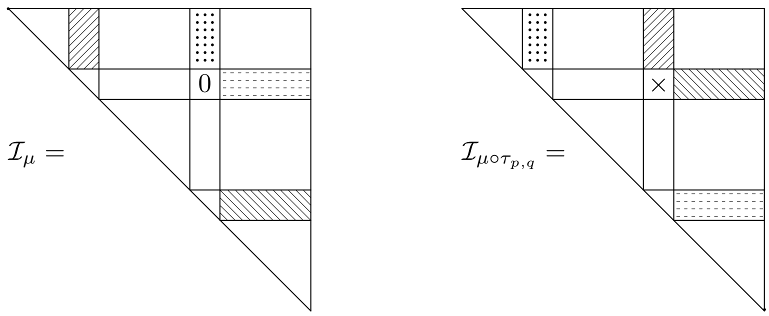
This last requirement can be restated as follows: for each $i \in]p,q[$, $\mu(i) \notin [\mu(p),\mu(q)]$. Graphically, the shapes of permutations $\mu\circ\tau_{p,q}$ can be obtained from the shape of μ in the following way. Let us restrict our attention to the places occupied by the 0's in the marked rectangle with k rows



Fix one such place. Then consider the hook having its head in the fixed place and the remaining boxes in the left hand side of the row, and the bottom part of the column occupied by $\boxed{0}$.



If in the hook the number of the 0's is equal to the number of the $\times$'s plus one, then this place gives a contribution to the right-hand side of the Monk formula. The shape of this summand is obtained from the shape of μ by replacing our $\boxed{0}$ by $\boxed{\times}$ and by exchanging parts of appriopriate columns and rows as indicated in the following pictures:



Example 2.8 $X_{246315879\ldots} \cdot (x_1 + x_2) = X_{346215879\ldots} + X_{264315879\ldots} + X_{256314879\ldots}$. Note that multiplication by x_1 becomes

$$X_\mu \cdot x_1 = \sum X_{\mu\circ\tau_{1,q}},$$

where q is such that if $i \in]1, q[$ then $\mu(i) \notin [\mu(1), \mu(q)]$. In the above graphical description, we consider only the places occupied by the 0's in the first row of the shape of μ.

From the Monk formula, we obtain:

2.9 *One has* $X_\mu \cdot x_k = \sum_j \operatorname{sign}(j-k)\cdot X_{\mu\circ\tau_{j,k}}$ *where the summation ranges over all* j *such that*

1° $\operatorname{sign}(j-k) = \operatorname{sign}(\mu(j)-\mu(k))$,
2° *for each* $i \in]j,k[$, $\mu(i) \notin [\mu(j),\mu(k)]$.

Example 2.10

$$X_{31425\ldots}\cdot x_1 = X_{41325\ldots}$$
$$X_{31425\ldots}\cdot x_2 = X_{34125\ldots} + X_{32415\ldots}$$
$$X_{31425\ldots}\cdot x_3 = -X_{41325\ldots} + X_{315246\ldots}$$
$$X_{31425\ldots}\cdot x_4 = -X_{32415\ldots} + X_{314526\ldots}$$

For the purposes of Section 4, the following special case of 2.9 will be of particular importance.

Transition Formula 2.11 ([8]) *Let* (j,s) *be a pair of positive integers such that*

1° $j < s$ *and* $\mu(j) > \mu(s)$,
2° *for every* i *in* $]j,s[$, $\mu(i)$ *is outside of* $[\mu(s),\mu(j)]$,
3° *for every* $r > j$, *if* $\mu(s) < \mu(r)$ *then there exists* $i \in]j,r[$ *such that* $\mu(i) \in [\mu(s),\mu(r)]$,

then

$$X_\mu = X_\lambda \cdot x_j + \sum X_{\psi_t}$$

where $\lambda = \mu\circ\tau_{j,s}$, $\psi_t = \mu\circ\tau_{j,s}\circ\tau_{k_t,j}$ *and the sum ranges over the set of numbers* k_t *satisfying*

4° $k_t < j$ *and* $\mu(k_t) < \mu(s)$,
5° *if* $i \in]k_t,j[$ *then* $\mu(i) \notin [\mu(k_t),\mu(s)]$.

Observe that if (j,s) is the maximal pair (in the lexicographical order) satisfying condition 1° then conditions 2°–3° are also satisfied. A transition determined by this pair will be called *maximal*. In particular, for every nontrivial permutation there exists at least one transition.

Example 2.12 $\mu = 5,2,1,8,6,3,4,7,9,\ldots,$

$$X_\mu = X_{521843679\ldots}\cdot x_5 + X_{524813679\ldots} + X_{541823679\ldots}$$
$$X_\mu = X_{521763489\ldots}\cdot x_4 + X_{527163489\ldots} + X_{571263489\ldots} + X_{721563489\ldots}$$
$$X_\mu = X_{512864379\ldots}\cdot x_2,$$

and the first equation gives the maximal transition.

A graphical interpretation of the transition formula will be given in detail in the course of the proof of Theorem 4.1. We finish this section with the following analog of Lemma 1.8 (see also [8]).

Lemma 2.13 *Assume that a shape $\mathcal{I}_\mu = [i_{k,l}] = 1, 2, \ldots$ satisfies the following condition: there exists p such that $i_{k,l} = 0$ for $k \leq p$ and $l > p$. Let $\mathcal{I}' = [i'_{k,l}]$ be a shape defined as $i'_{k,l} = i_{k,l}$ for $k \leq p$ and $i'_{k,l} = 0$ for $k > p$. Similarly let $\mathcal{I}'' = [i''_{k,l}]$ be a shape defined as $i''_{k,l} = 0$ for $k \leq p$ and $i''_{k,l} = i_{k,l}$ for $k > p$. Then there exist permutations μ' and μ'' such that $\mathcal{I}_{\mu'} = \mathcal{I}'$, $\mathcal{I}_{\mu''} = \mathcal{I}''$, and the following equality holds*

$$X_\mu = X_{\mu'} \cdot X_{\mu''} .$$

PROOF. Consider a transition for μ':

$$X_{\mu'} = X_{\lambda'} \cdot x_j + \sum X_{\psi'_t} .$$

Then

$$X_{\mu'} \cdot X_{\mu''} = \left(X_{\lambda'} \cdot x_j + \sum X_{\psi'_t}\right) \cdot X_{\mu''} = X_{\lambda'} \cdot X_{\mu''} \cdot x_j + \sum X_{\psi'_t} \cdot X_{\mu''}.$$

Let λ be the permutation corresponding to the shape $\left[i^{\lambda'}_{k,l} + i^{\mu''}_{k,l}\right]$ and let ψ_t be the permutation corresponding to the shape $\left[i^{\psi'}_{k,l} + i^{\mu''}_{k,l}\right]$. Then by induction on the length of a permutation we get

$$X_{\mu'} \cdot X_{\mu''} = X_\lambda \cdot x_j + \sum X_{\psi_t} .$$

It is easy to check that the right-hand side of this equality is a transition for X_μ. This proves that $X_{\mu'} \cdot X_{\mu''} = X_\mu$ as needed.

3 Schubert functors for vexillary permutations

In this section, we will deal with vexillary permutations. We will prove an analog of Theorem 2.6 in the context of Schubert functors and flagged Schur functors.

A flag $E.$ of free R-modules will be called a *splitting flag* provided $E_1 \simeq R$ and the ith inclusion in the flag is given by $E_i \hookrightarrow E_i \oplus R \simeq E_{i+1}$.

Theorem 3.1 *Let $E.$ be a splitting flag of free R-modules with* $\operatorname{rank} E_i = i$. *Assume that μ is a vexillary permutation and $I_1, I_2, \ldots, I_n$ are all nonempty sets among the sets $I_s(\mu)$, $s = 1, 2, \ldots$, counted with multiplicities. Then $S_\mu(E.)$ is the flagged Schur module associated with the sequence $(|I_k|)^{\geq}$, $k =$*

$1, 2, \ldots, n$, *and with the (sub)flag of modules of ranks* $(\min(I_k) - 1)^{\leq}$, $k = 1, 2, \ldots, n$.

PROOF. Each permutation $\mu = \mu(1), \mu(2), \mu(3), \ldots$ has a unique decomposition into maximal increasing sequences

$$\mu = a_1 < a_2 < \cdots < a_k > b_1 < b_2 < \cdots < b_l > c_1 < \cdots .$$

With a given permutation μ we associate the permutation μ' as follows.

1° From the set $\{a_1, a_2, \ldots, a_k\}$ we subtract all the numbers which are larger than b_1. Assume that their cardinality is equal to s.
2° Let $\sigma : \{1, 2, \ldots\} \setminus \{a_{k-s+1}, \ldots, a_k\} \to \{s+1, s+2, \ldots\}$ be the unique bijection preserving the order.

Define μ' as follows

$$\mu' := 1, 2, \ldots, s, \sigma(a_1), \ldots, \sigma(a_{k-s}), \sigma(b_1), \ldots, \sigma(b_l), \ldots .$$

Example 3.2 If $\mu = 2, 6, 4, 5, 1, 3, 7, \ldots$, then $\mu' = 1, 3, 5, 6, 2, 4, 7, \ldots$.

$$\mathcal{I}_\mu = \begin{array}{ccccc} 0 & 0 & 0 & \times & 0 \\ & \times & \times & \times & \times \\ & & 0 & \times & \times \\ & & & \times & \times \\ & & & & 0 \end{array} \qquad \mathcal{I}_{\mu'} = \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & \times & 0 \\ & & 0 & \times & \times \\ & & & \times & \times \\ & & & & 0 \end{array}$$

Observe that the shape of μ' is obtained from $\mathcal{I}_\mu$ by pushing down the first $k - s$ rows by s rows and introducing the first s rows with the 0's only. In particular, $l(\mu') < l(\mu)$, and μ' is vexillary if μ has this property. By definition, we have the following decomposition of $\Phi_\mu = \Phi_\mu(E.)$:

$$\begin{array}{ccccccc} & \bigotimes_{p \le k} S_{i_p}(E_p) & \xrightarrow{\Delta'_S} & \bigotimes_q \bigotimes_{p \le k} S_{i_{p,q}}(E_p) & \xrightarrow{m'_\wedge} & \bigotimes_q \bigwedge^{\sum_{p \le k} i_{p,q}}(E_q) \\ \bigotimes_p S_{i_p}(E_p) = & \otimes & & \otimes & & \bigotimes \\ & \bigotimes_{p > k} S_{i_p}(E_p) & \xrightarrow{\Delta''_S} & \bigotimes_q \bigotimes_{p > k} S_{i_{p,q}}(E_p) & \xrightarrow{m''_\wedge} & \bigotimes_q \bigwedge^{\sum_{p > k} i_{p,q}}(E_q) \\ & & & & & \downarrow m_\wedge \\ & & \xrightarrow{\Phi_\mu} & & & \bigotimes_q \bigwedge^{\widetilde{i_q}}(E_q) \end{array}$$

Observe that the sequence $(I_s(\mu))$, $s = 1, \ldots, k$, is nondecreasing because $a_1 < \cdots < a_k$. By a standard basis theorem for Schur functors (see [14], [1]), we can replace our initial flag by an arbitrary flag between $E_1 \subset \cdots \subset E_k \subset E_{k+1} \subset \cdots$ and $E_k \subset \cdots \subset E_k \subset E_{k+1} \subset \cdots$ without changing the image. Let us use the flag

$$E_{s+1} \subset \cdots \subset E_{s+(k-s)} \subset \underbrace{E_k \subset \cdots \subset E_k}_{s} \subset E_{k+1} \subset E_{k+2} \subset \cdots$$

The image of Φ_μ is now equal to the image of the following map Φ'_μ:

$$\begin{array}{ccccc}
\bigotimes_{p=k-s+1}^{k} S_{i_p}(E_k) & \xrightarrow{\Delta_S'''} & \bigotimes_{p,q} S_{i_{p,q}}(E_k) & \xrightarrow{m_\wedge'''} & \bigotimes_q \bigwedge^{(\sum_{p=k-s+1}^{k} i_{p,q})}(E_k) \\
\bigotimes & & \bigotimes & & \bigotimes \\
\begin{pmatrix} \bigotimes_{p=1}^{k-s} S_{i_p}(E_{s+p}) \\ \otimes \\ \bigotimes_{p>k} S_{i_p}(E_p) \end{pmatrix} & \xrightarrow{\Delta_S^{\mathrm{IV}}} & \bigotimes_{p,q} S_{i_{p,q}}(E_p) & \xrightarrow{m_\wedge^{\mathrm{IV}}} & \bigotimes_q \bigwedge^{(\sum_{p=1}^{k-s} i_{p,q} + \sum_{p>k} i_{p,q})}(E_q) \\
& & & & \big\downarrow m_\wedge \\
& \xrightarrow{\Phi'_\mu} & & & \bigotimes_q \bigwedge^{\widetilde{i_q}}(E_q)
\end{array}$$

The composition $m_\wedge^{\mathrm{IV}} \circ \Delta_S^{\mathrm{IV}}$ is nothing but $\Phi_{\mu'}(E.)$. Therefore, by induction on $l(\mu)$, we can replace $m_\wedge^{\mathrm{IV}} \circ \Delta_S^{\mathrm{IV}}$ by the corresponding map defining the flagged Schur module associated with the sequence and the (sub)flag of modules described by nonempty inversion sets of μ'. After this exchange, we get in the above diagram the map defining the flagged Schur module associated with the desired sequence and flag because $k+1 = \min I_j(\mu)$ for $j = k-s+1, \ldots, k$, and these sets are the only inversion sets of μ that are not inversion sets for μ'.

This completes the proof of the theorem.

4 A character formula relating Schubert modules and Schubert polynomials

In this section, $E. : E_1 \subset E_2 \subset \cdots$ will denote a flag of vector spaces over a field K of characteristic zero, $\dim E_i = i$. We choose a basis for $E = \bigcup E_i$ as in Remark 1.6. Then the group B of linear automorphisms of E preserving $E.$ may be identified with the group of upper triangular matrices. Let T be the subgroup of diagonal matrices in B

$$\begin{pmatrix} x_1 & & & \\ & x_2 & & 0 \\ & & x_3 & \\ & 0 & & \ddots \end{pmatrix}$$

Consider the action of T on $S_\mu(E.)$ induced from the canonical action of B by restriction. The main result of this paper is:

Theorem 4.1 *The trace of the action of T on $S_\mu(E.)$ is equal to X_μ.*

The proof is divided into several steps.

Step 1. Let

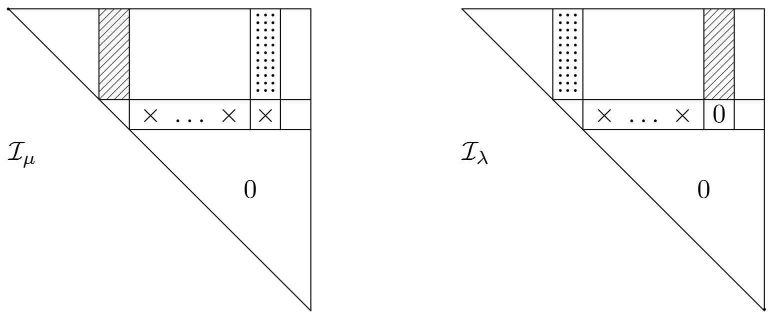
$$X_\mu = X_\lambda \cdot x_j + \sum_{t=1}^{k} X_{\psi_t}$$

be the maximal transition for μ. We claim that there exists a filtration of B-modules

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \mathcal{F} = S_\mu(E.),$$

together with surjections $\mathcal{F}/\mathcal{F}_k \to S_\lambda(E.) \otimes E_j/E_{j-1}$ and $\mathcal{F}_t/\mathcal{F}_{t-1} \to S_{\psi_t}(E.)$ for every $t = 1, \ldots, k$.

PROOF. Assume that λ is obtained from μ by exchanging $\mu(j)$ and $\mu(s)$ as described in 2.7. Recall that $\mu(j) > \mu(s)$ and that the sequence $\mu(j+1)$, $\mu(j+2)$, ..., $\mu(s)$, ... is increasing. Therefore the sth row of the shape of μ has the $\times$'s in columns with numbers $j+1, j+2, \ldots, s$. Moreover, the shape $\mathcal{I}_\lambda$ is obtained from the shape $\mathcal{I}_\mu$ by omitting the last $\boxed{\times}$ in the jth row and by exchanging the jth and sth columns.



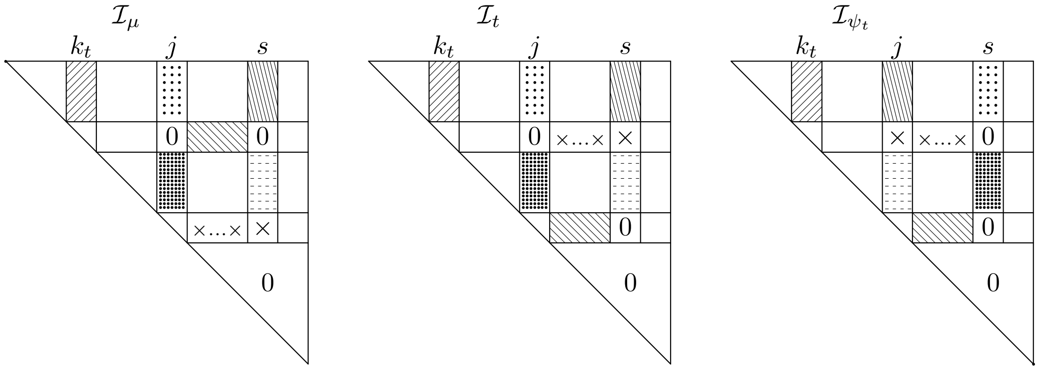
Let $\Delta_\wedge : \bigwedge^{\widetilde{i}_s} E_s \to \bigwedge^{\widetilde{i}_s - 1} E_s \otimes E_s$ be the diagonalization and let $p : E_s \to E_s/E_{j-1}$ be the projection. Denote by φ the following composition of B-module homomorphisms:

$$\begin{array}{c}
\bigwedge^{\widetilde{i}_1} E_1 \otimes \cdots \otimes \bigwedge^{\widetilde{i}_j} E_j \otimes \cdots \otimes \bigwedge^{\widetilde{i}_s} E_s \otimes \cdots \\
\Big\downarrow {\scriptstyle 1\otimes\cdots\otimes 1\otimes\cdots\otimes\Delta_\wedge\otimes\cdots} \\
\bigwedge^{\widetilde{i}_1} E_1 \otimes \cdots \otimes \bigwedge^{\widetilde{i}_j} E_j \otimes \cdots \otimes \bigwedge^{\widetilde{i}_s - 1} E_s \otimes E_s \otimes \cdots \\
\Big\downarrow \\
\left(\bigwedge^{\widetilde{i}_1} E_1 \otimes \cdots \otimes \bigwedge^{\widetilde{i}_s - 1} E_s \otimes \cdots \otimes \bigwedge^{\widetilde{i}_j} E_j \otimes \cdots\right) \otimes E_s
\end{array}$$

$$\Big\downarrow {\scriptstyle (1\otimes\cdots\otimes 1\otimes\cdots\otimes 1\otimes\cdots)\otimes p}$$

$$\Big(\bigwedge^{\widetilde{i}_1} E_1 \otimes \cdots \otimes \bigwedge^{\widetilde{i}_s - 1} E_s \otimes \cdots \otimes \bigwedge^{\widetilde{i}_j} E_j \otimes \cdots\Big) \otimes E_s/E_{j-1}$$

Observe that the image of φ restricted to $S_\mu(E.)$ is equal to $S_\lambda(E.) \otimes E_j/E_{j-1}$. Indeed, we have $\varphi(C_\mu) = C_\lambda \otimes e_j$. Assume now that ψ_t is obtained from μ by the cyclic exchange of $\mu(k_t)$, $\mu(j)$, $\mu(s)$ as described in 2.11. Define the shape $\mathcal{I}_t$ by exchanging the parts of the k_tth and jth rows corresponding to the columns $j+1$, $j+2$, ...



The shape of ψ_t can be obtained from the shape $\mathcal{I}_t$ by the following exchange of column-segments.

(a) The jth column goes into the place of the sth column.
(b) The upper $k_t - 1$ rows of the jth column of $\mathcal{I}_{\psi_t}$ are formed by the corresponding rows of the k_tth column of $\mathcal{I}_t$; the remaining part of the jth column of the shape of ψ_t is composed of the corresponding rows of the sth column.
(c) The upper $k_t - 1$ rows of the k_tth column are formed by the corresponding rows of the sth column.

We define the tth piece of the filtration by

$$\mathcal{F}_t := \sum_{r<t} S_{\mathcal{I}_r}(E.).$$

Let $p = \sum_{l<k_t} i_{l,s}$ and $q = \sum_{l>k_t} i_{l,s}$. We shall now define maps from $\mathcal{F}_t$ to $S_{\psi_t}(E.)$. Consider the following map

$$\bigwedge^i E_{k_t} \otimes \bigwedge^{p+q} E_j$$

$$\Big\downarrow {\scriptstyle 1\otimes\Delta_\wedge}$$

$$\bigwedge^i E_{k_t} \otimes \bigwedge^p E_j \otimes \bigwedge^q E_j$$

$$\Big\downarrow$$

$$
\begin{array}{c}
\bigwedge^p E_j \otimes \bigwedge^i E_{k_t} \otimes \bigwedge^q E_j \\
\Big\downarrow {\scriptstyle 1\otimes m_\wedge} \\
\bigwedge^p E_j \otimes \bigwedge^{i+q} E_j
\end{array}
$$

where $\Delta_\wedge$ is the corresponding diagonalization, $m_\wedge$ stands for multiplication in the exterior algebra and $i := \widetilde{i}_{k_t}$. This map tensored by identities corresponding to the remaining columns gives a map from the product of exterior powers associated with $\mathcal{I}_t$ (and also with all the shapes $\mathcal{I}_r$, $r < t$) to the product of exterior powers associated with $\mathcal{I}_{\psi_t}$ (note that we can use exterior powers of E_j instead of E_s because all the rows below the jth one are occupied by zeros). Denote the restriction of this map to $\mathcal{F}_t$ by φ_t. It is easy to see that $\varphi_t(C_{\mathcal{I}_t}) = C_{\psi_t}$ and $\varphi_t(C_{\mathcal{I}_r}) = 0$ for $r < t$. Therefore $\mathcal{F}_{t-1} \subset \operatorname{Ker} \varphi_t$. This proves that for every $t = 1, 2, \ldots, k$, $\mathcal{F}_t/\mathcal{F}_{t-1}$ surjects onto $S_{\psi_t}(E.)$.

Step 2. Define $d_\mu := \dim_K S_\mu(E.)$ and z_μ as the value of X_μ for the specialization $x_i = 1$, $i = 1, 2, \ldots$. We claim that $d_\mu \geq z_\mu$.

PROOF. Consider the maximal transition for μ:

$$X_\mu = X_\lambda \cdot x_j + \sum X_{\psi_t} \, ,$$

It follows from the description in 2.11 that

1) $l(\lambda) = l(\mu) - 1$,
2) for every t the index of ψ_t is strictly smaller than the index of μ.

Our claim now follows from the properties of the filtration of $S_\mu(E.)$ proved in Step 1 by (double) induction on the length and on the index of μ.

Step 3. We claim that $\operatorname{Ker} \varphi_t = \mathcal{F}_{t-1}$.

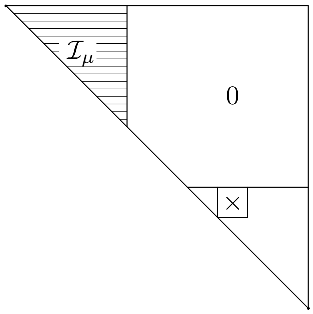
PROOF. We will say that a permutation μ (or alternatively its code) is *suitable* if $d_\mu = z_\mu$. We will prove by induction on the length of a permutation that every permutation is suitable. Then the claim will follow by comparing the dimensions of the both sides of the above equality. It suffices to show that if a code $(i_1, \ldots, i_p, 0, 0, \ldots)$ is suitable then the code $(i_1, \ldots, i_p + 1, 0, 0, \ldots)$ is suitable, and the code

$$(i_1, \ldots, i_p, \underbrace{0, \ldots, 0}_{l}, 1, 0, \ldots)$$

is suitable for every $l \geq 0$. First observe that the code of this type is suitable for $l \gg 0$. Indeed, let n be the smallest number such that $\mu(n) = n$, $\mu(n+1) = n+1, \ldots$. Then the permutation

$$\mu_r = \mu(1), \mu(2), \ldots, \mu(n-1), n, \ldots, n+r-1, n+r+1, n+r, \ldots$$

(where $r \geq 0$) has the shape



This shape satisfies the assumptions of Lemmas 1.8 and 2.13. These lemmas and the fact that μ is suitable imply that μ_r is suitable. To decrease l we will use 2.11. Observe that if μ is suitable then λ and ψ_t's appearing in the right-hand side of 2.11 are also suitable. Indeed, we have

$$z_\lambda + \sum z_{\psi_t} = z_\mu = d_\mu \geq d_\lambda + \sum d_{\psi_t},$$

and thus

$$(d_\lambda - z_\lambda) + \sum (d_{\psi_t} - z_{\psi_t}) \leq 0.$$

But by Step 2, we know that all the summands are nonnegative so they must be zero. Let us start with the permutation

$$\mu_0 = \mu(1), \mu(2), \ldots, \mu(n-1), n+1, n, n+2, \ldots.$$

The maximal transition for μ_0 contains in the right-hand side

$$\mu_{-1} = \mu(1), \mu(2), \ldots, \mu(n-2), n, \mu(n-1), n+1, \ldots.$$

The transition for μ_{-1} contains

$$\mu_{-2} = \mu(1), \mu(2), \ldots, \mu(n-1), \mu(n-2), \ldots.$$

By repeating this consideration we finally get

$$\mu(1), \mu(2), \ldots, \mu(p), \mu(p+2), \mu(p+1), \mu(p+3), \ldots.$$

In this way, we get permutations with the codes of the form

$$(i_1, \ldots, i_p, \underbrace{0, \ldots, 0}_{l}, 1, 0, \ldots), \qquad l \geq 0.$$

Moreover, if r is the smallest number bigger than p such that $\mu(p) > \mu(r)$, then the transition for the permutation

$$\mu(1), \ldots, \mu(p), \ldots, \mu(r+1), \mu(r), \ldots$$

contains the permutation

$$\mu(1), \ldots, \mu(p-1), \mu(r), \mu(p+1), \ldots, \mu(r-1), \mu(p), \mu(r+1), \ldots$$

with the code $(i_1, \ldots, i_p + 1, 0, \ldots)$. This proves our claim.

Step 4. From the above considerations, we get a filtration of B-modules

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \mathcal{F} = S_\mu(E.)$$

such that

$$\mathcal{F}/\mathcal{F}_k \simeq S_\lambda(E.) \otimes E_j/E_{j-1}, \qquad \mathcal{F}_t/\mathcal{F}_{t-1} \simeq S_{\psi_t}(E.)$$

for every $t = 1, \ldots, k$. In particular, we obtain an isomorphism of T-modules

$$S_\mu(E.) \simeq S_\lambda(E.) \otimes E_j/E_{j-1} \oplus \bigoplus_{t=1}^{k} S_{\psi_t}(E.).$$

By comparing this with the transition formula for Schubert polynomials (see 2.7), the assertion of Theorem 4.1 now follows by double induction on the length and the index of μ, as in Step 2.

The proof of Theorem 4.1 is finished.

5 A filtration associated with the Monk formula

Recall (see 2.7) that

$$X_\mu \cdot x_1 = \sum_q X_{\mu \circ \tau_{1,q}}$$

where q is such that for $i \in]1, q[$, $\mu(i)$ is outside of $[\mu(1), \mu(q)]$. Denote the set of all such q's by $\{q_1 < \cdots < q_k\}$.

Proposition 5.1 *For a given flag $E.$ of vector spaces over a field K of characteristic zero,* $\dim E_i = i$*, there exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = S_\mu(E.) \otimes E_1$$

of B-modules such that $\mathcal{F}_t/\mathcal{F}_{t-1} \simeq S_{\mu \circ \tau_{1,q_t}}(E.)$ for $t = 1, \ldots, k$.

PROOF. Fix $t = 1, \dots, k$. Set $\mu_t := \mu \circ \tau_{1,q_t}$ and let $\mathcal{I} = [i_{p,q}]$ (resp. $\mathcal{I}^t = [i^t_{p,q}]$) be the shape of μ (resp. μ_t). We define an element C_t in $\bigotimes_l \bigwedge^{\widetilde{i_l}} E_l$ as follows. Let $\{e_i : i = 1, 2, \dots\}$ be a basis of $E = \bigcup E_i$ such that $e_1, \dots, e_k$ span the vector space E_k. Set $C_t := \bigotimes_l e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_{i_l}}$, where for $l \neq q_t$ (resp. $l = q_t$) k_r's are precisely those indices (taken in ascending order) for which $i^t_{k_r,l} = 1$ (resp. those increasing indices different from 1 for which $i^t_{k_r,l} = 1$); note that $i^t_{1,q_t} = 1$. Let $\mathcal{F}_t$ be the B-module generated by $C_1 \otimes e_1, \dots, C_t \otimes e_1$; of course, $\mathcal{F}_k = S_\mu(E.) \otimes E_1$. We have a map

$$\varphi_t : \bigotimes_l \bigwedge^{\widetilde{i_l}} E_l \otimes E_1 \to \bigotimes_l \bigwedge^{\widetilde{i^t_l}} E_l$$

determined by the exterior multiplication

$$\bigwedge^{\widetilde{i_{q_t}}} E_{q_t} \otimes E_1 \to \bigwedge^{\widetilde{i^t_{q_t}}} E_{q_t} .$$

This map induces a surjection because $\varphi_t(C_t \otimes e_1) = C_{\mu_t}$. Moreover, it is easy to see that $\mathcal{F}_{t-1} \subset \operatorname{Ker} \varphi_t$. By Theorem 4.1, we know that $d_\mu = z_\mu$ for every permutation. Thus

$$d_\mu \geq \sum_{t=1}^{k} d_{\mu_t} = \sum_{t=1}^{k} z_{\mu_t} = z_\mu$$

(the inequality follows from the above filtration). Therefore $d_\mu = \sum d_{\mu_t}$, and this shows that $\mathcal{F}_t / \mathcal{F}_{t-1} \simeq S_{\mu_t}(E.)$ for $t = 1, \dots, k$. The proof is complete.

Remark 5.2 (i) A similar filtration can be associated with the Monk formula for multiplication by $\sum_{k=1}^{n} x_k$ when $\mu(r) = r$ for $r > n$. We omit the details. Does there exist an analogous filtration associated with the Monk formula for every μ?

(ii) Observe that the first part of the proof of Proposition 5.1 gives us another justification of the inequality $d_\mu \geq z_\mu$. It suffices to use descending induction on the length of a permutation and the Monk formula for Schubert polynomials.

Remark 5.3 Suppose, as we did in [4], that R is *any* commutative ring. Replace the symmetric powers (resp. symmetrizations) by divided powers (resp. diagonalizations in the algebra of divided powers) in the definition of a module $S_{\mathcal{I}}(E.)$ associated with flag $E.$ and shape $\mathcal{I}$. The proofs of Theorem 3.1, Theorem 4.1, and Proposition 5.1 suitably adapted to this modified definition, show that the so obtained flagged Schur functors and Schubert functors have the following properties:

- For any vexillary permutation and any splitting flag (in the sense of Section 3), Theorem 3.1 remains true; consequently these flagged Schur functors are *universally free*, following the terminology of [1].
- If K is an infinite field (of arbitrary characteristic), then Theorem 4.1 remains valid.

- If K is an infinite field, then Proposition 5.1 also holds true.

6 Further developments

The ideas and results of the present paper (in particular, Theorem 4.1) were developed mainly by V. Reiner & M. Shimozono, and P. Magyar.

A detailed study of different aspects of flagged Schur modules was made by the former authors in the following papers: *Key polynomials and a flagged Littlewood-Richardson rule* [J. Combin. Theory Ser. A 70 (1995), 107–143], *Percentage-avoiding, northwest shapes and peelable tableaux* [J. Combin. Theory Ser. A 82 (1998), 1–73], *Flagged Weyl modules for two column shapes* [J. Pure Appl. Algebra 141 (1999), 59–100]. On another hand, in the paper *Specht modules for column-convex diagrams* [J. Algebra 174 (1995), 489–522], the present article was linked with the study of Specht modules associated with column-convex diagrams and some related diagrams. In the paper *Balanced labellings and Schubert polynomials* [European J. Combin. 18 (1997), 373–389], Fomin-Greene-Reiner-Shimozono constructed an explicit basis of Schubert modules.

Magyar developed a more geometric approach to Theorem 4.1 in the papers: *Four new formulas for Schubert polynomials* [preprint, 1995], *Borel-Weil theorem for configuration varieties and Schur modules* [Adv. Math. 134 (1998), 328–366], *Schubert polynomials and Bott-Samelson varieties* [Comment. Math. Helv. 73 (1998), 603–636]. This last paper contains another, geometric proof of Theorem 4.1 obtained in collaboration with Reiner and Shimozono. In the paper *Standard monomial theory for Bott-Samelson varieties of* $\mathrm{GL}(n)$ [Publ. Res. Inst. Math. Sci. 34 (1998), 229–248], Lakshmibai-Magyar mention a link of Theorem 4.1 with Standard monomial theory.

In a recent paper by Buch-Kresch-Tamvakis-Yong *Schubert polynomials and quiver formulas* [to appear in Duke Math. J.], the authors discuss a relationship between a representation-theoretic interpretation of their work and Theorem 4.1.

See also a paper by Lascoux-Schützenberger: *Fonctorialité des polynômes de Schubert* [Contemp. Math. 88 (1989), 585–598].

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