Diagonals of flag bundles

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Abstract

We express the diagonals of projective, Grassmann and, more generally, flag bundles of type (A) using the zero schemes of some vector bundle sections, and do the same for their single point subschemes. We discuss diagonal and point properties of these flag bundles. We study when the complex manifolds G/B for other groups have the point and diagonal properties. We discuss explicit formulas for the classes of diagonals of the varieties G/B.

1 Introduction

For a map of varieties $\pi : F \to X$, it is useful to study the diagonal in the fibre square $F \times_X F$. The classes of such diagonals for fibre bundles in topology and smooth proper morphisms in algebraic geometry were investigated by Graham [13], Fulton and the second author [24], [12, Appendix G]. As explained in [24, Section 5], knowing such a class, one can compute the class of a subscheme of F. For an overview of applications, see [12, Chapter 7].

In the present paper, we shall rather study the diagonals in the Cartesian squares $F \times F$ of the total spaces of flag bundles $\pi : F \to X$. Suitable resolutions of the structure sheaves of the diagonals over the structure sheaves of

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the Cartesian squares of some homogeneous spaces were used by Kapranov in [18] to give descriptions of their derived categories. Many schemes can be realized as degeneracy loci of vector bundle homomorphisms. It turns out that to understand degeneracy loci, it is useful to study diagonals of flag bundles (cf. [24], [5], [11], [26], [12]).

In [27], Pati, Srinivas and the second author investigated which varieties X have the following "diagonal property" (D): there exists a vector bundle of rank $\dim(X)$ on $X \times X$ with a section whose zero scheme is the diagonal. If X has (D), then it is nonsingular.

Also, the following "weak point property" (P) was investigated: for some point $x \in X$, there exists a vector bundle of rank $\dim(X)$ on X with a section whose zero scheme is x. If a variety X has (D), then it has (P) for any $x \in X$. Any nonsingular curve has (D). The product of varieties having (D), has (D). In [27, Section 3], we gave several detailed results on surfaces with (D). In particular, it was shown (*loc.cit.*, Proposition 4) that a ruled surface (cf., e.g., [15, Chap. V, Sect. 2]) has (D), i.e., the projectivization of any rank 2 vector bundle on a nonsingular curve has (D). This result was one of the starting points of the present paper.

It was shown by Fulton [9] and the second author [25] that the flag varieties of the form SL_n/P over any field have (D). Samuelsson and Seppänen [28] gave recently an application of the diagonal property of flag varieties to global complex analysis. The interest to (D) for flag varieties was related to the theory of Schubert polynomials of Lascoux-Schützenberger [22]. These authors defined on the polynomial ring in several variables a scalar product [21, Chapter 10], for which the (single) Schubert polynomials and their "duals" form adjoint bases (*loc.cit.*, Corollary 10.2.4). The reproducing kernel for this scalar product is equal to the top (double) Schubert polynomial (*loc.cit.*, Section 10.2), which, in turn, is equal to the top Chern class of the vector bundle, realizing (D) for the variety SL_n/B of complete flags – a result of Fulton [9].

A natural question emerged then: do the flag varieties for other groups have the diagonal property? (see [27, p. 1235], [25, Conjecture 8.2] and [27, Proposition 12]). In the present paper, we answer this question almost completely for the manifolds G/B, where G is a simple, simply connected, complex algebraic group (see Section 6). We show that for G of type $(B_i)(i \geq 3), (D_i)(i \geq 4), (G_2), (F_4)$ and $(E_i)(i = 6, 7, 8)$, the flag manifold G/B has not the diagonal property (see Theorem 17). The main tools are the Atiyah-Hirzebruch homomorphism and the Borel characteristic homomorphism. In Section 7, we recall several explicit formulas for the classes of diagonals of G/B; we do not use, however, these formulas in the proof of Theorem 17.

Apart from the question about generalized flag manifolds, the present paper arose from our attempts to understand relations between the diagonals of the base spaces and those of the total spaces of flag bundles.

We shall also study the following variant of (D). Given a section s of a vector bundle, we write $Z(s)$ for its zero scheme. We say that X has property (D'), if there exist two vector bundles A and B on $X \times X$ such that $\mathrm{rank}(A) + \mathrm{rank}(B) = \dim(X)$, a section s of A and a section t on $Z(s)$ of the restriction $B_{Z(s)}$ of B to $Z(s)$ such that $Z(t)$ is the diagonal of X. Thus for $A = (0)$, we recover (D). Note that (D') is a slight weakening of (D) as the key property that the rank of the bundle is $\dim(X)$ holds also for (D'). A variety X with (D') is nonsingular because its cotangent sheaf is locally free: it is isomorphic to the restriction of $A^\vee \oplus B^\vee$ to the diagonal of X.

The main results of the paper are Theorems 4, 9, 11 and 17.

The paper is organized as follows. In Section 2, we discuss properties of flag bundles and construct a certain vector bundle on the Cartesian square of a flag bundle. Using this vector bundle, we prove in Theorem 4 that if the base space of a flag bundle has (D), then its total space has (D'). In Section 5, we discuss the analogs of this result for topological properties (D_r) and (D_c) from [27, Section 6]. The topological situation is easier than that in algebraic geometry: if X has (D_r) or (D_c) then the corresponding flag bundles also have these properties (see Theorem 14).

In Section 4 we show that if a quasiprojective base of a flag bundle has (P), then its total space has (P) (see Theorem 9). For more general schemes, we prove in Theorem 11 that if the base space of a flag bundle has (P), then its total space has (P'), a property analogous to (D'). In Section 5, we discuss the analogs for topological properties (P_r) and (P_c) from [27, Section 6].

In Section 6, we investigate which complex manifolds G/B for other groups G have (P_c), see Theorem 17 and Corollary 23. An absence of (P_c) implies for many of them the absence of (D_c).

In Appendix, we discuss explicit formulas for the classes of diagonals of the varieties G/B due to Fulton, the second author and Ratajski, Graham, and De Concini, adding one for the type (G_2) (see Lemma 27). We also disprove an integrality conjecture from [13] (see Remark 26).

2 Flag bundles of type (A_{n-1})

Let E be a vector bundle of rank n on a variety X over a field. Fix an increasing sequence of integers

$$d_\bullet : 0 < d_1 < d_2 < \ldots < d_{k-1} < d_k = n\,.$$

By a $d_\bullet$-flag, we mean an increasing sequence of subbundles of E

$$E_1 \subset E_2 \subset \cdots \subset E_{k-1} \subset E_k = E$$

such that $\mathrm{rank}(E_i) = d_i$ for $i = 1, \ldots, k$. Let

$$\pi : Fl_{d_\bullet}(E) \to X$$

be the flag bundle parametrizing all $d_\bullet$-flags. For example, the sequence

$$d_1 = d < d_2 = n$$

gives rise to the Grassmann bundle $G_d(E)$, parametrizing subbundles of rank d of E (see [8, B.5.7]). For $d = 1$, we get the projectivization of E: $P(E) = G_1(E)$ (see [15, Section 7] and [8, B.5.5]).

It is well-known that

$$\dim(G_d(E)) = \dim(X) + d(n-d)\,. \tag{1}$$

On $Fl_{d_\bullet}(E)$, there exists the following tautological sequence of vector bundles:

$$S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_{k-1} \hookrightarrow S_k = \pi^*(E) \overset{q_1}{\twoheadrightarrow} Q_1 \overset{q_2}{\twoheadrightarrow} Q_2 \overset{q_3}{\twoheadrightarrow} \cdots \overset{q_k}{\twoheadrightarrow} Q_k = 0\,, \tag{2}$$

where $\mathrm{rank}(S_i) = d_i$ for $i = 1, \ldots, k$, and Q_i is the quotient of π^*E by S_i, so that $\mathrm{rank}(Q_i) = n - d_i$. On $G_d(E)$, the tautological sequence (with $S = S_1$, $Q = Q_1$)

$$0 \to S \to \pi^*E \to Q \to 0\,, \tag{3}$$

where $\mathrm{rank}(S) = d$, is a short exact sequence.

Regarding $Fl_{d_\bullet}(E) \to X$ as a tower of Grassmann bundles

$$Fl_{d_\bullet}(E) = G_{n-d_{k-1}}(Q_{k-1}) \to \cdots \to G_{d_3-d_2}(Q_2) \to G_{d_2-d_1}(Q_1) \to G_{d_1}(E) \to X\,, \tag{4}$$

and using (1), we see that with $d_0 = 0$, we have

$$\dim(Fl_{d_\bullet}(E)) = \dim(X) + \sum_{i=1}^{k-1}(d_i - d_{i-1})(n-d_i)\,. \tag{5}$$

Write $F = Fl_{d_\bullet}(E)$. Let $F_1 = F_2 = F$, and denote by

$$p_i : F_1 \times F_2 \to F_i$$

the two projections. We shall now construct a certain vector bundle of rank $\dim(F) - \dim(X)$ on $F_1 \times F_2$. If $k = 2$, using the notation of (3), we define the following vector bundle

$$H = \mathrm{Hom}(p_1^*S, p_2^*Q) = (p_1^*S)^\vee \otimes p_2^*Q\,. \tag{6}$$

Suppose now that $k \geq 3$. Using the notation of (2), consider the following homomorphism of vector bundles on $F_1 \times F_2$:

$$\varphi : \bigoplus_{i=1}^{k-1} \mathrm{Hom}(p_1^*S_i, p_2^*Q_i) \to \bigoplus_{i=1}^{k-2} \mathrm{Hom}(p_1^*S_i, p_2^*Q_{i+1})\,,$$

defined by

$$\varphi(\sum_{i=1}^{k-1} h_i) = \sum_{i=1}^{k-2}\big(h_{i+1}|p_1^* S_i - p_2^*(q_{i+1}) \circ h_i\big)\,, \tag{7}$$

where $h_i \in \mathrm{Hom}(p_1^* S_i, p_2^* Q_i)$.

Lemma 1 *The homomorphism φ is surjective.*

Proof. Let us fix $i = 1, \ldots, k-2$. Let $h \in \mathrm{Hom}(p_1^* S_i, p_2^* Q_{i+1})$. By subtracting from h a suitable homomorphism from $\mathrm{Hom}(p_1^* S_{i+1}, p_2^* Q_{i+1})$ restricted to the subbundle $p_1^* S_i$ of $p_1^* S_{i+1}$, we get a homomorphism from $p_1^* S_i$ to $p_2^* Q_{i+1}$, which factorizes through $p_2^* Q_i$. But such a homorphism belongs to $\varphi(\mathrm{Hom}(p_1^* S_i, p_2^* Q_i))$. The assertion follows. $\square$

Define the following vector bundle on $F_1 \times F_2$:

$$H = \mathrm{Ker}(\varphi)\,. \tag{8}$$

Using Lemma 1, we obtain (with $d_0 = 0$)

$$\mathrm{rank}(H) = \sum_{i=1}^{k-1} d_i(n-d_i) - \sum_{i=1}^{k-2} d_i(n-d_{i+1}) = \sum_{i=1}^{k-1}(d_i - d_{i-1})(n-d_i)\,. \tag{9}$$

Note 2 The present section is an expanded version of [25, pp. 107-8]. The bundle H, defined in (8), is modeled on the bundle K from [9, (7.6)].

Remark 3 Let X be a point. It is shown in [9, Section 7] that for

$$d_\bullet = 0 < 1 < 2 < \ldots < n-1 < n\,,$$

the top Chern class of H is the top double Schubert polynomial taken on first Chern classes of the tautological quotient bundles on the two copies of complete flag varieties. It will follow from Section 3 that it is actually the class of the diagonal of a complete flag variety.

3 Diagonal properties

We adopt the set-up from the previous section, and state the following result.

Theorem 4 *If X has (D), then for any vector bundle E and any $d_\bullet$, $Fl_{d_\bullet}(E)$ has (D').*

Proof. Let G be a vector bundle of rank $\dim(X)$ on $X \times X$ with a section whose zero scheme $Z(s)$ is the diagonal Δ_X of X. Fix $d_\bullet$, and follow the notation from Section 2. Let

$$G' = (\pi_1 \times \pi_2)^*(G)$$

be a bundle on $F_1 \times F_2$ together with a section $s' = (\pi_1 \times \pi_2)^*(s)$. Consider

$$Z := Z(s') = (\pi_1 \times \pi_2)^{-1}(\Delta_X) \subset F_1 \times F_2\,.$$

Let $r_1, r_2 : X \times X \to X$ be the two projections. The following two vector bundles on Δ_X are equal:

$$(r_1^* E)_{\Delta_X} = (r_2^* E)_{\Delta_X}\,. \tag{10}$$

Since

$$(\pi_1 \times \pi_2)^* r_i^* E = p_i^*(E_{F_i})$$

for $i = 1, 2$, we obtain from (10) that the following two vector bundles on Z are equal:

$$\big(p_1^* E_{F_1}\big)_Z = \big(p_2^* E_{F_2}\big)_Z\,. \tag{11}$$

Thanks to (11), we get, for any $i = 1, \ldots, k-1$, the following homomorphism:

$$h_i : \big(p_1^* S_i\big)_Z \to \big(p_1^* E_{F_1}\big)_Z = \big(p_2^* E_{F_2}\big)_Z \to \big(p_2^* Q_i\big)_Z \tag{12}$$

of vector bundles on Z. Here, the subbundle $S_i \hookrightarrow E_F$ and the quotient bundle $E_F \twoheadrightarrow Q_i$ are from (2). The family of homorphisms $\{h_i\}$ gives rise to the section

$$h = \sum h_i \in \Gamma\big(Z, \oplus_{i=1}^{k-1} \operatorname{Hom}(p_1^* S_i, p_2^* Q_i)_Z\big)\,.$$

Suppose $k \geq 3$. It follows from (12) that we have on Z

$$h_{i+1} | p_1^* S_i = p_2^*(q_{i+1}) \circ h_i$$

for $i = 1, \ldots, k-2$. Indeed, since h_i and h_{i+1} factorize through the bundle (11), the two homomorphisms

$$h_{i+1} | p_1^* S_i\ ,\quad p_2^*(q_{i+1}) \circ h_i \ :\ \big(p_1^* S_i\big)_Z \to \big(p_2^* Q_{i+1}\big)_Z$$

are equal. Invoking (7), we see that

$$\varphi \circ h = 0\,,$$

so h induces a section t of the bundle H_Z, where H is the vector bundle on $F_1 \times F_2$ from (8) and (6). By (5) and (9), we have

$$\operatorname{rank}(G') + \operatorname{rank}(H) = \dim(F)\,.$$

We claim that the section t of the bundle H_Z vanishes precisely (scheme theoretically) on the diagonal $\Delta_F \subset F_1 \times F_2$. It vanishes on Δ_F since the tautological sequence of vector bundles on $G_{d_i}(E)$ is a complex for any $i = 1, \ldots, k-1$ (cf. (3)).

Having defined the sections s', t globally, it is sufficient to check the converse assertion $Z(t) \subset \Delta_F$ locally, where $F_1 \times F_2$ is the product of the Cartesian square of the base space times the Cartesian square of the flag variety $Fl_{d_\bullet}(E_x) =: F_x$, where $x \in X$. In other words, this boils down to check the assertion over the point x, i.e. on $F_x \times F_x$. For the case of complete flags, see [9, p. 402]. For any $d_\bullet$, let $f \in Z$ with $\pi_1(x) = \pi_2(x) = x$, so we may regard f as a point

$$f = (L_1 \subset \cdots \subset L_{k-1} \subset L_k = E_x \ , \ M_1 \subset \cdots \subset M_{k-1} \subset M_k = E_x)$$

in $F_x \times F_x$. Let $F_{x,1} = F_{x,2} = F_x$. For any $i = 1, \dots, k-1$, the restriction of h_i (see 12) to $F_{x,1} \times F_{x,2}$ is

$$p_1^* S_i \to p_1^* V_{F_{x,1}} = V_{F_{x,1} \times F_{x,2}} = p_2^* V_{F_{x,2}} \to p_2^* Q_i \,, \tag{13}$$

where we write V for E_x, S_i and Q_i are the restrictions to F_x of the tautological bundles on F, and p_1, p_2 are the two projections from $F_{x,1} \times F_{x,2}$ to the factors. At the point $f = ((L_i), (M_i))$, (13) becomes the map

$$L_i \hookrightarrow V \twoheadrightarrow V/M_i \,,$$

whose vanishing implies $L_i = M_i$. This holds for any $i = 1, \dots, k-1$. We have proved that set-theoretically $Z(t) = \Delta_F$. It is not hard to verify that this equality holds scheme-theoretically. The assertion of the theorem follows. $\square$

We record the following simple fact.

Lemma 5 *Let E be a vector bundle on a variety X. Let $r_1, r_2 : X \times X \to X$ be the two projections. Suppose that the following two vector bundles on $X \times X$ are equal:*

$$r_1^* E = r_2^* E \,.$$

Then E is a trivial bundle.

Proof. Fix a point $x \in X$. By the assumption, we have

$$(r_1^* E)_{X \times \{x\}} = (r_2^* E)_{X \times \{x\}} \,.$$

Via the identification $X \times \{x\} \simeq X$, the LHS is the bundle $E \to X$. The RHS is the trivial bundle $(E_x)_X$. The assertion follows. $\square$

Remark 6 Let us speculate a bit about *this* proof of Theorem 4. To convert it to that of (D), we must extend the section t to the whole $F_1 \times F_2$. This can be done only if $p_1^*(E_{F_1}) = p_2^*(E_{F_2})$; so, by virtue of Lemma 5, only if the bundle E_F is trivial.

4 Point properties

We first record the following result.

Proposition 7 *Suppose that X is a quasiprojective variety with (P). Then for any vector bundle E on X and any $1 \le d \le n-1$, the Grassmann bundle $G_d(E)$ has (P).*

Proof. By the assumption, for a certain point $x \in X$, there exists a vector bundle G of rank $\dim(X)$ and $s \in \Gamma(X, G)$ such that $Z(s) = x$.

Suppose that $\operatorname{rank}(E) = n$. We realize X as an open subset in a projective variety X'. By [4, Proposition 2], there exists a coherent sheaf E' on X' whose restriction to X is E. Let L be the restriction of $\mathcal{O}_{X'}(1)$ to X.

We claim that there exists an integer m such that $E \otimes L^{\otimes m}$ has n global sections which are independent at x. Indeed, this follows (by restriction from X' to X) from [29, Théorème 2(a), p. 259] which asserts that there exists an integer m such that the $\mathcal{O}_{x,X'}$-module $E'(m)_x$ is generated by the elements of $\Gamma(X', E'(m))$. Choose any d sections out of these n global sections of $E \otimes L^{\otimes m}$. Using a canonical isomorphism

$$G_d(E \otimes L^{\otimes m}) \simeq G_d(E)\,,$$

and the assumption on E, we can reduce to the situation when E has d sections $\{s_i\}$ which are independent at x.

Set $F = G_d(E)$, and denote by $\pi : F \to X$ the projection. Let Q be the tautological quotient rank $n - d$ bundle on F. Consider the following rank $\dim(F) = \dim(X) + d(n-d))$ vector bundle H on F:

$$H = \pi^* G \oplus Q^{\oplus d}\,.$$

We define the following section $t \in \Gamma(F, H)$. On the first summand of H, we take the pullback via π^* of the section $s : X \to G$. On the last d summands, we take the following sections: we compose the sections

$$\pi^*(s_i) : F \to \pi^* E$$

with the canonical surjection $\pi^* E \twoheadrightarrow Q$. We have

$$Z(t) = Z(\pi^*(s)) \cap Z(\oplus \pi^*(s_i))\,. \tag{14}$$

But

$$Z(\pi^*(s)) = \pi^{-1}(x) = G_d(E_x)\,,$$

so that (14) as a point of $G_d(E_x)$ corresponds to the d-dimensional vector subspace of E_x spanned by $(s_i)_x$. We get that $Z(t)$ is a single point in $G_d(E)$. Hence $G_d(E)$ has (P). $\square$

Remark 8 The projective bundle

$$P\big(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)\big) \to P^n ,$$

$a_i \in \mathbb{Z}$, is a toric variety (cf. [23]). Thus the proposition gives some support to the conjecture (cf. [25, p. 115]) that a nonsingular toric variety has (P) (and perhaps even (D) – which is known in the surface case [27]).

It is well-known (cf., e.g., [14, p. 142]) that the projective, Grassmann and flag bundles on quasiprojective varieties are quasiprojective. Realizing a flag bundle as a tower (4) of Grassmann bundles, and using an easy induction, we infer from the proposition the following result.

Theorem 9 *Suppose that X is a quasiprojective variety with (P). Then for any vector bundle E on X and any $d_\bullet$, the flag bundle $Fl_{d_\bullet}(E)$ has (P).*

Remark 10 Sometimes, one studies the following "strong point property" of a variety X: for *any* $x \in X$, there exists a bundle on X of rank $\dim(X)$ with a section whose zero scheme is x. Granting this property for a quasiprojective variety X, the above reasoning shows that $F = Fl_{d_\bullet}(E)$ has (P) for any point $f \in F$. Indeed, given $f \in F$, put $x = \pi(f)$, and argue as above. Thus F also has the strong point property.

For *any* scheme, we still have a result explained in the following theorem. We need a definition. We say that X has property (P'), if for some $x \in X$, there exist two vector bundles A and B on X such that $\operatorname{rank}(A)+\operatorname{rank}(B) = \dim(X)$, a section s of A and a section t of $B_{Z(s)}$ such that $Z(t)$ is x. If X has (D'), then for any $x \in X$, (P') holds by restricting the data giving (D') to $X \times \{x\}$.

Theorem 11 *If X has (P), then $Fl_{d_\bullet}(E)$ has (P') for any $d_\bullet$.*

Proof. Fix $d_\bullet$, and write $F = Fl_{d_\bullet}(E)$. Let $\pi : F \to X$ be the projection. Suppose that for the fixed point $x \in X$, there exists a vector bundle G of rank $\dim(X)$ on X with a section s whose zero scheme is x. We shall show that F has (P') for any point $f \in \pi^{-1}(x)$. Let $G' = \pi^*(G)$ and $s' = \pi^*(s)$. Consider $W = Z(s') \subset F$. In other words $W = \pi^{-1}(x)$.

Using the vector bundle H from (8) and (6), we define the following vector bundle:

$$H' = H_{F\times\{f\}}$$

on $F \simeq F \times \{f\}$. Note that $\operatorname{rank}(G') + \operatorname{rank}(H') = \dim(F)$. Invoke $Z = (\pi \times \pi)^{-1}(\Delta_X) \subset F \times F$ from the proof of Theorem 4. In this proof, we constructed the section t of the bundle $H_Z \to Z$ whose zero scheme is the diagonal of F. We have $W \simeq W \times \{f\} \subset Z$. The restriction to $W \times \{f\}$ of the section t, gives rise to a section, denoted t', of the bundle

$H'_{W\times\{f\}} \to W \times \{f\}$. We claim that $Z(t') = f$. The section t' vanishes at f because (f, f) belongs to the diagonal.

It is sufficient to check the converse assertion locally. Let $g \in Z(t')$. Since $\pi(g) = \pi(f) = x$, we may regard

$$f = (L_1 \subset \cdots \subset L_{k-1} \subset L_k = E_x) \quad \text{and} \quad g = (M_1 \subset \cdots \subset M_{k-1} \subset M_k = E_x)$$

as points in $Fl_{d_\bullet}(E_x) = F_x$. Write $V = E_x$. For $i = 1, \ldots, k-1$, we consider (13) restricted to $F_x \times \{f\}$:

$$p_1^* S_i \to p_1^* V_{F_x} = V_{F_x \times \{f\}} = p_2^* V_f \to p_2^* (Q_i)_f \,, \tag{15}$$

where $p_1 : F_x \times \{f\} \to F_x$, $p_2 : F_x \times \{f\} \to f$ are the two projections, and S_i (resp. Q_i) are the restrictions of the tautological bundles from F to F_x (resp. f). Restricted to the point g, (15) becomes the map

$$M_i \hookrightarrow V \twoheadrightarrow V/L_i \,,$$

whose vanishing implies $M_i = L_i$. This holds for every $i = 1, \ldots, k-1$. We have proved that $g = f$, i.e., $Z(t') = f$, and hence F has (P') for any $f \in \pi^{-1}(x)$. $\square$

Remark 12 Granting the strong point property for X, the above reasoning shows that F has (P') for any point $f \in F$. Indeed, given $f \in F$, put $x = \pi(f)$, and argue as above.

5 Topological properties

We now pass to topology. We first recall some definitions from [27, Section 6]. Let X be a (smooth) compact connected oriented manifold, and Δ be the diagonal submanifold of $X \times X$. We say that X has property (D_r) if there exists a smooth real vector bundle of rank $\dim(X)$ on $X \times X$ with a smooth section s which is transverse to the zero section of the bundle and whose zero locus is Δ. If $\dim_{\mathbb{R}} X = 2m$ and the above vector bundle is a complex vector bundle of complex rank m, then we say that X has property (D_c). If X has (D_c), then it is almost complex (*loc.cit.*, p. 1259). For a complex manifold, we have the following relation between the diagonal properties: $(D) \Rightarrow (D_c) \Rightarrow (D_r)$.

Remark 13 In [27, Section 6], the diagonal property (D_o) is also studied (one requires that the bundle involved in the definition of (D_r) is orientable). It is proved there that a real projective space of odd dimension does not have (D_o).

Let E be a smooth real vector bundle on X. For any $d_\bullet$ like in Section 2, there is an associated flag bundle $\pi : Fl^{\mathbb{R}}_{d_\bullet}(E) \to X$ parametrizing $d_\bullet$-flags of real subbundles of E. It is endowed with the tautological sequence (2) of real bundles. Similarly, if E is a smooth complex vector bundle on X, then there is an associated flag bundle $\pi : Fl^{\mathbb{C}}_{d_\bullet}(E) \to X$ parametrizing $d_\bullet$-flags of complex subbundles of E, endowed with the tautological sequence (2) of complex bundles.

Theorem 14 *(i) If X has (D_r) and $E \to X$ is a smooth real bundle, then $Fl^{\mathbb{R}}_{d_\bullet}(E)$ has (D_r).*
(ii) If X has (D_c) and $E \to X$ is a smooth complex bundle, then $Fl^{\mathbb{C}}_{d_\bullet}(E)$ has (D_c).

Proof. Both cases of the theorem can be proved by the construction using the tautological bundles from the proof of Theorem 4. Using the proof of this theorem and its notation, we have

$$\Delta_F \subset Z \subset F_1 \times F_2 \,,$$

and we have the section s' of G' and the section t of H_Z. By a partition of unity argument (cf. [1, Lemma 1.4.1]), t can be extended to a global section of H. Then, $s' \oplus t$ is a global section of $G' \oplus H$ which vanishes exactly on Δ_F (compare with Remark 6). $\square$

We say, following [27, Section 6], that X as above has property (P_r) if there exists a smooth real vector bundle of rank $\dim(X)$ on X with a smooth section s which is transverse to the zero section of the bundle and whose zero locus is a point. If $\dim_{\mathbb{R}} X = 2m$ and the above vector bundle is a complex vector bundle of complex rank m, then we say that X has property (P_c).

Remark 15 It was shown in [27, Remark 6] that if a bundle E of rank $\dim(X)$ on a (connected) manifold X has $e(E) = \pm 1$ (resp. $c_m(E) = \pm 1$), then we can use this bundle to realize (P_r) (resp. (P_c)).

Remark 16 By the argument from Theorem 11, we see that if X has (P_r) (resp. (P_c)), then $Fl^{\mathbb{R}}_{d_\bullet}(E)$ (resp. $Fl^{\mathbb{C}}_{d_\bullet}(E)$) has (P_r) (resp. (P_c)). The same holds for the corresponding strong point properties.

6 Manifolds G/B for other groups

A general reference for group-theoretic notions used in this section is [16]. Let G be a simple, simply connected algebraic group over $\mathbb{C}$, B its Borel subgroup, and T a maximal torus contained in B. Denote by G/B the generalized flag manifold.

In this section, we work in topological category and all vector bundles are complex. Suppose that the complex dimension of G/B is m. We shall study when G/B has (P_c), i.e. when there exists a vector bundle E of complex rank m on G/B such that $c_m(E)$ is the class of a point in $H^{2m}(G/B;\mathbb{Z})$ (cf. Remark 15).

Our main result in this section is

Theorem 17 *For G of type $(B_i)(i \geq 3), (D_i)(i \geq 4), (G_2), (F_4)$ and $(E_i)(i = 6, 7, 8)$, the flag manifold G/B has not (P_c), and consequently it has not the diagonal property (D_c).*

To prove the theorem, we need several results. Let $\mathcal{X}(T)$ be the group of characters of T and let $K(G/B)$ be the Grothendieck group of G/B (cf. [1, Section 2.1]). Consider the *Atiyah-Hirzebruch homomorphism* (see [19, Definition 3.17(a)]):

$$\beta_1 : S(\mathcal{X}(T)) \to K(G/B)$$

such that for $\lambda \in \mathcal{X}(T)$, $e^\lambda \mapsto$ class of $L_\lambda = G \times_B \mathbb{C}_\lambda$, a line bundle on G/B. Here, we regard the T-representation $\mathbb{C}_\lambda$ as a B-representation by letting the nilradical of B act trivially. Then, we extend this definition multiplicatively to the entire symmetric algebra $S(\mathcal{X}(T))$.

We record (see [19, Theorem 4.6] and the references therein):

Theorem 18 *The homomorphism β_1 is surjective.*

Since in $S(\mathcal{X}(T))$ any element is a $\mathbb{Z}$-linear combination of monomials $e^{\lambda_1} \cdots e^{\lambda_k}$, where $\lambda_i \in \mathcal{X}(T)$, and

$$\begin{aligned}\beta_1(e^{\lambda_1} \cdots e^{\lambda_k}) = \beta_1(e^{\lambda_1}) \cdots \beta_1(e^{\lambda_k}) \\ =[L_{\lambda_1}] \cdots [L_{\lambda_k}] = [L_{\lambda_1} \otimes \cdots \otimes L_{\lambda_k}] = [L_{\lambda_1 + \cdots + \lambda_k}]\,,\end{aligned} \tag{16}$$

the theorem implies the following

Corollary 19 *In $K(G/B)$, the class of any vector bundle is a $\mathbb{Z}$-linear combination of the classes of line bundles L_μ for some $\mu \in \mathcal{X}(T)$.*

Remark 20 It is shown by Kumar in [20, Corollary 2.12] that for G/B the present K-group is the same as the algebraic geometric K-group discussed in [8, Section 15.1].

Recall now the following *Borel characteristic homomorphism*:

$$c : S(\mathcal{X}(T)) \to H^*(G/B;\mathbb{Z})$$

such that for $\lambda \in \mathcal{X}(T)$, $e^\lambda \mapsto c_1(L_\lambda)$. Then, we extend this definition multiplicatively to all $S(\mathcal{X}(T))$ (see [2] and [7] for more details).

It follows from Corollary 19 that

Corollary 21 *The Chern classes of any vector bundle on G/B are in the image of c.*

The smallest positive integer t_G such that

$$t_G \cdot (\text{class of a point})$$

is in the image of c is called the *torsion index* of G. We record (see [3, Proposition 4.2], [7, Proposition 7] and also [30]):

Theorem 22 *We have $t_G = 1$ if and only if G is of type (A_i) or (C_i).*

Combining Corollary 21 and Theorem 22, the assertion of Theorem 17 follows. □

Let G be a complex reductive group. Recall that by replacing G with its universal covering group, the flag variety G/B can be regarded as a product of flag varieties associated to simple, simply-connected groups of type (A_i), (B_i) $(i \geq 3)$, (C_i) $(i \geq 2)$, (D_i) $(i \geq 4)$, (G_2), (F_4) and (E_i) $(i = 6, 7, 8)$.

Corollary 23 Let G be a complex reductive group containing either type $(B_i)(i \geq 3)$, (D_i) $(i \geq 4)$, (G_2), (F_4) or (E_i) $(i = 6, 7, 8)$ as a factor. Then, its flag variety G/B has not (P_c), and consequently it has not (D_c).

Proof. The class of a point in $G_1/B_1 \times G_2/B_2$ is the product of the classes of points in G_1/B_1 and G_2/B_2, and

$$K(G_1/B_1 \times G_2/B_2) \simeq K(G_1/B_1) \otimes K(G_2/B_2)$$

by the Künneth theorem. Therefore, our argument for the proof of Theorem 18 applies straightforwardly. □

We pass now to type (C_n). We have an identification of $F = Sp(2n, \mathbb{C})/B$ with the space of complete isotropic flags

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset \mathbb{C}^{2n}.$$

Let

$$(0) = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n \subset \mathbb{C}^{2n}_F$$

be the tautological flag on F. We set for $i = 1, \ldots, n$, $L_i = S_i/S_{i-1}$. Let $x_i = c_1(L_i)$. Then, we have

$$H^*(F; \mathbb{Z}) = \frac{\mathbb{Z}[x_1, x_2, \ldots, x_n]}{(e'_i(x))},$$

where $e'_i(x)$ is the i-th elementary symmetric polynomial in $x_1^2, x_2^2, \ldots, x_n^2$. Now, the class of a point is $x_1 x_2^3 \cdots x_n^{2n-1}$, which is the top Chern class of the bundle

$$L_1 \oplus {L_2}^{\oplus 3} \oplus \cdots \oplus {L_n}^{\oplus 2n-1}.$$

This shows that the following result holds.

Proposition 24 *The flag manifold $Sp(2n,\mathbb{C})/B$ has (P_c).*

Remark 25 In the paper [17], we shall investigate the diagonal and point properties of the spaces G/P.

7 Appendix: Explicit formulas for the classes of diagonals

Several authors worked out algebraic expressions for the classes of diagonals of the spaces G/B in the cohomology rings $H^*(G/B \times G/B;\mathbb{Z})$ (or equivalently in the Chow rings $A^*(G/B \times G/B)$). Let us mention them and relevant references: Fulton [9, 10], the second author and Ratajski [26], Graham [13], De Concini [6] (see also [12]).

Let BG and BB denote the classifying spaces of G and B. Consider the sequence

$$G/B \times G/B \to BB \times_{BG} BB \to BB \times BB\,,$$

which yields the following sequence of homomorphisms of their cohomology rings:

$$S(\mathcal{X}(T)) \otimes S(\mathcal{X}(T)) \to H^*(BB \times_{BG} BB;\mathbb{Z}) \to H^*(G/B \times G/B;\mathbb{Z})\,.$$

The first map is the Borel characteristic homomorphism and it is surjective after tensoring with $\mathbb{Q}$. Thus we shall realize the representatives of the classes of diagonals of G/B in

$$S(\mathcal{X}(T))_{\mathbb{Q}} \otimes_{\mathbb{Q}} S(\mathcal{X}(T))_{\mathbb{Q}} = \mathbb{Q}[x_1,\dots,x_n;y_1,\dots,y_n]\,, \tag{17}$$

where $x_i \in \mathcal{X}(T)$ and $y_i \in \mathcal{X}(T)$ are coordinates on $T \times T$[1].

In [13, Theorem 1.1], the author established a criterion for an element in (17) to represent the class of the diagonal Δ of G/B. Among other methods, we mention Gysin maps ([24, 26, 13, 12]) and equivariant cohomology ([6], cf. also the end of the introduction to [13]).

For type (A_{n-1}), $[\Delta]$ is represented by

$$\prod_{1\le i<j\le n} (x_i - y_j)\,,$$

the top (double) Schubert polynomial of Lascoux-Schützenberger (see [9]). (Note that our convention for numbering the variables is different from that in [9].)

[1] Our convention is that x_i and y_i represent the same coordinate on T; moreover, on $T \times T$, x_i corresponds to the first factor and y_i to the second.

For type (C_n), $[\Delta]$ is represented by

$$\prod_{i<j}(x_i - y_j) \cdot W(x,y)\,,$$

where denoting by $e_i(x)$ the ith elementary symmetric polynomial in x,

$$W(x,y) = |e_{n+1+j-2i}(x) + e_{n+1+j-2i}(y)|_{1\le i,j\le n}$$

(see [10]), or (see [26]):

$$W(x,y) = \sum_{I\subset(n,\ldots,1)} \widetilde{Q}_I(x)\cdot\widetilde{Q}_{(n,\ldots,1)\smallsetminus I}(y)\,,$$

where the sum is over strict partitions I, and $\widetilde{Q}_I(x)$ is defined in [26, Section 4]. In [6], the following expression representing $[\Delta]$ was given:

$$\prod_{i<j}(x_i^2 - y_j^2)\prod_i(x_i + y_i)\,. \tag{18}$$

For type (B_n) we have analogous expressions (differing by powers of 2) (cf. [10, 11, 6, 26]). This is also the case of the formulas for type (D_n) from [10, 26]. The author of [6] stated that $[\Delta]$ is represented by

$$\prod_{i=1}^{n-1} W_i(x,y)\,,$$

where

$$W_i(x,y) = \frac{1}{2}\Big((1+\frac{y_i}{x_i})\prod_{j>i}(x_i^2 - y_j^2) + (-1)^{n-i}(x_i\cdots x_n - y_i\cdots y_n)\frac{y_{i+1}\cdots y_n}{x_i}\Big)\,.$$

We can regard this expression in the following way. First, we multiply $W_i(x,y)$ by x_i. Then the first summand is the one for type (B_n), which is almost good but of degree by one greater. So we add a zero class so that it makes the sum divisible by x_i. This gives rise to the second summand. Notice that the only term in the first summand which is not divisible by x_i is $y_i y_{i+1}^2\cdots y_n^2$. Hence we have to add another term involving the y_i's to cancel out the term in cohomology. This gives the above expression.

For type (G_2), in [13], the following expression was given for a representative of $[\Delta]$:

$$-\frac{27}{2}(x_1 - y_2)(x_1 - y_3)(x_2 - y_3)(x_1x_2x_3 + y_1y_2y_3).$$

Note that here $S(\mathcal{X}(T)) = \mathbb{Z}[x_1,x_2,x_3]/(x_1+x_2+x_3)$.

Remark 26 In [13, p. 483], the author conjectured that

$$\frac{1}{2}(x_1x_2x_3 + y_1y_2y_3)$$

is integral. A Schubert calculus computation gives that this class is a $\mathbb{Q}$-linear combination of Schubert classes

$$-\frac{2}{9}\sigma_{s_2s_1s_2} + \text{lower terms}^2 \tag{19}$$

The expression (19) disproves the conjecture.

We add another expression for the type (G_2). We now identify

$$S(\mathcal{X}(T))_{\mathbb{Q}} \otimes_{\mathbb{Q}} S(\mathcal{X}(T))_{\mathbb{Q}} = \mathbb{Q}[a_1, a_2; b_1, b_2]\,,$$

where a_1 and b_1 (resp. a_2 and b_2) the two copies of the simple short root (resp. long root).

Proposition 27 *The expression*

$$\frac{1}{2}(a_1+b_1)(a_1-(2b_1+b_2))(a_1+(2b_1+b_2))(a_1-(b_1+b_2))(a_1+(b_1+b_2))(a_2-(3b_1+b_2)) \tag{20}$$

represents $[\Delta]$.

Proof. We use Graham's criterion [13, Theorem 1.1]. The Weyl group of G_2 is the dihedral group of order 12. The orbit of a_1 under W is

$$\{\pm a_1, \pm(2a_1 + a_2), \pm(a_1 + a_2)\}\,.$$

This accounts for the first five factors. The stabilizers of a_1 are the identity and $s_{3a_1+2a_2}$, and the latter takes a_2 to $3a_1 + a_2$. This accounts for the last factor. If we evaluate the polynomial (20) at $a_1 = b_1, a_2 = b_2$, we obtain the product of all the positive roots. By Grahams's criterion, the expression (20) represents $[\Delta]$. $\square$

A similar expression was stated in [6]. This representative

$$\frac{1}{2}(a_1 + b_1)(a_1^2 - (2b_1 + b_2)^2)(a_1^2 - (b_1 + b_2)^2)(a_2 - (3b_1 + b_2))$$

has property that the global coefficient equals the inverse of the torsion index, which is the best possible.

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[2] Here, we use the identification $H^*(BB \times_{BG} BB) = H^*_T(G/B)$ with the bi-grading on the RHS given by the degree in the coefficient $H^*(BT)$ and the degree of the Schubert basis. So "lower terms" mean a $H^*(BT)$-linear combination of Schubert classes corresponding to shorter Weyl group elements.

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