DIVIDED DIFFERENCES OF TYPE D AND THE GRASSMANNIAN OF COMPLEX STRUCTURES

HAIBAO DUAN AND PIOTR PRAGACZ

Dedicated to Professor Tatsuo Suwa on his 60th birthday

Contents

1. Introduction

Divided differences are discrete analogues of derivations. They were introduced by Newton in his famous interpolation formula in "Principia Mathematica" (1686). Their importance in geometry was shown in the early 1970's by Bernstein-Gelfand-Gelfand [BGG] and Demazure [De] in the context of Schubert calculus for generalized flag varieties associated with semisimple algebraic groups. More recently, simple divided differences, interpreted as correspondences in flag bundles, were used by Fulton in his study of the classes of degeneracy loci. Divided differences admit still another interpretation as Gysin maps in the cohomology of flag bundles associated with semisimple algebraic groups (cf., e.g., [P2]). We refer to the lecture notes [FP] for a systematic discussion of these issues. The case of $SL(n)$ has been recently developed extensively by Lascoux and Schützenberger (cf., e.g., [LSc]), and serves nowadays as an important and useful tool for multivariate polynomials (cf. [L2]).

The Grassmannian of complex structures parametrizes orthogonal automorphisms of the Euclidean space $\mathbf{R}^{2n}$ whose square is the minus identity. Equivalently, it parametrizes minimal geodesics from the identity to the minus identity in the orthogonal group $SO(2n, \mathbf{R})$ [Mi]. This space is usually denoted by CS_n. It played a significant role in several important achievements in topology: in the investigation of orthonormal vector fields on spheres by Hurewicz and Adams, in the study of the existence of complex structures on even dimensional spheres by Borel and Serre, and in the Bott's discovery of the eight-periodicity of homotopy groups of the stable real orthogonal groups.

Also, CS_n serves as the classifying space of all complex bundles whose real reduction is trivial, by a result of the first author [Du1].

The goal of the present paper is to develop in a systematic way a Schubert calculus for CS_n. We hope that it will be useful also for topologists.

The space CS_n has two connected components, each isomorphic to the homogeneous space

$$SO(2n, \mathbf{C})/U(n) \quad \text{or} \quad SO(2n, \mathbf{C})/P\,,$$

where P is the maximal parabolic corresponding to omitting the "right end root". This space is a connected component of the Grassmannian of all isotropic subspaces of $\mathbf{C}^{2n}$ w.r.t. to the orthogonal form induced by the scalar product, and as such, it is also known as the orthogonal Grassmannian.

With the help of the group-theoretic description, we can use the characteristic map of Borel [Bo], and – via the theory of Bernstein-Gelfand-Gelfand [BGG] and Demazure [De] – divided differences of type D to study the intersection theory on the space in question. In order to make the work with the characteristic map efficient, one needs a proper family of "invariant" polynomials that are well suited to divided differences, and also to geometry/topology at the same time.

A result of the second author [P1] identified Schubert classes in the homogeneous space $SO(2n, \mathbf{C})/U(n)$ with suitable Schur P-polynomials. In [P1] this identification used a geometric argument, namely an isomorphism $SO(2n, \mathbf{C})/U(n) \simeq SO(2n-1, \mathbf{C})/U(n-1)$, and an identification of the Schubert classes for the latter Grassmannian with Schur P-functions. (This last identification was based on comparison of the Pieri-type formulas from [HB] and [Mo].)

In the present paper we revisit the identification for the Schubert classes for $SO(2n, \mathbf{C})/U(n)$ with Schur P-functions via a direct *group-theoretic* argument based on the calculus of divided differences of type D. More precisely, we give a group-theoretic proof of a Pieri-type formula that is based on some vanishing results for operators composed of divided differences of type D and simple reflections from the Weyl group of type D. These last results form the most technical part of the present work. Our proof of the Pieri-type formula follows a strategy for deriving similar formulas for various homogeneous spaces worked out by Ratajski and the second autor in a series of papers summarized in [P2]. This particular proof was promised in [PR2] – a paper that is now under revision. The proof uses esentially an iteration of the Leibniz-type formula for a simple divided difference applied to the product of two functions.

Combining the Pieri formula with a combinatorial lemma of Schur [S] for the projective characters of the symmetric groups, we get a formula for the degree of Schubert varieties in $SO(2n, \mathbf{C})/U(n)$. (Occasionally, we discuss some alternative derivations of the lemma of Schur with the help of a specialization result from [DP].)

We remark that there exists now a refinement of Schur P-functions that seems to be even better adapted for some aspects of geometry. These are the so called $\widetilde{P}$-functions of [PR1], which are modeled on Schur P-functions. In [LP], Lascoux and the second author worked out a connection of orthogonal divided differences to $\widetilde{P}$-functions using vertex operators. This has led to *orthogonal Schubert polynomials* that are useful in various cohomological computations (cf. a recent work of Kresch and Tamvakis [KT],[T2], and Buch [BKT]).

After presenting the Schubert varieties in a group-theoretic way, we also describe them via Schubert-type conditions relative to some flag of linear subspaces, and finally we study them in terms of complex structures. To this end, we are guided by the Mahowald-Vassiljev-type formula ([DV],[V]):

$$H_i(CS_n) = \oplus_{k=0}^n H_{i-k(k-1)}(G_k(\mathbf{C}^n))\,,$$

where $G_k(\mathbf{C}^n)$ is the Grassmannian of all complex k-planes through zero in $\mathbf{C}^n$.

We end the paper by illustrating how the Schubert calculus developed here can be used to solve problems about enumeration of complex structures which satisfy some natural conditions of "partial overlapping" with a certain number of complex structures in general position in $\mathbf{R}^{2n}$. One of the applications leads to an interesting algebraic conjecture about homomorphisms between the cohomology ring of CS_n^+ and that of the Grassmannian $G_k(\mathbf{C}^n)$.

Acknowledgements. This work was partially supported by grant No.5031 of French-Polish cooperation C.N.R.S. - P.A.N. (the "French connection" of this grant being A. Lascoux whose comments clarified our Remark 4.11). While revising the paper and preparing its final version, the authors were supported by Polish KBN grant No.2 P03A 024 23. The present paper is a revised version of the Max-Planck-Institut für Mathematik in Bonn Preprint 2001-107. The second author is grateful to the MPIM for its hospitality during the preparation of the first draft of this paper.

2. Preliminaries, notation, and conventions

We start with some algebraic preliminaries on even orthogonal groups. We fix a positive integer n. Suppose that $H = SO(2n, \mathbf{C})$ is the orthogonal group (of type D_n) over the field of complex numbers. Our standard reference for the group-theoretic terminology, is [FH]. We shall use the following notation: B – a fixed Borel subgroup of H, $T \subset B$ – a fixed maximal torus, $\mathcal{R}$ – the root system of H associated with T, Σ – a set of simple roots of $\mathcal{R}$ associated with B, and finally W – the Weyl group of (H, T). In a standard realization of [Bu], we have:

$$\mathcal{R} = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\} \subset \mathbf{R}^n = \oplus_{i=1}^n \mathbf{R}\varepsilon_i,$$

$$\Sigma = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\},$$

$$W = S_n \ltimes \mathbf{Z}_2^{n-1}.$$

A typical element of W can be written as a pair (τ, ϵ), where $\tau \in S_n$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ is a sequence of elements of $\mathbf{Z}_2 = \{-1, 1\}$ such that $\#\{i\ :\ \epsilon_i = -1\}$ is even. Multiplication in W is given by

$$(\tau, \epsilon) \cdot (\tau', \epsilon') = (\tau \circ \tau', \delta),$$

where "$\circ$" denotes the composition of permutations and $\delta_i = \epsilon_{\tau'(i)} \cdot \epsilon'_i$. The following lemma can be easily verified (and is pretty well-known). For $w \in W$, let $l(w)$ denote the length of w taken w.r.t. to the above Σ.

Lemma 2.1. *For any $w \in W$, $l(w)$ is equal to:*

$$\sum_{i=1}^{n} \#\{j\ :\ j > i\ \&\ w(j) < w(i)\} + 2 \sum_{\epsilon_p = -1} \#\{q\ :\ q > p\ \&\ w(q) > w(p)\}\,.$$

We will use the "barred-permutation" notation, indicating by a bar a place in the permutation $w = [w(1), \ldots, w(n)]$ where $\epsilon_i = -1$.

The following lemma, that is easy to prove from Lemma 2.1, gives us the lengths of some barred permutations basic to this paper.

Lemma 2.2. *Let $y_1 < \cdots < y_{n-k}$ and $z_k > \cdots > z_1$ be sequences of integers that are complementary in $\{1, \ldots, n\}$. Assume that k is even. Then in W we have*

$$l([y_1, y_2, \ldots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1]) = \sum_{j=1}^{k} (n - z_j)\,.$$

The barred permutations of this type form the poset, denoted by W^*, of the minimal length left coset representatives of S_n in W.

Our terminology and all unexplained notation concerning partitions will follow [Ma].

We set $\rho(k) := (k, k-1, \ldots, 1)$, a "triangular partition" of length k.

Given a strict partition $\alpha = (\alpha_1 > \cdots > \alpha_l > 0) \subset \rho(n-1)$, we set

$$\alpha^+ := (\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_l + 1)$$

if l is even, and

$$\alpha^+ := (\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_l + 1, 1)$$

if l is odd. Note that α^+ is of even length.

Given a strict partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_k > 0) \subset \rho(n)$ of even length k, we associate with it the following element w_μ of W^*. We set

$$w_\mu := [y_1, y_2, \ldots, y_{n-k}, \overline{n+1-\mu_k}, \overline{n+1-\mu_{k-1}}, \ldots, \overline{n+1-\mu_1}]\,.$$

Note that $l(w_\mu) = |\mu| - k$ by Lemma 2.2. (Recall that the symbol $|\mu|$ denotes the sum of the parts of μ.)

Setting for a strict partition $\alpha \subset \rho(n-1)$, $\lambda := \alpha^+$, we have $l(w_\lambda) = |\alpha|$.

Finally, we adopt a convention that all homology or cohomology groups in the present paper are taken with integer coefficients.

3. Combinatorics of divided differences of type D_n

We define simple divided differences of type D_n which are operators $\partial_i : \mathbf{Z}[X] \to \mathbf{Z}[X]$, $i = 1, \ldots, n$, of degree -1 acting on the ring of polynomials $\mathbf{Z}[X]$ where X is a fixed set of indeterminates $X = \{x_1, x_2, \ldots, x_n\}$. To this end, we denote by s_i, $1 \le i \le n-1$, the transposition

$$[1, \ldots, i-1, i+1, i, i+2, \ldots, n] \in S_n \subset W\,,$$

acting on X by interchanging x_i and x_{i+1}. Moreover, let

$$s_n = [1, \ldots, n-2, \overline{n}, \overline{n-1}]$$

be the reflection which transposes x_{n-1} with x_n and changes the signs of both the variables. The remaining variables are invariant under the action of these transpositions. This action is extended multiplicatively to the ring $\mathbf{Z}[X]$. Note that s_n commutes with s_i, $i \ne n-2$, and

$$s_{n-2} \cdot s_n \cdot s_{n-2} = s_n \cdot s_{n-2} \cdot s_n\,.$$

Simple divided differences of type D_n are defined as follows:

$$\partial_i(f) = (f - s_i f)/(x_i - x_{i+1}), \qquad i = 1, \ldots, n-1;$$
$$\partial_n(f) = (f - s_n f)/(x_{n-1} + x_n).$$

For every $f, g \in \mathbf{Z}[X]$ and any i, we have

$$\partial_i(f \cdot g) = f \cdot (\partial_i g) + (\partial_i f) \cdot (s_i g) \tag{1}$$

(a Leibniz-type formula).

For a given $\mathbf{a} = (a_n, a_{n-1}, \dots, a_2, a_1) \in \{-1, 0, 1\}^n$, we define the generating function:

$$E_{\mathbf{a}} = \prod_{i=1}^{n} (1 + a_i x_i). \tag{2}$$

In particular, for $\mathbf{a} = (1, \dots, 1)$, the resulting generating function, denoted by E, is the generating function for the elementary symmetric polynomials $e_i(X) = e_i(x_1, \dots, x_n)$, $i = 1, \dots, n$.

Lemma 3.1. *a) We have $s_i(E_{\mathbf{a}}) = E_{\mathbf{a}'}$, where*

$$\mathbf{a}' = \begin{cases} (a_n, \dots, a_{i+2}, a_i, a_{i+1}, a_{i-1}, \dots, a_1) & i < n\,, \\ (-a_{n-1}, -a_n, a_{n-2}, \dots, a_1) & i = n\,. \end{cases}$$

b) For $i = 1, 2, \dots, n-1$,

$$\partial_i(E_{\mathbf{a}}) = d \cdot E_{\mathbf{a}'} \qquad \text{if} \;\; a_i = a_{i+1} + d \qquad (d = -2, -1, 0, 1, 2),$$

where $\mathbf{a}' = (a_n, \dots, 0, 0, \dots, a_1)$ is the sequence $\mathbf{a}$ with a_{i+1}, a_i replaced by zeros.
c) $\partial_n(E_{\mathbf{a}}) = (a_n + a_{n-1}) \cdot E_{(0,0,a_{n-2},\dots,a_1)}$.
In particular if Δ is a composition of some s- and ∂-operations, then for every $\mathbf{a}$, $\Delta(E_{\mathbf{a}}) = (scalar) \cdot E_{\mathbf{a}'}$, where $\mathbf{a}'$ is uniquely determined if this scalar is not zero.

Proof. We prove e.g. c). We have, with $\mathbf{a}' = (0, 0, a_{n-2}, \dots, a_1)$,

$$\begin{aligned} \partial_n(E_{\mathbf{a}}) &= \frac{(1 + a_{n-1}x_{n-1})(1 + a_n x_n) - (1 - a_{n-1}x_n)(1 - a_n x_{n-1})}{x_{n-1} + x_n} \cdot E_{\mathbf{a}'} \\ &= \frac{(a_{n-1} + a_n)(x_{n-1} + x_n)}{x_{n-1} + x_n} \cdot E_{\mathbf{a}'} = (a_n + a_{n-1}) \cdot E_{\mathbf{a}'}\,, \end{aligned}$$

as desired. □

We now recall the following fact from [BGG] and [De]. For any $w \in W$ and any reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ one can define $\partial_w = \partial_{i_1} \circ \cdots \circ \partial_{i_l}$ – an operator on $\mathbf{Z}[X]$ of degree $-l(w)$. In fact, since divided differences satisfy the braid relations, ∂_w does not depend on the chosen reduced decomposition of w.

Suppose a strict partition $\mu \subset \rho(n)$ with even length is given. Let us use the following coordinates for boxes in the Ferrers diagram D_μ of μ:

$$\begin{array}{c|ccccc} & n & n-1 & \dots & 2 & 1 \\ \hline 1 & & & & & \\ \vdots & & & & & \\ n & & & & & \end{array}$$

We associate with μ a certain distinguished reduced decomposition of $w_\mu \in W$. To this end, let us modify the diagram D_μ in the following way. Remove one box from each row of D_μ : from rows with even numbers remove the box in the n-th column, and from rows with odd numbers remove the box in the $(n-1)$-st column.

We display the removed boxes in the picture using the symbol $\times$ and denote the so obtained set of boxes by $\mathring{D}_\mu$. For example, $\mathring{D}_{(8,7,4,2)}$ is:

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & \times & & & & & & \\
\hline
\times & & & & & & & \\
\cline{1-7}
 & \times & & & \multicolumn{4}{c}{} \\
\cline{1-4}
\times & & \multicolumn{6}{c}{} \\
\cline{1-2}
\end{array}
$$

Assume now, that a subset $D \subset \mathring{D}_\mu$ is given. A box belonging to D will be called *a D-box* and a box from the difference $\mathring{D}_\mu \setminus D$ will be called *a $\sim D$-box.* D-boxes will be depicted using "$\bullet$" and $\sim D$-boxes will be depicted either as white boxes or using "$\circ$".

Definition 3.2. Read $\mathring{D}_\mu$ row by row from left to right and from top to bottom. Every D-box (resp. $\sim D$-box) in the i-th column gives us s_i (resp. ∂_i). Then ∂_μ^D is the composition of the resulting s_i's and ∂_i's (the composition written from right to left).

Definition 3.3. Read $\mathring{D}_\mu$. Every D-box in the i-th column gives us s_i. $\sim D$-boxes give no contribution. Then, r_D is the word obtained by writing the resulting s_i's from right to left. (In other words, one obtains r_D by erasing all the ∂_i's from ∂_μ^D.)

For example, for $n = 9$ and $\mu = (8, 7, 4, 2)$,

$$
\begin{array}{cccccccc}
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \;\; 1 \\
\end{array}
$$
$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bullet & \times & \bullet & \bullet & & \bullet & & \\
\hline
\times & \bullet & \bullet & & \bullet & & & \\
\cline{1-7}
 & \times & & \bullet & \multicolumn{4}{c}{} \\
\cline{1-4}
\times & & \multicolumn{6}{c}{} \\
\cline{1-2}
\end{array}
$$

$$\partial_\mu^D = \partial_8 \circ s_6 \circ \partial_7 \circ \partial_9 \circ \partial_3 \circ \partial_4 \circ s_5 \circ \partial_6 \circ s_7 \circ s_8 \circ \partial_2 \circ \partial_3 \circ s_4 \circ \partial_5 \circ s_6 \circ s_7 \circ s_9$$

$$r_D = s_6 \cdot s_5 \cdot s_7 \cdot s_8 \cdot s_4 \cdot s_6 \cdot s_7 \cdot s_9\,.$$

One can easily prove that for $D = \mathring{D}_\mu$, we have $r_D \in R(w_\mu)$ – the set of reduced decompositions of w_μ. This is our *distinguished reduced decomposition* of w_μ. For example, for $n = 9$ and $\mu = (8, 7, 4, 2)$,

$$
\begin{array}{cccccccc}
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \;\; 1 \\
\end{array}
$$
$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bullet & \times & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\times & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\cline{1-7}
\bullet & \times & \bullet & \bullet & \multicolumn{4}{c}{} \\
\cline{1-4}
\times & \bullet & \multicolumn{6}{c}{} \\
\cline{1-2}
\end{array}
$$

$$w_\mu = s_8 \cdot s_6 \cdot s_7 \cdot s_9 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_9\,.$$

Let now $\mu \subset \rho(n)$ be a strict partition with even length. We will examine subsets $D \subset \mathring{D}_\mu$ for which $\partial_\mu^D(E) = 0$ (we say: "D causes vanishing").

In many computations in this section, we will apply compositions of the operators of boxes of D_μ to the generating functions $E_{\mathbf{a}}$. With the following example we illustrate how such operators act.

Example 3.4. Let $n = 9$. We apply the operators of boxes from left to right to $E_{\mathbf{a}}$, where $\mathbf{a} = (a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1)$, and obtain $E_{\mathbf{a}'}$. We give 2 examples of the action of the operators associated with a row in D_μ:

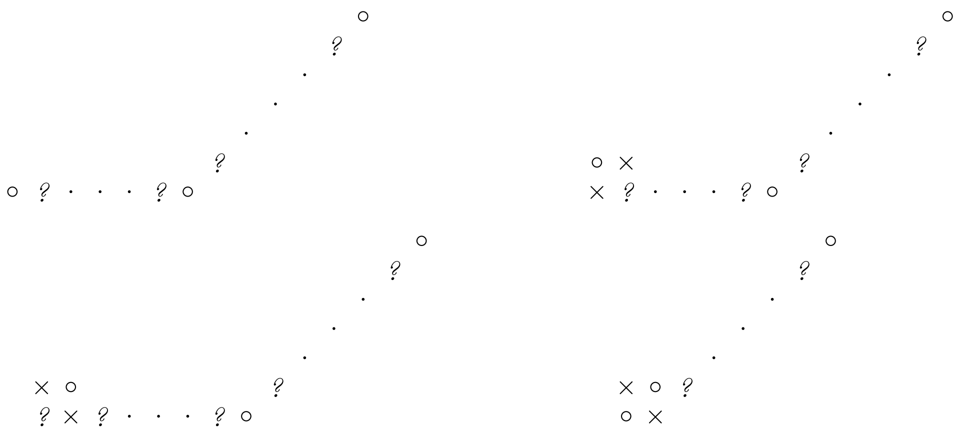
9	8	7	6	5	4	3	2	1
$\bullet$	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	

– we get: $\mathbf{a}' = (-a_8, a_7, a_6, a_5, a_4, a_3, a_2, 0, 0)$

9	8	7	6	5	4	3	2	1
$\times$	$\bullet$	$\bullet$	$\bullet$		$\bullet$	$\bullet$		$\bullet$

– we get: $\mathbf{a}' = (a_8, a_7, a_6, 0, a_4, a_3, 0, a_1, 0)$

Lemma 3.5. *The following configurations of three $\sim$D-boxes in D_μ give vanishing:*



(Above, "?" can be $\circ, \bullet$, or $\times$; and the skew directions are all parallel to the antidiagonal.)

Proof. Direct calculation using Lemma 3.1. $\square$

Let us fix an element $w = [y_1, y_2, \ldots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1] \in W^*$. Recall that k is even. We treat a given reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ as a sequence of simple transposition operations, which produces the element in question from the identity permutation:

$$[y_1, y_2, \ldots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1] = (\cdots([1, 2, \ldots, n] \cdot s_{i_1}) \cdots) \cdot s_{i_l}.$$

In the following, the simple transpositions involved will be called the "s_{i_h}-operations" ($h = 1, \ldots, l$).

Proposition 3.6. *We have the following two possibilities for the action of s_{i_h}-operations on the z's:*

1) If $i_h = n$ then this operation is:

$$[\ldots, z, z'] \ \to \ [\ldots, \overline{z}', \overline{z}],$$

where $z = z_{p-1}$ and $z' = z_p$ for some even p.

2) If $i_h < n$, then this operation is:

$$[\ldots, z, x, \ldots] \to \ [\ldots, x, z, \ldots]$$

where $x \neq z_j$ *for* $j = 1, \ldots, k$.

Proof. We must transpose each pair (z_j, y_i), for $z_j < y_i$, at least once, because the y's preced the (barred) z's in w. Also, we must transpose each pair (z_i, z_j), for $i < j$, at least once, because the (barred) z's appear in w in an descending order. In sum, we need at least

$$\sum \#\{(z_j, y_i) : z_j < y_i\} + \sum \#\{(z_i, z_j) : i < j\}$$

s_{i_h}-operations to reach the sequence w. But by Lemma 2.1 this last number is equal to $l(w)$. This means that the mentioned transpositions exchaust the family of all s_{i_h}-operations under consideration. As a consequence, no s_{i_h}-operation, for $i_h < n$, interchanges two (bar-free) z's (on their way towards the end of the permutation). Moreover, we see that exactly $k/2$ s_{i_h}-operations with $i_h = n$ appear. This implies immediately both assertions of the proposition. □

Now we assume that λ is another strict partition, that $D \subset \overset{\circ}{D}_\mu$, and that $r_D \in R(w_\lambda)$. Suppose that a D-box appears in the i-th column where $i < n$. We define the *mark* of this box to be p, if the corresponding s_{i_h}-operation acts on the i-th and $(i+1)$-st places as follows:

$$[\ldots, z_p, x, \ldots] \to [\ldots, x, z_p, \ldots],$$

where $x \neq z_j$, $j = 1, \ldots, k$. A $D - box$ in the n-th column has mark $p - 1$ if the corresponding $(s_{i_h} = s_n)$-operation acts via

$$[\ldots, z_{p-1}, z_p] \to [\ldots, \overline{z}_p, \overline{z}_{p-1}].$$

In particular, boxes in the n-th column have only odd marks. In the following lemma, we collect some simple properties of marks.

Lemma 3.7. *(i) (Connectedness) The D-boxes with a fixed mark in one row form a connected set; by this we understand that the numbers of their columns form an interval in $\{n, n-2, n-1, \ldots, 1\}$ (resp. in $\{n-1, n-2, \ldots, 1\}$) for a row with odd (resp. even) number.*
(ii) (Separation) In a fixed row, the two sets of D-boxes equipped with different marks are separated (i.e. there is at least one $\sim$$D$-box between them).
(iii) The sequence of boxes with odd mark p is of the form:

$$(t_n, n), (t_{n-2}, n-2), \ldots, (t_{z_p}, z_p),$$

where $p \leq t_n \leq t_{n-2} \leq \cdots \leq t_{z_p}$.
(iv) The sequence of boxes with even mark p is of the form:

$$(t_{n-1}, n-1), (t_{n-2}, n-2), \ldots, (t_{z_p}, z_p),$$

where $p \leq t_{n-1} \leq t_{n-2} \leq \cdots \leq t_{z_p}$.
(v) The marks of boxes in a fixed column (strictly) increase from top to bottom.
(vi) The marks of boxes in a fixed row (weakly) decrease from left to right.

Definition 3.8. The set of D-boxes with mark p is called the *ribbon with mark p*.

We have two basic operations of deforming ribbons.

- ("Push down") Let i be odd and suppose that the boxes

$$(i, n), (i, n-2), \ldots, (i, j)$$

form an entire ribbon. The operation transforms them to

$$(i+2,n),(i+2,n-2),\dots,(i+2,j)\,.$$

Let i be even and suppose that the boxes

$$(i,n-1),(i,n-2),\dots,(i,j)$$

form an entire ribbon. The operation transforms them to

$$(i+2,n-1),(i+2,n-2),\dots,(i+2,j)\,.$$

(We assume that the $(i+1)$-st and $(i+2)$-nd row contain no D-boxes before this operation.)

For example, the ribbons

can be pushed down to

- ("Breaking a ribbon") Let $j \le n-2$. The operation transforms a final segment

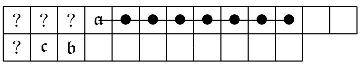
$$(i,j),(i,j-1),\dots,(i,h)$$

of a ribbon to the $\sim D$-boxes:

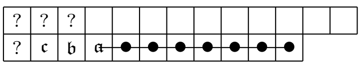
$$(i+1,j),(i+1,j-1),\dots,(i+1,h)\,,$$

provided $(i+1,j+1)$ is a $\sim D$-box, or it is $\times$ and $(i+1,j+2)$ is a $\sim D$-box. The box (i,j) (before the operation) is called the *breaking box.*

For example, for a breaking box $\mathfrak{a}$, $\mathfrak{b}$ a $\sim D$-box, or $\mathfrak{b} = \times$ and $\mathfrak{c}$ a $\sim D$-box:



can be broken at $\mathfrak{a}$ and transformed to



Suppose $r_D \in R(w_\lambda)$. Using the braid relations in W one easily shows that after breaking a ribbon in D, we get D' such that $r_{D'} \in R(w_\lambda)$. In the case of the push down operation, it is clear that we get D' with $r_D = r_{D'}$. Note that any configuration of boxes $D \subset D_\mu$ such that $r_D \in R(w_\lambda)$ can be obtained from $\mathring{D}_\lambda$ by a sequence of operations of the above described two types. Consequently, by considering the inverse operations, we infer that if D_λ is not contained in D_μ then there is no $D \subset \mathring{D}_\mu$ such that $r_D \in R(w_\lambda)$.

Definition 3.9. ("Maximal deformation" of $\mathring{D}_\lambda \subset \mathring{D}_\mu$)

- Pick the lowest ribbon. Push it down as many times as possible. Then choose the leftmost breaking box on this ribbon (if it exists) and break the ribbon.
- Pick a ribbon and suppose that lower ribbons in $\mathring{D}_\lambda$ have been already deformed. Push down this ribbon as many times as possible. Let $\mathfrak{a}$ be the leftmost breaking box on the ribbon. Break the ribbon at $\mathfrak{a}$ as many times as possible. Then choose the next leftmost breaking box $\mathfrak{b}$ and break the ribbon at $\mathfrak{b}$ as many times as possible etc.

For some examples of maximal deformations of diagrams, see Example 4.7.

Proposition 3.10. *Let $D \subset \mathring{D}_\mu$ be such that $r_D \in R(w_\lambda)$. If $\partial^D_\mu(E) \neq 0$ then D is the maximal deformation of $\mathring{D}_\lambda \subset \mathring{D}_\mu$.*

Proof. The proof is by descending induction on the mark of a ribbon. Pick the ribbon with mark p. Assume that the ribbons with marks $p+1, \ldots, l(\lambda)$ have been already maximally deformed. Suppose that we have either a possibility of pushing down of a ribbon or breaking a ribbon. In the case of the former operation we will refer to boxes of the three involved rows; in the case of the latter operation, we will refer to the two involved rows. We note that

- any box directly to the right or directly below of the rightmost box of a row of the (deformed) ribbon is a $\sim\!D$-box;
- any box directly to the left or directly above of the leftmost box of a row of the (deformed) ribbon is a $\sim\!D$-box;
- $\sim\!D$-boxes in the n-th or $(n-1)$-st column cannot be supplied by marks smaller than p.

This implies that if we not perform the operations in a maximal way, then we will either obtain a configuration:

$$\begin{array}{ccccccc} & & & & & & \circ \\ \circ & ? & \cdot & \cdot & \cdot & ? & \circ \end{array}$$

or, we will get one of the following two possibilities:

$$\begin{array}{ccccccc} \bullet & \times & & & & & \\ \times & \mathfrak{b} & \mathfrak{c} & & & & \\ \mathfrak{a} & \times & \bullet & \cdot & \cdot & \cdot & \bullet \end{array} \qquad \begin{array}{ccccccc} \times & & & & & & \\ \mathfrak{a} & \times & \mathfrak{b} & & & & \\ \times & \mathfrak{c} & \bullet & \cdot & \cdot & \cdot & \bullet \end{array}$$

where $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ are $\sim D$-boxes. By Lemma 3.5 all these three configurations of $\sim D$-boxes cause vanishing. The obtained contradiction means that the maximal deformation of $\mathring{D}_\lambda \subset \mathring{D}_\mu$ is necessary to avoid vanishing. □

As a corollary of this proposition, we get the following two results bounding the size of μ w.r.t. λ, if one wants to avoid vanishing. We will need some additional notation. Given a strict partition μ of even length l, we denote by μ^- the strict partition $(\mu_1 - 1, \ldots, \mu_l - 1)$. Note that $(\mu^-)^+ = \mu$. We also set $\mathring{l}(\mu) := l(\mu^-)$. Setting for a strict partition $\alpha \subset \rho(n-1)$, $\lambda := \alpha^+$, we have $\mathring{l}(\lambda) = l(\alpha) = \#D_\alpha = \#\mathring{D}_\lambda$.

Proposition 3.11. *If, for the maximal deformation D of $\mathring{D}_\lambda \subset \mathring{D}_\mu$, one has $\partial^D_\mu(E) \neq 0$, then $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$. (In, particular there is is no push down operation in this maximal deformation.)*

Proof. Suppose that $\overset{\circ}{l}(\mu) \geq \overset{\circ}{l}(\lambda)+2$. Pick the highest, say i-th, row which contains a $\sim D$-box in the n-th or $(n-1)$-st column. This is the highest row from which some ribbon has been pushed down in the maximal deformation, or, if there was no pushing down, it is the row with number $l(\lambda)+1$. Since $\overset{\circ}{l}(\mu) \geq \overset{\circ}{l}(\lambda)+2$ and by the construction of maximal deformation, we see that the $(i+1)$-st row contains also a $\sim D$-box in its $(n-1)$-st or n-th row respectively.

After breaking some higher ribbons we get:

$$\begin{array}{ccccccccc} & & & & & \circ & \mathfrak{a} & \cdots & \circ \\ & & & & \cdot & & & & \\ & & & \cdot & & & & & \\ & & \cdot & & & & & & \\ \times & \circ & & & & & & & \\ \circ & \times & & & & & & & \end{array} \qquad \text{or} \qquad \begin{array}{ccccccccc} & & & & & & \circ & \mathfrak{a} & \cdots \circ \\ & & & & & \cdot & & & \\ & & & & \cdot & & & & \\ & & & \cdot & & & & & \\ & & \circ & & & & & & \\ \circ & \times & & & & & & & \\ \times & \circ & & & & & & & \end{array}$$

The boxes marked by $\mathfrak{a}$ exist and they are $\sim D$-boxes (because μ is a strict partition). We get vanishing by Lemma 3.5 –a contradiction. $\square$

Proposition 3.12. *Assume that $\overset{\circ}{l}(\mu) \leq \overset{\circ}{l}(\lambda)+1$. If for the maximal deformation D of $\overset{\circ}{D}_\lambda \subset \overset{\circ}{D}_\mu$ one has $\partial^D_\mu(E) \neq 0$, then $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip.*

Proof. Suppose that $\lambda_i < \mu_{i+1}$ for some i. We can assume that for some $j < i$,

$$\mu_i = \lambda_{i-1}, \mu_{i-1} = \lambda_{i-2}, \ldots, \mu_{j+2} = \lambda_{j+1} \quad \text{but} \quad \lambda_j > \mu_{j+1}\,.$$

After the maximal deformation, we get in the consecutive rows with numbers $i+1, i, \ldots, j+1$:

$$\begin{array}{ccccccccccccc} & & & & \circ & \cdot & \cdot & \cdot & \circ & \cdot & \cdot & \cdot & \circ \\ & & & \cdot & & & & \cdot & & & & & \\ & & \cdot & & & & \cdot & & & & & & \\ & \cdot & & & & \cdot & & & & & & & \\ \circ & & & & \mathfrak{a} & & & & & & & & \end{array} \qquad \text{or} \qquad \begin{array}{ccccccccccccccc} & & & & & & \circ & \cdot & \cdot & \cdot & \circ & \cdot & \cdot & \cdot & \circ \\ & & & & & \cdot & & & & \cdot & & & & & \\ & & & & \cdot & & & & \cdot & & & & & & \\ & & & \cdot & & & & \cdot & & & & & & & \\ \times & \bullet & \circ & & & & \cdot & & & & & & & & \\ \circ & \times & & & & \mathfrak{a} & & & & & & & & & \end{array}$$

where $\mathfrak{a}$ displays a $\sim D$-box. We get vanishing by Lemma 3.5 – a contradiction. $\square$

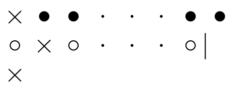
The maximal deformation is obtained by breaking each row such that $\lambda_i = \mu_{i+1}$, at one breaking point.

In the following discussion, by a *connected component* of $\overset{\circ}{D}_\mu \setminus D$ we shall mean a subset of $\overset{\circ}{D}_\mu \setminus D$ which, after removing all the $\times$'s and reshifting the rows of $\overset{\circ}{D}_\mu$ to the ones of D_{μ^-}, gives rise to a connected component of $D_{\mu^-} \setminus D$. (Two boxes in $D_{\mu^-} \setminus D$ are connected if they share a vertex or an edge; this defines the connected components of $D_{\mu^-} \setminus D$.)

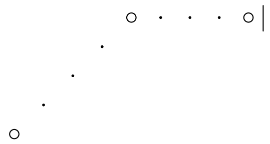
Among the connected components of $\overset{\circ}{D}_\mu \setminus \overset{\circ}{D}_\lambda$ we have those which do not meet the n-th column: they are ordinary horizontal strips [Ma]. Those which meet the n-th component are of the form:

(3) 

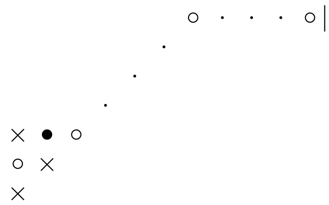
Note the following particular case of (3):

(4) 

(By "|" we visualize the end of a row.) After the maximal deformation an ordinary horizontal strip and configuration (3) which is not of the form (4), becomes respectively:



and



Of course, configuration (4) does not change under the maximal deformation.

Proposition 3.13. *Suppose that $\lambda \subset \mu \subset \rho(n)$ are strict partitions such that $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip (in particular, $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$). Let D be the maximal deformation of $\mathring{D}_\mu \setminus \mathring{D}_\lambda$. Then $\partial^D_\mu(E) = 2^m$, where m is the number of connected components of $\mathring{D}_\mu \setminus D$.*

Proof. Different connected components of $\mathring{D}_\mu \setminus D$ lie in separate rows and separate columns. Let us number these components from top to bottom. Pick a connected component of $\mathring{D}_\mu \setminus D$. The part of ∂^D_μ associated with the boxes in the rows preceding the rows of the component, transform E into $2^{m'} E_{\mathbf{a}}$, where m' is the number of components preceding the given one. If the first row of its appearance has odd (resp. even) number, then $\mathbf{a} = (1, 1, \ldots, 1, *, \ldots, *)$ (resp. $\mathbf{a} = (-1, 1, \ldots, 1, *, \ldots, *)$) and the cardinality of displayed ± 1's is the length of the first row of μ supporting the component, the count including the "$\times$". In turn, the operators of rows supporting the component transform $E_{\mathbf{a}}$ to $2E_{\mathbf{a}'}$ for some $\mathbf{a}'$. The multiplicity 2 comes from the highest leftmost box of the component; the operators of all remaining boxes give the multiplicity 1. If such highest leftmost box lies in the h-th column where $h < n$, then one gets the multiplicity 2 by applying ∂_h to $E_{\mathbf{b}}$ where $\mathbf{b} = (\ldots, -1, 1, \ldots)$, the displayed entries being in $(h+1)$-st and h-th places. If the component is of the form (4), then we get the multiplicity 2 by applying ∂_n to $E_{\mathbf{c}}$ where $\mathbf{c} = (1, 1, \ldots)$. This proves the proposition. □

We summarize the results of this section in the following theorem.

Theorem 3.14. *Let $\lambda \subset \mu \subset \rho(n)$ be strict partitions. Then for $D \subset \mathring{D}_\mu$, one has $\partial_\mu^D(E) \neq 0$ iff $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip (in particular, $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$), and D is obtained by the maximal deformation of $\mathring{D}_\lambda \subset \mathring{D}_\mu$. In this case, $\partial_\mu^D(E) = 2^m$, where m is the number of connected components of $D_{\mu^-} \setminus D_{\lambda^-}$.*

4. A group-theoretic approach to Schubert calculus for CS_n

We first introduce some notation. Recall that $H = SO(2n, \mathbf{C})$ and $B \subset H$ is a Borel subgroup of H. We denote by P the maximal parabolic subgroup of H containing B and corresponding to the subset Σ of simple roots minus the right end root $\varepsilon_{n-1} + \varepsilon_n$, by F – an "isotropic" orthogonal flag manifold H/B, and by G – the orthogonal Grassmannian H/P.

Moreover, the Schubert variety X_w, $w \in W$, is defined as the closure of the Schubert cell $B^- wB/B$ in H/B (B^- is the opposite Borel subgroup to B). We record the following well-known result:

Lemma 4.1. *X_w is a (closed) subvariety of H/B of (complex) codimension $l(w)$.*

Let $X = \{x_1, \ldots, x_n\}$ be a sequence of variables. For brevity, we denote also by the symbol X_w the class of the variety X_w in $H^{2l(w)}(F)$. Let $\alpha \subset \rho(n-1)$ be a strict partition and put $\lambda := \alpha^+$; one has $X_{w_\lambda} \in H^{2|\alpha|}(F)$. Since $w_\lambda \in W^*$, it follows from [BGG] that X_{w_λ} belongs already to $H^{2|\alpha|}(G) \subset H^{2|\alpha|}(F)$. Let us denote this element in $H^{2|\alpha|}(G)$ (as well as the representing it Schubert variety in G), by σ_α.

There exists a surjective ring homomorphism $c : \mathbf{Z}[1/2][X] \to H^*(F)$ (called the *Borel characteristic map*) such that for a homogeneous $f \in \mathbf{Z}[X]$ one has

$$c(f) = \sum_{l(w) = \deg f} \partial_w(f)\ X_w \,. \tag{5}$$

(The original Borel's definition [Bo] of the characteristic map was different; the present description comes from [BGG] and [De].)

Note (cf. e.g. [Bo]) that the ring $H^*(G)$ can be identified algebraically as

$$H^*(G) = \mathbf{Z}[X]^{S_n} / (e_i(X^2), i = 1, \ldots, n-1; x_1 x_2 \cdots x_n) \,,$$

where $X^2 = (x_1^2, \ldots, x_n^2)$.

We have also another identification of $H^*(G)$ stemming from [Du1] and [P1]:

Lemma 4.2. *Let S be the tautological rank n subbundle on G. The Chern classes $c_i(S)$ are all divisible by 2, and one has the identification $\sigma_i = \frac{1}{2} c_i(S^*)$ for $i = 1, \ldots, n-1$. Moreover,*

$$H^*(G) = \mathbf{Z}[\sigma_1, \ldots, \sigma_{n-1}] / (R_i, 1 \leq i \leq n-1) \,,$$

where, with the convention $\sigma_i = 0$ for $k > n-1$, the relations R_i are given by

$$R_i = \sigma_i^2 - 2\sigma_{i-1}\sigma_{i+1} + 2\sigma_{i-2}\sigma_{i+2} - \cdots + 2(-1)^{i-1}\sigma_1\sigma_{2i-1} + (-1)^i\sigma_{2i}\,.$$

It turns out that after restriction to $\mathbf{Z}[X]^{S_n} \otimes \mathbf{Z}[1/2]$, the map c goes onto $H^*(G)$, and we have the following fact. Let, from now on, $e_r = e_r(X)$ denote the r-th elementary symmetric function in $X = \{x_1, \ldots, x_n\}$.

Lemma 4.3. *For every $r = 1, \ldots, n-1$, one has $c(e_r) = 2\sigma_r$.*

Proof. We have

$$\partial_{n-r} \cdots \partial_{n-2}\partial_n(e_r) = 2\,.$$

Any other divided difference operator of degree r applied to e_r gives 0. This implies the assertion. □

For a strict partition $\alpha \subset \rho(n-1)$, we choose a homogeneous $f_\alpha \in \mathbf{Z}[1/2][X]$ such that $c(f_\alpha) = \sigma_\alpha$. Then, for $w \in W^*$ with $l(w) = l(\alpha)$, one has $\partial_w(f_\alpha) \neq 0$ iff $w = w_\lambda$ and $\partial_{w_\lambda}(f_\alpha) = 1$ for $\lambda = \alpha^+$. We want to find the coefficients d_β in the expansion

$$c(f_\alpha \cdot e_r) = \sum d_\beta \sigma_\beta\,. \tag{6}$$

Proposition 4.4. *In the above notation, setting $\mu := \beta^+$, one has*

$$d_\beta = \sum \partial_\mu^D(e_r),$$

where the sum is over all $D \subset \mathring{D}_\mu$ such that $r_D \in R(w_\lambda)$ (here, $\lambda = \alpha^+$).

Proof. We have $d_\beta = \partial_{w_\mu}(f_\alpha \cdot e_r)$, and $\partial_{w_\mu} = \partial_\mu^\emptyset$. The integer $d_\beta = \partial_\mu^\emptyset(f_\alpha \cdot e_r)$ is computed by a consecutive application of the Leibniz-type formula (1): we apply only the ∂_i's (and the identity operators) to f_α, and both the s_i's and ∂_i's to the factor e_r. We get

$$d_\beta = \sum \partial_{r_D}(f_\alpha) \cdot \partial_\mu^D(e_r)\,,$$

the sum over all $D \subset \mathring{D}_\mu$. The summand corresponding to a subset $D \subset \mathring{D}_\mu$ is not zero only if $\#D = \deg f_\alpha$ and $\#(D_\mu \setminus D) = r$. By the choice of f_α, $\partial_{r_D}(f_\alpha) = 0$ if $r_D \notin R(w_\lambda)$, and equals 1 if $r_D \in R(w_\lambda)$, and thus we get the desired equality. □

Remark 4.5. This use of an iterated Leibniz-type formula to compute the multiplicities d_β stems from a series of papers of Ratajski and the second author (cf. [P2]). It was also known to Kostant and Kumar – see [KK].

Combining this proposition with Theorem 3.14, and taking into account Lemma 4.3, we get a group-theoretic proof of the following result (that is refered to as a “Pieri-type formula”):

Theorem 4.6. *Let $\alpha \subset \rho(n-1)$ be a strict partition. Then for any $1 \leq r \leq n-1$,*

$$\sigma_\alpha \cdot \sigma_r = \sum_\beta 2^{m_\beta}\ \sigma_\beta\,,$$

where the sum is over all strict partitions $\beta \subset \rho(n-1)$ such that $D_\beta \setminus D_\alpha$ is a horizontal strip of length r and m_β is the number of connected components of $D_\beta \setminus D_\alpha$ minus 1.

(Cf. also [P1, Theorem 6.17'].)

Example 4.7. Let $n = 8$. We examine the product $\sigma_{5,3} \cdot \sigma_4$:

$$\begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet \end{array} \qquad \text{times} \qquad \begin{array}{l} \circ\ \circ\ \circ\ \circ \end{array}$$

On the LHS we depict the β's; on the RHS we display the unique $D \subset \overset{\circ}{D}_\mu$ ($\mu = \beta^+$) such that $\partial_\mu^D(E) \neq 0$:

$$\begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \circ\ \circ \\ \bullet\ \bullet\ \bullet\ \circ\ \circ \end{array} \qquad \begin{array}{l} \bullet\ \times\ \bullet\ \bullet\ \bullet\ \circ\ \circ\ \circ \\ \times\ \bullet\ \bullet\ \bullet\ \circ\ \bullet \end{array}$$

$$\begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \circ\ \circ \\ \bullet\ \bullet\ \bullet\ \circ \\ \circ \\ \end{array} \qquad \begin{array}{l} \bullet\ \times\ \bullet\ \bullet\ \bullet\ \bullet\ \circ\ \circ \\ \times\ \bullet\ \bullet\ \bullet\ \circ \\ \circ\ \times \\ \times \end{array}$$

$$\begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \circ\ \circ \\ \bullet\ \bullet\ \bullet \\ \circ\ \circ \\ \end{array} \qquad \begin{array}{l} \bullet\ \times\ \bullet\ \bullet\ \bullet\ \bullet\ \circ\ \circ \\ \times\ \bullet\ \bullet\ \bullet \\ \circ\ \times\ \circ \\ \times \end{array}$$

$$\begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \circ \\ \bullet\ \bullet\ \bullet\ \circ\ \circ \\ \circ \\ \end{array} \qquad \begin{array}{l} \bullet\ \times\ \bullet\ \bullet\ \bullet\ \circ\ \circ \\ \times\ \bullet\ \bullet\ \bullet\ \circ\ \bullet \\ \circ\ \times \\ \times \end{array}$$

$$\begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \circ \\ \bullet\ \bullet\ \bullet\ \circ \\ \circ\ \circ \\ \end{array} \qquad \begin{array}{l} \bullet\ \times\ \bullet\ \bullet\ \bullet\ \bullet\ \circ \\ \times\ \bullet\ \bullet\ \bullet\ \circ \\ \circ\ \times\ \circ \\ \times \end{array}$$

$$\begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \circ \\ \circ\ \circ\ \circ \\ \end{array} \qquad \begin{array}{l} \bullet\ \times\ \bullet\ \bullet\ \bullet\ \bullet \\ \times\ \bullet\ \circ\ \circ\ \circ \\ \circ\ \times\ \bullet\ \bullet \\ \times \end{array}$$

Thus we get:

$$\sigma_{5,3} \cdot \sigma_4 = \sigma_{7,5} + 4\sigma_{7,4,1} + 2\sigma_{7,3,2} + 2\sigma_{6,5,1} + 4\sigma_{6,4,2} + \sigma_{5,4,3} \,.$$

A fundamental invariant of a projective variety $X \subset \mathbf{P}^N_{\mathbf{C}}$ is its *degree*, defined by

$$\deg X = \int_X {\omega_X}^n \,,$$

where $n = \dim_{\mathbf{C}} X$ and ω_X is the restriction of the standard Kähler form on $\mathbf{P}^N$ to X. The importance of this invariant is seen from its various interpretations [GH, p.171]:

(i) The number $\deg X$ equals to the number of intersection points of X with a general linear subspace in $\mathbf{P}^N$ of complementary dimension.

(ii) The number $n! \deg X$ agrees with the volume of X .

It is well known that σ_1 is the generator of $\mathrm{Pic}(G)$ and also σ_1 is the Kähler class of G. So by (i),

$$\deg(\sigma_\alpha) = \sigma_\alpha \cdot {\sigma_1}^{n(n-1)/2-|\alpha|} \,.$$

We invoke Schur P-functions $P_\lambda = P_\lambda(X)$ of [S] whose definition reads:

1) For a nonnegative integer i, $P_i := \sum s_\lambda$, where the sum is over all hook partitions λ of i, and s_λ denotes the corresponding Schur S-function (cf., e.g., [Ma]).

2) For integers $i > j > 0$,

$$P_{(i,j)} := P_i P_j + 2 \sum_{1 \le q \le i-1} (-1)^q P_{j+q} P_{i-q} + (-1)^{i+j} P_{i+j} \,.$$

3) For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ written with an even k (by putting $\lambda_k = 0$ if necessary),

$$P_{(\lambda_1,\ldots,\lambda_k)} := \mathrm{Pf}[P_{(\lambda_p,\lambda_q)}]_{1 \le p < q \le k} \,,$$

where Pf denotes the Pfaffian. See [S], [Ma], [HH], and [P1] for more on Schur P-functions. Sometimes it is more handy to work with Schur Q-functions defined by $Q_\lambda = Q_\lambda(X) = 2^{l(\lambda)} P_\lambda$ for a strict partition λ.

Comparing the Pieri-type formula for P-functions [Ma,III.8.15] (extracted in [P1] from [Mo]) with Theorem 4.6, we get that $\deg \sigma_\alpha$ is the coefficient of $P_{\rho(n-1)}$ in

$$P_\alpha \cdot P_1^{n(n-1)/2 - |\alpha|} \,,$$

or the coefficient of $P_{\bar\alpha}$ in

$$P_1^{n(n-1)/2 - |\alpha|} = P_1^{|\bar\alpha|} \,.$$

Here $\bar\alpha$ is the partition whose part complement the parts of α in $\{1, \ldots, n-1\}$.

We define for a partition $\gamma = (\gamma_1, \gamma_2, \ldots)$,

$$g^\gamma = \frac{|\gamma|!}{\gamma!} \prod_{i<j} \frac{\gamma_i - \gamma_j}{\gamma_i + \gamma_j} \,, \tag{7}$$

where $\gamma! = \gamma_1! \gamma_2! \cdots$.

Proposition 4.8. *One has $deg(\sigma_\alpha) = g^\gamma$ for $\gamma = \bar\alpha$.*

Remark 4.9. Certain special cases of this formula were obtained by Hiller [Hi]. Some related computations were performed by Tamvakis [Ta2] in the context of heights of homogeneous spaces in arithmetic intersection theory.

The proposition follows from the following lemma due essentially to Schur [S].

Lemma 4.10. *One has*

$$P_1^k = \sum g^\gamma P_\gamma \,,$$

the sum over strict partitions γ of k.

Proof. We give here a proof using a specialization result from [DP] and the following formula (8). Let $p_i(X) = x_1^i + \cdots + x_n^i$ be the power sum. For a partition $\mu = (\mu_1, \mu_2, \ldots)$ we set $p_\mu(X) = \prod_i p_{\mu_i}(X)$ and $z_\mu = \prod_{i \ge 1} i^{m_i} m_i!$, where $m_i = \#\{j : \mu_j = i\}$. Moreover, by an *odd* partition we understand the one whose all parts are odd. Let $Y = \{y_1, \ldots, y_n\}$ be another set of indeterminates. Then we have [Ma, III.8.13], [HH, Cor.7.15]:

$$\sum_{\lambda \ strict} P_\lambda(X) Q_\lambda(Y) = \sum_{\mu \ odd} 2^{l(\mu)} z_\mu^{-1} p_\mu(X) p_\mu(Y) \,. \tag{8}$$

We use the following specialization. We set $p_1(Y) = 1/2$ and $p_i(Y) = 0$ for $i \geq 2$. Using

$$Q_i(Y) = \sum_{\nu\ odd} z_\nu^{-1} 2^{l(\nu)} p_\nu(Y)$$

[Ma, p.260], [HH, (7.9)], we see that under this specialization, we have $Q_i(Y) = 1/i!$. The following equality was proved in [DP]: via this specialization, for a strict partition λ,

$$Q_\lambda(Y) = g^\lambda/|\lambda|!\ . \tag{9}$$

Therefore the specialization under consideration transforms equation (8) into the assertion of the lemma. □

Remark 4.11. As a matter of fact the key in the original Schur's calculation [S] (see also [Ma, p.267] and in more detail [HH]) is the proof of the following equality: for a strict partition $\gamma = (\gamma_1 > \cdots > \gamma_l > 0)$,

$$g^\gamma = \sum_{i=1}^{l} g^{\gamma^{(i)}}\ , \tag{10}$$

where $\gamma^{(i)}$ is the strict partition obtained from γ by subtracting 1 from the i-th part γ_i of γ. The original argument rests on the expansion into partial fractions of the function

$$(2t-1)\prod_i \frac{(t+\gamma_i)(t-\gamma_i-1)}{(t+\gamma_i-1)(t-\gamma_i)}\ .$$

Here is another way of obtaining (10) for those who prefer the Lagrange interpolation to the expansion into partial fractions. Suppose that $\gamma_1, \ldots, \gamma_l$ are l indeterminates. We start with the equation:

$$(\gamma_1 + \cdots + \gamma_l)\prod_{i<j}(\gamma_i - \gamma_j) = \sum_p (-1)^{p-1}\gamma_p \prod_{\{i,j\neq p;\ i<j\}}(\gamma_i - \gamma_j)\prod_{i\neq p}(\gamma_p + \gamma_i)\ , \tag{11}$$

which, as Lascoux points out, is exactly the content of the Lagrange interpolation for $\gamma_1 + \cdots + \gamma_l$ (cf.[L2]). (Equation (11) is easy to prove, e.g. by showing that its RHS is skew-symmetric.)

Letting $\overline{Q}_\lambda$ be a function in the γ's given by the expression for $Q_\lambda(Y)$ in (9), we rewrite (11) as

$$(\gamma_1 + \cdots + \gamma_l)\overline{Q}_\gamma = \sum_{i=1}^{l}(-1)^{i-1}\overline{Q}_{\gamma_i - 1}\overline{Q}_{\gamma_1,\ldots,\gamma_{i-1},\gamma_{i+1},\ldots,\gamma_l}\ . \tag{12}$$

We now record the following identity for general Q-functions that follows rather easily from their definition by induction:

$$\sum_{i=1}^{l}(-1)^{i-1}Q_{\gamma_i-1}Q_{\gamma_1,\ldots,\gamma_{i-1},\gamma_{i+1},\ldots,\gamma_l} = \sum_{j=1}^{l} Q_{\gamma_1,\ldots,\gamma_j-1,\ldots,\gamma_l}\ . \tag{13}$$

Comparing (12) and (13), we get

$$(\gamma_1 + \cdots + \gamma_l)\overline{Q}_\gamma = \sum_{j=1}^{l}\overline{Q}_{\gamma_1,\ldots,\gamma_j-1,\ldots,\gamma_l}\ ,$$

which gives (10).

Remark 4.12. Here is still another derivation of the lemma for the reader knowing Hall-Littlewood functions: combine [Ma, Ex.III.8.1 p.259; formula III.7.1 p.246; and Ex.III.8.12 p.266].

Up to now, we have considered Schubert varieties as purely group-theoretic objects. We end this section by recalling their interpretation in terms of Schubert-type conditions. This description is a recollection from [LSe] and [P1], and we will need it in the next section.

Let U be a $2n$-dimensional vector space endowed with a nondegenerate orthogonal form $\xi : U \times U \to \mathbf{C}$. Consider

$$Z = \{L \subset U : L \text{ is maximal isotropic subspace in } U\}.$$

This subvariety is canonically embedded in the Grassmannian $G_n(U)$. This last variety has Schubert (sub)varieties which are defined w.r.t. to flag

$$U_1 \subset U_2 \subset \cdots \subset U_{2n} = U$$

(where $\dim(U_i) = i$), in the following way: Given a sequence $1 \le i_1 < \cdots < i_n \le 2n$, we set

$$\Omega(i_1, \ldots, i_n) = \{L \in G_n(U) : \dim(L \cap U_{i_p}) \ge p \ \forall p = 1, \ldots, n\}.$$

One has

$$\dim \Omega(i_1, \ldots, i_n) = i_1 + \cdots + i_n - n(n+1)/2\,.$$

It is known that Z has two connected components which are isomorphic to $G = H/P$. Let $v_1, \ldots, v_n, w_1, \ldots, w_n$ be a basis of U such that $\xi(v_i, v_j) = \xi(w_i, w_j) = 0, \xi(v_i, w_j) = \xi(w_j, v_i) = \delta_{i,j}$. Let V_i be the vector space spanned by the first i vectors of the above basis. Then the Schubert varieties in $G_n(U)$ (determined by the flag $V_1 \subset \cdots \subset V_{2n} = V$) which *give rise* to the Schubert varieties in G (in the sense of [LSe] and [P1]) are indexed by the sequences $(i_1, \ldots, i_n)$ where $i_p \ne 2n + 1 - i_q$ for $p, q = 1, \ldots, n$, and if k denotes the largest number such that $i_k \le n$, then $n - k$ is even. Let us denote by $\Omega[i_1, \ldots, i_k]$ the Schubert variety in G determined (via restriction to G) by this Schubert variety in $G_n(U)$, that is:

$$\Omega[i_1, \ldots, i_k] := \{L \in G : \dim(L \cap V_{i_p}) \ge p \ \forall \ p = 1, \ldots, k\}.$$

(Instead refering to the flag $V_1 \subset \cdots \subset V_n$, we will also say that this Schubert variety is defined w.r.t. the ordered basis $\{v_1, \ldots, v_n\}$.) One has

$$\dim \Omega[i_1, \ldots, i_k] = i_1 + \cdots + i_k + n(n-k) - n(n+1)/2\,.$$

The Schubert classes in $H^*(G)$ determined by these Schubert varieties are related in following way to the Schubert classes σ_α considered earlier in this section. For a strict partition $\alpha = (\alpha_1, \ldots, \alpha_k) \subset \rho(n-1)$, one has $\sigma_\alpha = \Omega[n - \alpha_1, \ldots, n - \alpha_k]$ if $n - k$ is even, and $\sigma_\alpha = \Omega[n - \alpha_1, \ldots, n - \alpha_k, n]$ if $n - k$ is odd.

The corresponding Schubert variety σ_α in G can be defined in the following way w.r.t. the above flag $V_1 \subset \cdots \subset V_n$; it is:

$$\{L \in G : \dim(L \cap V_{n-\alpha_p}) \ge i \ \forall \ p = 1, \ldots, k \ \& \ \mathrm{codim}_{V_n}(L \cap V_n) \text{ is even}\}.$$

5. Schubert cycles of complex structures on $\mathbf{R}^{2n}$

We will adopt the following convention. Let $\mathbf{R}^{2n}$ be the real Euclidean $2n$-space with the standard orthonormal basis $\{e_1, \dots, e_{2n}\}$.

The $2n$-dimensional complex Euclidean space $\mathbf{C}^{2n}$ will be considered as the complexification of $\mathbf{R}^{2n}$; $\mathbf{C}^{2n} = \mathbf{R}^{2n} \otimes_{\mathbf{R}} \mathbf{C}$. Note the following simple facts:

a) The set $\{e_1 \otimes 1, \dots, e_{2n} \otimes 1\}$ is an orthonormal basis for $\mathbf{C}^{2n}$.

b) If $L, K \subset \mathbf{R}^{2n}$ are two linear subspaces satisfying $\dim_{\mathbf{R}}(L \cap K) \geq i$, then their complexifications $L^{\mathbf{C}}, K^{\mathbf{C}} \subset \mathbf{C}^{2n}$ satisfy $\dim_{\mathbf{C}}(L^{\mathbf{C}} \cap K^{\mathbf{C}}) \geq i$.

c) Corresponding to an orthogonal decomposition $L = L_1 \oplus L_2$ of a subspace $L \subset \mathbf{R}^{2n}$, one has the orthogonal decomposition $L^{\mathbf{C}} = L_1^{\mathbf{C}} \oplus L_2^{\mathbf{C}}$ of $L^{\mathbf{C}} \subset \mathbf{C}^{2n}$.

d) An $\mathbf{R}$-linear endomorphism of a subspace $L \subset \mathbf{R}^{2n}$ induces a $\mathbf{C}$-linear endomorphism of the subspace $L^{\mathbf{C}} \subset \mathbf{C}^{2n}$.

Let V be an oriented even dimensional real Euclidean space and $Iso(V)$, the group of orientation preserving isometries of V. Consider

$$CS(V) = \{A \in Iso(V) \ : \ A^2 = -Id_V\}\,.$$

It is the space of complex structures on V.

If $V = \mathbf{R}^{2n}$, one has the identification $Iso(R^{2n}) = SO(2n, \mathbf{R})$, the special orthogonal group of order $2n$, and

$$CS(\mathbf{R}^{2n}) = \{A \in SO(2n, \mathbf{R}) \mid A^2 = -I_{2n}\}\,.$$

Note that if $A \in CS(\mathbf{R}^n)$ then A is a skew-symmetric matrix. Let us abbreviate $CS(\mathbf{R}^{2n})$ by CS_n as is common. The space CS_n has two connected components which are distinguished by the Pfaffian function

$$\mathrm{Pf} : CS_n \to \{\pm 1\}\,.$$

We write $CS_n = CS_n^+ \sqcup CS_n^-$ with $\mathrm{Pf}(CS_n^{\pm}) = \pm 1$.(The symbol $\sqcup$ denotes the disjoint union.) Both manifolds $CS_n^{\pm}$ are isometric to $G = SO(2n, \mathbf{C})/U(n)$.

Define the complex structure J_n in the initial basis $\{e_1, \dots, e_{2n}\}$ by

$$J_n = J_1 \oplus J_1 \oplus \cdots \oplus J_1 \quad n \text{ times}\,,$$

where

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example 5.1. Suppose that $J_0 = J_n$. For a sequence $1 \leq i_1 < \cdots < i_k \leq n$, we put

$$J(i_1, \dots, i_k) = \epsilon_1 J_1 \oplus \epsilon_2 J_1 \oplus \cdots \oplus \epsilon_n J_1$$

where $\epsilon_h = -1$ if $h = i_p$ for $1 \leq p \leq k$. Then $J(i_1, \dots, i_k) \in CS_n^+$ iff k is even.

Our goal in this section is to interpret Schubert varieties in $SO(2n, \mathbf{C})/U(n)$ in terms of complex structures. We will give an interpretation, in terms of Schubert varieties, of the following Mahowald-Vassiljev-type formula

$$H_p(CS_n) = \oplus_{k=0}^{n} H_{p-k(k-1)}(G_k(\mathbf{C}^n))\,,$$

where $G_k(\mathbf{C}^n)$ is the Grassmannian of all complex k-planes through zero in $\mathbf{C}^n$ (see [DV] and also [V]).

We will now define Schubert varieties of complex structures. Let us fix a complex structure $J_0 \in CS_n$. By convention, we will denote by CS_n^+ the connected component of CS_n that contains J_0. We will work here with the component CS_n^+, leaving to the reader details concerning the other component CS_n^-. The results about the component CS_n^- will be summarized in Proposition 5.5.

Let

$$\mathbf{R}^{2n} = L_1 \oplus L_2 \oplus \cdots \oplus L_n ,$$

where $\dim_{\mathbf{R}} L_i = 2$, be an invariant subspace decomposition of the orthogonal operator $J_0 : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$. This yields a flag in $\mathbf{R}^{2n}$

$$F_1 \subset F_2 \subset \cdots \subset F_n = \mathbf{R}^{2n} , \tag{14}$$

where $F_i = L_1 \oplus L_2 \oplus \cdots \oplus L_i$. Furnishing $\mathbf{R}^{2n}$ with the complex structure J_0, we get an n-dimensional complex space $\mathbf{C}^n = (\mathbf{R}^{2n}, J_0)$. Since each F_i is an invariant subspace w.r.t. J_0, the flag (14) gives rise to a complex flag

$$W_1 \subset W_2 \subset \cdots \subset W_n = \mathbf{C}^n , \tag{15}$$

where $\dim_{\mathbf{C}} W_i = i$.

Consider the Grassmannian $G_l(\mathbf{C}^n)$ of all complex l-planes through zero in $\mathbf{C}^n$. For a sequence $1 \le j_1 < \cdots < j_l \le n$, one defines a Schubert variety

$$\Omega(j_1, \dots, j_l) = \{L \in G_l(\mathbf{C}^n) : \dim(L \cap W_{i_p}) \ge p \ \ \forall p = 1, \dots, l\} .$$

One has

$$\dim_{\mathbf{C}} \Omega(j_1, \dots, j_l) = j_1 + \cdots + j_l - l(l+1)/2 .$$

Following Dynnikov-Veselov [DV], for even l, we set

$$CS_n^+(j_1, \dots, j_l) = \{A \in CS_n^+ : \exists L \in \Omega(j_1, \dots, j_l) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|M = J_0|M\} ,$$

where $M = L_{\mathbf{R}}^{\perp}$ is the orthogonal complement of the real reduction $L_{\mathbf{R}}$ of L. This is a closed subvariety in CS_n^+. One has

$$\dim_{\mathbf{C}} CS_n^+(j_1, \dots, j_l) = j_1 + \cdots + j_l - l .$$

One verifies easily that the class of the variety $CS_n^+(j_1, \dots, j_l)$ is independent of the choice of J_0. Indeed, a path in CS_n^+ joining J_0 to another $J \in CS_n^+$ yields a one-parameter family of varieties from $CS_n^+(j_1, \dots, j_l)$ attached to J_0, to that attached to J.

We now want to identify the variety $CS_n^+(j_1, \dots, j_l)$ with a suitable Schubert variety $\Omega[i_1, \dots, i_k]$ in G. To this end, we first describe an imbedding $\iota : CS_n \to G_n(\mathbf{C}^{2n})$. Every $A \in CS_n$ has two eigenvalues $\pm i$ (here i is the pure imaginary complex number) with equal multiplicities n. Thus, as an endomorphism of $\mathbf{C}^{2n}$, (cf. d)), A has the eigensubspace decomposition

$$\mathbf{C}^{2n} = L(A,+) \oplus L(A,-) \ \text{ where } \ \dim L(A,+) = \dim L(A,-) = n ,$$

with

$$A(v) = iv \text{ for all } v \in L(A,+) \ \text{ and } \ A(v) = -iv \ \text{ for all } v \in L(A,-) .$$

The embedding $\iota : CS_n \to G_n(\mathbf{C}^{2n})$ defined by $A \to L(A,+)$ has as its image the Grassmannian of all isotropic subspaces of $\mathbf{C}^{2n}$ w.r.t. the orthogonal form induced by the scalar product.

Let us fix the complex structure J_0 to be J_n. By using a simple linear algebra, one shows that w.r.t. the flag (15), associated with this complex structure, the following identification takes place.

Proposition 5.2. *Let l be even. Then the embedding ι restricts to an isomorphism of varieties:*

$$CS_n^+(j_1,\dots,j_l) \quad \text{and} \quad \Omega[n+1-t_{n-l},\dots,n+1-t_1]\,,$$

where $t_1<\cdots<t_{n-l}$ is the complement of $j_1<\cdots<j_l$ in $\{1,\dots,n\}$.

Remark 5.3. The isomorphism in this proposition gives the cellular decomposition of CS_n^+ announced by Dynnikov and Veselov in [DV].

As a consequence of Propositions 4.8 and 5.2 we get:

Corollary 5.4. *The degree of $CS_n^+(j_1,\dots,j_l)$ is equal to the number g^γ, where $\gamma=(j_l-1,\dots,j_1-1)$.*

Since, for even l, $\Omega[n+1-t_{n-l},\dots,n+1-t_1]$ is equal to

$$\{A:\exists K\in\Omega(j_1,\dots,j_l) \text{ s.t. } A(K_{\mathbf{R}})=K_{\mathbf{R}} \ \&\ A|K_{\mathbf{R}}^\perp=J_0|K_{\mathbf{R}}^\perp\}\,,$$

then, rewriting it for even $n-k$, $\Omega[i_1,\dots,i_k]$ is equal to

$$\{A:\exists K\in\Omega(n+1-r_{n-k},\dots,n+1-r_1) \text{ s.t. } A(K_{\mathbf{R}})=K_{\mathbf{R}} \ \&\ A|K_{\mathbf{R}}^\perp=J_0|K_{\mathbf{R}}^\perp\}\,,$$

where $r_1<\cdots<r_{n-k}$ is the complement of $i_1<\cdots<i_k$ in $\{1,\dots,n\}$. By taking $L=K^\perp$, we can present this $\Omega[i_1,\dots,i_k]$ as

$$\{A:\exists L\in\Omega(i_1,\dots,i_k) \text{ s.t. } A(L_{\mathbf{R}})=L_{\mathbf{R}} \ \&\ A|L_{\mathbf{R}}=J_0|L_{\mathbf{R}}\}\,. \tag{16}$$

This last identification (16) seems to be the most handy for applications.

In the following proposition, keeping the notation from this section, we collect properties of Schubert varieties in CS_n^-. For an odd l, we define $CS_n^-(j_1,\dots,j_l)$ as

$$\{A\in CS_n^-:\exists L\in\Omega(j_1,\dots,j_l) \text{ s.t. } A(L_{\mathbf{R}})=L_{\mathbf{R}} \ \&\ A|L_{\mathbf{R}}^\perp=J_0|L_{\mathbf{R}}^\perp\}\,,$$

where $L_{\mathbf{R}}^\perp$ is the orthogonal complement of the real reduction $L_{\mathbf{R}}$ of L.

Proposition 5.5. *(i) $CS_n^-(j_1,\dots,j_l)$ is a closed subvariety in CS_n^- of dimension $j_1+\cdots+j_l-l$.*
(ii) $CS_n^-(j_1,\dots,j_l)$ can be identified with the restriction to CS_n^-, properly embedded in $G_n(\mathbf{C}^{2n})$, of the Schubert variety

$$\Omega(n+1-t_{n-l},\dots,n+1-t_1,n+j_1,\dots,n+j_l)$$

in this last Grassmannian.
(iii) $CS_n^-(j_1,\dots,j_l)$ is also identified with

$$\{A\in CS_n^-:\exists L\in\Omega(i_1,\dots,i_k) \text{ s.t. } A(L_{\mathbf{R}})=L_{\mathbf{R}} \ \&\ A|L_{\mathbf{R}}=J_0|L_{\mathbf{R}}\}\,.$$

(iv) The degree of $CS_n^-(j_1,\dots,j_l)$ is equal to g^γ, where $\gamma=(j_l-1,\dots,j_1-1)$.

Example 5.6. We describe the Schubert varieties in CS_n^+ which are divisors. We have different description according to the parity of n. If n is odd, then the divisor $\sigma_1=\Omega[n-1]$ is

$$\{A:\exists L\subset W_{n-1} \ s.t.\ \dim_{\mathbf{C}}L=1,\ A(L_{\mathbf{R}})=L_{\mathbf{R}} \ \&\ A|L_{\mathbf{R}}=J_0|L_{\mathbf{R}}\}\,.$$

If n is even, then the divisor $\sigma_1=\Omega[n-1,n]$ is

$$\{A:\exists L\subset W_n \ s.t.\ \dim_{\mathbf{C}}L=2,\ A(L_{\mathbf{R}})=L_{\mathbf{R}} \ \&\ A|L_{\mathbf{R}}=J_0|L_{\mathbf{R}}\}\,.$$

We end this paper with some applications. The identification made in (16) allows us to solve enumerative problems about the number of general complex structures satisfying some natural conditions of "partial overlapping" with a certain number of complex structures in general position in $\mathbf{R}^{2n}$. To this end, we need the following definition.

Definition 5.7. Let A and B be two orthogonal operators on $\mathbf{R}^{2n}$. A linear subspace $L \subset \mathbf{R}^{2n}$ is said to be a *common k-space* of A and B iff

$$A(L) = B(L) = L \ , \ A|L = B|L \ \& \ \dim_{\mathbf{R}} L = k \, .$$

We will work in CS_n^+. Let n and $2 \le k \le n$ be even integers. Let $1 \le i_1 < \cdots < i_k \le n$ be a sequence of integers. Set $d = \dim \Omega[i_1, \ldots, i_k]$. Suppose that a list $\{B_i\}$, $0 \le i \le d$, of general complex structures on $\mathbf{R}^{2n}$ is given. Then the number of complex structures $A \in CS_n^+$ s.t. A has a common $2k$-space from $\Omega(i_1, \ldots, i_k)$ with B_0, and A has a common 4-space with any other B_i from the list, is equal to $\deg \Omega[i_1, \ldots, i_k]$.

As a particular case, we have:

Proposition 5.8. *Let n and $2 \le k \le n$ be even integers. Suppose that a list $\{B_i\}$ of general complex structures on $\mathbf{R}^{2n}$ is given, where*

$$0 \le i \le 1 + (n-k)(n-k-1)/2 \, .$$

Then the number of complex structures $A \in CS_n^+$ having a common fixed $2k$-space $(W_k)_{\mathbf{R}}$ with B_0 and a common 4-space with any other B_i is given by

$$g^{(n-k-1, n-k-2, \ldots, 2, 1)} = [(n-k)(n-k-1)/2]! \prod_{i=1}^{n-k-1} \frac{(i-1)!}{(2i-1)!} \, .$$

Indeed, this is a restatement of the formula about the degree of

$$\Omega[1, 2, \ldots, k-1, k] = CS_n^+(1, 2, \ldots, n-k) \, ,$$

which is simplified in this case of a triangular partition, cf. [DP].

Example 5.9. For $n = 8$ and $k = 4$, a list of 7 complex structures $\{B_0, B_1, B_2, B_3, B_4, B_5, B_6\}$ is given. There exist exactly 2 complex structures $A \in CS_n^+$ s.t. A has a common fixed 8-space with B_0 and at least a common 4-space with every B_i, where $1 \le i \le 6$. If $n = 10$ and $k = 4$, a list of 16 structures $\{B_0, B_1, \ldots, B_{15}\}$ is given. There exist exactly 286 complex structures $A \in CS_n^+$ s.t. A has a common fixed 8-space with B_0 and at least a common 4-space with every B_i, where $1 \le i \le 15$.

Let now n and $2 < k \le n$ be odd integers. Let $1 \le i_1 < \cdots < i_k \le n$ be a sequence of integers. Put $d = \dim \Omega[i_1, \ldots, i_k]$. Suppose that a list $\{B_i\}$, $0 \le i \le d$, of general complex structures on $\mathbf{R}^{2n}$ is given. Then the number of complex structures $A \in CS_n^+$ s.t. A has a common $2k$-space from $\Omega(i_1, \ldots, i_k)$ with B_0, and A has a common 2-space from $(W_{n-1})_{\mathbf{R}}$ with any other B_i from the list, is equal to $\deg \Omega[i_1, \ldots, i_k]$.

We leave it to the reader to deduce from it a result analogous to the one in the last proposition.

We will give now another example of enumerating complex structures satisfying some constraints and state some conjecture.

For a complex structure $B_0 \in CS_n$, we have an n-dimensional complex space $\mathbf{C}^n = (\mathbf{R}^{2n}, B_0)$. Note that for all $L \in G_k(\mathbf{C}^n)$, both $L_{\mathbf{R}}$ and $L_{\mathbf{R}}^{\perp}$ are invariant subspaces of B_0. We have then an embedding $\alpha : G_k(\mathbf{C}^n) \to CS_n$ defined by

$$\alpha(L) = (B_0|L_{\mathbf{R}}) \oplus (-B_0|L_{\mathbf{R}}^{\perp}).$$

Without loss of generality, we can assume that the image of α lies in CS_n^+.

Let R be the canonical complex k-bundle over $G_k(\mathbf{C}^n)$ and $R^{\perp}$ its orthogonal complement in the trivial complex n-bundle. Let S be the canonical complex n-bundle over CS_n^+. From the definition of α we have

$$\alpha^* S = R \oplus \overline{R^{\perp}},$$

where $\overline{R^{\perp}}$ denotes the complex conjugation of $R^{\perp}$. We infer that the pullback of the total Chern class of S, $\alpha^*(1 + c_1(S) + \cdots + c_n(S))$, is equal to

$$(1 + c_1(R) + \cdots + c_k(R))(1 - c_1(R) + c_2(R) - \cdots + (-1)^k c_k(R))^{-1}.$$

From this we get that the induced homomorphism

$$\alpha^* : H^*(CS_n) \to H^*(G_k(\mathbf{C}^n))$$

satisfies $\alpha^*(\frac{1}{2}c_1(S)) = -\alpha^*(\sigma_1) = c_1(R)$. That is, the embedding α preserves the classes of hyperplane sections, or, the Kähler classes of both varieties.

As a consequence, we get results summarized in the following proposition.

Proposition 5.10. *(i) Let n be an even integer. Suppose that $2 \le k \le n$ is another integer. Let $\{B_i\}$, $0 \le i \le k(n-k)$, be a list of general complex structures on $\mathbf{R}^{2n}$. Then the number of complex structures $A \in CS_n^+$ s.t. A and B_0 have a common $2k$-space, A and $(-B_0)$ have a common $2(n-k)$-space, and A and each B_i, $i \ge 1$ have at least a common 4-space, is equal to the degree of $G_k(\mathbf{C}^n)$.*
(ii) Let now n be an odd integer. Suppose that $2 \le k \le n$ is another integer. Let $\{B_i\}$, $0 \le i \le k(n-k)$, be a list of general complex structures on $\mathbf{R}^{2n}$. Then the number of complex structures $A \in CS_n^+$ s.t. A and B_0 have a common $2k$-space, A and $(-B_0)$ have a common $2(n-k)$-space, and A and each B_i, $i \ge 1$ have at least a common 2-space in $(W_{n-1})_{\mathbf{R}}$, is equal to the degree of $G_k(\mathbf{C}^n)$.

(Recall that

$$\deg G_k(\mathbf{C}^n) = \frac{1!2!\cdots(k-1)![k(n-k)]!}{(n-k)!(n-k+1)!\cdots(n-1)!},$$

a result which goes back to Schubert (1886).)

It is well known that the Grassmannian $G_k(\mathbf{C}^n)$ is an approximation space for the classifying space $BU(k)$ of all complex k-bundles. On the other hand, the space CS_n serves as the classifying space for all complex n-bundles with a trivial real reduction [Du1]. Thus a homotopy classification of continuous maps $G_k(\mathbf{C}^n) \to CS_n$ may suggest possible interesting operators between these two vector bundle theories.

Let $\beta : G_k(\mathbf{C}^n) \to CS_n$ be a continuous map. Since $\sigma_1 = -\frac{1}{2}c_1(S) \in H^*(CS_n^{\pm}) = \mathbf{Z}$ and $c_1(R) \in H^*(G_k(\mathbf{C}^n)) = \mathbf{Z}$ are the only generators in dimension 2, then the induced map $\beta^* : H^*(CS_n^{\pm}) \to H^*(G_k(\mathbf{C}^n))$ satisfies

$$\beta^*(\frac{1}{2}c_1(S)) = -\beta^*(\sigma_1) = m \cdot c_1(R)$$

for some $m \in \mathbf{Z}$.

We finish this paper by stating the following conjecture.

Conjecture 5.11. *If $m \neq 0$, then the map $\beta^* : H^*(CS_n^{\pm}) \to H^*(G_k(\mathbf{C}^n))$ is given by*

$$\beta^*(x) = m^p \alpha^*(x)\,,$$

for $x \in H^{2p}(CS_n^{\pm})$.

We refer the reader to [Du2] and [Ho] for some background related to this conjecture.

References

[BGG] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, *Schubert cells and cohomology of the spaces G/P,* Russian Math. Surveys **28:3** (1973), 1–26.

[Bo] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts,* Annals of Math. **57** (1953), 115–207.

[Bu] N. Bourbaki, *Groupes et Algèbrés de Lie,* Chapters 4,5, and 6, Herrmann, Paris, 1968.

[BKT] A.S. Buch, A. Kresch, H. Tamvakis, *Gromov-Witten invariants on Grassmannians,* preprint (April 2002).

[DP] C. De Concini, P. Pragacz, *On the class of Brill-Noether loci for Prym varieties,* Math. Ann. **302** (1995), 687–697.

[De] M. Demazure, *Désingularisation des variétés des Schubert géneralisées,* Ann. E.N.S. **7** (1974), 53–88.

[Du1] H. Duan, *The secondary Chern characteristic classes,* Proc. A.M.S. **128(8)** (2000), 2465-2471.

[Du2] H. Duan, *Self-maps of the Grassmannian of complex structures,* Compositio Math. **132** (2002), 159–175.

[DV] I.A. Dynnikov and A.P. Veselov, *Integrable gradient flows and Morse Theory,* Algebra i Analiz **8** (1996), 78-103 (Russian); English translation: St. Petersburgh Math. J., **8** (1997), 429–446.

[FH] W. Fulton, J. Harris, *Representation theory, a first course,* Springer-Verlag, 1991.

[FP] W. Fulton, P. Pragacz, *Schubert varieties and degeneracy loci,* Springer LNM **1689**, 1998.

[GH] P. Griffith, J. Harris, *Principles of algebraic geometry,* Wiley, New York, 1978.

[HP] W.V.D. Hodge, D. Pedoe, *Methods of Algebraic Geometry,* Cambridge University Press, 1954.

[Hi] H. Hiller, *Combinatorics and intersection of Schubert varieties,* Comment. Math. Helvetici **57** (1982), 41–59.

[HB] H. Hiller, B. Boe, *Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n,* Adv. in Math. **62** (1986), 49–67.

[Ho] M. Hoffman, *Endomorphisms of the cohomology of complex Grassmannians,* Trans. A.M.S. **281** (1984), 745–760.

[HH] P.N. Hoffman, J.F. Humphreys, *Projective representations of the symmetric groups,* Oxford University Press, 1992.

[KK] B. Kostant, S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G^*,* Adv. in Math. **62** (1986), 187–237.

[KT] A. Kresch, H. Tamvakis, *Quantum cohomology of orthogonal Grassmannians,* Preprint (2001).

[LSe] V. Lakshmibai and C.S. Seshadri, *Geometry of G/P II,* Proc. Indian Acad. Sci. A **87** (1978), 1–54.

[L1] A. Lascoux, *Notes on interpolation,* Nankai University, Tjanjin (1996).

[L2] A. Lascoux, *Notes for the CBMS Conference on Algebraic Combinatorics,* N.C.S.U., June 2001.

[LP] A. Lascoux, P. Pragacz, *Orthogonal divided differences and Schubert polynomials, $\widetilde{P}$-functions, and vertex operators,* Michigan Math. J., Fulton's volume **48** (2000), 417–441.

[LSc] A. Lascoux, M.-P. Schützenberger, *Décompositions dans l'algèbre des differences divisées,* Discrete Math. **99** (1992), 165–179.

[Ma] I.G. Macdonald, *Symmetric functions and Hall polynomials,* (Second edition), Clarendon Press, Oxford, 1995.

[Mi] J. Milnor, *Morse Theory,* Annals of Mathematics Studies **51**, Princeton, 1963.

[Mo] A. Morris, *A note on the multiplication of Hall functions,* J. London Math. Soc. **39** (1964), 481–488.

[P1] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials,* Séminaire d'Algèbre Dubreil-Malliavin 1988-89, (M.-P. Malliavin, ed), Lecture Notes in Math., Vol. **1478**, Springer-Verlag, Berlin and New York, 1991, 130–191.

[P2] P. Pragacz, *Symmetric polynomials and divided differences in formulas of intersection theory,* in "Parameter Spaces", Banach Center Publications **36** (1996), 125–177.

[PR1] P. Pragacz, J. Ratajski, *Formulas of Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach,* Compositio Math. **107** (1997), 11–87.

[PR2] P. Pragacz, J. Ratajski, *A Pieri-type theorem for even orthogonal Grassmannians,* MPIM-Preprint 1996-83.

[S] I. Schur, *Uber die Darstellung der symmetrischen und alternierenden Gruppe durch gebrochene lineare Substitutionen,* J. Reine Angew. Math. **139** (1911), 155-250.

[T1] H. Tamvakis, *Height formulas for homogeneous spaces,* Michigan Math. J., Fulton's volume **48** (2000), 593–610.

[T2] H. Tamvakis, *Quantum cohomology of Lagrangian and orthogonal Grassmannians,* "Arbeitstagung 2001", Bonn, MPIM-Preprint 2001-50.

[V] V.A. Vassiljev, *A geometric realization of homology of classical groups* Algebra & Analysis **3(4)** (1991), 113-120 (Russian); English translation: St. Petersburg Math. J. **3** (1992), 809–815.

Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, P.R. of China.

E-mail address: `dhb@math.ac.cn`

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, PL 00-956 Warszawa 10, P.O. Box 21, Poland

E-mail address: `pragacz@impan.gov.pl`