# FORMULAS FOR LAGRANGIAN AND ORTHOGONAL DEGENERACY LOCI; $\widetilde{Q}$-Polynomial Approach[1]

**Piotr Pragacz**[2]
Max-Planck Institut für Mathematik, Gottfried-Claren Strasse 26, D-53225 Bonn, Germany.

**Jan Ratajski**[3]
Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, PL-00950 Warsaw, Poland.

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## Introduction

In this paper we give formulas for the fundamental classes of Schubert subschemes in Lagrangian and orthogonal Grassmannians of maximal subbundles as well as some globalizations of them. Our motivation to deal with this subject came essentially from

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3 examples where such degeneracy loci appear in algebraic geometry: $1^o$ The Brill-Noether loci for Prym varieties, as defined by Welters [W], $2^o$ The loci of curves with sufficiently many theta characteristics, as considered by Harris [Har], $3^o$ Some "higher" Brill-Noether loci in the moduli spaces of higher rank vector bundles over curves, considered by Bertram and Feinberg [B-F] and, independently, by Mukai [Mu].

The common denominator of these 3 situations is a simple and beautiful construction of Mumford [M]. With a vector bundle over a curve equipped with a nondegenerate quadratic form with values in the sheaf of 1-differentials, Mumford associates an even dimensional vector space endowed with a nondegenerate quadratic form and 2 maximal isotropic subspaces such that the space of global sections of the initial bundle is the intersection of the two isotropic subspaces. A globalization of this construction allows one to present in a similar way the varieties in $1^o$ and $2^o$ above as loci where two isotropic rank $n$ subbundles of a certain rank $2n$ bundle equipped with a quadratic nondegenerate form, intersect in dimension exceeding a given number. On the other hand, the locus in $3^o$ admits locally this kind of presentation using an appropriate symplectic form.

These varieties are particular cases of Schubert subschemes in Lagrangian and orthogonal Grassmannian bundles and their globalizations. The formulas for such loci are the main theme of this paper. More specifically, given a vector bundle $V$ on a variety $X$ endowed with a nondegenerate symplectic or orthogonal form, we pick $E$ and $F_1 \subset F_2 \subset ... \subset F_n = F$ – isotropic subbundles of $V$ $(rank\ E = n, rank\ F_i = i)$, and for a given sequence $a_\bullet = (1 \leqslant a_1 < ... < a_k \leqslant n)$, we look at the locus:

$$D(a_\bullet) := \left\{\ x \in X |\ dim\big(E \cap F_{a_p}\big)_x \geqslant p,\ p = 1, ..., k\right\}.$$

We distinguish three cases:

1. Lagrangian: $rank\ V = 2n$, the form is symplectic;
2. odd orthogonal: $rank\ V = 2n+1$, the form is orthogonal;
3. even orthogonal: $rank\ V = 2n$, the form is orthogonal.

(In this last case the definition of $D(a_\bullet)$ must be slightly modified - see Section 9.)

Let us remark that the loci $D(a_\bullet)$ (for the Lagrangian case) admit an important specialization to the loci introduced by Ekedahl and Oort in the moduli space of abelian varieties with fixed dimension and polarization, in characteristic $p$ (see, e.g. [O], the references therein and [E-vG]). This comes from certain filtrations on the de Rham cohomology defined with the help of the Frobenius- and "Verschiebung"-maps. The formulas of the present paper are well suited to computations of the fundamental classes of such loci in the Chow groups of the moduli spaces - for details see a forthcoming paper by T. Ekedahl and G. van der Geer [E-vG].

The goal of this paper is to give an algorithm for computing the fundamental classes of $D(a_\bullet)$ as polynomials in the Chern classes of $E$ and $F_i$. Formulas given here can be thought of as Lagrangian and orthogonal analogs of the formulas due independently to Kempf-Laksov [K-L] and Lascoux [L1] (notice, however, that the formulas given in [K-L] are proved under a weaker assumption of "expected" dimension).

The method for computing the fundamental class of a subscheme of a given (smooth) scheme which we use here stems from a paper by the first author [P3, Sect.5]. It depends on a desingularization of the subscheme in question and the knowledge of the class of the diagonal of the ambient space. It appears that the diagonals in the fibre products of Lagrangian or orthogonal Grassmannian- and flag bundles are not given as the subschemes of zeros of sections of bundles over the corresponding products. This makes an additional difficulty (e.g. in comparison with [K-L]) which is overcomed here using again a result from [P3, Sect.5] allowing to compute the class od the diagonal with the help of an appropriate "orthogonality" property of Gysin maps.

To establish formulas for the classes of these diagonals, we use essentially two tools. The first one is Theorem 6.17 of [P2] interpreting (cohomology dual to) the classes of Schubert subvarieties in Lagrangian and orthogonal Grassmannians as Schur's $Q$- and $P$-polynomials. The importance of these polynomials to algebraic geometry was illuminated by the first author in [P1] and then developed in [P2]. In fact in [P2, Sect.6], a variant of these polynomials was used to give a full description of Schubert Calculus on Grassmannians of maximal isotropic subspaces associated with a nondegenerate symplectic and orthogonal form. These familes of symmetric polynomials are called $\widetilde{Q}$- and $\widetilde{P}$-polynomials in the present paper. Perhaps the "orthogonality" proved in Theorem 5.23 is their central property. This is, in fact, the second tool in our computation of the classes of the diagonals in isotropic Grassmannian bundles which allows us to apply the technique of [P3, Sect.5]. The results of [P2, Sect.6], recalled in Theorem 2.1 below, are a natural source of the ubiquity of $\widetilde{Q}$- and $\widetilde{P}$-polynomials in various formulas of this paper. As a general rule, these are $\widetilde{Q}$-polynomials that appear in the Lagrangian case and $\widetilde{P}$-polynomials that appear in the orthogonal cases.

In general, our approach gives an efficient algorithm for computing formulas for Lagrangian and orthogonal Schubert subschemes. In several cases, however, we are able to give "closed" expressions. At first, these are the cases of a single Schubert condition and two Schubert conditions. The corresponding formulas are given in Sections 6 and 7.

The derivation of those formulas uses a formula for the push-forward of $\widetilde{Q}$-polynomials (Theorems 5.10, 5.14, 5.20) from isotropic Grassmannian bundles. For instance, in the Lagrangian case, $\pi : LG_nV \to X$ with the tautological subbundle $R$, the element $\widetilde{Q}_I R^\vee$ has a nonzero image under $\pi_*$ only if each number $p$, $1 \leqslant p \leqslant n$, appears as a part of $I$ with an odd multiplicity $m_p$. If this last condition holds then

$$\pi_*\widetilde{Q}_I R^\vee = \prod_{p=1}^{n} \big((-1)^p c_{2p}V\big)^{(m_p-1)/2}.$$

We also give formulas for the push-forward of $S$-polynomials (Theorems 5.13, 5.15, 5.21) from isotropic Grassmannian bundles. For example, in the Lagrangian case, the element $s_I R^\vee$ has a nonzero image under $\pi_*$ only if the partition $I$ is of the form $2J+\rho_n$ for some partition $J$ (here, $\rho_n = (n, n-1, ..., 1)$). If $I = 2J + \rho_n$ then

$$\pi_* s_I R^\vee = s_J^{[2]} V \quad ,$$

where the right hand side is defined as follows: if $s_J = P(e.)$ is a unique presentation of $s_J$ as a polynomial in the elementary symmetric functions $e_i$, $E-$ a vector bundle, then $s_J^{[2]}(E) := P$ with $e_i$ replaced by $(-1)^i c_{2i}E,\ i = 1, 2, \dots$ .

Another case (corresponding to the Schubert condition $a_\bullet = (n-k+1, ..., n)$) that leads to closed formulas is the variety of maximal isotropic subbundles which intersect a fixed maximal isotropic subbundle in dimension exceeding a given number (Proposition 3.2 and its analogs). Thanks to the Cohen-Macaulayness of Schubert subschemes in isotropic Grassmannians proved in [DC-L], one gets globalizations of those formulas (as well as the other ones) to more general loci. For instance, this last case $a_\bullet = (n-k+1, ..., n)$ globalizes to the Mumford type locus discussed above where two maximal isotropic subbundles $E$ and $F$ intersect in dimension greater than or equal to $k$.[4]

Our formulas (see Theorems 9.1, 9.5 and 9.6) are quadratic expressions in $\widetilde{Q}$- and $\widetilde{P}$-polynomials of the subbundles. More explicitly in the corresponding cases we have

1. Lagrangian: $\sum \widetilde{Q}_I E^\vee \cdot \widetilde{Q}_{(k,k-1,\dots,1)\smallsetminus I} F^\vee$;
2. odd orthogonal: $\sum \widetilde{P}_I E^\vee \cdot \widetilde{P}_{(k,k-1,\dots,1)\smallsetminus I} F^\vee$;
3. even orthogonal: $\sum \widetilde{P}_I E^\vee \cdot \widetilde{P}_{(k-1,k-2,\dots,1)\smallsetminus I} F^\vee$;

where in 1. and 2. the sum is over all subsequences $I$ in $(k, k-1, \dots, 1)$, in 3. the sum is over all subsequences $I$ in $(k-1, k-2, \dots, 1)$ and $(k, k-1, \dots, 1) \smallsetminus I$ denotes the strict partition whose parts complements the ones of $I$ in $\{k, k-1, \dots, 1\}$.

Formula 3. has been recently used by C. De Concini and the first named author in [DC-P] to compute the fundamental classes of the Brill-Noether loci $V^r$ for the Prym varieties (see [W]), thus solving a problem of Welters, left open since 1985. The formula of [DC-P] asserts that if either $V^r$ is empty or of pure codimension $r(r+1)/2$ in the Prym variety then its fundamental class in the numerical equivalence ring, or its cohomology class is equal to

$$2^{r(r-1)/2} \prod_{i=1}^{r} \big((i-1)!/(2i-1)!\big)\ [\Xi]^{r(r+1)/2},$$

where $\Xi$ is the theta divisor on the Prym variety.

The paper is organized as follows.

Section 1 contains definitions and properties of Schubert varieties in Lagrangian and orthogonal Grassmannian bundles. Also, some desingularizations of these varieties, used in later sections, are described.

Section 2 contains some recollections of Schubert calculus for Lagrangian and orthogonal Grassmannians from [P2, Sect.6] and computation of the classes of the diagonals in

[4] It is mentioned in [F1,2] that the problem of finding formulas for the classes in this case was posed originally by Professor J. Harris several years ago.

the Chow rings of Lagrangian and orthogonal Grassmannian bundles. This computation relies on the Gysin maps technique from [P3, Sect.5] and on the orthogonality theorem 5.23 which is proved independently later.

Section 3 contains an explicit computation of Gysin maps needed to determine the formulas for the fundamental classes of Schubert varieties $\Omega(n-k+1,\dots,n)$ parametrizing subbundles intersecting an $n$-subbundle in dimension exceeding $k$. This is done using an elementary Schubert Calculus–type technique based on linear algebra.

In Section 4 we introduce a family of symmetric polynomials called $\widetilde{Q}$-polynomials which is modelled on Schur's $Q$-polynomials (but is different from the latter family). These polynomials are the basic algebraic tools of the present paper. We prove several elementary but useful properties of $\widetilde{Q}$-polynomials and give some examples.

In Section 5 we establish some new algebraic properties of of $\widetilde{Q}$-polynomials and $S$-polynomials; these are either certain determinantal identities like Propositions 5.2 and 5.11, or the computation of the values of these polynomials under some divided differences operators. These algebraic results are then interpreted using Gysin maps for Lagrangian and orthogonal Grassmannian bundles. Perhaps the most important result of this section is the "orthogonality" Theorem 5.23. This theorem, interpreted geometrically (using a result of [P3, Sect.5]), gives us the classes of the diagonals of Lagrangian and orthogonal Grassmannian bundles which are crucial for our computations.

Sections 6., 7. and 8. have a supplementary character. They contain some examples and a certain alternative (to the content of the previous sections) way of computing. Section 6 contains formulas for Schubert varieties defined by one Schubert condition in Lagrangian and orthogonal cases. Section 7 contains similar computations for two Schubert conditions in the Lagrangian and odd orthogonal cases. Section 8 contains another (purely algebraic) proof, using divided differences, of Proposition 3.1 that describes the Gysin maps for some flag bundles.

In Section 9 we formulate previous results in the general setup of degeneracy loci and give some examples. A special emphasis is put on formulas answering J. Harris' problem concerning the Mumford-type degeneracy loci described above.

In Appendix A we collect a number of useful results about Quaternionic Grassmannians. We use them to reprove some results proved earlier using different methods and to show how some problems concerning Grassmannians of nonmaximal Lagrangian subspaces can be reduced to those of maximal Lagrangian subspaces; this sort of applications we plan to develop elsewhere.

Finally, in Appendix B, we give an introduction to a theory of symplectic Schubert polynomials which has grown up from the present work. This theory (see [L-P-R]) seems to be well suited to the needs of algebraic geometry because it generalizes in a natural way $\widetilde{Q}$-polynomials which govern the Schubert calculus on Lagrangian Grassmannians.

In Sections 2, 3, 5, 6, 7, 8 and 9 we work in the Chow rings; all results therein, however, are equally valid in the cohomology rings.

Some of the results of this paper were announced in [P-R0].

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Background

Several results of this paper: e.g. Propositions 3.2, 3.4 and 3.6 as well as their globalizations in Theorems 9.1, 9.5 and 9.6 were obtained already in Spring 1993 when we tried to deduce formulas for the loci $D(a_\bullet)$ by combining the ideas of the paper of Kempf and Laksov [K-L] with the $Q$-polynomials technique developed in [P1,2]. These results were announced together with outlines of their proofs in [P-R0].

In summer '93, we received an e-mail message from Professor W. Fulton informing us about his (independent) work on the same subject and announcing another expressions for the loci considered in Proposition 3.2, 3.4 and 3.6 of the present paper. Responding, we informed Professor Fulton about our results of [P-R0] mentioned above. In February '94 we obtained from Professor W. Fulton his preprints [F1,2] containing details of his e-mail announcement. Both the form of the formulas obtained as well as the approach used in [F1,2] are totally different from the content of our work and just a simple comparison of the results of [F1,2] with ours leads to very nontrivial new identities which are interesting in themselves. It would be desirable to develop, in a systematic way, the comparison of formulas given in [F1,2] from one side with those in the present paper and [P-R0] - from the other one.

Conventions

Partitions are weakly decreasing sequences of positive integers (as in [Mcd1] and are denoted by capital Roman letters (as in [L-S1]). We identify partitions with their Ferrers' diagrams visualized as in [L-S1]. The relation "$\subset$" for partitions is induced from that for diagrams.

For a given partition $I = (i_1, i_2, \dots)$ we denote by $|I|$ (the *weight* of $I$) the partitioned number (i.e. the sum of all parts of $I$) and by $l(I)$ (the *length* of $I$) the number of nonzero parts of $I$. Moreover, $I^\sim$ denotes the dual partition of $I$, i.e. $I^\sim = (j_1, j_2, \dots)$ where $j_p = card\{h | i_h \geqslant p\}$, and $(i)^k$ - the partition $(i, \dots, i)$ ($k$-times).

Given sequences $I = (i_1, i_2, \dots)$ and $J = (j_1, j_2, \dots)$ we denote by $I \pm J$ the sequence $(i_1 \pm j_1, i_2 \pm j_2, \dots)$.

By *strict* partitions we mean those whose (positive) parts are all different.

In this paper, we denote by $s_i(E)$ the complete symmetric polynomial of degree $i$ with variables specialized to the Chern roots of a vector bundle $E$.

The reader should be careful with our notion of $\widetilde{Q}$-polynomials here. Namely, since we are mainly interested in the polynomials in the Chern classes of vector bundles, we introduce $\widetilde{Q}$-polynomials given by the Pfaffian of an antisymmetric matrix whose entries are quadratic expressions in the elementary symmetric polynomials rather than in the "one row" Schur's $Q$-polynomials. Therefore these polynomials are different from the original Schur's $Q$-polynomials. Note that nonzero $\widetilde{Q}$-polynomials $\widetilde{Q}_I(x_1, \dots, x_n)$ are indexed by "usual" partitions $I$ but the parts of these partitions cannot exceed the number of variables; on the contrary, nonzero Schur's $Q$-polynomials $Q_I(x_1, \dots, x_n)$ are indexed by strict partitions $I$ only but the parts of these partitions can be bigger than the number of variables.

Also, the specialization of $\widetilde{Q}_I(x_1, \dots, x_n)$ with $(x_i)$ equal to the sequence of the Chern roots of a rank $n$ vector bundle $E$, denoted here - accordingly - by $\widetilde{Q}_I E$, is a different cohomology class than the one associated with $E$ in [P1] and [P2, Sect.3 and 5], and denoted by $Q_I E$ therein. (Notice, however, that the $\widetilde{Q}$-polynomials appeared already in an implicit way in [P2, Sect.6].) The reader should make a proper distinction between Schur's $Q$-polynomials and $\widetilde{Q}$-polynomials that are mainly used in the present paper.

For a vector bundle $V$, by $G_n V$ we denote the usual Grassmannian bundle parametrizing rank $n$ subbundles of $V$. Moreover, $\mathbb{P}(V) = G_1 V$. We follow mostly [F] for the terminology in algebraic geometry. In many situations when the notation starts to be too cumbersome, we omit some pullback-indices of the induced vector bundles.

A good reference for "changes of alphabets" in the $\lambda$-ring sense is [L-S1].

## 1. Schubert subschemes and their desingularizations

We start with the Lagrangian case. Let $K$ be an arbitrary ground field.

Assume that $V$ is a rank $2n$ vector bundle over a smooth scheme $X$ over $K$ equipped with a nondegenerate symplectic form. Moreover, assume that a flag $V_\bullet : V_1 \subset V_2 \subset \dots \subset V_n$ of Lagrangian (i.e. isotropic) subbundles w.r.t. this form is fixed, with $rank\ V_i = i$. Let $\pi : LG_n(V) \to X$ denote the Grassmannian bundle parametrizing Lagrangian rank

$n$ subbundles of $V$. $G = LG_n(V)$ is endowed with the tautological Lagrangian bundle $R \subset V_G$. Given a sequence $a_\bullet = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$ we consider in $G$ a closed subset:

$$\Omega(a_\bullet) = \Omega(a_\bullet; V_\bullet) = \{g \in G |\ dim\big(R \cap V_{a_i}\big)_g \geqslant i,\ i = 1, \ldots, k\}.$$

The locus $\Omega(a_\bullet)$, called a *Schubert subscheme* is endowed with a reduced scheme structure induced from the reduced one of the corresponding Schubert subscheme in the Grassmannian $G_nV$ - this is discussed in detail , e.g., in [L-Se].

The following desingularization of $\Omega = \Omega(a_\bullet)$ should be thought of as a Lagrangian analogue of the construction used in [K-L]. Let $\mathcal{F} = \mathcal{F}(a_\bullet) = \mathcal{F}(V_{a_1} \subset \ldots \subset V_{a_k})$ be the scheme parametrizing flags $A_1 \subset A_2 \subset \ldots \subset A_k \subset A_{k+1}$ such that $rank\ A_i = i$ and $A_i \subset V_{a_i}$ for $i = 1, \ldots, k$ ; $rank\ A_{k+1} = n$ and $A_{k+1}$ is Lagrangian. $\mathcal{F}$ is endowed with the tautological flag $D_1 \subset D_2 \ldots \subset D_k \subset D_{k+1}$, where $rank\ D_i = i,\ i = 1, \ldots, k$ and $rank\ D_{k+1} = n$ . We will write $D$ instead of $D_{k+1}$.

We have a fibre square:

$$\begin{array}{ccc} G \times_X \mathcal{F} & \xrightarrow{p_2} & \mathcal{F} \\ {\scriptstyle p_1}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ G & \longrightarrow & X \end{array}$$

Let $\alpha : \mathcal{F} \to G$ be the map defined by: $(A_1 \subset A_2 \subset \ldots \subset A_{k+1}) \mapsto A_{k+1}$, in other words $\alpha$ is a "classifying map" such that $\alpha^* R = D$. It is easily verified that $\alpha$ maps $\mathcal{F}$ onto $\Omega$ and $\alpha$ is an isomorphism over the open subset of $\Omega$ parametrizing rank $n$ Lagrangian subbundles $A$ of $V$ such that $rank(A \cap V_{a_i}) = i,\ i = 1, \ldots, k$. Moreover, $\alpha$ induces a section $s$ of $p_2$. Set $Z := s(\mathcal{F}) \subset G \times_X \mathcal{F}$. Alternatively, we can describe $Z$ as $(1 \times \alpha)^{-1}(\Delta)$ where $\Delta$ is the diagonal in $G \times_X G$. The map $p_1$ restricted to $Z$ is a desingularization of $\Omega$. Therefore $[\Omega] = (p_1)_*([Z])$. On the other hand, $[Z] = (1 \times \alpha)^*([\Delta])$ (see [K-L, Lemma 9]). Note that $\mathcal{F}$ is obtained as a composition of the following flag- and Grassmannian bundles. Let $Fl = Fl(a_\bullet) = Fl(V_{a_1} \subset \ldots \subset V_{a_k})$ be the "usual" flag bundle parametrizing flags $A_1 \subset \ldots \subset A_k$ where $rank\ A_i = i$ and $A_i \subset V_{a_i},\ i = 1, \ldots, k$. Let $C_1 \subset \ldots \subset C_k$ be the tautological flag on $Fl$. We will write $C$ instead of $C_k$. Then $\mathcal{F}$ is the Lagrangian Grassmannian bundle $LG_{n-k}(C^\perp/C)$ over $Fl$, where $C^\perp$ is the subbundle of $V_{Fl}$ consisting of all $v$ that are orthogonal to $C$ w.r.t. the given symplectic form. Note that $C \subset C^\perp$ because $C$ is Lagrangian, $rank(C^\perp/C) = 2(n-k)$ and the vector bundle $C^\perp/C$ is endowed with a nondegenerate symplectic form induced from the one on $V$. Of course the tautological Lagrangian rank $n-k$ subbundle on $LG_{n-k}(C^\perp/C)$ is identified with $D/C_{\mathcal{F}}$. In other words, $\mathcal{F}$ is a composition of a flag bundle (with the fiber being $Fl(K^{a_1} \subset \ldots \subset K^{a_k})$) and a Lagrangian Grassmanian bundle (with the fiber being $LG_{n-k}(K^{2(n-k)})$. In particular,

$$dim\ \Omega = dim\ \mathcal{F} = dim\ Z = \sum_{i=1}^{k} (a_i - i) + (n-k)(n-k+1)/2 + dim\ X.$$

The following particular cases will be treated in a detailed way in this paper: $a_\bullet = (n-k+1,\ n-k+2, \ldots, n)$ (then $\Omega(a_\bullet)$ parametrizes Lagrangian rank $n$ subbundles $L$ of

$V$ such that $rank(L \cap V_n) \geqslant k$ ); $a_\bullet = (n+1-i)$ , i.e. $k = 1$ ; and $a_\bullet = (n+1-i, n+1-j)$, i.e. $k = 2$.

Now consider the odd orthogonal case. Let $K$ be a ground field of characteristic different from 2. Assume, that $V$ is a rank $2n+1$ vector bundle over a smooth scheme $X$ over $K$ equipped with a nondegenerate orthogonal form. We assume throughout this paper that the form restricts to a hyperbolic form on each fiber (i.e. each fiber has an $n$-dimensional isotropic subspace; if $K$ is algebraically closed, this is automatically satisfied.) Let $OG_nV$ be the Grassmannian bundle parametrizing rank $n$ isotropic subbundles of $V$. Whenever, in this paper, we speak about Schubert subschemes in $OG_nV$, we assume that there exists a completely filtered rank $n$ isotropic subbundle in $V$. All definitions, notions and notation concerning Schubert subschemes and their desingularizations are used mutatis mutandis (just instead of "symplectic" use "orthogonal" and instead of "Lagrangian" use "isotropic"). The formula for the dimension of $\Omega(a_\bullet)$ in the odd orthogonal case is the same as in the Lagrangian case. Of course, $\mathcal{F}$ is now a composition of the same flag bundle $Fl$ and the odd orthogonal Grassmannian bundle $OG_{n-k}(C^\perp/C)$, where $C$ is the rank $k$ tautological subbundle on $Fl$.

Assume now that $V$ is a rank $2n$ vector bundle over a smooth connected scheme $X$ over a field $K$ of characteristic different from 2 equipped with a nondegenerate orthogonal form. We assume throughout this paper that there exists an isotropic rank $n$ subbundle of $V$. The scheme parametrizing isotropic rank $n$ subbundles of $V$ breaks up into two connected components denoted $OG'_nV$ and $OG''_nV$. Let $V_n$ be a rank $n$ isotropic subbundle of $V$ fixed once and for all. Then $OG'_nV$ (resp. $OG''_nV$) parametrizes rank $n$ isotropic subbundles $E \subset V$ such that $dim(E \cap V_n)_x \equiv n (mod\, 2)$ (resp. $dim(E \cap V_n)_x \equiv n+1 (mod\, 2)$) for every $x \in X$. Write $G' := OG'_nV$ and $G'' := OG''_nV$. Two isotropic rank $n$ subbundles are in the same component iff they intersect fiberwise in dimension congruent to $n$ modulo 2.

Let $V_\bullet : V_1 \subset V_2 \subset \ldots \subset V_n$ be a flag of isotropic subbundles of $V$ with $rank\ V_i = i$. Given a sequence $a_\bullet = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$ such that $k \equiv n\ (mod\ 2)$, we consider in $G'$ a Schubert subvariety:

$$\Omega(a_\bullet) = \Omega(a_\bullet; V_\bullet) = \Big\{ g \in G' |\ dim(R \cap V_{a_i})_g \geqslant i,\ i = 1, \ldots, k \Big\}$$

($R \subset V_{G'}$ is here the tautological bundle). Similarly, given a sequence $a_\bullet = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$ such that $k \equiv n+1\ (mod\ 2)$, we consider in $G''$ a Schubert subvariety

$$\Omega(a_\bullet) = \Omega(a_\bullet; V_\bullet) = \Big\{ g \in G'' |\ dim(R \cap V_{a_i})_g \geqslant i,\ i = 1, \ldots, k \Big\}.$$

(Over a point, say, the interiors of the $\Omega(a_\bullet)$'s form a cellular decomposition of $G'$ and respectively $G''$.) Here, the definition of the scheme structure is more delicate than in the previous two cases (roughly speaking, instead of minors one should use the Pfaffians of the "coordinate" antisymmetric matrix of $G'$ and $G''$). We refer the reader for details to [L-Se] and references therein.

The Schubert subvarieties $\Omega(a_\bullet)$ in $G'$ (resp. $\Omega(a_\bullet)$ in $G''$) are desingularized using the same construction as above but instead of the scheme $\mathcal{F}$ one must now use the following scheme $\mathcal{F}'$ (resp. $\mathcal{F}''$). Let $\mathcal{F}' = \mathcal{F}'(a_\bullet) = \mathcal{F}'(V_{a_1} \subset \ldots \subset V_{a_k})$ be a scheme parametrizing flags $A_1 \subset A_2 \subset \ldots \subset A_k \subset A_{k+1}$ such that $rank\ A_i = i$ and $A_i \subset V_{a_i}$ for $i = 1, \ldots, k$; $rank\ A_{k+1} = n$, $A_{k+1}$ is isotropic and $rank(A_{k+1} \cap V_n)_x \equiv n\ (mod\ 2)$ for any $x \in X$. The definition of $\mathcal{F}'' = \mathcal{F}''(a_\bullet)$ is the same with exception of the last condition now replaced by: $rank(A_{k+1} \cap V_n)_x \equiv n+1\ (mod\ 2)$ for any $x \in X$. Let $p' : \mathcal{F} \to X$ (resp. $p'' : \mathcal{F} \to X$) denote the projection maps. Of course, $\mathcal{F}'$ (resp. $\mathcal{F}''$) now is a composition of the same flag bundle $Fl$ and the even orthogonal Grassmannian bundle $OG'_{n-k}(C^\perp/C)$ (resp. $OG''_{n-k}(C^\perp/C)$), where $C$ is the rank $k$ tautological subbundle on $Fl$ and $V_n/C$ is the rank $n-k$ isotropic bundle used to define $OG'_{n-k}(C^\perp/C)$ and $OG''_{n-k}(C^\perp/C)$.

The formula for dimension now is different:

$$dim\ \mathcal{F}' = dim\ \mathcal{F}'' = \sum_{i=1}^{k}(a_i - i) + (n-k)(n-k-1)/2 + dim\ X.$$

We finish this section with the following lemma which will be of constant use in this paper.

**Lemma 1.1.** *Consider cases 1., 2., 3. of a vector bundle endowed with a nondegenerate form $\Phi$ that are specified in the Introduction. Let $C \subset V$ be an isotropic subbundle and $C^\perp$ be the subbundle of $V$ consisting of all $v \in V$ such that $\Phi(v, c) = 0$ for any $c \in C$.*

*1. Then one has an exact sequence*

$$0 \longrightarrow C^\perp \longrightarrow V \xrightarrow{\ \phi\ } C^\vee \longrightarrow 0$$

*where the map $\phi$ is defined by $v \mapsto \Phi(v, -)$. In particular, in the Grothendieck group, $[V] = [C^\perp] + [C^\vee]$, $[C^\perp/C] = [V] - [C] - [C^\vee]$ and the Chern classes of $C^\perp/C$ are the same as the ones of $[V] - [C \oplus C^\vee]$.*

*2. Assume now that $C$ is a maximal isotropic subbundle of $V$. Then in cases 1. and 3. we have $C = C^\perp$ and $c(V) = c(C \oplus C^\vee)$; in case 2. one has $rank(C^\perp/C) = 1$ and $2c(V) = 2c(C \oplus C^\vee)$.*

The latter equality of assertion 2 in case 2. follows from the fact that the form $\Phi$ induces an isomorphism $(C^\perp/C)^{\otimes 2} \cong \mathcal{O}_X$. This assertion will be used in the proof of Theorem 5.14 and 5.15 and is well suited for this purpose because of the appearance of the factor "$2^n$" on the right hand side of the formulas of the theorems.

## 2. Isotropic Schubert Calculus and the class of the diagonal

Let us first recall the following result on Lagrangian and orthogonal Schubert Calculus from [P1,2].

We need two families of polynomials in the Chern classes of a vector bundle $E$ over a smooth variety $X$. Their construction is inspired by I. Schur's paper [S]. The both families will be indexed by partitions (i.e. by sequences $I = (i_1 \geqslant \ldots \geqslant i_k \geqslant 0)$ of integers). Set, in the Chow ring $A^*(X)$ of $X$, for $i \geqslant j \geqslant 0$:

$$\widetilde{Q}_{i,j}E := c_iE \cdot c_jE + 2\sum_{p=1}^{j}(-1)^p c_{i+p}E \cdot c_{j-p}E,$$

so, in particular $\widetilde{Q}_iE := \widetilde{Q}_{i,0}E = c_iE$ for $i \geqslant 0$. In general, for a partition $I = (i_1, \dots, i_k)$, $k-$even (by putting $i_k = 0$ if necessary), we set in $A^*(G)$:

$$\widetilde{Q}_IE := Pf\Big(\widetilde{Q}_{i_p,i_q}E\Big)_{1\leqslant p<q\leqslant k},$$

where $Pf$ means the Pfaffian of the given antisymmetric matrix. For the definition and basic properties of Pfaffians we send interested readers to [A], [Bou] and [B-E, Chap.2]. Also, we refer the reader to the beginning of Section 4 for an alternative recurrent definition of $\widetilde{Q}_IE$: just replace the polynomial $\widetilde{Q}_I(X_n)$ from Section 4 by the element $\widetilde{Q}_IE$.

The member of the second family, associated with a partition $I$, is defined by

$$\widetilde{P}_IE := 2^{-l(I)}\widetilde{Q}_IE.$$

Observe that in particular $\widetilde{P}_iE = c_iE/2$ (so here we must assume that $c_iE$ is divisible by 2), and

$$\widetilde{P}_{i,j}E = \widetilde{P}_iE \cdot \widetilde{P}_jE + 2\sum_{p=1}^{j-1}(-1)^p \widetilde{P}_{i+p}E \cdot \widetilde{P}_{j-p}E + (-1)^j\widetilde{P}_{i+j}E.$$

It should be emphasized that $\widetilde{Q}$- and $\widetilde{P}$-polynomials are especially important and useful for isotropic (sub)bundles.

The following result from [P1, (8.7)] and [P2, Sect.6], gives a basic geometric interpretation of $\widetilde{Q}$- and $\widetilde{P}$-polynomials. (It can be interpreted as a Giambelli-type formula for isotropic Grassmannians; recall that a Pieri-type formula for these Grassmannians was given in [H-B] - consult also [P-R1] for a simple proof of the latter result.)

**Theorem 2.1.** [P2, Sect.6]

*(i)* *Let $V$ be a $2n-$dimensional vector space over a field $K$ endowed with a nondegenerate symplectic form. Then, one has in $A^*(LG_nV)$,*

$$\big[\Omega(a_\bullet)\big] = \widetilde{Q}_IR^\vee \ ,$$

*where $R$ is the tautological subbundle on $LG_nV$ and $i_p = n+1-a_p$, $p = 1, \dots, k$.*

*(ii)* *Let $V$ be a $(2n+1)-$dimensional vector space over a field $K$ of char. $\neq 2$ endowed with a nondegenerate orthogonal form. Then, one has in $A^*(OG_nV)$,*

$$\big[\Omega(a_\bullet)\big] = \widetilde{P}_I R^\vee \ ,$$

*where $R$ is the tautological subbundle on $OG_nV$ and $i_p = n+1-a_p$, $p=1,\ldots,k$.*

*(iii)* *Let $V$ be a $2n-$dimensional vector space over a field $K$ of char. $\neq 2$ endowed with a nondegenerate orthogonal form. Then one has in $A^*(OG'_nV)$ (resp. $A^*(OG''_nV)$),*

$$\big[\Omega(a_\bullet)\big] = \widetilde{P}_I R^\vee \ ,$$

*where $R$ is the tautological subbundle on $OG'_nV$ (resp. $OG''_nV$) and $i_p = n - a_p$, $p = 1,...,k$. ( Notice that the indexing family of $I$'s runs here over all strict partitions contained in $\rho_{n-1}$. )*

*Observe that by Lemma 1.1, $R^\vee$ is the tautological quotient bundle on $LG_nV$, $OG'_nV$ and $OG''_nV$. Moreover, the Chern classes of the tautological quotient bundle on $OG_nV$ and $R^\vee$ are equal.*

Note that this result has been reproved recently by S. Billey and M. Haiman in [B-H].

Assume now that $V$ is a vector bundle over a smooth variety $X$ and $V_\bullet$ is a flag of isotropic bundles on $X$. Then, using Noetherian induction, one shows that $\{\widetilde{Q}_I R^\vee\}_{I\subset\rho_n}$, $\{\widetilde{P}_I R^\vee\}_{I\subset\rho_n}$ and $\{\widetilde{P}_I R^\vee\}_{I\subset\rho_{n-1}}$ are $A^*(X)$-bases respectively of $A^*(LG_nV)$, $A^*(OG_nV)$ and $A^*(OG'_nV)$ (resp. $A^*(OG''_nV)$). Moreover, there is an expression for $\Omega(a_\bullet;V_\bullet)$ as a polynomial in the Chern classes of $R^\vee$ and $V_i$. (This follows, e.g., from the existence of desingularizations given in Section 1 and formulas for Gysin push forwards – for "usual" flag bundles they are obtained by iterating a well known projective bundle case; for isotropic Grassmannian bundles, they are given for the first time in Section 5 of the present paper). Then the maximal degree term in the $c_i(R^\vee)$'s of this expression, in respective cases (i), (ii), (iii), coincides with that in Theorem 2.1. We will call it *the dominant term* (w.r.t. $R$).

Let $G_1, G_2$ be two copies of the Lagrangian Grassmannian bundle $LG_nV$ over a smooth variety $X$, equipped with the tautological subbundles $R_1$ and $R_2$. Write $GG := G_1 \times_X G_2$ . Consider the following diagonal

$$\Delta = \Big\{(g_1,g_2)\in GG|\ \big((R_1)_{GG}\big)_{(g_1,g_2)} = \big((R_2)_{GG}\big)_{(g_1,g_2)}\Big\}.$$

Our goal is to write down a formula for the class of this diagonal. We first record:

**Lemma 2.2.** *Let $G$ be a smooth complete variety such that the "$\times$-map"* (cf. [F, end of Sect.1]) *gives an isomorphism $A^*(G\times G)\cong A^*(G)\otimes A^*(G)$ . Assume that there exists a family $\{b_\alpha\}$ , $b_\alpha\in A^{n_\alpha}(G)$ , such that $A^*(G)=\oplus\mathbb{Z}b_\alpha$ , and for every $\alpha$ there is a unique $\alpha'$ such that $n_\alpha+n_{\alpha'} = dim\ G$ and $\int_X b_\alpha\cdot b_{\alpha'}\neq 0$ . Suppose $\int_X b_\alpha\cdot b_{\alpha'}=1$ . Then the class $[\Delta]$ in $A^*(G\times G)$ is given by $\sum\limits_\alpha b_\alpha\times b_{\alpha'}$ .*

*Proof.* It follows from the assumptions that in $A^*(G \times G)$ , $[\Delta] = \sum m_{\alpha\beta} b_\alpha \times b_\beta$ , for some integers $m_{\alpha\beta}$ and $n_\alpha + n_\beta = dim\ G$ for all pairs $(\alpha, \beta)$ indexing the sum. We have by a standard property of intersection theory for $g, h \in A^*(G)$

$$\int_{X\times X} [\Delta] \cdot (g \times h) = \int_X g \cdot h.$$

Hence the coefficients $m_{\alpha\beta}$ satisfy:

$$m_{\alpha\beta} = \int_{X\times X} [\Delta] \cdot (b_{\alpha'} \times b_{\beta'}) = \int_X b_{\alpha'} \cdot b_{\beta'}.$$

The latter expression, according to our assumption is not zero only if $\alpha' = (\beta')'$ i.e. $\beta = \alpha'$, when it equals 1. This proves the lemma. $\square$

For a given positive integer $k$, put $\rho_k = (k, k-1, \ldots, 2, 1)$ . For a strict partition $I \subset \rho_k$ (i.e. $i_1 \leqslant k,\ i_2 \leqslant k-1, \ldots$) we denote by $\rho_k \smallsetminus I$ the strict partition whose parts complement the parts of $I$ in the set $\{k, k-1, \ldots, 2, 1\}$.

The Lagrangian Grassmannian (over a point, say) satisfies the assumptions of the lemma with $\{\widetilde{Q}_I R^\vee\}_{strict\ I\subset\rho_n}$ playing the role of $\{b_\alpha\}$ and for $\alpha = I$ we have $\alpha' = \rho_n \smallsetminus I$. This is a direct consequence the existence of a well known cellular decomposition of such a Grassmannian into Schubert cells and the results of [P2] recalled in Theorem 2.1(i) together with a description of Poincaré duality in $A^*(LG_nV)$ from loc.cit. Thus in this situation we get by the lemma:

**Lemma 2.3.** *The class of the diagonal $\Delta$ of the Lagrangian Grassmannian equals*

$$[\Delta] = \sum \widetilde{Q}_I(R_1^\vee) \times \widetilde{Q}_{\rho_n \smallsetminus I}(R_2^\vee),$$

*the sum over all strict $I \subset \rho_n$ .*

We now want to show that the same formula holds true for an arbitrary smooth base space $X$ of a vector bundle $V$. Our argument is based on the following result expressing the class of the (relative) diagonal in terms of Gysin maps. This result was due to the first author in [P3, Sect.5] and is accompanied here by its proof for the reader's convenience.

**Lemma 2.4.** [P3] *Let $\pi : G \to X$ be a proper morphism of smooth varieties such that $\pi^*$ makes $A^*(G)$ a free $A^*(X)$-module, $A^*(G) = \oplus_{\alpha\in\Lambda} A^*(X) \cdot b_\alpha$, where $b_\alpha \in A^{n_\alpha}(G)$ and for any $\alpha$ there is a unique $\alpha'$ such that $n_\alpha + n_{\alpha'} = dim\ G - dim\ X$ and $\pi_*(b_\alpha \cdot b_{\alpha'}) \neq 0$; suppose $\pi_*(b_\alpha \cdot b_{\alpha'}) = 1$. Moreover, denoting by $p_i : G \times_X G \to G$ $(i = 1, 2)$ the projections, assume that, for a smooth $G \times_X G$, the homomorphism $A^*(G) \otimes_{A^*(X)} A^*(G) \to A^*(G \times_X G)$, defined by $g \otimes h \mapsto p_1^* g \cdot p_2^* h$, is an isomorphism. Then*

*(i) The class of the diagonal $\Delta$ in $G \times_X G$ equals $[\Delta] = \sum_{\alpha,\beta} m_{\alpha\beta} b_\alpha \otimes b_\beta$, where, for any $\alpha, \beta$, $m_{\alpha\beta} = P_{\alpha\beta}(\{\pi_*(b_\kappa \cdot b_\lambda)\})$ for some polynomial $P_{\alpha\beta} \in \mathbb{Z}[\{x_{\kappa\lambda}\}]$.*

*(ii) If $\pi_*(b_\alpha \cdot b_\beta) \neq 0$ iff $\beta = \alpha'$, then the class of the diagonal $\Delta \subset G \times_X G$ equals $[\Delta] = \sum_\alpha b_\alpha \otimes b_{\alpha'}$.*

*Proof.* Denote by $\delta : G \to G \times_X G$, $\delta' : G \to G \times_K G$ (the Cartesian product) the diagonal embeddings and by $\gamma$ the morphism $\pi \times_X \pi : G \times_X G \to X$. For $g, h \in A^*(G)$ we have

$$\pi_*(g \cdot h) = \pi_*\Big((\delta')^*(g \times h)\Big) = \pi_*\Big(\delta^*(g \otimes h)\Big) = \gamma_*\delta_*\Big(\delta^*(g \otimes h)\Big) = \gamma_*\Big([\Delta] \cdot (g \otimes h)\Big),$$

using $\pi = \gamma \circ \delta$ and standard properties of intersection theory ([F]). Hence, writing $[\Delta] = \sum m_{\mu\nu} b_\mu \otimes b_\nu$, we get

$$\begin{aligned} \pi_*(b_\alpha \cdot b_\beta) &= \gamma_*\big([\Delta] \cdot (b_\alpha \otimes b_\beta)\big) = (\pi_* \otimes \pi_*)\Big(\big(\sum m_{\mu\nu} b_\mu \otimes b_\nu\big) \cdot (b_\alpha \otimes b_\beta)\Big) \\ &= \sum_{\mu,\nu} m_{\mu\nu} \pi_*(b_\mu \cdot b_\alpha) \cdot \pi_*(b_\nu \cdot b_\beta). \end{aligned} \tag{*}$$

(i) By the assumption and (*) with $\alpha$ replaced $\alpha'$ and $\beta$ – by $\beta'$, we get

$$m_{\alpha\beta} = \pi_*(b_{\alpha'} \cdot b_{\beta'}) - \sum_{\mu \neq \alpha, \nu \neq \beta} m_{\mu\nu} \pi_*(b_\mu \cdot b_{\alpha'}) \cdot \pi_*(b_\nu \cdot b_{\beta'}). \tag{**}$$

where the degree of $m_{\mu\nu} \in A^*(X)$ such that $\mu \neq \alpha$ or $\nu \neq \beta$ and $\pi_*(b_\mu \cdot b_{\alpha'}) \cdot \pi_*(b_\nu \cdot b_{\beta'}) \neq 0$, is smaller than the degree of $m_{\alpha\beta}$. The assertion now follows by induction on the degree of $m_{\alpha\beta}$.

(ii) By virtue of the assumption, Equation (**) now reads $\pi_*(b_{\alpha'} \cdot b_{\beta'}) = m_{\alpha\beta}$ and immediately implies the assertion. $\square$

Let us remark that in [DC-P, Proposition 2], where a weaker variant of Theorem 9.6 of the present paper is used, already assertion (i) (plus results of Section 5) are sufficient to conclude the proof.

Let now $G = LG_n V \to X$ denote a Lagrangian Grassmannian bundle.

**Proposition 2.5.** *The class of the diagonal of the Lagrangian Grassmannian bundle in $A^*(G \times_X G)$ equals*

$$[\Delta] = \sum \widetilde{Q}_I(R_1^\vee)_{GG} \cdot \widetilde{Q}_{\rho_n \smallsetminus I}(R_2^\vee)_{GG},$$

*the sum over all strict $I \subset \rho_n$, $GG = G \times_X G$ and $R_i$, $i = 1, 2$, are the tautological (sub)bundles on the corresponding factors.*

*Proof.* The assertion follows from Lemma 2.4(ii) applied to $b_I = \widetilde{Q}_I R^\vee$ ($I$ strict $\subset \rho_n$) and Theorem 5.23(i) which will be proved (independently) later. $\square$

**Corollary 2.6.** *With the notation of Section 1 and $G\mathcal{F} := G \times_X \mathcal{F}$, the class of $Z$ in $A^*(G\mathcal{F})$ (i.e. the image of the class of the diagonal of $G \times_X G$ via $(1 \times \alpha)^*$) equals*

$$\sum_{strict\ I \subset \rho_n} \widetilde{Q}_I(D_{G\mathcal{F}}^\vee) \cdot \widetilde{Q}_{\rho_n \smallsetminus I}(R_{G\mathcal{F}}^\vee).$$

Thus the problem of computing the classes of the $\Omega(a_\bullet)$'s is essentially that of calculation $p_*(\widetilde{Q}_I D^\vee)$ where $p : \mathcal{F} \to X$ is the projection map; then we use a base change.

Consider now the case of the orthogonal Grassmannian parametrizing rank $n$ subbundles of $V$, where *rank* $V = 2n+1$. The results of Lemma 2.3, Proposition 2.5 and Corollary 2.6 translate mutatis mutandis to this case with $\widetilde{Q}$-polynomials replaced by $\widetilde{P}$-polynomials (using essentially Theorem 5.23(ii) ). Thus the problem of computing the classes of the $\Omega(a_\bullet)$'s is essentially that of calculation $p_*(\widetilde{P}_I D^\vee)$ where $p : \mathcal{F} \to X$ is the projection.

Finally, consider the even orthogonal case. Supppose that $V$ is a vector bundle of *rank* $V = 2n$ endowed with a nondegenerate orthogonal form. Let $G = OG'_n V$ or $G = OG''_n V$ following the notation of Section 1. The even orthogonal analog of Proposition 2.5 and Corollary 2.6 is obtained using Theorem 5.23(iii) and reads as follows:

**Proposition 2.7.** *The class of the diagonal of the Grassmannian bundle in* $A^*(G \times_X G)$ *equals*

$$[\Delta] = \sum \widetilde{P}_I(R_1^\vee)_{GG} \cdot \widetilde{P}_{\rho_{n-1} \smallsetminus I}(R_2^\vee)_{GG},$$

*the sum over all strict* $I \subset \rho_{n-1}$, $GG = G \times_X G$ *and* $R_i$, $i = 1, 2$, *are the tautological (sub)bundles on the corresponding factors. With* $G\mathcal{F} := G \times_X \mathcal{F}'$ *(resp.* $G\mathcal{F} := G \times_X \mathcal{F}''$*), the class of* $Z$ *in* $A^*(G\mathcal{F})$ *(i.e. the image of the class of the diagonal of* $G \times_X G$ *via* $(1 \times \alpha)^*$*) equals*

$$\sum_{strict\ I \subset \rho_{n-1}} \widetilde{P}_I(D_{G\mathcal{F}}^\vee) \cdot \widetilde{P}_{\rho_{n-1} \smallsetminus I}(R_{G\mathcal{F}}^\vee).$$

Thus the problem of computing the classes of the $\Omega(a_\bullet)$'s is essentially that of calculation $p'_*(\widetilde{P}_I D^\vee)$ and $p''_*(\widetilde{P}_I D^\vee)$ where $p' : \mathcal{F}' \to X$ and $p'' : \mathcal{F}'' \to X$ are the projection maps.

## 3. Subbundles intersecting an $n$-subbundle in dim $\geqslant k$

We will now show an explicit computation in case $a_\bullet = (n{-}k{+}1, n{-}k{+}2, \dots, n)$ . This computation relies on a simple linear algebra argument. Another proof of Proposition 3.1, using the algebra of divided differences, will be given in Section 8.

We start with the Lagrangian case and follow the notation from Section 1.

**Proposition 3.1.** *Assume* $a_\bullet = (n{-}k{+}1, \dots, n)$ . *Let* $I \subset \rho_n$ *be a strict partition. If* $(n, n{-}1, \dots, k{+}1) \not\subset I$, *then* $p_* \widetilde{Q}_I D^\vee = 0$ . *In the opposite case, write* $I = (n, n{-}1, \dots, k+1, j_1, \dots, j_l)$, *where* $j_l > 0$ *and* $l \leqslant k$ . *Then* $p_* \widetilde{Q}_I D^\vee = \widetilde{Q}_{j_1, \dots, j_l} V_n^\vee$ .

*Proof.* It suffices to prove the formula for a vector bundle $V \to B$ endowed with a symplectic form, $X$ equal to $LG_n V$ and $V_n$ equal to the tautological subbundle on $LG_n V$. (Recall that $\Omega(n-k+1, \dots, n; V_\bullet)$ depends only on $V_n$ ; more precisely, it parametrizes Lagrangian rank $n$ subbundles $L$ of $V$ such that *rank*$(L \cap V_n) \geqslant k$ .) The variety $\mathcal{F}$ in this case parametrizes triples $(L, M, N)$ of vector bundles over $B$ such that $L$ and $N$

are Lagrangian rank $n$ subbundles of $V$ and $M$ is a rank $k$ subbundle of $L\cap N$. Let $W_\bullet$: $W_1\subset W_2\subset\ldots\subset W_n$ be a flag of Lagrangian subbundles of $V$ with $rank\ W_i=i$. For a partition $J=(j_1>\ldots>j_l>0)\subset\rho_k$ ,

$$\alpha_J=\Omega(n+1-j_1,\ldots,n+1-j_l;W_\bullet)=\big\{L\in X|\ rank(L\cap W_{n+1-j_h})\geqslant h,\ h=1,\ldots,l\big\}$$

defines a Schubert cycle whose class has the dominant term (w.r.t. $V_n$) equal to $\widetilde{Q}_JV_n^\vee\in A^*(X)$. It is well known that $\alpha_J$ is an irreducible subvariety of $X$ provided $B$ is irreducible.

Similarly, for a partition $I=(i_1>\ldots>i_l>0)\subset\rho_n$, $q:\mathcal{F}\to LG_nV$ the projection on the third factor,

$$A_I=q^*\Omega(n+1-i_1,\ldots,n+1-i_l;W_\bullet)=$$
$$=\big\{(L,M,N)\in\mathcal{F}|\ rank(N\cap W_{n+1-i_h})\geqslant h,\ h=1,\ldots,l\big\}$$

defines a cycle whose class has the dominant term (w.r.t. $D$) equal to $\widetilde{Q}_ID^\vee\in A^*(\mathcal{F})$. Also, $A_I$ is an irreducible subvariety of $\mathcal{F}$ provided $B$ is irreducible.

We will show (the push-forward is taken on the cycles level) that:

**1)** If $I\not\supset(n,n-1,\ldots,k+1)$ then $p_*A_I=0$. Passing to the rational equivalence classes, this implies $p_*\widetilde{Q}_ID^\vee=0$.

**2)** If $I\supset(n,n-1,\ldots,k+1)$ i.e. $I=(n,n-1,\ldots,k+1,j_1,\ldots,j_l)$, where $j_l>0$ and $l\leqslant k$, then $p_*A_I=\alpha_J$ where $J=(j_1,...,j_l)$. Then, passing to the rational equivalence classes (and using the projection formula), we get the following equality involving the dominant terms: $p_*\widetilde{Q}_ID^\vee=\widetilde{Q}_JV_n^\vee$.

Observe that **1)** holds if $l(I)\leqslant n-k$ because we then have $codim_{\mathcal{F}}A_I=|I|<n+(n-1)+\ldots+(k+1)$, which is the dimension of the fiber of $p$. We will need the following:

Claim Let $I\subset\rho_n$ be a strict partition. Let $l=card\{h\mid i_{n-k+h}\neq 0\}$. Assume that $l>0$. Then one has

$$(*)\qquad p(A_I)\subset\alpha_{i_{n-k+1},i_{n-k+2},\ldots,i_{n-k+l}}.$$

Indeed, for $(L,-,N)\in A_I$, since $rank(L\cap N)\geqslant k$, the inequality $rank(N\cap W_r)\geqslant h$ implies $rank(L\cap W_r)\geqslant h-(n-k)$ for every $h,r$; this gives (*).

**1)** To prove this assertion we first use (*) (by the above remark we can assume that $l(I)>n-k$) and thus get

$$codim_{\mathcal{F}}A_I-codim_Xp(A_I)\leqslant(i_1+\ldots+i_{n-k+l})-(i_{n-k+1}+\ldots+i_{n-k+l})=i_1+\ldots+i_{n-k}.$$

Then, since $I\not\supset(n,n-1,\ldots,k+1)$, we have

$$i_1+\ldots+i_{n-k}<n+\ldots+(k+1),$$

where the last number is the dimension of the fiber of $p$. Hence comparison of the latter inequality with the former yields $p_*A_I = 0$.

**2)** To prove this, it suffices to show $p(A_I) \subset \alpha_J$, $dim\ A_I = dim\ \alpha_J$; and if $p_*A_I = d \cdot \alpha_J$ for some $d \in \mathbb{Z}$ then $d = 1$. We have:

$p(A_I) \subset \alpha_J$ : this is a direct consequence of (*).

$dim\ A_I = dim\ \alpha_J$ : this results from comparison of the following three formulas $dim\ \mathcal{F} = dim\ X + k(n-k) + (n-k)(n-k+1)/2$, $codim_X \alpha_J = |J|$, and $codim_{\mathcal{F}} A_I = n + \ldots + (k+1) + |J|$.

Therefore $p_*A_I = d \cdot \alpha_J$ for some integer $d$. To show $d = 1$ it suffices to find an open subset $U \subset \alpha_J$ such that $p|_{p^{-1}U}\ :\ p^{-1}U \to U$ is an isomorphism. We define the open subset $U$ in question as $\alpha_J \setminus \Omega(n-k; W_\bullet)$. More explicitly, $U$ is defined by the conditions:

$$rank(L \cap W_{n+1-j_1}) \geqslant 1,\ \ldots,\ rank(L \cap W_{n+1-j_l}) \geqslant l \ \text{ and } \ L \cap W_{n-k} = (0).$$

Observe that these conditions really define an open nonempty subset of $\alpha_J$ because $\Omega(n+1-j_1, \ldots, n+1-j_l; W_\bullet) \not\subset \Omega(n+1-(k+1); W_\bullet)$ for $J \subset \rho_k$. (Recall that for $I = (i_1 > \ldots > i_l > 0)$, $J = (j_1 > \ldots > j_{l'} > 0)$ one has $\Omega(n+1-i_1, \ldots, n+1-i_l; W_\bullet) \subset \Omega(n+1-j_1, \ldots, n+1-j_{l'}; W_\bullet)$ iff $I \supset J$. )

Since our problem of showing that $d = 1$ is of local nature, we can assume that $B$ is a point and deal with vector spaces instead of vector bundles. Let us choose a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ such that, denoting the symplectic form by $\Phi$, we have $\Phi(e_i, e_j) = 0 = \Phi(f_i, f_j)$ and $\Phi(e_i, f_j) = -\Phi(f_j, e_i) = \delta_{i,j}$. Assume that $W_i$ is generated by the first $i$ vectors of $\{e_j\}$. Let $W^i$ be the subspace generated by the last $i$ vectors of $\{e_j\}$. Moreover, let $\widetilde{W}_i$ be the subspace generated by the first $i$ vectors of $\{f_j\}$ and $\widetilde{W}^i$ be the subspace generated by the last $i$ vectors of $\{f_j\}$.

Observe that for a strict partition $\rho_n \supset I \supset (n, n-1, \ldots, k+1)$ a necessary condition for "$(-,-,N) \in A_I$" is "$N \supset W_{n-k}$". (This corresponds to the first $(n-k)$ Schubert conditions defining $A_I$.) On the other hand, if $L \in U$ then $L \cap W_{n-k} = (0)$ and consequently $L$ must contain $\widetilde{W}_{n-k}$ (from the rest, i.e. $W^k \oplus \widetilde{W}^k$, we can get at most $k-$dimensional isotropic subspace). Hence also $|L \cap \big(W^k \oplus \widetilde{W}^k\big)| = k$ ($| - |$ denotes the dimension). We conclude that a necessary choice for an $n$-dimensional Lagrangian subspace $N$ such that $(L, M, N) \in A_I$ for some $M$, is

$$N := W_{n-k} \oplus \big(L \cap (W^k \oplus \widetilde{W}^k)\big).$$

It follows from the above discussion that $N$ is really a Lagrangian subspace of dimension $n$ and it satisfies the first $(n-k)$ Schubert conditions defining $A_I$. $N$ also satisfies the last $l$ ($\leqslant k$) Schubert conditions defining $A_I$ : since $|L \cap W_{n+1-j_h}| \geqslant h$ and $L \cap W_{n-k} = (0)$, we have $|N \cap W_{n+1-j_h}| = |W_{n-k}| + h \geqslant n-k+h$ for $h = 1, \ldots, l$.

Moreover, since $|L \cap N| = k$, the subspace $M$ above is determined uniquely: $M = L \cap N$.

Summing up, we have shown that $d = 1$; this ends the proof of **2)**.

Thus the proposition has been proved. $\square$

**Proposition 3.2.** *One has in $A^*(G)$,*

$$[\Omega(n-k+1,\dots,n)] = \sum_{strict\ I\subset\rho_k} \widetilde{Q}_I(V_n^\vee)_G \cdot \widetilde{Q}_{\rho_k\smallsetminus I}(R^\vee)\ .$$

*Proof.* This formula is obtained directly by pushing forward via $(p_1)_*$ the class of $Z$ in $A^*(G\mathcal{F})$ given by

$$\sum_{strict\ I\subset\rho_n} \widetilde{Q}_I(D_{G\mathcal{F}}^\vee)\cdot \widetilde{Q}_{\rho_n\smallsetminus I}(R_{G\mathcal{F}}^\vee)$$

(see Corollary 2.6), with the help of Proposition 3.1. □

**Example 3.3.** For successive $k$ (and any $n$) the formula reads (with $D = D_{G\mathcal{F}},\ R = R_{G\mathcal{F}}$ for brevity):

k=1 $\widetilde{Q}_1D^\vee + \widetilde{Q}_1R^\vee$;

k=2 $\widetilde{Q}_{21}D^\vee + \widetilde{Q}_2D^\vee\cdot\widetilde{Q}_1R^\vee + \widetilde{Q}_1D^\vee\cdot\widetilde{Q}_2R^\vee + \widetilde{Q}_{21}R^\vee$;

k=3 $\widetilde{Q}_{321}D^\vee + \widetilde{Q}_{32}D^\vee\cdot\widetilde{Q}_1R^\vee + \widetilde{Q}_{31}D^\vee\cdot\widetilde{Q}_2R^\vee + \widetilde{Q}_{21}D^\vee\cdot\widetilde{Q}_3R^\vee + \widetilde{Q}_3D^\vee\cdot\widetilde{Q}_{21}R^\vee + \widetilde{Q}_2D^\vee\cdot\widetilde{Q}_{31}R^\vee + \widetilde{Q}_1D^\vee\cdot\widetilde{Q}_{32}R^\vee + \widetilde{Q}_{321}R^\vee$.

In the odd orthogonal case, the analogs of Propositions 3.1 and 3.2 are obtained by replacing $\widetilde{Q}$-polynomials by $\widetilde{P}$-polynomials.

**Proposition 3.4.** *(i) Assume $a_\bullet = (n-k+1, n-k+2,\dots,n)$. Let $I\subset\rho_n$ be a strict partition. If $(n,n-1,\dots,k+1)\not\subset I$, then $p_*\widetilde{P}_ID^\vee = 0$ . In the opposite case, write $I = (n,n-1,\dots,k+1,j_1,\dots,j_l)$, where $j_l>0$ and $l\leqslant k$ . Then $p_*\widetilde{P}_ID^\vee = \widetilde{P}_{j_1,\dots,j_l}V_n^\vee$ .*
*(ii) One has in $A^*(OG_nV)$,*

$$[\Omega(n-k+1,\dots,n)] = \sum_{strict\ I\subset\rho_k} \widetilde{P}_I(V_n^\vee)_G \cdot \widetilde{P}_{\rho_k\smallsetminus I}(R^\vee)\ .$$

Assertion (ii) follows from (i) like Proposition 3.2 follows from Proposition 3.1. The proof of (i) is essentially the same as the one of Proposition 3.1. More precisely, in the proof of (i), $\alpha_J$ and $A_I$ are defined in the same way as in the proof of this proposition. Also, the whole reasoning is the same, word by word, except of the following one point. To prove that $d=1$ one chooses now a basis $e_1,\dots,e_n,f_1,\dots,f_n,g$ such that denoting the orthogonal form by $\Phi$, we have $\Phi(e_i,e_j) = \Phi(f_i,f_j) = \Phi(e_i,g) = \Phi(f_j,g) = 0$, $\Phi(e_i,f_j) = \Phi(f_j,e_i) = \delta_{i,j}$ and $\Phi(g,g)=1$. Then $W_i$, $W^i$, $\widetilde{W}_i$ and $\widetilde{W}^i$ defined like in the proof of Proposition 3.1 allow us to show that $d=1$ exactly in the same way as in the proof of this proposition.

Let us pass now to the even orthogonal case. So let $V\to X$ ($X$ is connected) be a rank $2n$ vector bundle endowed with a nondegenerate quadratic form. Fix an isotropic rank $n$ subbundle $V_n$ of $V$. Recall that for $k\equiv n\ (mod\ 2)$ by $p' : \mathcal{F}'\to X$ we denote the flag bundles parametrizing flags $A_1\subset A_2$ of subbundles of $V$ such that $rank\ A_1 = k$,

$rank\ A_2 = n$, $A_1 \subset V_n$ and $A_2$ is isotropic with $dim(A_2 \cap V_n)_x \equiv n\ (mod\ 2)$ for every $x \in X$. Similarly, for $k \equiv n+1\ (mod\ 2)$ by $p'' : \mathcal{F}'' \to X$ we denote the flag bundle parametrizing flags $A_1 \subset A_2$ of subbundles of $V$ such that $rank\ A_1 = k$, $rank\ A_2 = n$, $A_1 \subset V_n$ and $A_2$ is isotropic with $dim(A_2 \cap V_n)_x \equiv n+1\ (mod\ 2)$ for every $x \in X$.

In the even orthogonal case the analog of Proposition 3.1 reads:

**Proposition 3.5.** *Let $I \subset \rho_{n-1}$ be a strict partition. If $(n-1, n-2, \ldots, k) \not\subset I$ then $p'_* \widetilde{P}_I D^\vee = 0$. In the opposite case, write $I = (n-1, n-2, \ldots, k, j_1, \ldots, j_l)$, where $j_l > 0$ and $l \leqslant k-1$. Then*

$$p'_* \widetilde{P}_I D^\vee = \widetilde{P}_{j_1,\ldots,j_l} V_n^\vee .$$

*The same formula is valid for $p''_*$.*

*Proof.* We consider first the case of $p'_*$ i.e. $k \equiv n\ (mod\ 2)$. It suffices to prove the formula for a rank $2n$ vector bundle $V \to B$ (we assume that $B$ is irreducible) endowed with a nondegenerate orthogonal form, $X$ equal to $OG'_n V$ or $OG''_n V$ and $V_n$ equal to the tautological subbundle on $X$. Then the variety $\mathcal{F}'$ parametrizes triples $(L, M, N)$ such that $dim(L \cap N)_b \equiv n\ (mod\ 2)$ for every $b \in B$ (i.e. $L$ and $N$ either belong together to $OG'_n V$ or together to $OG''_n V$) and $M$ is a rank $k$ subbundle of $L \cap N$.

We will now prove the proposition for $X = OG'_n V$. (Obvious modifications lead to a proof in the case $X = OG''_n V$.) Since the strategy of proof is the same as in the Lagrangian case, we will skip those parts of the reasoning which have appeared already in the proof of Proposition 3.1. Let $W_\bullet : W_1 \subset W_2 \subset \ldots \subset W_n$ be an isotropic flag in $V$.

For $J = (j_1 > \ldots > j_l > 0) \subset \rho_{k-1}$ we define

$$\alpha_J = \Omega(n-j_1, \ldots, n-j_l; W_\bullet) \quad if \quad l \equiv n\ (mod\ 2)\ and$$
$$\alpha_J = \Omega(n-j_1, \ldots, n-j_l, n; W_\bullet) \quad if \quad l \equiv n+1\ (mod\ 2).$$

Similarly for $I = (i_1 > \ldots > i_l > 0) \subset \rho_{n-1}$, $q : \mathcal{F}' \to OG'_n V$ the projection on the third factor, we define

$$A_I = q^* \Omega(n-i_1, \ldots, n-i_l; W_\bullet) \quad if \quad l \equiv n\ (mod\ 2)\ and$$
$$A_I = q^* \Omega(n-i_1, \ldots, n-i_l, n; W_\bullet) \quad if \quad l \equiv n+1\ (mod\ 2).$$

It is known that $\alpha_J$ and $A_I$ are irreducible subvarieties provided $B$ is. The dominant terms of the classes of $\alpha_J$ and $A_I$ are equal to $\widetilde{P}_J V_n^\vee$ and $\widetilde{P}_I D^\vee$ respectively.

The proposition now follows from:

1) If $I \not\supset (n-1, n-2, \ldots, k)$ then $p'_* A_I = 0$.
2) If $I \supset (n-1, n-2, \ldots, k)$ i.e $I = (n-1, n-2, \ldots, k+1, k, j_1, \ldots, j_l)$, where $j_l > 0$ and $l \leqslant k-1$, then $p'_* A_I = \alpha_J$ where $J = (j_1, \ldots, j_l)$.

Assertion 1) (being obvious if $l(I) < n-k$) is a consequence of:

Claim: For every strict partition $I \subset \rho_{n-1}$, let $l = card\{h |\ i_{n-k+h} \neq 0\}$. Assume that $l > 0$. Then one has

$$p'(A_I) \subset \alpha_{i_{n-k+1},\ldots,i_{n-k+l}} . \tag{*}$$

Inclusion (*) also implies $p'(A_I) \subset \alpha_J$ in 2). The equality $dim\ p'(A_I) = dim\ \alpha_J$ now follows from: $dim\ \mathcal{F}' = dim\ X + k(n-k) + (n-k)(n-k-1)/2$, $codim_X \alpha_J = |J|$ and $codim_{\mathcal{F}'} A_I = (n-1) + \ldots + k + |J|$.

Therefore $p'_* A_I = d \cdot \alpha_J$ for some integer $d$. To prove that $d = 1$ it is sufficient to show an open subset $U \subset \alpha_J$ such that $p'|_{(p')^{-1}U}\colon (p')^{-1}U \to U$ is an isomorphism. The open subset $U$ in question parametrizes those $L \in \alpha_J$ for which $L \cap W_{n-k} = (0)$.

The problem being local, we can assume that $B$ is a point. Let $e_1, \dots, e_n, f_1, \dots, f_n$ be a basis of $V$ such that denoting the form by $\Phi$ we have $\Phi(e_i, e_j) = \Phi(f_i, f_j) = 0$, $\Phi(e_i, f_j) = \Phi(f_j, e_i) = \delta_{i,j}$ and $W_i$ is spanned by $e_1, \dots, e_i$. Define $W^i, \widetilde{W}_i$ and $\widetilde{W}^i$ as in the proof of Proposition 3.1.

Now, given $L \in U$, the unique $N$ such that $(L, M, N) \in A_I$ for some $M$, is defined also as in the proof of Proposition 3.1: $N := W_{n-k} \oplus \big(L \cap (W^k \oplus \widetilde{W}^k)\big)$.

This $N$ is isotropic and satisfies the first $n-k$ Schubert conditions because it contains $W_{n-k}$. Moreover, it satisfies the last $l (\leqslant k-1)$ Schubert conditions defining $A_I$: since $|L \cap W_{n-j_h}| \geqslant h$ and $L \cap W_{n-k} = (0)$, we have $|N \cap W_{n-j_h}| = |W_{n-k}| + h \geqslant n - k + h$ for $h = 1, \dots, l$. Finally, the $M$ above is determined uniquely: $M = L \cap N$, and $p'|_{(p')^{-1}U}$ is an isomorphism.

We next consider the case of $p''_*$, i.e. $k \equiv n+1$ $(mod\ 2)$. It suffices to prove the formula for a rank $2n$ vector bundle $V \to B$ ($B$ is irreducible) endowed with a nondegenerate orthogonal form, $X$ equal to $OG'_n V$ or $OG''_n V$ and $V_n$ equal to the tautological subbundle on $X$. Then the variety $\mathcal{F}''$ parametrizes triples $(L, M, N)$ such that $dim(L \cap N)_b \equiv n+1$ $(mod\ 2)$ for every $b \in B$ (i.e. $L$ and $N$ belong to different components $OG'_n V$ and $OG''_n V$) and $M$ is a rank $k$ subbundle of $L \cap N$.

We will prove the proposition for $X = OG''_n V$. (Obvious modifications lead to a proof in the case $X = OG'_n V$.) Let $W_\bullet : W_1 \subset W_2 \subset \ldots \subset W_n$ be an isotropic flag in $V$. For $J = (j_1 > \ldots > j_l > 0) \subset \rho_{k-1}$ we define

$$\alpha_J = \Omega(n - j_1, \dots, n - j_l; W_\bullet) \quad if \quad l \equiv n+1 \ (mod\ 2) \ and$$
$$\alpha_J = \Omega(n - j_1, \dots, n - j_l, n; W_\bullet) \quad if \quad l \equiv n \ (mod\ 2).$$

Similarly for $I = (i_1 > \ldots > i_l > 0) \subset \rho_{n-1}$, $q : \mathcal{F}'' \to OG'_n V$ the projection on the third factor, we define

$$A_I = q^* \Omega(n - i_1, \dots, n - i_l, n; W_\bullet) \quad if \quad l \equiv n+1 \ (mod\ 2) \ and$$
$$A_I = q^* \Omega(n - i_1, \dots, n - i_l; W_\bullet) \quad if \quad l \equiv n \ (mod\ 2).$$

The dominant terms of the classes of $\alpha_J$ and $A_I$ are equal to $\widetilde{P}_J V_n^\vee$ and $\widetilde{P}_I D^\vee$ respectively. The proposition now follows from:

1) If $I \not\supset (n-1, n-2, \dots, k)$ then $p''_* A_I = 0$.
2) If $I \supset (n-1, n-2, \dots, k)$ i.e $I = (n-1, n-2, \dots, k+1, k, j_1, \dots, j_l)$, where $j_l > 0$ and $l \leqslant k-1$, then $p''_* A_I = \alpha_J$ where $J = (j_1, \dots, j_l)$.

The proof of these assertions is analogous to the one above. For every strict partition $I \subset \rho_{n-1}$ and $l = card\{h|\ i_{n-k+h} \neq 0\} > 0$, one has $p''(A_I) \subset \alpha_{i_{n-k+1}, \dots, i_{n-k+l}}$, which implies 1) and $p''(A_I) \subset \alpha_J$ in 2); moreover, for the dimensions reasons we have $p''_* A_I = d \cdot \alpha_J$ for some integer $d$. One finishes the proof like in the case of $p'_*$, by showing that $p''|_{(p'')^{-1}U}\colon (p'')^{-1}U \to U$ is an isomorphism, where an open subset $U \subset \alpha_J$ parametrizes those $L \in \alpha_J$ for which $L \cap W_{n-k} = (0)$. Hence $d = 1$ and the proof is complete. $\square$

**Proposition 3.6.** *If $k \equiv n$ (mod 2) (resp. $k \equiv n+1$ (mod 2) ) then one has in $A^*(OG'_nV)$ (resp. in $A^*(OG''_nV)$ ),*

$$[\Omega(n-k+1,\dots,n)] = \sum_{strict\ I\subset\rho_{k-1}} \widetilde{P}_I(V_n^\vee)_G \cdot \widetilde{P}_{\rho_{k-1}\smallsetminus I}(R^\vee).$$

*Proof.* This formula is obtained directly by pushing forward via $p'_*$ (resp. $p''_*$) the class of $Z$ in $A^*(G\mathcal{F})$ given by

$$\sum_{strict\ I\subset\rho_{n-1}} \widetilde{P}_I\big(D_{G\mathcal{F}}^\vee\big) \cdot \widetilde{P}_{\rho_{n-1}\smallsetminus I}\big(R_{G\mathcal{F}}^\vee\big),$$

where $G\mathcal{F} = G\times_X \mathcal{F}'$ (resp. $G\mathcal{F} = G\times_X \mathcal{F}''$) , using Propositions 2.7 and 3.5. □

## 4. $\widetilde{Q}$-polynomials and their properties

In this section we define a family of symmetric polynomials modelled on Schur's $Q$-polynomials. In Schur's Pfaffian-definition (see [S]), we replace $Q_i$ by $e_i$ – the $i$-th elementary symmetric polynomial. After this modification one gets an interesting family of symmetric polynomials $\widetilde{Q}_I$ (indexed by all partitions) whose properties are studied in this section and then applied in the next ones. It turns out that $\widetilde{Q}_I$ is the Young dual (in sense of the involution $\omega$ of [Mcd1, I.2.(2.7)] to the Hall–Littlewood polynomial $\widetilde{Q}_I(Y;q)$ where the alphabet $Y$ is equal to $X_n/(1-q)$ in the sense of $\lambda$-rings, specialized with $q=-1$ ([L-L-T], [D-L-T]). Though most of the properties of the $\widetilde{Q}_I$ given in this section can be deduced from the theory of Hall–Littlewood polynomials, we give here their proofs using the Pfaffian definition. The only exception is made for the Pieri-type formula which is deduced from the one for Hall–Littlewood polynomials.

Let $X=(x_1,x_2,\dots)$ be a sequence of independent variables. Denote by $X_n$ the subsequence $(x_1,\dots,x_n)$. We set $\widetilde{Q}_i(X_n) := e_i(X_n)$ – the $i$-th elementary symmetric polynomial in $X_n$. Given two nonnegative integers $i,j$ we define

$$\widetilde{Q}_{i,j}(X_n) = \widetilde{Q}_i(X_n)\widetilde{Q}_j(X_n) + 2\sum_{p=1}^{j}(-1)^p\widetilde{Q}_{i+p}(X_n)\widetilde{Q}_{j-p}(X_n)$$

Finally, for any (i.e. not necessary strict) partition $I=(i_1\geqslant i_2\geqslant\dots\geqslant i_k\geqslant 0)$, with even $k$ (by putting $i_k=0$ if necessary), we set

$$\widetilde{Q}_I(X_n) = Pf\Big(\widetilde{Q}_{i_p,i_q}(X_n)\Big)_{1\leqslant p<q\leqslant k}.$$

Equivalently (in full analogy to [S, pp.224–225]), $\widetilde{Q}_I(X_n)$ is defined recurrently on $l(I)$, by putting for odd $l(I)$,

$$(*)\qquad \widetilde{Q}_I(X_n) = \sum_{j=1}^{l(I)}(-1)^{j-1}\widetilde{Q}_{i_j}(X_n)\widetilde{Q}_{I\smallsetminus\{i_j\}}(X_n),$$

and for even $l(I)$,

$$(**)\qquad \widetilde{Q}_I(X_n) = \sum_{j=2}^{l(I)} (-1)^j \widetilde{Q}_{i_1,i_j}(X_n) \widetilde{Q}_{I\smallsetminus\{i_1,i_j\}}(X_n).$$

The latter case, with $l = l(I)$, can be rewritten as

$$(***)\qquad \widetilde{Q}_I(X_n) = \sum_{j=1}^{l-1} (-1)^{j-1} \widetilde{Q}_{i_j,i_l}(X_n) \widetilde{Q}_{I\smallsetminus\{i_j,i_l\}}(X_n).$$

Note that assuming formally $i_l = 0$, the relation (***) specializes to (*). We will refer to the above equations as to Laplace-type developements or simply recurrent formulas. (Invoking the raising operators $R_{ij}$ ([Mcd1,I], [D-L-T]) the above definition is rewritten

$$\widetilde{Q}_I(X_n) = \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}} e_I(X_n),$$

where $e_I(X_n)$ is the product of the elementary symmetric polynomials in $X_n$ associated with the parts of $I$.)

We start with a useful linearity-type formula for $\widetilde{Q}$-polynomials.

**Proposition 4.1.** *For any strict partition $I$ one has*

$$\widetilde{Q}_I(X_n) = \sum_{j=0}^{l(I)} x_n^j \Big( \sum_{|I|-|J|=j} \widetilde{Q}_J(X_{n-1}) \Big),$$

*where the sum is over all (i.e. not necessary strict) partitions $J \subset I$ such that $I/J$ has at most one box in every row. (Using the terminology of [Mcd1], this is equivalent to saying that $I/J$ is a vertical strip; note that $I/J$ is here also a horizontal strip.)*

*Proof.* We use induction on $l(I)$.

1° $l(I) = 1$. Since we have: $e_i(X_n) = e_i(X_{n-1}) + x_n e_{i-1}(X_{n-1})$, the assertion follows.

2° $l(I) = 2$. We have for $i > j > 0$ and with $e_i = e_i(X_n)$, $\bar{e}_i = e_i(X_{n-1})$, $\bar{e}_{-1} = 0$,

$$\widetilde{Q}_{i,j}(X_n) = e_i e_j + 2 \sum_{p=1}^{j} (-1)^p e_{i+p} e_{j-p} =$$

$$= (\bar{e}_i + x_n \bar{e}_{i-1})(\bar{e}_j + x_n \bar{e}_{j-1}) + 2 \sum_{p=1}^{j} (-1)^p (\bar{e}_{i+p} + x_n \bar{e}_{i+p-1})(\bar{e}_{j-p} + x_n \bar{e}_{j-p-1})$$

$$= \Big( \bar{e}_i \bar{e}_j + 2 \sum_{p=1}^{j} (-1)^p \bar{e}_{i+p} \bar{e}_{j-p} \Big) + x_n \Big[ \Big( \bar{e}_{i-1} \bar{e}_j + 2 \sum_{p=1}^{j} (-1)^p \bar{e}_{i-1+p} \bar{e}_{j-p} \Big) +$$

$$+ \Big( \bar{e}_i \bar{e}_{j-1} + 2 \sum_{p=1}^{j-1} (-1)^p \bar{e}_{i+p} \bar{e}_{j-1-p} \Big) \Big] + x_n^2 \Big( \bar{e}_{i-1} \bar{e}_{j-1} + 2 \sum_{p=1}^{j-1} (-1)^p \bar{e}_{i-1+p} \bar{e}_{j-1-p} \Big)$$

$$= \widetilde{Q}_{i,j}(X_{n-1}) + x_n\Big[\widetilde{Q}_{i-1,j}(X_{n-1}) + \widetilde{Q}_{i,j-1}(X_{n-1})\Big] + x_n^2\widetilde{Q}_{i-1,j-1}(X_{n-1}).$$

3° By the remarks before the proposition, to prove the assertion in general it suffices to show it by using the recurrent relation (***). (Note that the R.H.S. of the formula of the proposition specializes after the formal replacement $i_l := 0$ ($l = l(I)$) to the expression asserted for ($i_1 > i_2 > \ldots > i_{l-1}$).

So, let us assume that $l$ is even and set $\widetilde{Q}_I := \widetilde{Q}_I(X_n)$, $\bar{Q}_I = \widetilde{Q}_I(X_{n-1})$. Moreover, let $\mathcal{P}(I,j)$ be the set of all partitions $J \subset I$ such that $I \backslash J$ has at most one box in every row and $|I| - |J| = j$. We have by induction on $l$,

$$\widetilde{Q}_{I\backslash\{i_j,i_l\}} = \sum_{r=0}^{l-2} x_n^r \Bigg(\sum_{J\in\mathcal{P}(I\backslash\{i_j,i_l\},r)} \bar{Q}_J\Bigg).$$

Therefore, using 2° we have

$$\begin{aligned}\widetilde{Q}_I = &\sum_{j=1}^{l-1}(-1)^{j-1}\Big[\bar{Q}_{j_j,i_l} + x_n(\bar{Q}_{i_j-1,i_l} + \bar{Q}_{i_j,i_l-1}) + x_n^2\bar{Q}_{i_j-1,i_l-1}\Big] \\ &\times \left[\sum_{r=0}^{l-2} x_n^r\Big(\sum_{J\in\mathcal{P}(I\backslash\{i_j,i_l\},r)} \bar{Q}_J\Big)\right]\end{aligned}$$

On the other hand, apply the relation (***) to the R.H.S. of the formula in the proposition. One gets:

$$\sum_{j=0}^{l} x_n^j\Big(\sum_{J\in\mathcal{P}(I,j)} \bar{Q}_J\Big) = \sum_{j=0}^{l} x_n^j\left[\sum_{J\in\mathcal{P}(I,j)}\Big(\sum_{q=1}^{l-1}(-1)^{q-1}\bar{Q}_{j_q,j_l}\cdot\bar{Q}_{J\backslash\{j_q,j_l\}}\Big)\right].$$

It is straightforward to verify that both these sums contain $2^l(l-1)$ terms of the form

$$(-1)^s x^j \bar{Q}_{a,b}\bar{Q}_{c_1,\ldots,c_{l-2}},$$

and such a term appears in both sums if and only if

$$(c_1, \ldots, c_s, a, c_{s+1}, \ldots, c_{l-2}, b) \in \mathcal{P}(I,j).$$

Thus the assertion follows and the proof of the proposition is complete. □

**Proposition 4.2.** *:* $\widetilde{Q}_{i,i}(X_n) = e_i(x_1^2, \ldots, x_n^2)$.

*Proof.* By definition we have $\big(e_i = e_i(X_n)\big)$:

$$\widetilde{Q}_{i,i}(X_n) = e_ie_i - 2e_{i+1}e_{i-1} + 2e_{i+2}e_{i-2} - \ \ldots = \sum_{p=0}^{2i}(-1)^{p+i}e_pe_{2i-p}.$$

On the other hand, with an indeterminate $t$, we have

$$(1+x_1t)\dots(1+x_nt)(1-x_1t)\dots(1-x_nt) = (1-x_1^2t^2)\dots(1-x_n^2t^2),$$

or equivalently,

$$\Big(\sum e_pt^p\Big)\Big(\sum(-1)^qe_qt^q\Big) = \sum(-1)^ie_i(x_1^2,\dots,x_n^2)t^{2i},$$

which implies

$$(-1)^ie_i(x_1^2,\dots,x_n^2) = \sum_{p=0}^{2i}(-1)^pe_pe_{2i-p}.$$

Comparison of the last equation with the first one gives the assertion. $\square$

**Proposition 4.3.** *For partitions $I' = (i_1,i_2,\dots,j,j,\dots,i_{k-1},i_k)$ and $I = (i_1,\dots,i_k)$, the following equality holds*

$$\widetilde{Q}_{I'}(X_n) = \widetilde{Q}_{j,j}(X_n)\widetilde{Q}_I(X_n).$$

*Proof.* Write $\widetilde{Q}_I$ for $\widetilde{Q}_I(X_n)$. We use induction on $k$. For $k=0$, the assertion is obvious. For $k=1$, we have $\widetilde{Q}_{i,j,j} = \widetilde{Q}_i\widetilde{Q}_{j,j}$ and $\widetilde{Q}_{j,j,i} = \widetilde{Q}_{j,j}\widetilde{Q}_i$ by the Laplace type developements, so the assertion follows.

In general, it suffices to show the assertion inductively, using the relation (***), if the marked "$j$" does not appear on the last place; and independently, to prove it (inductively) for $I' = (i_1,\dots,i_k,j,j)$. In both instances $k$ is assumed to be even.

In the former case, using (***) we get

$$\begin{aligned}\widetilde{Q}_{I'} = {} & \widetilde{Q}_{i_1,i_k}\widetilde{Q}_{i_2,\dots,j,j,\dots,i_{k-1}} - \dots \pm \widetilde{Q}_{j,i_k}\widetilde{Q}_{i_1,i_2,\dots,j,\dots,i_{k-1}} \\ & \mp \widetilde{Q}_{j,i_k}\widetilde{Q}_{i_1,i_2,\dots,j,\dots,i_{k-1}} \pm \dots - \widetilde{Q}_{i_{k-1},i_k}\widetilde{Q}_{i_1,\dots,j,j,\dots,i_{k-2}}\end{aligned}$$

and the assertion follows from the induction assumption by using the relation (***) w.r.t. $\widetilde{Q}_{i_1,\dots,i_k}$ once again.

In the latter case we use the relation (**). We have

$$\begin{aligned}\widetilde{Q}_{i_1,\dots,i_k,j,j} = {} & \widetilde{Q}_{i_1,i_2}\widetilde{Q}_{i_3,\dots,i_k,j,j} - \dots + \widetilde{Q}_{i_1,i_k}\widetilde{Q}_{i_2,\dots,i_{k-1},j,j} \\ & - \widetilde{Q}_{i_1,j}\widetilde{Q}_{i_2,\dots,i_k,j} + \widetilde{Q}_{i_1,j}\widetilde{Q}_{i_2,\dots,i_k,j}\end{aligned}$$

and the assertion follows from the induction assumption by using the relation (**) w.r.t. $\widetilde{Q}_{i_1,\dots,i_k}$ once again. $\square$

**Lemma 4.4.** *Let $I = (i_1,i_2,\dots,i_k)$ be a partition. If $i_1 > n$ then $\widetilde{Q}_I(X_n) = 0$.*

*Proof.* We use induction on $l(I)$. For $l(I) = 1,2$ the assertion is obvious because $e_p(x_1,\dots,x_n) = 0$ for $p > n$. For bigger $l(I)$ one uses induction on the length and the recurrent formulas, which immediately imply the assertion. $\square$

**Example 4.5.** The following equalities hold: (in 1° and 2° we set $\widetilde{Q}_I := \widetilde{Q}_I(X_n)$ for brevity)

1° $\widetilde{Q}_{5544441} = \widetilde{Q}_{55}\widetilde{Q}_{44441} = \widetilde{Q}_{55}\widetilde{Q}_{44}\widetilde{Q}_{441} = \widetilde{Q}_{55}\widetilde{Q}_{44}\widetilde{Q}_{44}\widetilde{Q}_1 = \widetilde{Q}_{55}\widetilde{Q}_{4444}\widetilde{Q}_1$;

2° $\widetilde{Q}_{5554443331} = \widetilde{Q}_{55}\widetilde{Q}_{44}\widetilde{Q}_{33}\widetilde{Q}_{5431} = \widetilde{Q}_{554433}\widetilde{Q}_{5431}$;

3° Here, we set $\bar{Q}_I := \widetilde{Q}_I(x_1, x_2)$, $\bar{Q}'_I := \widetilde{Q}_I(x_1)$. Then

$$
\begin{aligned}
&\widetilde{Q}_{321}(x_1, x_2, x_3) = \\
&\quad = x_3\bar{Q}_{221} + x_3^2(\bar{Q}_{211} + \bar{Q}_{22}) + x_3^3\bar{Q}_{21} = x_3\bar{Q}_{22}\bar{Q}_1 + x_3^2(\bar{Q}_{11}\bar{Q}_2 + \bar{Q}_{22}) + x_3^3\bar{Q}_{21} \\
&\quad = x_3 e_2(x_1^2, x_2^2)(x_1 + x_2) + x_3^2\big[e_1(x_1^2, x_2^2)x_1x_2 + e_2(x_1^2, x_2^2)\big] + x_3^3\big(x_2\bar{Q}'_{11} + x_2^2\bar{Q}'_1\big) \\
&\quad = x_3(x_1^2x_2^2)(x_1 + x_2) + x_3^2\big[(x_1^2 + x_2^2)x_1x_2 + x_1^2x_2^2\big] + x_3^3\big(x_2x_1^2 + x_2^2x_1\big).
\end{aligned}
$$

By iterating the linearity formula for $\widetilde{Q}_I(X_n)$ (Proposition 4.1), we get the following algorithm for decomposition of $\widetilde{Q}_I = \widetilde{Q}_I(X_n)$ into a sum of monomials:

1. If $I$ is not strict, we factorize

$$\widetilde{Q}_I = \widetilde{Q}_{k_1,k_1} \cdot \widetilde{Q}_{k_2,k_2} \cdot \ldots \cdot \widetilde{Q}_{k_l,k_l} \cdot \widetilde{Q}_L,$$

where $L$ is strict (we use Proposition 4.3).

2. We apply the linearity formula to $\widetilde{Q}_L(X_n)$ and $x_n$. Also, we decompose

$$
\begin{aligned}
\widetilde{Q}_{k_p,k_p}(X_n) &= e_{k_p}\big(x_1^2, \ldots, x_n^2\big) \\
&= e_{k_p}\big(x_1^2, \ldots, x_{n-1}^2\big) + e_{k_p-1}\big(x_1^2, \ldots, x_{n-1}^2\big)x_n^2 \\
&= \widetilde{Q}_{k_p,k_p}(X_{n-1}) + \widetilde{Q}_{k_p-1,k_p-1}(X_{n-1})x_n^2.
\end{aligned}
$$

We then repeat 1 and 2 with the so obtained $\widetilde{Q}_I(X_{n-1})$'s, thus extracting $x_{n-1}$; then, we proceed similarly with the so obtained $\widetilde{Q}_I(X_{n-2})$'s etc.

Note that if we stop this procedure after extracting the variables $x_n, x_{n-1}, \ldots, x_{m+1}$ we get a development:

$$(*) \qquad \widetilde{Q}_I(X_n) = \sum_J \widetilde{Q}_J(X_m)F_J(x_{m+1}, ..., x_n) \ ,$$

where the sum is over $J \subset I$ (this follows from the linearity formula; $J$ are not necessary strict).

It follows from the above algorithm that $\widetilde{Q}_I(X_n)$ is a positive sum of monomials. It is, in general, not a positive sum of Schur $S$-polynomials in $X_n$ (we refer the reader to [Mcd1, I] and [L-S1] for a definition and properties of Schur $S$-polynomials). Here comes an example computed with the help of SYMMETRICA [K-K-L].

**Example 4.6.** Let $n = 5$, $\widetilde{Q}_I = \widetilde{Q}_I(X_5)$ and $s_J = s_J(X_5)$. We have:

$$\widetilde{Q}_{54} = s_{22221} \quad \widetilde{Q}_{53} = s_{22211} \quad \widetilde{Q}_{52} = s_{22111} \quad \widetilde{Q}_{51} = s_{21111}$$

$$\widetilde{Q}_{43} = s_{2221} - s_{22111} \quad \widetilde{Q}_{42} = s_{2211} - s_{21111} \quad \widetilde{Q}_{41} = s_{2111} - s_{11111}$$

$$\widetilde{Q}_{32} = s_{221} - s_{2111} + s_{11111} \quad \widetilde{Q}_{31} = s_{211} - s_{1111}$$

$$\widetilde{Q}_{21} = s_{21} - s_{111}$$

$$\widetilde{Q}_{543} = s_{33321} - s_{33222} \quad \widetilde{Q}_{542} = s_{33221} - s_{32222} \quad \widetilde{Q}_{541} = s_{32221} - s_{22222}$$

$$\widetilde{Q}_{532} = s_{33211} - s_{32221} + s_{22222} \quad \widetilde{Q}_{531} = s_{32211} - s_{22221}$$

$$\widetilde{Q}_{521} = s_{32111} - s_{22211}$$

$$\widetilde{Q}_{432} = s_{3321} - s_{3222} - s_{33111} \quad \widetilde{Q}_{431} = s_{3221} - s_{32111} - s_{2222}$$

$$\widetilde{Q}_{421} = s_{3211} - s_{31111} - s_{2221}$$

$$\widetilde{Q}_{321} = s_{321} - s_{3111} - s_{222}$$

$$\widetilde{Q}_{5432} = s_{44321} - s_{44222} - s_{43331} \quad \widetilde{Q}_{5431} = s_{43321} - s_{43222} - s_{33331}$$

$$\widetilde{Q}_{5421} = s_{43221} - s_{42222} - s_{33321}$$

$$\widetilde{Q}_{5321} = s_{43211} - s_{42221} - s_{33311}$$

$$\widetilde{Q}_{4321} = s_{4321} - s_{43111} - s_{4222} - s_{3331} + s_{32221} - 2s_{22222}$$

$$\widetilde{Q}_{54321} = s_{54321} - s_{54222} - s_{53331} - s_{44421} + s_{43332} - 2s_{33333}.$$

We denote by $S\mathcal{P}(X_n)$ the ring of symmetric polynomials in $X_n$.

**Proposition 4.7.** *The set* $\{\widetilde{Q}_I(X_n)\}$ *indexed by all partitions such that* $i_1 \leqslant n$ *forms an additive basis of* $S\mathcal{P}(X_n)$*. Moreover, for any commutative ring* $\mathcal{R}$*, the same set is a basis of a free* $\mathcal{R}$*-module* $S\mathcal{P}(X_n) \otimes \mathcal{R}$*.*

*Proof.* Let us compare the family $\{\widetilde{Q}_I(X_n)\}$ with the $\mathcal{R}$-basis $\{e_I(X_n)\}$ of $S\mathcal{P}(X_n)\otimes\mathcal{R}$, where $I$ runs over all partitions such that $i_1 \leqslant n$ (and for a partition $I = (i_1, \dots, i_k)$ we write $e_I(X_n) = e_{i_1}(X_n) \cdot \ldots \cdot e_{i_k}(X_n)$). Consider the reverse lexicographic ordering of [Mcd1, I] on the set of such partitions, inducing a linear ordering on the above set $\{e_I(X_n)\}$. By the definition of $\widetilde{Q}_I(X_n)$, one has

$$\widetilde{Q}_I(X_n) = e_I(X_n) + (\text{combination of earlier monomials in the } e_i(X_n)\text{'s}).$$

Since $\{e_I(X_n)\}$ is a $\mathcal{R}$-basis of $S\mathcal{P}(X_n) \otimes \mathcal{R}$, $\{\widetilde{Q}_I(X_n)\}$ forms another $\mathcal{R}$-basis of $S\mathcal{P}(X_n) \otimes \mathcal{R}$. $\square$

**Corollary-Definition 4.8.** *For every $m \leqslant n$ and any partitions $J \subset I$, there exist uniquely defined polynomials $\widetilde{Q}_{I/J}(x_{m+1}, \ldots, x_n) \in S\mathcal{P}(x_{m+1}, \ldots, x_n)$ such that the following equality holds*

$$\widetilde{Q}_I(X_n) = \sum_{J \subset I} \widetilde{Q}_J(X_m)\widetilde{Q}_{I/J}\big(x_{m+1}, \ldots, x_n\big).$$

*Proof.* The existence of such polynomials $Q_{I/J}(x_{m+1}, \ldots, x_n) \in \mathbb{Z}[x_{m+1}, \ldots, x_n]$ follows from the above discussion and (*); we put $\widetilde{Q}_{I/J} := F_J$.

Since $S\mathcal{P}(X_n) \subset S\mathcal{P}(X_m) \otimes S\mathcal{P}(x_{m+1}, \ldots, x_n)$ and $\{\widetilde{Q}_J(X_m) |\ j_1 \leqslant m\}$ is a $\mathbb{Z}$-basis of $S\mathcal{P}(X_m)$ (Proposition 4.7), we have the corresponding development:

$$\widetilde{Q}_I(X_n) = \sum_J \widetilde{Q}_J(X_m) G_J(x_{m+1}, ..., x_n)\ .$$

Using Proposition 4.6 once again with $n$ replaced by $m$ and $\mathcal{R} = \mathbb{Z}[x_{m+1}, \ldots, x_n]$, we infer that $F_J (= G_J)$ are symmetric in $x_{m+1}, \ldots, x_n$ (and defined uniquely). □

We will need also a family of $\widetilde{P}$-polynomials in $S\mathcal{P}(X_n) \otimes \mathbb{Z}[1/2]$ defined by $\widetilde{P}_I(X_n) := 2^{-l(I)}\widetilde{Q}_I(X_n)$ for a partition $I$. Also, in analogy to the above, for every $m \leqslant n$ and any partitions $J \subset I$ there exists uniquely defined polynomials $\widetilde{P}_{I/J}(x_{m+1}, \ldots, x_n) \in S\mathcal{P}(x_{m+1}, \ldots, x_n) \otimes \mathbb{Z}[1/2]$ such that

$$\widetilde{P}_I(X_n) = \sum_{J \subset I} \widetilde{P}_J(X_m)\, \widetilde{P}_{I/J}(x_{m+1}, \ldots, x_n)\,.$$

$\widetilde{P}$-polynomials satisfy properties which can be automatically gotten from the above established properties of $\widetilde{Q}$-polynomials. For instance, an analogue of Proposition 4.1 for $\widetilde{P}$-polynomials reads:

$$\widetilde{P}_I(X_n) = \sum_{j=0}^{l(I)} x_n^j \Big( \sum_{|I|-|J|=j} 2^{l(J)-l(I)} \widetilde{P}_J(X_{n-1}) \Big)\,,$$

the sum as in Proposition 4.1.

Given a rank $n$ vector bundle $E$ with the Chern roots $(r_1, \ldots, r_n)$ we set $\widetilde{Q}_I E := \widetilde{Q}_I(X_n)$ and $\widetilde{P}_I E := \widetilde{P}_I(X_n)$ with $x_i$ specialized to $r_i$. Note that this notation is consistent with that used in Section 2 and 3. Similarly, given a subbundle $E' \subset E$ with the Chern roots $(r_{m+1}, \ldots, r_n)$, we define $\widetilde{Q}_{I/J}E' = \widetilde{Q}_{I/J}(x_{m+1}, \ldots, x_n)$ and $\widetilde{P}_{I/J}E' = \widetilde{P}_{I/J}(x_{m+1}, \ldots, x_n)$ with $x_i$ specialized to $r_i$.

In the next section we will need the following Pieri-type formula for the $\widetilde{Q}_I$'s.

**Proposition 4.9.** *Let $I=(i_1,...,i_k)$ be a strict partition of length $k$. Then*

$$\widetilde{Q}_I(X_n)\widetilde{Q}_r(X_n)=\sum 2^{m(I,r;J)}\widetilde{Q}_J(X_n),$$

*where the sum is over all partitions (i.e. not necessary strict) $J\supset I$ such that $|J|=|I|+r$ and $J/I$ is a horizontal strip. Moreover, $m(I,r;J)=card\{1\le p\le k|\ j_{p+1}<i_p<j_p\}$ or, equivalently, it is expressed as the number of connected components of the strip $J/I$ not meeting the first column.*

(A skew diagram $D$ is connected if each of the sets $\{i:\exists_j(i,j)\in D\}$ and $\{j:\exists_i(i,j)\in D\}$ is an interval in $\mathbb{Z}$.)

*Proof.* Let after [L-L-T], $Q'_I(X_n;q)$ denote the Hall–Littlewood polynomial $Q_I(Y;q)$ where the alphabet $Y$ is equal to $X_n/(1-q)$ (in the sense of $\lambda$-rings). Using raising operators $R_{ij}$ ([Mcd1,I] we have (see, e.g., [D-L-T])

$$Q'_I(X_n;q)=\prod_{i<j}(1-qR_{ij})^{-1}s_I(X_n).$$

Specialize $q=-1$ and invoke the well known Jacobi–Trudi formula

$$s_I(X_n)=\prod_{i<j}(1-R_{ij})h_I(X_n),$$

where $h_I(X_n)$ is the product of complete homogeneous symmetric polynomials in $X_n$ associated with the parts of $I$. We have

$$Q'_I(X_n;-1)=\prod_{i<j}\frac{1-R_{ij}}{1+R_{ij}}h_I(X_n).$$

Therefore, denoting by $\omega$ the Young duality-involution we get $\widetilde{Q}_I(X_n)=\omega\big(Q'_I(X_n;-1)\big)$.

The required assertion now follows by an appropriate specialization of the Pieri-type formula for Hall–Littlewood polynomials ([Mo], [Mcd1, III.3.(3.8)]). $\square$

## 5. Divided differences and isotropic Gysin maps; orthogonality of $\widetilde{Q}$-polynomials

Let $V\to X$ be a vector bundle of rank $2n$ endowed with a nondegenerate symplectic form. Let $\pi: LG_n(V)\to X$ and $\tau: LFl(V)\to X$ denote respectively the Grassmannian bundle parametrizing Lagrangian subbundles of $V$ and the flag bundle parametrizing flags of rank 1, rank 2, ..., rank $n$ Lagrangian subbundles of $V$. We have $\tau=\pi\circ\omega$ where $\omega: LFl(V)\to LG_n(V)$ is the projection map. The main goal of this section is to derive several formulas for the Gysin map $\pi_*: A^*(LG_n(V))\to A^*(X)$.

We start by recalling the Weyl group $W_n$ of type $C_n$. This group is isomorphic to $S_n\ltimes\mathbb{Z}_2^n$. We write a typical element of $W_n$ as $w=(\sigma,\tau)$ where $\sigma\in S_n$ and $\tau\in\mathbb{Z}_2^n$; so that if $w'=(\sigma',\tau')$ is another element, their product in $W_n$ is $w\cdot w'=(\sigma\circ\sigma',\delta)$

where "$\circ$" denotes the composition of permutations and $\delta_i = \tau_{\sigma'(i)} \cdot \tau'_i$. To represent elements of $W_n$ we will use the standard "barred-permutation" notation, writing them as permutations equipped with bars on those places (numbered with "$i$") where $\tau_i = -1$. Instead of using a standard system of generators of $W_n$ given by simple reflections $s_i = (1, 2, \dots, i+1, i, \dots, n)$, $1 \leqslant i \leqslant n-1$, and $s_n = (1, 2, \dots, n-1, \overline{n})$, we will use the following system of generators $S = \big\{ s_o = (\overline{1}, 2, \dots, n), \quad s_1, \dots, s_{n-1} \big\}$ corresponding to the basis: $(-2\varepsilon_1), \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, ..., \varepsilon_{n-1} - \varepsilon_n$. It is easy to check that $(W_n, S)$ is a Coxeter system of type $C_n$. This "nonstandard" system of generators has several advantages over the standard one: it leads to easier reasonings by induction on $n$ and the divided differences associated with it produce "stable" symplectic Schubert type polynomials (for the details concerning this last topic - consult a recent work of S. Billey and M. Haiman [B-H]). Let us record first the formula for the length of an element $w = (\sigma, \tau) \in W_n$ w.r.t. $S$. This formula can be proved by an easy induction on $l(w)$ and we leave this to the reader.

**Lemma 5.1.** *$l(w) = \sum_{i=1}^{n} a_i + \sum_{\tau_j = -1} (2b_j + 1)$, where $a_i := card\big\{ p \mid \, p > i \ \& \ \sigma_p < \sigma_i \big\}$ and $b_j := card\big\{ p \mid \, p < j \ \& \ \sigma_p < \sigma_j \big\}$.*

In the sequel, whenever we will speak about the "length" of an element $w \in W_n$, we will have in mind the length w.r.t. $S$.

Let $X_n = (x_1, \dots, x_n)$ be a sequence of indeterminates.

We now define symplectic divided differences $\partial_i : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$, $i = 0, 1, \dots, n-1$, setting

$$\partial_0(f) = (f - s_0 f)/(-2x_1) \quad ,$$
$$\partial_i(f) = (f - s_i f)/(x_i - x_{i+1}) \quad , \qquad i = 1, \dots, n-1 \quad ,$$

where $s_0$ acts on $\mathbb{Z}[X_n]$ by sending $x_1$ to $-x_1$ and $s_i$ – by exchanging $x_i$ with $x_{i+1}$ and leaving the remaining variables invariant. For every $w \in W_n$, $l(w) = l$, let $s_{i_1} \cdot \ldots \cdot s_{i_l}$ be a reduced decomposition w.r.t. S. Following the theory in [B-G-G] and [D1,2] we define $\partial_w := \partial_{s_{i_1}} \cdot \ldots \cdot \partial_{s_{i_l}}$. By loc.cit. we get a well-defined operator of degree $-l(I)$ acting on $\mathbb{Z}[X_n]$ (here, "well-defined" means: independent on the reduced decomposition chosen).

We want first to study the operator $\partial_{w_o}$ where $w_o = (\overline{1}, \overline{2}, \dots, \overline{n})$ is the maximal length element of $W_n$. To this end we need some preliminary considerations.

Let $Q\mathcal{P}(X_n)$ denote the ring of Schur's $Q$-polynomials in $X_n$. We record the following (apparently new) identity in this ring. In 5.2 – 5.4 below we will write: $e_i = e_i(X_n)$, $s_I = s_I(X_n)$, $Q_I = Q_I(X_n)$ and $\widetilde{Q}_I = \widetilde{Q}_I(X_n)$ for brevity.

**Proposition 5.2.** *In $Q\mathcal{P}(X_n)$,*

$$Q_{\rho_k} = Det(a_{i,j})_{1 \leqslant i,j \leqslant k},$$

*where $a_{i,j} = Q_{k+1+j-2i}$ if $k+1+j-2i \neq 0$ (with $Q_i = 0$ for $i < 0$) and $a_{i,j} = 2$ if $k+1+j-2i = 0$.*

*Proof.* We have from the theory of symmetric polynomials (see [P2] and the references therein),

$$Q_{\rho_k} = 2^k s_{\rho_k} = Det\Big(2e_{k+1+j-2i}\Big)_{1\leqslant i,j\leqslant k}.$$

It follows from the Pieri formula (see [Mcd] and [LS1]) that

$$\Bigg(2 \sum_{\substack{hooks\ I,\\ |I|=i-2}} s_I\Bigg) \cdot s_2 + 2e_i = 2 \sum_{\substack{hooks\ J,\\ |J|=i}} s_J.$$

Hence, by multiplying the $p$-th row by $s_2$ and adding it to the $(p-1)$-th one successively for $p = k, k-1, \ldots, 3, 2$, the above determinant is rewritten in the form:

$$Det\Bigg(2 \sum_{\substack{hooks\ I,\\ |I|=k+1+j-2i}} s_I\Bigg)_{1\leqslant i,j\leqslant k}.$$

Notice that the degree 0 entries in this determinant are equal to 2 and the negative degree entries vanish. Since $Q_i = 2 \sum_{hooks\ I, |I|=i} s_I$, the assertion follows. $\square$

Let $\mathcal{J}$ be the ideal in $S\mathcal{P}(X_n)$ generated by $e_i(x_1^2, \ldots, x_n^2)$, $1 \leqslant i \leqslant n$. We now invoke a corollary of [P2, Theorem 6.17] combined with [B-G-G, Theorem 5.5] and [D2, 4.6(a)]: there is a ring isomorphism $S\mathcal{P}(X_n)/\mathcal{J} \to Q\mathcal{P}(X_n)/\oplus \mathbb{Z}Q_I(X_n)$, where $I$ runs over all strict partitions $I \not\subset \rho_n$, given by $e_i(X_n) \mapsto Q_i(X_n)$ (see the remark after Theorem 6.17 in [P2, pp.181-182]). We thus get from the proposition:

**Corollary 5.3.** *In $S\mathcal{P}(X_n)$, $\widetilde{Q}_{\rho_k}$ is congruent to $Det(b_{i,j})_{1\leqslant i,j\leqslant k}$ modulo $\mathcal{J}$, where $b_{i,j} = e_{k+1+j-2i}$ if $k+1+j-2i \neq 0$ (with $e_i = 0$ for $i < 0$) and $b_{i,j} = 2$ if $k+1+j-2i = 0$.*

We now state:

**Lemma 5.4.** *In $S\mathcal{P}(X_n)$, $\widetilde{Q}_{\rho_n} \equiv e_n e_{n-1} \ldots e_1 \equiv s_{\rho_n}$ (mod $\mathcal{J}$).*

*Proof.* By the corollary it is sufficient to prove that $Det(b_{i,j})_{1\leqslant i,j\leqslant n} \equiv e_n e_{n-1} \ldots e_1 \equiv s_{\rho_n}$ (mod $\mathcal{J}$). Recall that $s_{\rho_n} = Det(c_{i,j})_{1\leqslant i,j\leqslant n}$ where $c_{i,j} = e_{n+1+j-2i}$ if $n+1+j-2i \neq 0$ and $c_{i,j} = 1$ if $n+1+j-2i = 0$, i.e. the matrices $(b_{i,j})$ and $(c_{i,j})$ are the same modulo the degree 0 entries.

Let us write the determinants $Det(b_{i,j})$ and $Det(c_{i,j})$ as the sums of the standard $n!$ terms (some of them are zero). It is easy to see that apart from the "diagonal" term $e_n e_{n-1} \ldots e_1$, every other term appearing in both sums is divisible by $e_n e_{n-1} \ldots e_{p+1} e_p^2$ for some $p \geqslant 1$. We claim that, $e_n e_{n-1} \ldots e_{p+1} e_p^2 \in \mathcal{J}$. Indeed, $e_n^2 \in \mathcal{J}$ and suppose, by induction, that we have shown $e_n e_{n-1} \ldots e_{q+1} e_q^2 \in \mathcal{J}$ for $q > p$. Then

$$e_n e_{n-1} \ldots e_{p+1} e_p^2 = e_n e_{n-1} \ldots e_{p+1} \Big[\widetilde{Q}_{p,p} + 2\sum_{i=1}^{p} (-1)^{i-1} e_{p+i} e_{p-i}\Big]$$

belongs to $\mathcal{J}$ by the induction assumption, because $\widetilde{Q}_{p,p} \in \mathcal{J}$ (see Proposition 4.2). This shows that

$$Det(b_{i,j}) \equiv e_n e_{n-1} \dots e_1 \equiv Det(c_{i,j}) \ (mod\ \mathcal{J}).$$

Thus the lemma is proved. $\square$

The following known result (see, e.g., [D1] where the result is given also for other root systems) is accompanied by a proof for the reader's convenience.

**Proposition 5.5.** *One has for $f \in \mathbb{Z}[X_n]$,*

$$\partial_{w_o}(f) = (-1)^{n(n+1)/2}\Big(2^n x_1 \cdot \ldots \cdot x_n \prod_{i<j}(x_i^2 - x_j^2)\Big)^{-1} \sum_{w\in W_n} (-1)^{l(w)} w(f).$$

*Proof.* By the definition of $\partial_{w_o}$ we infer that $\partial_{w_o} = \sum\limits_{w\in W_n} \alpha_w w$ where the coefficients $\alpha_w$ are rational functions in $x_1, \dots, x_n$. Since $w_o$ is the maximal length element in $W_n$, $\partial_i \circ \partial_{w_o} = 0$ for all $i = 0, 1, \dots, n-1$. Consequently $s_i\partial_{w_o} = \partial_{w_o}$ for $i = 0, 1, \dots, n-1$ and hence $v\partial_{w_o} = \partial_{w_o}$ for all $v \in W_n$. In particular, for every $v \in W_n$, $\partial_{w_o} = \sum\limits_{w\in W_n} v(\alpha_w)vw$. Thus $\alpha_{vw} = v(\alpha_w)$ for all $v, w \in W_n$, and we see that, e.g., $\alpha_{w_o}$ determines uniquely all the $\alpha_w$'s.

Claim $\alpha_{w_o} = (-1)^{n(n-1)/2}\Big(2^n x_1 \cdot \ldots \cdot x_n \prod\limits_{i<j}(x_i^2 - x_j^2)\Big)^{-1}$.

*Proof of the claim*: Denote now the maximal length element in $W_n$ by $w_o^{(n)}$. We argue by induction on $n$. For $n = 1$, we have $\alpha_{w_o^{(1)}} = \frac{1}{2x_1}$. We now record the following equality:

$$w_o^{(k+1)} = s_k \cdot s_{k-1} \cdot \ldots \cdot s_1 \cdot s_0 \cdot s_1 \cdot \ldots \cdot s_{k-1} \cdot s_k \cdot w_o^{(k)},$$

that implies

$$\partial_{w_o^{(k+1)}} = \partial_k \circ \partial_{k-1} \circ \ldots \circ \partial_1 \circ \partial_0 \circ \partial_1 \circ \ldots \circ \partial_{k-1} \circ \partial_k \circ \partial_{w_o^{(k)}}.$$

It follows easily from the latter equality that

$$\alpha_{w_o^{(k+1)}} = (-1)^k \Big(2x_{k+1} \prod_{i\leqslant k}(x_i - x_{k+1}) \prod_{i\leqslant k}(x_i + x_{k+1})\Big)^{-1} \alpha_{w_o^{(k)}}.$$

This allows us to perform the induction step $n \to n+1$, thus proving the claim.

Finally, for arbitrary $w \in W_n$,

$$\alpha_w = ww_o(\alpha_{w_o}) = (-1)^{n(n+1)/2+l(w)}\Big(2^n x_1 \cdot \ldots \cdot x_n \prod_{i<j}(x_i^2 - x_j^2)\Big)^{-1}$$

because, for $w = (\sigma, \tau)$, $l(\sigma, \tau) = \sum a_i + \sum_{\tau_j=-1}(2b_j + 1) \equiv l(\sigma) + card\{p|\ \tau_p = -1\}$ $(mod\ 2)$ (see Lemma 5.1). $\square$

**Corollary 5.6.** *(i) $\partial_{w_o}(x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n})=0$ if $\alpha_p$ is even for some $p=1,\dots,n$.*
*(ii) If all $\alpha_p$ are odd then*

$$\partial_{w_o}(x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n})=(-1)^{n(n+1)/2}s_{\rho_n}(X_n)^{-1}\partial(x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}),$$

*where here and in the sequel $\partial$ denotes the Jacobi symmetrizer*

$$\Big(\sum_{\sigma\in S_n}(-1)^{l(\sigma)}\sigma(-)\Big)/\prod_{i<j}(x_i-x_j).$$

*Proof.* (i) Let us fix $\sigma\in S_n$ and look at all elements of $W_n$ of the form $(\sigma,\tau)$ where $\tau\in\mathbb{Z}_2^n$. Then, writing $x^\alpha$ for $x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}$, we have

$$\sum_\tau(-1)^{l(\sigma,\tau)}(\sigma,\tau)x^\alpha=(-1)^{l(\sigma)}\sigma(x^\alpha)\sum_\tau(-1)^{card\{p|\ \tau_p=-1\}}\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n},$$

because $l(\sigma,\tau)\equiv l(\sigma)+card\{p|\ \tau_p=-1\}\ (mod\ 2)$. Suppose that some numbers among $\alpha_1,\dots,\alpha_n$ are even. We will show that this implies

$$\sum_\tau(-1)^{card\{p|\ \tau_p=-1\}}\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n}=0.$$

We can assume that $\alpha_1,\dots,\alpha_k$ are odd and $\alpha_{k+1},\dots,\alpha_n$ are even for some $k<n$ (by permuting the $\tau_p$'s if necessary). We have

$$\begin{aligned}&\sum_\tau(-1)^{card\{p|\ \tau_p=-1\}}\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n}=\\&\qquad=\sum_\tau(-1)^{card\{p|\ \tau_p=-1\}}(-1)^{card\{p|\ \tau_p=-1,\ p\leqslant k\}}\\&\qquad=\sum_\tau(-1)^{card\{p|\ \tau_p=-1,\ p>k\}}\\&\qquad=2^k\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}=2^k(1-1)^{n-k}=0.\end{aligned}$$

(ii) Let us now compute $\partial_{w_o}(x_1^{\alpha_1}\dots x_n^{\alpha_n})$ where all $\alpha_p$ are odd. Then

$$\sum_\tau(-1)^{card\{j|\ \tau_j=-1\}}\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n}=2^n,\ \text{and}$$

$$\begin{aligned}\partial_{w_o}(x^\alpha)&=(-1)^{n(n+1)/2}\Big(2^nx_1\dots x_n\prod_{i<j}(x_i^2-x_j^2)\Big)^{-1}2^n\sum_{\sigma\in S_n}(-1)^{l(\sigma)}\sigma(x^\alpha)\\&=(-1)^{n(n+1)/2}s_{\rho_n}(X_n)^{-1}\partial(x^\alpha).\quad\square\end{aligned}$$

We now record the following properties of the operator $\nabla=\partial_{(\bar n,\dots,\bar 2,\bar 1)}$. In the following, let $\mathcal{I}=\mathcal{J}\mathbb{Z}[X_n]$.

**Lemma 5.7.** *(i) If $f \in S\mathcal{P}(x_1^2, \dots, x_n^2)$ then $\nabla(f \cdot g) = f \cdot \nabla(g)$.*
*(ii) $\nabla\big(\widetilde{Q}_{\rho_n}(X_n)\big) = (-1)^{n(n+1)/2}$.*

*Proof.* (i) This assertion is clear because every polynomial in $S\mathcal{P}(x_1^2, \dots, x_n^2)$ is $W_n$-invariant. Observe that it implies that if $f \equiv g \pmod{\mathcal{I}}$ then $\nabla(f) \equiv \nabla(g) \pmod{\mathcal{I}}$.

(ii) (This can be also deduced from the Chow ring of the Lagrangian Grassmannian. We present here a direct algebraic argument.) In this part we will use the following properties of the Jacobi symmetrizer $\partial$ (see [L-S2], [Mcd2]):

1. If $f \in S\mathcal{P}(X_n)$, $g \in \mathbb{Z}[X_n]$ then $\partial(f \cdot g) = f \cdot \partial(g)$.
2. For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\partial(x^\alpha) = s_{\alpha - \rho_{n-1}}(X_n)$. In particular, if $\alpha_i = \alpha_j$ for some $i \neq j$ then $\partial(x^\alpha) = 0$.
3. $\partial = \partial_{(n,n-1,\dots,1)}$.

Let $e_i = e_i(X_n)$. Since $\widetilde{Q}_{\rho_n}(X_n) \equiv e_n e_{n-1} \dots e_1 \pmod{\mathcal{I}}$ (by Lemma 5.4), we have

$$\nabla(\widetilde{Q}_{\rho_n}(X_n)) = \nabla(e_n e_{n-1} \dots e_1) = (\nabla \circ \partial)(x^{\rho_{n-1}} e_n e_{n-1} \dots e_1)$$

by properties 1 and 2 above. Since

$$(\overline{n}, \overline{n-1}, \dots, \overline{1}) \circ (n, n-1, \dots, 1) = w_0,$$

the above expression equals $\partial_{w_0}(x^{\rho_{n-1}} e_n e_{n-1} \dots e_1)$ by property 3. The degree of the polynomial $x^{\rho_{n-1}} e_n e_{n-1} \dots e_1$ is $n^2$. Assuming that $\alpha_1 + \ldots + \alpha_n = n^2$, we have $\partial_{w_0}(x^\alpha) \neq 0$ only if

$$x^\alpha = x_{\sigma(1)}^{2n-1} x_{\sigma(2)}^{2n-3} \dots x_{\sigma(n)}$$

for some $\sigma \in S_n$. Indeed, it follows from Corollary 5.6(i) that $\partial_{w_0}(x^\alpha) \neq 0$ only if all the $\alpha_i$'s are odd. Moreover, they must be all different; otherwise $\partial(x^\alpha) = 0$ (and consequently $\partial_{w_o}(x^\alpha) = 0$) by property 2. We conclude that $\{\alpha_1, \dots, \alpha_n\} = \{2n-1, 2n-3, \dots, 1\}$. But there is only one such a monomial $x^\alpha$ in $x^{\rho_{n-1}} e_n e_{n-1} \dots e_1$, namely the one with $(\alpha_1, \dots, \alpha_n) = (2n-1, 2n-3, \dots, 1)$. Therefore

$$\partial_{w_o}(x^{\rho_{n-1}} e_n e_{n-1} \dots e_1) = \partial_{w_0}(x_1^{2n-1} x_2^{2n-3} \dots x_n) = (-1)^{n(n+1)/2}$$

by Corollary 5.6(ii) and property 2. $\square$

We now pass to a geometric interpretation of the operator $\nabla$.

**Proposition 5.8.** *Specializing the variables $x_1, \dots, x_n$ to the Chern roots $r_1, \dots, r_n$ of the tautological subbundle $R$ on $LG_nV$, one has the equality*

$$\pi_*\big(f(r_1, \dots, r_n)\big) = \Big(\partial_{(\overline{n}, \overline{n-1}, \dots, \overline{2}, \overline{1})} f\Big)(r_1, \dots, r_n),$$

*where $f(-)$ is a polynomial in $n$ variables.*

(A symmetrization operator variant of this proposition follows also from a recent paper by M. Brion [Br]. We give here a short proof using only divided differences interpretation of Gysin maps for complete (usual and Lagrangian) flag bundles.)

*Proof.* We invoke a result saying that the Gysin map associated with $\omega$ and $\tau$ is induced by the following divided differences operators:

$$\tau_*\big(f(r_1,\dots,r_n)\big) = \Big(\partial_{(\overline{1},\overline{2},\dots,\overline{n})} f\Big)(r_1,\dots,r_n) \quad \text{and}$$
$$\omega_*\big(f(r_1,\dots,r_n)\big) = \Big(\partial_{(n,n-1,\dots,1)} f\Big)(r_1,\dots,r_n).$$

As for the latter equality, see [P1, Sect.2] , as for the former compare [Br] where the author gives a symmetrizing operator expression for $G/B$-fibrations (over a point, say, this expression was given in [A-C]). The needed divided differences interpretation of those symmetrizing operators follows, e.g., from [D1].
Since

$$(\overline{1},\overline{2},\dots,\overline{n}) = (\overline{n},\overline{n-1},\dots,\overline{1})\circ(n,n-1,\dots,1),$$

we get

$$\partial_{(\overline{1},\overline{2},\dots,\overline{n})} = \partial_{(\overline{n},\overline{n-1},\dots,\overline{1})}\circ\partial_{(n,n-1,\dots,1)}.$$

Of course, $\tau_* = \pi_*\circ\omega_*$. Since $\omega_*$ is surjective, comparison of the latter equation with the former implies the desired assertion about $\pi_*$. $\square$

We now show how to compute the images via $\pi_*$ of $\widetilde{Q}$-polynomials in the Chern classes of $R^\vee$. Let us write $X_n^\vee = (-x_1,\dots,-x_n)$ for brevity.

**Proposition 5.9.** *One has $\nabla(\widetilde{Q}_I(X_n^\vee)) \neq 0$ only if the set of parts of $I$ is equal to $\{1,2,\dots,n\}$ and each number $p$ $(1\leqslant p\leqslant n)$ appears in $I$ with an odd multiplicity $m_p$. Then, the following equality holds in $\mathbb{Z}[X_n]$,*

$$\nabla\big(\widetilde{Q}_I(X_n^\vee)\big) = \prod_{p=1}^{n} e_p(x_1^2,\dots,x_n^2)^{(m_p-1)/2}.$$

*Proof.* By Proposition 4.3 we can express $\widetilde{Q}_I(X_n^\vee)$ as

$$\widetilde{Q}_I(X_n^\vee) = \widetilde{Q}_{j_1,j_1}(X_n^\vee)\ \dots\ \widetilde{Q}_{j_l,j_l}(X_n^\vee)\widetilde{Q}_L(X_n^\vee),$$

where $L$ is a strict partition. (We divide the elements of the multiset $I$ into pairs of equal elements and the set $L$ whose elements are all different.) Some of the $j_p$'s can be mutually equal.

By Proposition 4.2, $\widetilde{Q}_{j,j}(X_n^\vee) = e_j(x_1^2,\dots,x_n^2)$ is a scalar w.r.t. $\nabla$.

By Lemma 4.4, $\widetilde{Q}_L(X_n^\vee)\neq 0$ only if $L\subset\rho_n$. On the other hand, for a strict partition $L\subset\rho_n$, $\nabla(\widetilde{Q}_L(X_n^\vee))\neq 0$ only if $L=\rho_n$, when it is equal to 1 (see Lemma 5.7(ii)).

Putting this information together, the assertion follows. $\square$

Consequently, specializing $(x_i)$ to the Chern roots $(r_i)$ of the tautological subbundle on $LG_n(V)$ we have

**Theorem 5.10.** *The element $\widetilde{Q}_I R^\vee$ has a nonzero image under $\pi_* : A^*(LG_nV) \to A^*(X)$ only if each number $p$, $1 \leqslant p \leqslant n$, appears as a part of $I$ with an odd multiplicity $m_p$. If this last condition holds then*

$$\pi_*\widetilde{Q}_I R^\vee = \prod_{p=1}^{n} \big((-1)^p c_{2p}V\big)^{(m_p-1)/2}.$$

*Proof.* This follows from Proposition 5.9 and the equality $c_{2p}V = (-1)^p e_p(r_1^2, ..., r_n^2)$. □

Our next goal will be to show how to compute the images via $\pi_*$ of $S$-polynomials in the Chern classes of the tautological Lagrangian bundle. To this end we record the following identity of symmetric polynomials. We have found this simple and remarkable identity during our work on isotropic Gysin maps and have not seen it in the literature.

**Proposition 5.11.** *For every partition $I = (i_1, \dots, i_n)$ and any positive integer $p$, one has in $S\mathcal{P}(X_n)$,*

$$s_I\big(x_1^p, \dots, x_n^p\big) \cdot s_{(p-1)\rho_{n-1}}(X_n) = s_{pI+(p-1)\rho_{n-1}}(X_n).$$

*Here, given a partition $I = (i_1, i_2, \dots)$, we write $pI = (pi_1, pi_2, \dots)$.*

*Proof.* We use the Jacobi presentation of a Schur polynomial as a ratio of two alternants (see [Mcd1], [L-S1]). We have:

$$s_I\big(x_1^p, \dots, x_n^p\big) = \frac{Det\big(x_k^{(i_l+n-l)p}\big)_{1\leqslant k,l\leqslant n}}{Det\big(x_k^{p(n-l)}\big)_{1\leqslant k,l\leqslant n}}$$

$$= \frac{Det\big(x_k^{pi_l+(n-l)(p-1)+(n-l)}\big)_{1\leqslant k,l\leqslant n}}{Det\big(x_k^{n-l}\big)_{1\leqslant k,l\leqslant n} \cdot \left(\dfrac{Det\big(x_k^{(p-1)(n-l)+(n-l)}\big)_{1\leqslant k,l\leqslant n}}{Det\big(x_k^{n-l}\big)_{1\leqslant k,l\leqslant n}}\right)}$$

$$= \frac{s_{pI+(p-1)\rho_{n-1}}(X_n)}{s_{(p-1)\rho_{n-1}}(X_n)}. \quad \square$$

**Corollary 5.12.** *For $p = 2$ we get*

$$s_I(x_1^2, \dots, x_n^2) \cdot s_{\rho_{n-1}}(X_n) = s_{2I+\rho_{n-1}}(X_n).$$

(For another derivation of this identity with the help of Quaternionic Grassmannians see Appendix A.)

We now give a geometric translation of this last formula, or rather its consequence

$$(*) \qquad s_I(x_1^2, \dots, x_n^2) \cdot s_{\rho_n}(X_n) = s_{\rho_n+2I}(X_n).$$

**Theorem 5.13.** *The element $s_I R^\vee$ has a nonzero image under $\pi_*$ only if the partition $I$ is of the form $2J+\rho_n$ for some partition $J$. If $I = 2J+\rho_n$ then*

$$\pi_* s_I R^\vee = s_J^{[2]} V \quad ,$$

*where the right hand side is defined as follows: if $s_J = P(e.)$ is a unique presentation of $s_J$ as a polynomial in the elementary symmetric functions $e_i$, $E$ – a vector bundle, then $s_J^{[2]}(E) := P$ with $e_i$ replaced by $(-1)^i c_{2i} E \quad (i = 1, 2, \dots)$.*

*Proof.* Since $s_I R^\vee = \omega_*(q^{I+\rho_{n-1}})$ where $q = (q_1, \dots, q_n)$ are the Chern roots of $R^\vee$ (this is a familiar Jacobi-Trudi formula restated using the Gysin map for the flag bundle – see, e.g., [P3] and the references therein), we infer from Corollary 5.6(i) that $s_I R^\vee$ has a nonzero image under $\pi_*$ only if all parts of $I + \rho_{n-1}$ are odd. This implies that $l(I) = n$ and $I$ is strict thus of the form $I' + \rho_n$ for some partition $I'$. Finally all parts of $I' + \rho_n + \rho_{n-1}$ are odd iff $I' = 2J$ for some partition $J$, as required.

Assume now that $I = 2J+\rho_n$ and specialize the identity (*) by replacing the variables $(x_i)$ by the Chern roots $(q_i)$. The claimed formula now follows since: $s_I(q_1^2, \dots, q_n^2)$ is a scalar w.r.t. $\pi_*$, $\pi_* s_{\rho_n}(q_1, \dots, q_n) = 1$ by Lemma 5.7(ii) combined with Lemma 5.4; finally $(-1)^i c_{2i} V = e_i(q_1^2, \dots, q_n^2)$ because of Lemma 1.1(2). $\square$

Observe that the theorem contains an explicit calculation of the ratio in Corollary 5.6(ii).

We now pass to the odd orthogonal case. The Weyl group $W_n$ of type $B_n$. is isomorphic to $S_n \ltimes \mathbb{Z}_2^n$ and its elements are "barred-permutations". We use the following system of generators of $W_n$: $S = \{s_o = (\bar{1}, 2, \dots, n), \quad s_1, \dots, s_{n-1}\}$ corresponding to the basis $(-\varepsilon_1), \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, ..., \varepsilon_{n-1} - \varepsilon_n$. Consequently, the divided differences $\partial_i$, $i = 1, \dots, n-1$, are the same but $\partial_0(f) = (f - s_0 f)/(-x_1)$.

The odd orthogonal analog of Proposition 5.5 reads:

$$\partial_{w_0}(f) = (-1)^{n(n+1)/2} \Big( x_1 \cdot \ldots \cdot x_n \prod_{i<j} (x_i^2 - x_j^2) \Big)^{-1} \sum_{w \in W_n} (-1)^{l(w)} w(f).$$

Arguing essentially as in the proof of Proposition 5.8 (with obvious modifications), one shows that the Gysin map associated with $\pi : OG_n V \to X$ is induced by the orthogonal divided difference operator $\partial_{(\overline{n}, \overline{n-1}, \dots, \overline{1})}$.

The odd orthogonal analog of Theorem 5.10 reads:

**Theorem 5.14.** *The element $\widetilde{Q}_I R^\vee$ has a nonzero image under $\pi_* : A^*(OG_n V) \to A^*(X)$ only if each number $p$, $1 \leqslant p \leqslant n$, appears as a part of $I$ with an odd multiplicity $m_p$. If this last condition holds then*

$$\pi_* \widetilde{Q}_I R^\vee = 2^n \prod_{p=1}^n \big( (-1)^p c_{2p} V \big)^{(m_p - 1)/2}.$$

This holds because the calculation in Proposition 5.9 now goes as follows: with the notation from the proof of Proposition 5.9, the polynomial

$$\widetilde{Q}_I(X_n^\vee) = 2^n \widetilde{Q}_{j_1,j_1}(X_n^\vee)\ \dots\ \widetilde{Q}_{j_l,j_l}(X_n^\vee)\widetilde{P}_{\rho_n}(X_n^\vee)$$

is mapped via $\partial_{(\overline{n},\overline{n-1},\dots,\overline{1})}$ to

$$2^n \prod_{h=1}^{l} e_{j_h}(x_1^2, ..., x_n^2),$$

since $\partial_{(\overline{n},\overline{n-1},\dots,\overline{1})}\big(\widetilde{P}_{\rho_n}(X_n^\vee)\big) = 1$. (The proof of the last statement is the same as that of Lemma 5.7(ii).)

Finally, the odd orthogonal analog of Theorem 5.13 reads:

**Theorem 5.15.** *The element $s_I R^\vee$ has a nonzero image under $\pi_*$ only if the partition $I$ is of the form $2J + \rho_n$ for some partition $J$. If $I = 2J + \rho_n$ then*

$$\pi_* s_I R^\vee = 2^n s_J^{[2]} V,$$

*where $s_J^{[2]}(-)$ is defined as in Theorem 5.13.*

This holds because $s_{\rho_n}(X_n^\vee)$ is congruent to $2^n \widetilde{P}_{\rho_n}(X_n^\vee)$ modulo $\mathcal{J}$ (Lemma 5.4) and $\pi_* \widetilde{P}_{\rho_n} R^\vee = 1$. Also, we use Lemma 1.1(2).

We now pass to the even orthogonal case.

In type $D_n$ the Weyl group $W_n$ is identified with the subgroup of the group of "barred permutations" $(w_1, \dots, w_n)$ whose elements have even number of bars only. Consider a system $S$ of generators of $W_n$ consisting of $s_{\overline{1}} = (\overline{2}, \overline{1}, 3, \dots, n)$ and $s_i = (1, 2, \dots, i-1, i+1, i, i+2, \dots, n)$, $i = 1, 2, \dots, n-1$. $(W_n, S)$ is a Coxeter system of type $D_n$ and the length function w.r.t. $S$ is

$$l(w) = \sum_{i=1}^{n} a_i + \sum_{\tau_j=-1} 2b_j,$$

where $a_i = card\{p \mid p > i \ \&\ w_p < w_i\}$ and $b_j = card\{p \mid p < j \ \&\ w_p < w_j\}$. The longest element $w_0$ in $W_n$ is equal to $(\overline{1}, ..., \overline{n})$ if $n$ is even and to $(1, \overline{2}, ..., \overline{n})$ if $n$ is odd. Following [B-G-G] and [D1,2] one defines the operators $\partial_w : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$ for $w \in W_n$ ( here,

$$\partial_{\overline{1}} f = \big(f - f(-x_2, -x_1, x_3, ..., x_n)\big)/(-x_1 - x_2).\ )$$

The even orthogonal analog of Proposition 5.5 reads:

$$\partial_{w_0}(f) = (-1)^{n(n-1)/2} \prod_{i<j} (x_i^2 - x_j^2)^{-1} \sum_{w \in W_n} (-1)^{l(w)} w(f).$$

The even orthogonal analog of Corollary 5.6 reads:

**Lemma 5.16.** *(i)* $\partial_{w_o}(x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n})=0$ *if* $\alpha_p$ *is odd for some* $p=1,\dots,n$.
*(ii) If all* $\alpha_p$ *are even then*

$$\partial_{w_o}(x^\alpha)=(-1)^{n(n-1)/2}2^{n-1}s_{\rho_{n-1}}(X_n)^{-1}\partial(x^\alpha),$$

*where* $\partial$ *denotes the Jacobi symmetrizer.*

*Proof.* (i) Let us fix $\sigma\in S_n$ and look at all elements of $W_n$ of the form $(\sigma,\tau)$ where $\tau\in\{+1,-1\}^n$ and $\prod_i\tau_i=1$. We have

$$\sum_\tau(-1)^{l(\sigma,\tau)}(\sigma,\tau)x^\alpha=(-1)^{l(\sigma)}\sigma(x^\alpha)\sum_\tau\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n},$$

because $l(\sigma,\tau)\equiv l(\sigma)$ $(mod\ 2)$. Suppose that some numbers among $\alpha_1,\dots,\alpha_n$ are odd. We can assume that $\alpha_1,\dots,\alpha_k$ are odd and $\alpha_{k+1},\dots,\alpha_n$ are even for some $k<n$ (by permuting the $\tau_p$'s if necessary). We have

$$\sum_\tau\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n}=\sum_\tau(-1)^{card\{p|\ \tau_p=-1,\ p>k\}}=2^k\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}=0.$$

(ii) Let us now compute $\partial_{w_o}(x_1^{\alpha_1}\dots x_n^{\alpha_n})$ where all $\alpha_p$ are even. Then

$$\sum_\tau\tau_1^{\alpha_1}\dots\tau_n^{\alpha_n}=2^{n-1},\ \text{and}$$

$$\begin{aligned}\partial_{w_o}(x^\alpha)&=(-1)^{n(n-1)/2}\prod_{i<j}(x_i^2-x_j^2)^{-1}2^{n-1}\sum_{\sigma\in S_n}(-1)^{l(\sigma)}\sigma(x^\alpha)\\&=(-1)^{n(n-1)/2}2^{n-1}s_{\rho_{n-1}}(X_n)^{-1}\partial(x^\alpha).\quad\Box\end{aligned}$$

Let us now denote by $\mathcal{J}$ the ideal in $S\mathcal{P}(X_n)\otimes\mathbb{Z}[1/2]$ generated by $e_i(x_1^2,\dots,x_n^2)$, $i=1,\dots,n-1$, and $x_1\cdot\dots\cdot x_n$. In the following analog of Lemma 5.4 we write $e_i=e_i(X_n)$, $s_I=s_I(X_n)$, $P_I=P_I(X_n)$ and $\widetilde{P}_I=\widetilde{P}_I(X_n)$ for brevity.

**Lemma 5.17.** *In* $S\mathcal{P}(X_n)\otimes\mathbb{Z}[1/2]$,

$$\widetilde{P}_{\rho_{n-1}}\equiv2^{-(n-1)}e_{n-1}e_{n-2}\dots e_1\equiv2^{-(n-1)}s_{\rho_{n-1}}\ (mod\ \mathcal{J}).$$

*Proof.* Proposition 5.2 implies that $P_{\rho_k}=Det(a_{i,j})_{1\leqslant i,j,\leqslant k}$ where $a_{i,j}=P_{k+1+j-2i}$ if $k+1+j-2i\neq0$ (with $P_i=0$ for $i<0$) and $P_{i,j}=1$ if $k+1+j-2i=0$. Similarly as in Corollary 5.3, this implies that in $S\mathcal{P}(X_n)\otimes\mathbb{Z}[1/2]$, $\widetilde{P}_{\rho_k}$ is congruent to $Det(b_{i,j})_{1\leqslant i,j\leqslant k}$ modulo $\mathcal{J}$, where $b_{i,j}=\widetilde{P}_{k+1+j-2i}$ if $k+1+j-2i\neq0$ ( with $\widetilde{P}_i=0$ for $i<0$) and $b_{i,j}=1$ if $k+1+j-2i=0$. Thus it is sufficient to prove that $Det(2b_{i,j})_{1\leqslant i,j\leqslant n-1}\equiv e_{n-1}\dots e_1\equiv s_{\rho_{n-1}}$ (mod $\mathcal{J}$). Recall that $s_{\rho_{n-1}}=Det(c_{i,j})_{1\leqslant i,j\leqslant n-1}$ where $c_{i,j}=$

$e_{n+1+j-2i}$ if $n+1+j-2i \neq 0$ and $c_{i,j} = 1$ if $n+1+j-2i = 0$, i.e. the matrices $(2b_{i,j})$ and $(c_{i,j})$ are the same modulo the degree 0 entries.

Let us write the determinants $Det(2b_{i,j})$ and $Det(c_{i,j})$ as the sums of the standard $n!$ terms (some of them are zero). It is easy to see that apart from the "diagonal" term $e_{n-1} \dots e_1$, every other term appearing in both the sums is divisible either by $e_n$ or by $e_{n-1}e_{n-2} \dots e_{p+1}e_p^2$ for some $p \geqslant 1$. We claim that, $e_{n-1}e_{n-2} \dots e_{p+1}e_p^2 \in \mathcal{J}$. To this end, it suffices to show that $e_{n-1}^2$ belongs to $\mathcal{J}$ and argue as in the proof of Lemma 5.4. The needed claim follows from the fact that $e_{n-1}^2 - e_{n-1}(x_1^2, \dots, x_n^2)$ is divisible by $e_n$.

This shows that

$$Det(2b_{i,j}) \equiv e_{n-1} \dots e_1 \equiv Det(c_{i,j}) \ (mod\ \mathcal{J}).$$

Thus the lemma is proved. $\square$

The even orthogonal analog of Lemma 5.7 for the operator $\nabla = \partial_{(\bar{n}, \dots, \bar{2}, \bar{1})}$ if $n$ is even and $\nabla = \partial_{(\bar{n}, \dots, \bar{2}, 1)}$ if $n$ is odd, reads as follows.

**Lemma 5.18.** *(i) If* $f \in S\mathcal{P}(x_1^2, \dots, x_n^2)[x_1 \cdot \dots \cdot x_n]$ *then* $\nabla(f \cdot g) = f \cdot \nabla(g)$.
*(ii)* $\nabla\big(\widetilde{P}_{\rho_{n-1}}(X_n)\big) = (-1)^{n(n-1)/2}$.

*Proof.* (i) This assertion is clear because every polynomial in $S\mathcal{P}(x_1^2, \dots, x_n^2)[x_1 \cdot \dots \cdot x_n]$ is $W_n$-invariant.
(ii) In this part we will use the Jacobi symmetrizer $\partial$ (see the proof of Lemma 5.7). In the following, $\mathcal{I} = \mathcal{J}\mathbb{Z}[X_n]$.

Let $e_i = e_i(X_n)$. Since $\widetilde{P}_{\rho_{n-1}}(X_n) \equiv 2^{-(n-1)}e_{n-1} \dots e_1$ (mod $\mathcal{I}$), we have

$$\begin{aligned} \nabla(\widetilde{P}_{\rho_{n-1}}(X_n)) &= \nabla(2^{-(n-1)}e_{n-1} \dots e_1) = (\nabla \circ \partial)(2^{-(n-1)}x^{\rho_{n-1}}e_{n-1} \dots e_1) \\ &= \partial_{w_0}(2^{-(n-1)}x^{\rho_{n-1}}e_{n-1} \dots e_1). \end{aligned}$$

The degree of the polynomial $x^{\rho_{n-1}}e_{n-1} \dots e_1$ is $n^2 - n$. Assuming that $\alpha_1 + \dots + \alpha_n = n^2 - n$, we have $\partial_{w_0}(x^\alpha) \neq 0$ only if

$$x^\alpha = x_{\sigma(1)}^{2n-2}x_{\sigma(2)}^{2n-4} \dots x_{\sigma(n-1)}^2 x_{\sigma(n)}^0$$

for some $\sigma \in S_n$. Indeed, it follows from Lemma 5.16 that $\partial_{w_0}(x^\alpha) \neq 0$ only if all the $\alpha_i$'s are even. Moreover, they must be all different; otherwise $\partial(x^\alpha) = 0$ and consequently $\partial_{w_o}(x^\alpha) = 0$. We conclude that $\{\alpha_1, \dots, \alpha_n\} = \{2n-2, 2n-4, \dots, 2, 0\}$. But there is only one such a monomial $x^\alpha$ in $x^{\rho_{n-1}}e_{n-1} \dots e_1$, namely the one with $(\alpha_1, \dots, \alpha_n) = (2n-2, 2n-4, \dots, 2, 0)$. Therefore

$$\partial_{w_0}(2^{-(n-1)}x^{\rho_{n-1}}e_{n-1} \dots e_1) = 2^{-(n-1)}\partial_{w_0}(x_1^{2n-2}x_2^{2n-4} \dots x_{n-1}^2 x_n^0) = (-1)^{n(n-1)/2}$$

by Lemma 5.16. $\square$

The even orthogonal analog of Proposition 5.9 reads (since $e_n(X_n^\vee)$ is a scalar for $\nabla$, it suffices to evaluate the images via $\nabla$ of $P_I(X_n^\vee)$, where all $i_p \leqslant n-1$):

**Proposition 5.19.** *Let $I$ be a partition with all parts not greater than $n-1$. One has $\nabla(\widetilde{Q}_I(X_n^\vee)) \neq 0$ only if the set of parts of $I$ is equal to $\{1,2,\ldots,n-1\}$ and each number $p$ $(1 \leqslant p \leqslant n-1)$ appears in $I$ with an odd multiplicity $m_p$. Then, the following equality holds in $\mathbb{Z}[X_n]$,*

$$\nabla\big(\widetilde{Q}_I(X_n^\vee)\big) = 2^{n-1}\prod_{p=1}^{n-1} e_p(x_1^2,\ldots,x_n^2)^{(m_p-1)/2}.$$

Arguing as in the proof of Proposition 5.8 one shows that the Gysin maps $\pi_*$ associated with $\pi : OG'_n V \to X$ (resp. $\pi : OG''_n V \to X$) are induced by the operator $\nabla$. The role of $LFl(V)$ is played now by the flag bundle parametrizing flags of rank 1, rank 2, ..., rank $n$ isotropic subbundles of $V$ whose rank $n$ subbundle $E$ satisfies $dim(E\cap V_n)_x \equiv n\ (mod\ 2)$ (resp. $dim(E\cap V_n)_x \equiv n+1\ (mod\ 2)$) for every $x \in X$. Consequently, the proposition whose proof is the same as the one of Proposition 5.9, has as its consequence:

**Theorem 5.20.** *Let $I$ be a partition with all parts not greater than $n-1$. The element $\widetilde{Q}_I R^\vee$ has a nonzero image under $\pi_*$ only if each number $p$, $1 \leqslant p \leqslant n-1$, appears as a part of $I$ with an odd multiplicity $m_p$. If this last condition holds then*

$$\pi_*\widetilde{Q}_I R^\vee = 2^{n-1}\prod_{p=1}^{n-1}\Big((-1)^p c_{2p}V\Big)^{(m_p-1)/2}.$$

**Theorem 5.21.** *The element $s_I R^\vee$ $(l(I) \leqslant n-1)$ has a nonzero image under $\pi_*$ only if the partition $I$ is of the form $2J+\rho_{n-1}$ for some partition $J$ $(l(J) \leqslant n-1)$. If $I = 2J+\rho_{n-1}$, then*

$$\pi_* s_I R^\vee = 2^{n-1} s_J^{[2]} V,$$

*where $s_J^{[2]}(-)$ is defined as in Theorem 5.13.*

*Proof.* Since $s_I R^\vee = \omega_*(q^{I+\rho_{n-1}})$ where $q = (q_1,\ldots,q_n)$ are the Chern roots of $R^\vee$, we infer from Lemma 5.16 that $s_I R^\vee$ has a nonzero image under $\pi_*$ only if all parts of $I+\rho_{n-1}$ are even. This implies that $l(I) = n-1$ and $I$ is strict thus of the form $I'+\rho_{n-1}$ for some partition $I'$. Finally all parts of $I'+\rho_{n-1}+\rho_{n-1}$ are even iff $I' = 2J$ for some partition $J$, as required.

Assume now that $I = 2J+\rho_{n-1}$ and specialize the identity from Corollary 5.12 by replacing the variables $(x_i)$ by the Chern roots $(q_i)$. The claimed formula now follows since: $s_I(q_1^2,\ldots,q_n^2)$ is a scalar w.r.t. $\pi_*$, $\pi_* s_{\rho_{n-1}}(q_1,\ldots,q_n) = 2^{n-1}$ by Lemma 5.17 and 5.18; moreover $2(-1)^i c_{2i}V = 2e_i(q_1^2,\ldots,q_n^2)$ by Lemma 1.1(2). $\square$

**Remark 5.22.** 1. Our desingularizations of Schubert subschemes are compositions of flag– and isotropic Grassmannian bundles (see Section 1). Therefore Corollary 2.6, the algebra of $\widetilde{Q}$-polynomials together with formulas for Gysin push forwards (Theorem 5.10 for Lagrangian Grassmannians and a well known formula for projective bundles) give an explicit algorithm for calculation the fundamental classes of Schubert subschemes in the Lagrangian Grassmannian bundles. One has analogous algorithms in the orthogonal cases. Examples of such calculations are given in Section 6 and 7.

2. In case $X$ is singular, by interpreting polynomials in Chern classes as operators acting on Chow groups (see [F]) or singular homology groups, the same formulas hold (after their obvious adaptation to the operator setup).

We finish this section with the following important "orthogonality" property for the Gysin maps associated with isotropic Grassmannian bundles.

**Theorem 5.23.** *(i) For $\pi : LG_nV \to X$ and any strict partitions $I, J$ ($\subset \rho_n$),*

$$\pi_*(\widetilde{Q}_I R^\vee \cdot \widetilde{Q}_J R^\vee) = \delta_{I, \rho_n \smallsetminus J}.$$

*(ii) For $\pi : OG_nV \to X$ ($\dim V = 2n+1$) and any strict partitions $I, J$ ($\subset \rho_n$),*

$$\pi_*(\widetilde{P}_I R^\vee \cdot \widetilde{P}_J R^\vee) = \delta_{I, \rho_n \smallsetminus J}.$$

*(iii) For $\pi : OG'_nV \to X$ (resp. $OG''_nV \to X$), and any strict partitions $I, J$ ($\subset \rho_{n-1}$),*

$$\pi_*(\widetilde{P}_I R^\vee \cdot \widetilde{P}_J R^\vee) = \delta_{I, \rho_{n-1} \smallsetminus J}.$$

*(Here, $\delta_{\cdot,\cdot}$ is the Kronecker delta.)*

*Proof.* We will prove first the Lagrangian case (i). (In case (ii), the proof goes mutatis mutandis using the divided differences operator $\partial_{(\overline{n}, \overline{n-1}, \dots, \overline{1})}$ for $SO(2n+1)$ instead of the operator $\nabla$ for $Sp(2n)$ . Case (iii) will be discussed separately at the end of the proof.

Let $X_n = (x_1, ..., x_n)$ be a sequence of variables. We show that the operator $\nabla : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$, satisfies the following formula for any strict partitions $I, J$ ($\subset \rho_n$):

$$\nabla\big(\widetilde{Q}_I(X_n^\vee) \cdot \widetilde{Q}_J(X_n^\vee)\big) = \delta_{I, \rho_n \smallsetminus J}.$$

Since $\pi_*$ is induced by $\nabla$ (Proposition 5.8), this implies the assertion. Observe that for the degree reasons $\nabla(\widetilde{Q}_I \cdot \widetilde{Q}_J) = 0$ for $|I| + |J| < n(n+1)/2$ (here and in the rest of the proof, $\widetilde{Q}_I = \widetilde{Q}_I(X_n^\vee)$). Also, because of the universality of the formula for $\pi_*$ (see e.g. Theorem 5.10), we know (Lemma 2.3, Lemma 2.4 and 5.8) that for $|I|+|J| = n(n+1)/2$, $\nabla(\widetilde{Q}_I \cdot \widetilde{Q}_J) = 0$ unless $J = \rho_n \smallsetminus I$, when $\nabla(\widetilde{Q}_I \cdot \widetilde{Q}_J) = 1$. So it remains to show that for $|I| + |J| > n(n+1)/2$, $\nabla(\widetilde{Q}_I \cdot \widetilde{Q}_J) = 0$. The proof is by double induction whose first parameter is $l(I)$ and the second one is $i_l$ where $l = l(I)$ (i.e. the shortest part of $I$).

Assume first that $I = (i)$ and use the Pieri-type formula from Proposition 4.9. A general partition $J'$ indexing the R.H.S. of the formula from Proposition 4.9 stems from $J$ by adding a horizontal strip of length $i$. Since $|J| + i > n(n+1)/2$, the only possibility for getting $\nabla(\widetilde{Q}_{J'}) \neq 0$ is the following (Theorem 5.10): there exist two equal parts $p$ in $J'$ such that after factoring out $\widetilde{Q}_{p,p}$ from $\widetilde{Q}_{J'}$ (Proposition 4.3) we obtain $\widetilde{Q}_{\rho_n}$ (recall that $\widetilde{Q}_{p,p}$ is a scalar w.r.t. $\nabla$). But $l(J') \leq l(J) + 1 \leq n+1$, so after factoring out the length of the so-obtained partition is not greater than $n-1$, i.e. this partition is not $\rho_n$.

To perform the induction step write $I' = (i_1, ..., i_{l-1})$ and $r = i_l$ where we assume that $l = l(I) \geq 2$. Using the Pieri-type formula again, we have:

$$\widetilde{Q}_I \cdot \widetilde{Q}_J = (\widetilde{Q}_{I'} \cdot \widetilde{Q}_r) \cdot \widetilde{Q}_J - (\sum_M 2^{m(I',r;M)} \widetilde{Q}_M) \cdot \widetilde{Q}_J = \widetilde{Q}_{I'} \cdot (\widetilde{Q}_J \cdot \widetilde{Q}_r) - (\sum_M 2^{m(I',r;M)} \widetilde{Q}_M) \cdot \widetilde{Q}_J$$

$$= \widetilde{Q}_{I'} \cdot (\sum_N 2^{m(J,r;N)} \widetilde{Q}_N) - (\sum_M 2^{m(I',r;M)} \widetilde{Q}_M) \cdot \widetilde{Q}_J.$$

Here $M$ runs over all partitions different from $I$ which contain $I'$ with $M/I'$ being a horizontal strip of length $r$. Observe that either $l(M) < l(I)$ or $l(M) = l(I)$ but $m_l < i_l = r$, so we can apply the induction assumption to $M_1$ defined below. The partitions $M$ and $N$ can have equal parts; if so, using the factorization property, we write:

$$\widetilde{Q}_M = \widetilde{Q}_{p_1,p_1} \cdot \ldots \cdot \widetilde{Q}_{p_s,p_s} \cdot \widetilde{Q}_{M_1} \quad \text{and} \quad \widetilde{Q}_N = \widetilde{Q}_{q_1,q_1} \cdot \ldots \cdot \widetilde{Q}_{q_t,q_t} \cdot \widetilde{Q}_{N_1},$$

where $M_1, N_1$ are strict partitions and $p_1 > \ldots > p_s$, $q_1 > \ldots > q_t$ are positive integers. Using the induction assumption or because of the degree reasons we see that the only possibility to get in the first sum a summand (corresponding to $N$) which is not annihilated by $\nabla$ is: after adding to $J$ a horizontal strip of length $r$ and factoring out all pairs of equal rows, we obtain the partition $N_1 = \rho_n \smallsetminus I'$. Similarly, the only possibility to get in the second sum a summand (corresponding to $M$) which is not annihilated by $\nabla$ is: after adding to $I'$ a horizontal strip of length $r$ and factoring out all pairs of equal rows, we obtain the partition $M_1 = \rho_n \smallsetminus J$.

Therefore to conclude the proof it is sufficient to define, for a fixed pair of strict partitions $I', J$ and fixed positive integers $r$ and $p. : p_1 > \ldots > p_s$, a bijection between the sets of partitions (with parts not exceeding $n$):

$\mathcal{N} = \{$ $N \mid N \supset J;$ $N/J$ is a horizontal strip of length $r$; $N$ has exactly $s$ parts occuring twice, equal to $p.$; after subtraction from $N$ the parts $p.$ one obtains $\rho_n \smallsetminus I'\}$

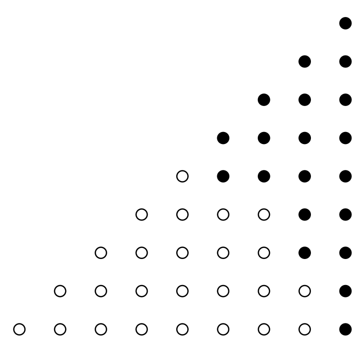
and

$\mathcal{M} = \{$ $M \mid M \supset I'; M/I'$ is a horizontal strip of length $r$; $M$ has exactly $s$ parts occuring twice, equal to $p.$; after subtraction from $M$ the parts $p.$ one obtains $\rho_n \smallsetminus J\}$

which preserves the cardinality of the connected components of the strip, not meeting the first column (compare the Pieri-type formula used).

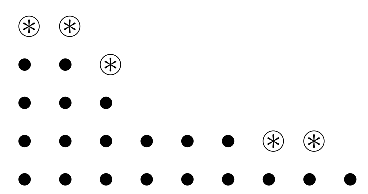
In order to define the bijection $\Phi : \mathcal{N} \to \mathcal{M}$ we first invoke the diagramatic presentation of the $\rho_n$-complementary partition from [P2, p.160]: for example $n = 9$, $I = (9, 6, 3, 2)$, $\rho_9 \smallsetminus I = (8, 7, 5, 4, 1)$,

Fig.1



(the collection of "•" gives the shifted diagram of $I$ (appropriately placed); the collection of "∘" gives the shifted diagram of $\rho_9 \setminus I$). The map $\Phi : \mathcal{N} \to \mathcal{M}$ is defined as follows. Having an element $N \in \mathcal{N}$, i.e. a strict partition $J$ with an added horizontal strip of length $r$, e.g. $J = (9,6,3,2), r = 5, N = (9,8,3,3,2), s = 1, p. : 3$ ( and $I' = (7,6,5,4,3,1)$ ):

Fig.2



we remove the $s$ bottom rows in all pairs of equal rows (in the example, the third row) and place the shift of the so-obtained diagram as in Fig.1 to get the diagram $\widehat{N}$, say. In our example we get the diagram in Fig.3 :

Fig.3

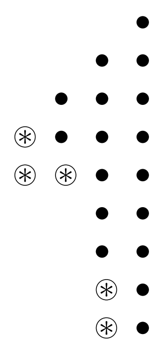
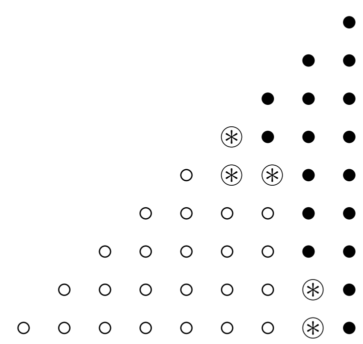


Fig.4



(We know, by the definition of $\mathcal{N}$, that if we would also remove from $\widehat{N}$ the remaining parts of lengths $p.$ then the resulting partition will be $\rho_n \setminus I'$. We preserve these parts, however, because we need them for the construction of $\Phi(N)$.) Then we construct the complement of the so-obtained diagram in $\rho_n$. In our example, using "∘" to visualize the complementary diagram we get the diagram in Fig.4. By reshifting the so-obtained complementary diagram plus *the same* horizontal strip (now added to this complementary diagram) - call it $\Phi(N)_0$, and inserting $s$ rows of lengths $p.$, we get the needed partition $\Phi(N)$. Observe that :

1) Since at the last stage we have inserted rows of lengths $p.$, $\Phi(N)$ consists of the diagram $I'$ with an added horizontal strip of length $r$.

2) $\Phi(N)$ has exactly $s$ parts occuring twice, equal to $p.$ (apart from the parts inserted at the last stage, the remaining $s$ parts are the rows whose the rightmost boxes are precisely the lowest boxes of the rows of length $p.$ in $\widehat{N}$).

3) After removing from $\Phi(N)$ the $2s$ parts equal to $p.$, we get $\rho_n \setminus J$ (this is the same as removing from $\Phi(N)_0$ the $s$ parts equal to $p.$ - but $\Phi(N)_0$ minus $s$ parts equal to $p.$ complements precisely $J$ in $\rho_n$).

Therefore $\Phi(N) \in \mathcal{M}$. Also, the cardinality of the connected components of the strip not meeting the first column is preserved by $\Phi$. In our example, we obtain

Fig.5
$$\begin{array}{llllllll}
\circledast \\
\circ & \circledast & \circledast \\
\circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circledast \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circledast
\end{array}$$

i.e. $\Phi(N) = (8, 7, 5, 4, 3, 3, 1)$.

Let us now define, by reversing the roles of $J$ and $I'$, the map $\Psi : \mathcal{M} \to \mathcal{N}$. If we define, by a complete analogy to the above, the partitions $\widehat{M}$ and $\Psi(M)_0$, then we have $\widehat{N} = \Psi(M)_0$ and $\Phi(N)_0 = \widehat{M}$; and clearly $\Psi \circ \Phi = id_{\mathcal{N}}$ and $\Phi \circ \Psi = id_{\mathcal{M}}$.

This proves the orthogonality theorem in the Lagrangian case.

Essentially the same proof, with $\widetilde{Q}$-polynomials replaced by $\widetilde{P}$-polynomials (for which a Pieri-type formula is given below), works in the odd orthogonal case (ii).

In the even orthogonal case the proof goes as follows. Let $\nabla : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$ be the even orthogonal divided differences operator inducing $\pi_*$.

We show that the operator $\nabla : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$, satisfies the following formula for any strict partitions $I, J$ $(\subset \rho_{n-1})$:

$$\nabla\big(\widetilde{P}_I(X_n^\vee) \cdot \widetilde{P}_J(X_n^\vee)\big) = \delta_{I,\rho_{n-1} \smallsetminus J}.$$

Since $\pi_*$ is induced by $\nabla$, this implies the assertion. Observe that for the degree reasons $\nabla(\widetilde{P}_I \cdot \widetilde{P}_J) = 0$ for $|I| + |J| < n(n-1)/2$ (here and in the rest of the proof, $\widetilde{P}_I = \widetilde{P}_I(X_n^\vee)$). Also, because of the universality of the formula for $\pi_*$ that for $|I| + |J| = n(n-1)/2$, $\nabla(\widetilde{P}_I \cdot \widetilde{P}_J) = 0$ unless $J = \rho_{n-1} \smallsetminus I$, when $\nabla(\widetilde{P}_I \cdot \widetilde{P}_J) = 1$. So it remains to show that for $|I| + |J| > n(n-1)/2$, $\nabla(\widetilde{P}_I \cdot \widetilde{P}_J) = 0$. The proof is by double induction whose first parameter is $l(I)$ and the second one is $i_l$ where $l = l(I)$ (i.e. the shortest part of $I$).

Assume first that $I = (i)$ and use the Pieri-type formula for $\widetilde{P}$-polynomials (see Proposition 4.9; a Pieri-type formula for $\widetilde{P}$-polynomials reads similarly:

$$\widetilde{P}_J \cdot \widetilde{P}_i = \sum 2^{m'(J,i;J')} \widetilde{P}_{J'},$$

the only difference being the exponent $m'(J,i;J')$ equal to $m(J,i;J')$ if $J'/J$ meets the first column and $m(J,i;J') - 1$ – if not.)

A general partition $J'$ indexing the R.H.S. of the Pieri formula stems from $J$ by adding a horizontal strip of length $i$. Since $|J| + i > n(n-1)/2$, the only possibility for getting $\nabla(\widetilde{P}_{J'}) \neq 0$ is the following: there exists a (single) row of length $n$ or there exist two equal parts $p$ in $J'$ $(1 \leqslant p \leqslant n-1)$ such that after factoring out $\widetilde{P}_n$ and $\widetilde{P}_{p,p}$ from $\widetilde{P}_{J'}$ (Proposition 4.3) we obtain $\widetilde{P}_{\rho_{n-1}}$. As in the proof of the Lagrangian case we see that it is impossible to get $\widetilde{P}_{\rho_{n-1}}$ after factoring out $\widetilde{P}_{p,p}$. Also, it is impossible to get $\widetilde{P}_{\rho_{n-1}}$ by factoring out $\widetilde{P}_n$. Indeed, using the Pieri-type formula, we should add one box to each of the first $n$ columns which is impossible because $i \leqslant n-1$.

To perform the induction step write $I' = (i_1, ..., i_{l-1})$ and $r = i_l$ where we assume that $l = l(I) \geq 2$. Using the Pieri-type formula, we have:

$$\widetilde{P}_I \cdot \widetilde{P}_J = \widetilde{P}_{I'} \cdot (\sum_N 2^{m'(J,r;N)} \widetilde{P}_N) - (\sum_M 2^{m'(I',r;M)} \widetilde{P}_M) \cdot \widetilde{P}_J.$$

Here $M$ runs over all partitions different from $I$ which contain $I'$ with $M/I'$ being a horizontal strip of length $r$. Observe that either $l(M) < l(I)$ or $l(M) = l(I)$ but $m_l < i_l = r$, so we can apply the induction assumption to $M_1$ defined below. The partitions $M$ and $N$ can have equal parts; if so, using the factorization property, we write:

$$\widetilde{P}_M = \widetilde{P}_{p_1,p_1} \cdot \ldots \cdot \widetilde{P}_{p_s,p_s} \cdot \widetilde{P}_{M_1} \quad \text{and} \quad \widetilde{P}_N = \widetilde{P}_{q_1,q_1} \cdot \ldots \cdot \widetilde{P}_{q_t,q_t} \cdot \widetilde{P}_{N_1},$$

where $M_1, N_1$ are strict partitions and $p_1 > \ldots > p_s$, $q_1 > \ldots > q_t$ are positive integers. Moreover $M$ and $N$ can contain a single row of length $n$, and if so, then the polynomial $\widetilde{P}_n$ can be factored out by a property of the operator $\nabla$. Using the induction assumption or because of the degree reasons we see that the only possibility to get in the first sum a summand (corresponding to $N$) which is not annihilated by $\nabla$ is: after adding to $J$ a horizontal strip of length $r$ and factoring out all pairs of equal rows and the row of length $n$ (if any), we obtain the partition $N_1 = \rho_{n-1} \smallsetminus I'$. Similarly, the only possibility to get in the second sum a summand (corresponding to $M$) which is not annihilated by $\nabla$ is: after adding to $I'$ a horizontal strip of length $r$ and factoring out all pairs of equal rows and the row of length $n$, if any, we obtain the partition $M_1 = \rho_{n-1} \smallsetminus J$.

Therefore to conclude the proof it is sufficient to give two bijections.

The data of the first bijection are: a pair of strict partitions $I', J \subset \rho_{n-1}$ and fixed positive integers $r$ and $p. : p_1 > \ldots > p_s$. One needs a bijection between the sets of partitions (with parts not exceeding $n-1$):

$\mathcal{N} = \{$ $N \mid N \supset J$; $N/J$ is a horizontal strip of length $r$; $N$ has exactly $s$ parts occuring twice, equal to $p.$; after subtraction from $N$ the parts $p.$ one obtains $\rho_{n-1} \smallsetminus I'\}$

and

$\mathcal{M} = \{$ $M \mid M \supset I'$; $M/I'$ is a horizontal strip of length $r$; $M$ has exactly $s$ parts occuring twice, equal to $p.$; after subtraction from $M$ the parts $p.$ one obtains $\rho_{n-1} \smallsetminus J\}$

which preserves the property that the strip meets or not the first column and preserves the cardinality of the connected components of the strip, not meeting the first column – compare the Pieri-type formula used).

Here the bijection $\Phi : \mathcal{N} \to \mathcal{M}$ from the proof of the Lagrangian case with $n$ replaced by $n-1$ does the job (i.e. to construct $\Phi(N)_0$ for $N \in \mathcal{N}$ we take the complement in $\rho_{n-1}$). Note that $\Phi$ preserves the property that the strip meets or not the first column by the construction.

The data of the second bijection are also: a pair of strict partitions $I', J \subset \rho_{n-1}$ and fixed positive integers $r$ and $p. : p_1 > \ldots > p_s$. One needs a bijection between the sets of partitions (with parts not exceeding $n$):

$\mathcal{N}' = \{$ $N \mid N \supset J$; $N/J$ is a horizontal strip of length $r$; $N$ has a single part equal to $n$; $N$ has exactly $s$ parts occuring twice, equal to $p.$; after subtraction from $N$ the parts $p.$ and $n$ one obtains $\rho_{n-1} \smallsetminus I'\}$

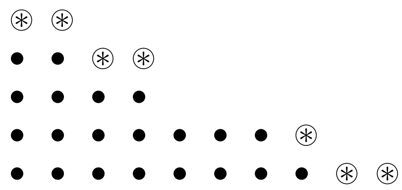
and

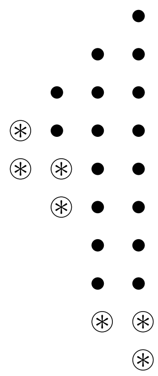
$\mathcal{M}' = \{$ $M \mid M \supset I'; M/I'$ is a horizontal strip of length $r$; $M$ has a single part equal to $n$; $M$ has exactly $s$ parts occuring twice, equal to $p.$; after subtraction from $M$ the parts $p.$ and $n$ one obtains $\rho_{n-1} \smallsetminus J\}$

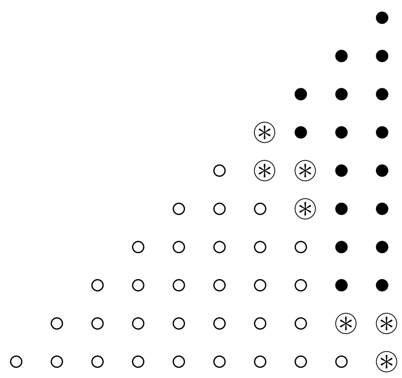
which preserves the property that the strip meets or not the first column and preserves cardinality of the connected components of the strip, not meeting the first column.

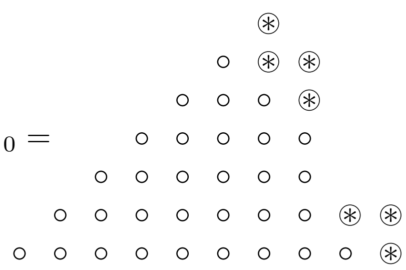
Here we also use the map $\Phi$ from the proof of case (i) (to construct $\Phi(N)_0$ for $N \in \mathcal{N}'$ we take the complement in $\rho_n$). We ilustrate the map $\Phi$ on the following example.

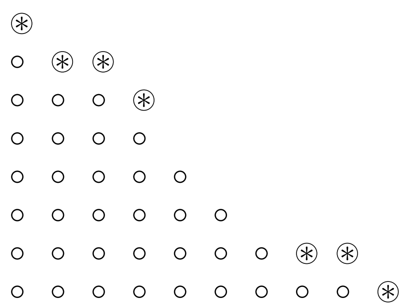
Let $n = 10$ , $J = (8, 7, 4, 2)$ , $N = (10, 8, 4, 4, 2)$ and $I' = (9, 7, 6, 5, 4, 3, 1)$ .

$N =$ 

$\widehat{N} =$ 



$\phi(N)_0 =$ 

$\phi(N) =$ 

$(N = \{10, 8, 4, 4, 2\}) \smallsetminus \{10, 4, 4\} = \{9, 8, 7, 6, 5, 4, 3, 2, 1\} \smallsetminus (\{9, 7, 6, 5, 4, 3, 1\} = I')$

$(\Phi(N) = \{10, 9, 6, 5, 4, 4, 3, 1\}) \smallsetminus \{10, 4, 4\} = \{9, 8, 7, 6, 5, 4, 3, 2, 1\} \smallsetminus (\{8, 7, 4, 2\} = J)$

We have $\Phi(\mathcal{N}') \subset \mathcal{M}'$. Indeed, if $N \in \mathcal{N}'$ then $\Phi(N)$ has a part equal to $n$ by the construction. Moreover, the equations:

$$N \smallsetminus \{n, p.\} = \rho_{n-1} \smallsetminus I' \ , \ \Phi(N) \smallsetminus \{n, p.\} = \rho_{n-1} \smallsetminus J$$

are equivalent to the equations:

$$N \smallsetminus \{p.\} = \rho_n \smallsetminus I' \ , \ \Phi(N) \smallsetminus \{p.\} = \rho_n \smallsetminus J,$$

so the assertion follows from the proof in case (i) above.

By reversing the roles of $J$ and $I'$, one defines (as in the proof of case (i)) the map $\Psi : \mathcal{M}' \to \mathcal{N}'$ which satisfies: $\Psi \circ \Phi = id_{\mathcal{N}'}$ and $\Phi \circ \Psi = id_{\mathcal{M}'}$.

This ends the proof of the theorem. $\square$

## 6. Single Schubert condition

We consider first the Lagrangian case $G = LG_nV$ and follow the notation introduced in Section 1.

**Proposition 6.1.** *The class of $\Omega(a)$ in $A^*(G)$, where $a = n + 1 - i$, is given by the formula*

$$\big[\Omega(a)\big] = \sum_{p=0}^{i} c_p R^\vee \cdot s_{i-p}(V_a^\vee).$$

*Proof.* The desingularization $\mathcal{F}$ of $\Omega(a) \subset G$ is given by the composition (recall that $Fl(a_\bullet)$ from Section 1 is here $\mathbb{P}(V_a)$ and $C$ is the tautological line bundle on it):

$$\mathcal{F} = LG_{n-1}(C^\perp/C) \xrightarrow{\pi_1} \mathbb{P}(V_a) \xrightarrow{\pi_2} G \ ,$$

where $\pi_1$ and $\pi_2$ denote the corresponding projection maps. By Corollary 2.6 we have

$$(*) \qquad [Z] = \sum_{strict \ I \subset \rho_n} \widetilde{Q}_I D^\vee \cdot \widetilde{Q}_{\rho_n \smallsetminus I} R^\vee.$$

Let $S$ be the tautological rank $n-1$ bundle on $\mathcal{F}$; $S = D/C_{\mathcal{F}}$. Let $c = c_1(C^\vee)$. Then, by Proposition 4.1,

$$(**) \qquad \widetilde{Q}_I D^\vee = \sum_{k=0}^{n} (\pi_1^* c)^k \cdot \sum_J \widetilde{Q}_J S^\vee \ ,$$

the sum over all partitions $J \subset I$ of weight $|J| = |I| - k$ and $I/J$ has at most one box in each row. By Theorem 5.10 the only $I$'s in (*) for which $(\pi_1)_* \widetilde{Q}_I D^\vee \neq 0$, are those containing $\rho_{n-1}$, i.e. $I$ must be equal to one of the partitions of the form

$I_p = \big(n, n-1, \dots, p+1, p-1, \dots, 1\big)$ for some $p = 0, 1, \dots, n$. For $I = I_p$ the only term in (**) which contributes after applying $(\pi_1)_*$ is the one with $J = \rho_{n-1}$ and $k = n - p$.

Since, by a well-known push forward formula for projective bundles, we have

$$(\pi_2)_*\big(c^{n-p}\big) = s_{n-p-(n-i)}\big(V_a^\vee\big) = s_{i-p}\big(V_a^\vee\big) \ ,$$

we infer that only $p = 0, 1, \dots, i$ give a nontrivial contribution from (**) (with $k = n-p$). Finally, we get

$$\big[\Omega(a)\big] = (\pi_2\pi_1)_*[Z] = \sum_{p=0}^{i} \widetilde{Q}_p R^\vee \cdot s_{i-p}(V_a^\vee) = \sum_{p=0}^{i} c_p R^\vee \cdot s_{i-p}(V_a^\vee)$$

as asserted. $\square$

Essentially the same computation gives the following formula in for $G = OG_n V$ where $dim\, V = 2n+1$.

**Proposition 6.2.** *The class of $\Omega(a)$ in $A^*(G)$, where $a = n+1-i$ is given by the formula*

$$\big[\Omega(a)\big] = 1/2 \cdot \sum_{p=0}^{i} c_p R^\vee\, s_{i-p}(V_a^\vee)\,.^5$$

Consider finally the even orthogonal case where the computation is slightly different.

**Proposition 6.3.** *The class of $\Omega(a)$ in $A^*(OG'_n V)$ for odd $n$, or in $A^*(OG''_n V)$ for even $n$, is given by the following expression where $i = n - a$:*

$$\big[\Omega(a)\big] = 1/2 \cdot \sum_{p=0}^{i} \big(c_p R^\vee + c_p V_n\big)\, s_{i-p}(V_a^\vee).$$

**Proof.** Suppose that $n$ is odd. Then the desingularization $\mathcal{F}'$ of $\Omega(a) \subset G = OG'_n V$ is given by the composition:

$$\mathcal{F}' = OG'_{n-1}(C^\perp/C) \xrightarrow{\pi_1} \mathbb{P}(V_a) \xrightarrow{\pi_2} G \ ,$$

where $\pi_1$ and $\pi_2$ denote the corresponding projection maps.

[5] Observe that though "1/2" appears in the formula, the integrality property of the class obtained holds true (i.e. we get the class in the Chow group with the integer coefficients). This follows directly from our way of computing it. Indeed, the (odd) Orthogonal version of Proposition 2.5 and consequently also of Corollary 2.6 holds true over integers. Also, the integrality is preserved by the Gysin maps in the odd Orthogonal analogs of Propositions 3.1 and 3.2. The same remark applies to Proposition 6.3 and Theorem 7.9 below.

If $n$ is even then the desingularization $\mathcal{F}''$ of $\Omega(a) \subset G = OG''_n V$ is given by the composition:

$$\mathcal{F}'' = OG''_{n-1}(C^{\perp}/C) \xrightarrow{\pi_1} \mathbb{P}(V_a) \xrightarrow{\pi_2} G \ ,$$

where $\pi_1$ and $\pi_2$ denote the corresponding projections. In the following we denote by $\mathcal{F}$ both $\mathcal{F}'$ and $\mathcal{F}''$ for brevity.

By Proposition 2.7 we have in both the cases:

$$[Z] = \sum_{strict \ I \subset \rho_{n-1}} \widetilde{P}_I D^{\vee} \cdot \widetilde{P}_{\rho_{n-1} \smallsetminus I} R^{\vee}.$$

Let $S$ be the tautological rank $n-1$ bundle on $\mathcal{F}$; $S = D/C_{\mathcal{F}}$. Let $c = c_1 C^{\vee}$. Then, by Proposition 4.1 interpreted now in terms of $\widetilde{P}$-polynomials we have

$$(***) \qquad \widetilde{P}_I D^{\vee} = \sum_{k=0}^{n} (\pi_1^* c)^k \cdot \Big( \sum_J 2^{l(J)-l(I)} \widetilde{P}_J S^{\vee} \Big) \ ,$$

the sum over all partitions $J \subset I$ of weight $|J| = |I| - k$ and $I/J$ has at most one box in each row. By Theorem 5.20, if $(\pi_1)_* \widetilde{Q}_I D^{\vee} \neq 0$ then $I \supset \rho_{n-2}$, so $I$ must be equal to the partition $I_p = \big(n-1, n-2, \ldots, p+1, p-1, \ldots, 1\big)$ for some $p = 0, 1, \ldots, n-1$. More precisely, the only terms in (***) which contribute nontrivially after applying $(\pi_1)_*$ correspond to the following two instances:

(1) $I = I_0 = \rho_{n-1}$, $k = 0$ and $J = \rho_{n-1}$ – this gives a difference between the odd orthogonal case and the present one.

(2) $I = I_p$, $k = n-p-1$ and $J = \rho_{n-2}$; here $p = 0, 1, \ldots, n-1$ but we will see that only $p = 0, 1, \ldots, i$ give a nontrivial contribution.

Let us first compute the contribution of (1). We claim that

$$(\pi_1)_* P_{\rho_{n-1}} S^{\vee} = 1/2 \cdot v,$$

where $v = c_{n-1}(V_n/C)$. Indeed, if $n$ is odd then

$$(\pi_1)_* P_{\rho_{n-1}} S^{\vee} = 1/2 \cdot (\pi_1)_* \big(c_{n-1} S \cdot P_{\rho_{n-2}} S^{\vee}\big) = 1/2 \cdot (\pi_1)_* \big(v \cdot P_{\rho_{n-2}} S^{\vee}\big) = 1/2 \cdot v$$

by Lemma 5.18 and a theorem of Edidin-Graham [E-G] asserting, in this case, the equality $c_{n-1} S = v$.

If $n$ is even then

$$(\pi_1)_* P_{\rho_{n-1}} S^{\vee} = -1/2 \cdot (\pi_1)_* \big(c_{n-1} S \cdot P_{\rho_{n-2}} S^{\vee}\big) = -1/2 \cdot (\pi_1)_* \big(-v \cdot P_{\rho_{n-2}} S^{\vee}\big) = 1/2 \cdot v$$

because the theorem of Edidin-Graham now asserts that $c_{n-1} S = -v$.

Therefore the contribution of (1) is equal to

$$\begin{aligned}(\pi_2\circ\pi_1)_*P_{\rho_{n-1}}S^\vee &= 1/2\cdot(\pi_2)_*c_{n-1}(V_n/C)\\ &= 1/2\cdot(\pi_2)_*\big(\sum_{p=0}^{n-1}(-1)^{n-p-1}c_pV_n\cdot s_{n-p-1}C\big)\\ &= 1/2\cdot(\pi_2)_*\big(\sum_{p=0}^{n-1}c_pV_n\cdot c^{n-p-1}\big) = 1/2\cdot\big(\sum_{p=0}^{n-1}c_pV_n\cdot s_{i-p}C^\vee\big)\end{aligned}$$

On the other hand the contribution of (2) is equal to

$$1/2\cdot(\pi_2\circ\pi_1)_*\big(c_pR^\vee\cdot c^{n-p-1}\cdot P_{\rho_{n-2}}S^\vee\big) = 1/2\cdot c_pR^\vee\cdot(\pi_2)_*(c^{n-p-1}) = 1/2\cdot c_pR^\vee\cdot s_{i-p}(V_a^\vee).$$

Summing up the contributions of (1) and (2) we infer

$$[\Omega(a)] = 1/2\cdot\sum_{p=0}^{i}\big(c_pV_n + c_pR^\vee\big)\cdot s_{i-p}(V_a^\vee),$$

which is the asserted formula. $\square$

## 7. Two Schubert conditions

In this section we treat the classes of Schubert subschemes defined by two Schubert conditions in the Lagrangian and odd orthogonal cases.

We consider first the Lagrangian case. Our desingularization of $\Omega(n+1-i, n+1-j)$ in $G = LG_nV$ is given by the composition (we use the notation of Section 1, *rank* $C = 2$):

$$\mathcal{F} = LG_{n-2}\big(C^\perp/C\big) \xrightarrow{\pi_1} Fl(V_a\subset V_b) \xrightarrow{\pi_2} G,$$

where $(a,b) = (n+1-i, n+1-j)$ and the element to be push forwarded via $(\pi_2\pi_1)_*$ is $\sum\widetilde{Q}_ID^\vee\cdot\widetilde{Q}_{\rho_n\smallsetminus I}R^\vee$, the sum over all strict $I\subset\rho_n$. Let $S$ be the tautological rank $(n-2)$ bundle on $LG_{n-2}\big(C^\perp/C\big)$. Using $[D^\vee] = [S^\vee] + [C^\vee_{\mathcal{F}}]$ and the linearity formula from Proposition 4.1 together with the factorization property from Proposition 4.3, we have $(\pi_1)_*\widetilde{Q}_ID^\vee \neq 0$ only if $(\pi_1)_*\widetilde{Q}_JS^\vee\neq 0$ for some $J\subset I$. By virtue of Theorem 5.10 (applied to $S^\vee$), $(\pi_1)_*\widetilde{Q}_JS^\vee\neq 0$ only if $J\supset\rho_{n-2}$. Consequently, the unique strict $I$'s for which $(\pi_1)_*\widetilde{Q}_ID^\vee\neq 0$ must contain $\rho_{n-2}$, i.e. they are of the form: $I=\rho_n$, $I = (n, n-1,\dots,\hat{p},\dots,1) =: I_p$, $I = (n,n-1,\dots,\hat{p},\dots,\hat{q},\dots,1) =: I_{p,q}$ (here, $p$ and $q$ run over $\{1,\dots,n\}$ and the symbol " ˆ " indicates the corresponding omission).

We need the following technical lemma.

**Lemma 7.1.** *If rank* $C = 2$ *then*

*(i)* $\widetilde{Q}_{I_p/\rho_{n-2}}(C^\vee) = s_{n-1,n-p}(C^\vee)$;

*(ii)* *For* $q < p$ , $\widetilde{Q}_{I_{p,q}/\rho_{n-2}}(C^\vee) = s_{n-q-1,n-p}(C^\vee)$;

*(iii)* *For* $0 \leqslant v \leqslant n-2$, $\widetilde{Q}_{\rho_n/(\rho_{n-2}+(2)^v)^{\sim}}(C^\vee) = s_{n-v,n-v-1}(C^\vee)$.

*Proof.* The proof is an easy application of the linearity formula from Proposition 4.1 and is given here in case (i) (the proofs of (ii) and (iii) being similar).

Denote the Chern roots of $C^\vee$ by $x_1, x_2$. Consider the skew Ferrers' diagram of $I_p/\rho_{n-2}$ and fill up with "1" the boxes, whose subtraction correspond to the summands in Proposition 4.1 applied to $x_1$ instead of $x_n$. Then fill up with "2" the boxes, whose subtraction correspond to the summands in Proposition 4.1 applied to $x_2$ instead of $x_n$. Of course it is impossible to have two "1" or two "2" in one row. Also, the following configuration cannot appear:

$$\begin{array}{cc} 2 & \\ \\ \mathrm{x} & 1 \end{array}$$

where the box "x" belongs to $D_{\rho_{n-2}}$ (Having two equal rows ending with $\substack{2\\x}$ we use Proposition 4.3, thus we must subtract both boxes instead of the higher one only). For example, for $n = 6,\ p = 3$ we get two Ferrers' diagrams, one contained in another (the smaller diagram is depicted with "x" and the difference between the bigger diagram and the smaller one is depicted with "$\bullet$"):

$$\begin{array}{cccccc} \bullet & & & & & \\ \mathrm{x} & \bullet & & & & \\ \mathrm{x} & \mathrm{x} & \bullet & \bullet & & \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \bullet & \bullet & \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \bullet & \bullet \end{array}$$

and we have 3 possibilities:

$$\begin{array}{cccccc} 2 & & & & & \\ & 2 & & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & 1 \end{array} \qquad \begin{array}{cccccc} 1 & & & & & \\ & 2 & & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & 1 \end{array} \qquad \begin{array}{cccccc} 1 & & & & & \\ & 1 & & & & \\ & & 2 & 1 & & \\ & & & 2 & 1 & \\ & & & & 2 & 1 \end{array}$$

giving $Q_{I_3/\rho_4}(x_1, x_2) = (x_1x_2)^3(x_1^2 + x_1x_2 + x_2^2) = s_{5,3}(x_1, x_2)$. In general, arguing in the same way, we get

$$\begin{aligned} Q_{I_p/\rho_{n-2}}(x_1, x_2) &= (x_1x_2)^{n-p}(x_1^{p-1} + x_1^{p-2}x_2 + \ldots + x_2^{p-1}) = \\ &= e_2(x_1, x_2)^{n-p} s_{p-1}(x_1, x_2) = s_{n-1,n-p}(x_1, x_2). \quad \square \end{aligned}$$

**Lemma 7.2.** *With the above notation we have:*

*(i)* $(\pi_1)_*\big(\widetilde{Q}_{I_p}D^\vee\big) = s_{n-1,n-p}(C^\vee)$;

*(ii)* *For* $q<p$, $(\pi_1)_*\big(\widetilde{Q}_{I_{p,q}}D^\vee\big) = s_{n-q-1,n-p}(C^\vee)$;

*(iii)* $$(\pi_1)_*\big(\widetilde{Q}_{\rho_n}D^\vee\big) = \sum_{k=0}^{n-2}(-1)^k c_{2k}V\cdot\Big[s_{n-k,n-k-1}(C^\vee) - s_{n-k+1,n-k-2}(C^\vee) + \ldots \\ \ldots + (-1)^{n-k}s_{2(n-k-1),1}(C^\vee)\Big].$$

*Proof.* Assertions (i) and (ii) follow immediately from Lemma 7.1(i),(ii) and Theorem 5.10. As for (iii), we have ( in the following, $(\pi_1)_*($ *other terms* $)=0$ ):

$$
\begin{aligned}
&(\pi_1)_*\big(\widetilde{Q}_{\rho_n}D^\vee\big) = \\
&\quad = (\pi_1)_*\Big[\sum_{v=0}^{n-2}\widetilde{Q}_{(\rho_{n-2}+(2)^v)^\sim}(S^\vee)\cdot\widetilde{Q}_{\rho_n/(\rho_{n-2}+(2)^v)^\sim}(C^\vee_{\mathcal{F}}) + \ (\mathit{other\ terms})\Big] \\
&\quad = \sum_{v=0}^{n-2}(-1)^v c_{2v}(C^\perp/C)\cdot\widetilde{Q}_{\rho_n/(\rho_{n-2}+(2)^v)^\sim}(C^\vee) \\
&\quad = \sum_{v=0}^{n-2}(-1)^v\Big[\sum_{k+l=v}c_{2k}V\cdot s_{2l}(C\oplus C^\vee)\Big]\cdot s_{n-v,n-v-1}(C^\vee) \\
&\quad = \sum_{k=0}^{n-2}(-1)^k c_{2k}V\cdot\Big[\sum_{l=0}^{n-2-k}(-1)^l s_{2l}(C\oplus C^\vee)\cdot s_{n-k-l,n-k-l-1}(C^\vee)\Big] \\
&\quad = \sum_{k=0}^{n-2}(-1)^k c_{2k}V\cdot\Big[s_{n-k,n-k-1}(C^\vee) - s_{n-k+1,n-k-2}(C^\vee) + \ldots \\
&\qquad\qquad \ldots + (-1)^{n-k}s_{2(n-k-1),1}(C^\vee)\Big],
\end{aligned}
$$

where the above equalities follow from: Theorem 5.10, Lemma 1.1 and Pieri's formula ([Mcd1], [L-S1]); recall that *rank* $C=2$. $\square$

**Lemma 7.3.** *Let* $0<a<b$ *and* $k\geqslant l\geqslant 0$ *be integers. Let* $C$ *be the rank 2 tautological (sub)bundle of* $\tau: Fl(a,b)\to X$. *Then*

$$\tau_* s_{k,l}(C^\vee) = s_{k-(b-2)}(V_b^\vee)\cdot s_{l-(a-1)}(V_a^\vee) - s_{k-(a-2)}(V_a^\vee)\cdot s_{l-(b-1)}(V_b^\vee).$$

*where we assume* $s_h(-)=0$ *for* $h<0$.

*Proof.* Let $C_1\subset C_2 = C$ be the tautological subbundles on $Fl(a,b)$, $C_1\subset V_a$, $C_2\subset V_b$; *rank* $C_h = h$, $h=1,2$. Let $x_1 = c_1(C_1^\vee)$ and $x_2 = c_1\big((C_2/C_1)^\vee\big)$. The flag bundle $\tau: Fl(a,b)\to X$ is equal to the composition:

$$\mathbb{P}\big(V_b/C_1\big)\xrightarrow{\tau_2}\mathbb{P}(V_a)\xrightarrow{\tau_1} X.$$

We have

$$\tau_* s_{k,l}(C^\vee) = \tau_*\Big[(x_1x_2)^l\big(x_1^{k-l} + x_1^{k-l-1}x_2 + \ldots + x_1x_2^{k-l-1} + x_2^{k-l}\big)\Big].$$

The assertion now follows by applying to all summands the well known formulas:

$$(\tau_2)_*(x_2^p) = s_{p-(b-2)}(V_b/C_1)^\vee = s_{p-(b-2)}(V_b^\vee) - s_{p-(b-2)-1}(V_b^\vee)\cdot x_1,$$
$$(\tau_1)_*(x_1^p) = s_{p-(a-1)}(V_a^\vee)$$

and simplifying. $\square$

**Theorem 7.4.** *For $n \geqslant i > j > 0$ one has in $A^*(G)$ with $a = n+1-i$, $b = n+1-j$,*

$$\big[\Omega(a,b)\big] = \sum_{\substack{p>q\geqslant 0\\ p\leqslant i,\, q\leqslant j}} \widetilde{Q}_{p,q}R^\vee \cdot \big(s_{i-p}(V_a^\vee)\cdot s_{j-q}(V_b^\vee) - s_{i-q}(V_a^\vee)\cdot s_{j-p}(V_b^\vee)\big) +$$
$$+ \sum_{p=0}^{i-1}\sum_{t\geqslant 1}(-1)^{p+t-1}c_{2p}V\cdot\big(s_{i-p-t}(V_a^\vee)\cdot s_{j-p+t}(V_b^\vee) - s_{i-p+t}(V_a^\vee)\cdot s_{j-p-t}(V_b^\vee)\big),$$

*where we assume $s_h(-) = 0$ for $h < 0$.*

*Proof.* It follows from Lemma 7.2 that

$$\big[\Omega(a,b)\big] = \sum_{0\leqslant q<p}(\pi_2)_*\big(s_{n-q-1,n-p}(C^\vee)\big)\cdot\widetilde{Q}_{p,q}R^\vee +$$
$$+\sum_{p=0}^{n-2}(-1)^p c_{2p}V\cdot(\pi_2)_*\Big[s_{n-p,n-p-1}(C^\vee) - s_{n-p+1,n-p-2}(C^\vee) + \ldots$$
$$\ldots + (-1)^{n-p}s_{2(n-p-1),1}(C^\vee)\Big].$$

Applying Lemma 7.3 to $\pi_2 : Fl(a,b)\to X$, the assertion follows. $\square$

**Example 7.5.** 1. For $i=2, j=1$ and any $n$ the formula reads:

$$\widetilde{Q}_{21}R^\vee + \widetilde{Q}_2R^\vee\cdot s_1V_n^\vee + \widetilde{Q}_1R^\vee\cdot\big(s_1V_{n-1}^\vee\cdot s_1V_n^\vee - s_2V_{n-1}^\vee\big) +$$
$$+\big(s_1V_{n-1}^\vee\cdot s_2V_n^\vee - s_3V_{n-1}^\vee - s_3V_n^\vee - c_2V\cdot s_1V_n^\vee\big) =$$
$$= \widetilde{Q}_{21}R^\vee + \widetilde{Q}_2R^\vee\cdot\widetilde{Q}_1V_n^\vee + \widetilde{Q}_1R^\vee\cdot\widetilde{Q}_2V_n^\vee + \widetilde{Q}_{21}V_n^\vee.$$

2. For $i=3, j=1$ and any $n$ one obtains, with $\widetilde{Q}_{p,q} = \widetilde{Q}_{p,q}R^\vee$, $s_k = s_k(V_{n-2}^\vee)$ and $s'_k = s_k(V_n^\vee)$, the expression:

$$\widetilde{Q}_{31} + \widetilde{Q}_3 \cdot s_1' + \widetilde{Q}_{21} \cdot s_1 + \widetilde{Q}_2 \cdot s_1 \cdot s_1' + \widetilde{Q}_1 \cdot (s_2 \cdot s_1' - s_3) + \\ + s_2 \cdot s_2' - s_4 - s_1 \cdot s_3' + s_4' - c_2 V \cdot (s_1 \cdot s_1' - s_2') + c_4 V.$$

3. For $i = 3, j = 2$ and any $n$ one obtains, with $\widetilde{Q}_{p,q} = \widetilde{Q}_{p,q} R^\vee$ and $s_{k,l} = s_{k,l}(V_{n-1}^\vee)$, the expression:

$$\widetilde{Q}_{32} + \widetilde{Q}_{31} \cdot s_1 + \widetilde{Q}_3 \cdot s_2 + \widetilde{Q}_{21} \cdot s_{11} + \widetilde{Q}_2 \cdot s_{21} + \widetilde{Q}_1 \cdot s_{22} + s_{32} - s_{41} + s_5 - c_2 V \cdot (s_{21} - s_3) + c_4 V \cdot s_1.$$

More generally we have:

**Corollary 7.6.** *With the above notation and $j = i-1$, $s_{k,l} = s_{k,l}(V_{n+2-i}^\vee)$, the class $[\Omega(a,b)]$ equals*

$$\sum_{i \geqslant p > q \geqslant 0} \widetilde{Q}_{p,q} R^\vee \cdot s_{i-1-q,i-p} + \sum_{p=0}^{i-1} (-1)^p c_{2p} V \cdot \sum_{h=0}^{i-1-p} (-1)^h s_{i-p+h,i-1-p-h}.$$

Consider now the odd orthogonal case. Our desingularization in case $a_\bullet = (a,b) := (n+1-i, n+1-j)$ is given by the composition ($rank\ C = 2$):

$$OG_{n-2}\big(C^\perp/C\big) \xrightarrow{\pi_1} Fl(V_a \subset V_b) \xrightarrow{\pi_2} G.$$

Then the analog of Lemma 7.1 reads (with the notation explained before this lemma):

**Lemma 7.7.** *:*

*(i)* $\widetilde{P}_{I_p/\rho_{n-2}}(C^\vee) = 1/2 \cdot s_{n-1,n-p}(C^\vee)$. *(ii)* $\widetilde{P}_{I_{p,q}/\rho_{n-2}}(C^\vee) = s_{n-q-1,n-p}(C^\vee)$.

*(iii)* $$\widetilde{P}_{\rho_n/(\rho_{n-2}+(2)^v)^\sim}(C^\vee) = s_{n-v,n-v-1}(C^\vee)\,, \quad 0 < v \leqslant n-2\,,$$ $$\textit{and} \quad \widetilde{P}_{\rho_n/\rho_{n-2}}(C^\vee) = 1/4 \cdot s_{n,n-1}(C^\vee).$$

The element to be push forwarded via $(\pi_2\pi_1)_*$ is $\sum \widetilde{P}_I D^\vee \cdot \widetilde{P}_{\rho_n \smallsetminus I} R^\vee$, the sum over all strict $I \subset \rho_n$ (the notation as in Section 1). The analog of Lemma 7.2 reads:

**Lemma 7.8.** *:*

*(i)* $(\pi_1)_*(\widetilde{P}_{I_p} D^\vee) = 1/2 \cdot s_{n-1,n-p}(C^\vee)$. *(ii)* $q < p$ $(\pi_1)_*(\widetilde{P}_{I_{p,q}} D^\vee) = s_{n-q-1,n-p}(C^\vee)$.

*(iii)* $$(\pi_1)_*(\widetilde{P}_{\rho_n} D^\vee) = 1/4 \cdot \sum_{k=0}^{n-2} (-1)^k c_{2k}\, V \cdot \big[s_{n-k,n-k-1}(C^\vee) - s_{n-k+1,n-k-2}(C^\vee) + \\ \dots (-1)^{n-k} \cdot s_{2(n-k-1),1}(C^\vee)\big]\,.$$

Consequently, the analog of Theorem 7.4 now reads:

**Theorem 7.9.** *For $n \geqslant i > j > 0$ one has in $A^*(G)$ with $a = n+1-i$, $b = n+1-j$,*

$$\begin{aligned}\big[\Omega(a,b)\big] = \sum_{\substack{p>q>0\\ p\leqslant i,q\leqslant j}} \widetilde{P}_{p,q}\, R^\vee \cdot \Big(s_{i-p}(V_a^\vee)\cdot s_{j-q}(V_b^\vee) - s_{i-q}(V_a^\vee)\cdot s_{j-p}(V_b^\vee)\Big) + \\ 1/2 \cdot \sum_{\substack{p>0\\ p\leqslant i}} \widetilde{P}_p\, R^\vee \cdot \Big(s_{i-p}(V_a^\vee)\cdot s_j(V_b^\vee) - s_i(V_a^\vee)\cdot s_{j-p}(V_b^\vee)\Big) + \\ +1/4 \cdot \sum_{p=0}^{i-1}\sum_{t\geqslant 1}(-1)^{p+t-1}c_{2p}\, V \cdot \Big(s_{i-p-t}(V_a^\vee)\cdot s_{j-p+t}(V_b^\vee) - s_{i-p+t}(V_a^\vee)\cdot s_{j-p-t}(V_b^\vee)\Big).\end{aligned}$$

For instance, invoking Example 7.5, the formula reads for $i = 2, j = 1$ and any $n$:

$$[\Omega(n-1,n)] = \widetilde{P}_{21}R^\vee + \widetilde{P}_2 R^\vee \cdot \widetilde{P}_1 V_n^\vee + \widetilde{P}_1 R^\vee \cdot \widetilde{P}_2 V_n^\vee + \widetilde{P}_{21} V_n^\vee.$$

## 8. An operator proof of Proposition 3.1

The goal of this section is to provide another proof of Proposition 3.1 and its odd orthogonal analogue by using divided differences operators. We start with the Lagrangian case. Let $X_n = (x_1, \dots, x_n)$ be a sequence of indeterminates. Recall (see Section 5) that the symplectic Weyl group $W_n$ is isomorphic to $S_n \ltimes \mathbb{Z}_2^n$ and the elements of $W_n$ are identified with "barred permutations": if $w = (\sigma, \tau)$, $\sigma \in S_n$, $\tau \in \mathbb{Z}_2^n$ then we write $w$ as the sequence $(w_1, \dots, w_n)$ endowed with bars on places where $\tau_i = -1$. In particular, $w_0 = (\overline{1}, \overline{2}, \dots, \overline{n})$ is the longest element of $W_n$. Consider in $W_n$ the poset $W^{(n)}$ of minimal length left coset representatives of $W_n$ modulo its subgroup generated by reflections corresponding to the simple roots $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n$ (in the standard notation):

$$W^{(n)} = \Big\{(\overline{z}_1 > \overline{z}_2 > \ldots > \overline{z}_l; y_1 < \ldots < y_{n-l}) \in W_n,\ l = 0, 1, \dots, n\Big\}.$$

The assignment $w = (\overline{z}_1, \dots, \overline{z}_l; y_1, \dots, y_{n-l}) \mapsto I = (z_1, \dots, z_l)$ establishes a bijection between the poset $W^{(n)}$ and the poset of all strict partitions contained in $\rho_n$.

One has divided differences $\partial_w : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$ $(w \in W_n)$ i.e. operators of degree $-l(w)$, whose definition has been explained in Section 5.

Fix now an integer $0 < k < n$ and denote:

$$w^{(k)} := (\overline{n}, \overline{n-1}, \dots, \overline{k+1}; 1, 2, \dots, k).$$

Observe first that for a strict partition $I \subset \rho_n$ of length $l(I)$, $\partial_{w^{(k)}} \widetilde{Q}_I(X_n) \neq 0$ only if $l(I) \geqslant n-k$. (This is because $\partial_{w^{(k)}}$ decreases the degree by $l(w^{(k)}) = n + (n-1) + \ldots + (k+1)$.) More precisely, writing $X_n^\vee = (-x_1, \dots, -x_n)$, we have:

**Proposition 8.1.** *For a strict partition $I$ of length $\geqslant n-k$, $\partial_{w^{(k)}}\widetilde{Q}_I(X_n^\vee) \neq 0$ iff $I \supset (n, n-1, \ldots, k+1)$. In this case, writing $I = (n, n-1, \ldots, k+1, j_1, \ldots, j_l)$, where $j_l > 0$ and $l \leqslant k$, one has in $\mathbb{Z}[X_n]$,*

$$\partial_{w^{(k)}}\widetilde{Q}_I(X_n^\vee) = \widetilde{Q}_{j_1,\ldots,j_l}(X_n^\vee).$$

*Proof.* Let $I$ be a strict partition of length $h \geqslant n-k$. Let

$$w_I = (\overline{\sigma}_1, \overline{\sigma}_2, \ldots, \overline{\sigma}_h; \sigma_{h+1}, \ldots, \sigma_n)$$

be the element of $W^{(n)}$ corresponding to $I$. Then taking into account that

$$(w^{(k)})^{-1} = (n-k+1, n-k+2, \ldots, n; \overline{n-k}, \overline{n-k-1}, \ldots, \overline{1}),$$

we get $\quad w_I \circ (w^{(k)})^{-1} =$

$$\big(\overline{\sigma}_{n-k+1} > \overline{\sigma}_{n-k+2} > \ldots > \overline{\sigma}_h, \sigma_{h+1} < \sigma_{h+2} < \ldots < \sigma_n, \sigma_{n-k} < \sigma_{n-k-1} < \ldots < \sigma_1\big).$$

We have $l(w_I) = \sigma_1 + \ldots + \sigma_h$, $\;l(w^{(k)}) = n + (n-1) + \ldots + (k+1)$, and

$$l\big(w_I \circ (w^{(k)})^{-1}\big) = \sigma_{n-k+1} + \sigma_{n-k+2} + \ldots + \sigma_h + \sum_{j=1}^{n-h} card\big\{1 \leqslant p \leqslant n-k \mid \sigma_p < \sigma_{h+j}\big\}$$

by Lemma 5.1. Thus, denoting the above sum $\sum\limits_{j=1}^{n-h}(\ldots)$ by $\sum$, we get:

$$l(w_I) - l(w^{(k)}) - l\big(w_I \circ (w^{(k)})^{-1}\big) =$$
$$= \sigma_1 + \ldots + \sigma_{n-k} - \big(n + (n-1) + \ldots + (k+1)\big) - \textstyle\sum.$$

Now, a necessary condition for $\partial_{w^{(k)}}\widetilde{Q}_I(X_n^\vee) \neq 0$ is:

$$\sigma_1 + \ldots + \sigma_{n-k} - \big(n + (n-1) + \ldots + (k+1)\big) - \sum = 0,$$

which implies $(\sigma_1, \ldots, \sigma_{n-k}) = (n, n-1, \ldots, k+1)$ and $\sum = 0$, i.e., $\sigma_n < \sigma_{n-k}$. (Using the theory from [B-G-G], [D1,2] and the result from [P2] recalled in Theorem 2.1, the just proved assertion easily implies that $\partial_{w^{(k)}}\widetilde{Q}_I(X_n^\vee) = \widetilde{Q}_{j_1,\ldots,j_l}(X_n^\vee)$ $(mod\, \mathcal{I})$). We will now prove directly that for $I = (n, n-1, \ldots, k+1, j_1, \ldots, j_l)$ one has $\partial_{w^{(k)}}\widetilde{Q}_I(X_n^\vee) = \widetilde{Q}_{j_1,\ldots,j_l}(X_n^\vee)$ already in $\mathbb{Z}[X_n]$. Observe that

$$\partial_{w^{(k)}} = (\partial_k \ldots \partial_1\partial_0)\ldots(\partial_{n-2}\ldots\partial_1\partial_0)(\partial_{n-1}\ldots\partial_1\partial_0).$$

The proof is by induction on $n-k-1$. For $n-k-1=0$, one has $(J = (j_1, \ldots, j_l))$:

$$\partial_{n-1}\ldots\partial_1\partial_0\big(\widetilde{Q}_{n,J}(X_n^\vee)\big) = \partial_{n-1}\ldots\partial_1\partial_0\big(e_n(X_n^\vee)\cdot\widetilde{Q}_J(X_n^\vee)\big)$$

$$= \partial_{n-1}...\partial_1\big((-x_2)...(-x_n)\widetilde{Q}_J(X_n^\vee) - e_n(X_n^\vee)\cdot\partial_0\widetilde{Q}_J(X_n^\vee)\big)$$
$$= \widetilde{Q}_J(X_n^\vee) - e_n(X_n^\vee)\cdot\partial_{n-1}...\partial_1\partial_0\big(\widetilde{Q}_J(X_n^\vee)\big) = \widetilde{Q}_J(X_n^\vee),$$

where the vanishing of the second summand in the last difference follows from the just proved first assertion.

The induction step goes as follows. By the equality proved above,

$$(\partial_k...\partial_1\partial_0)...(\partial_{n-2}...\partial_1\partial_0)(\partial_{n-1}...\partial_1\partial_0)\big(\widetilde{Q}_{n,n-1,...,k+1,J}(X_n^\vee)\big)$$
$$(\partial_k...\partial_1\partial_0)...(\partial_{n-2}...\partial_1\partial_0)\big(\widetilde{Q}_{n-1,n-2,...,k+1,J}(X_n^\vee)\big)$$
$$(\partial_k...\partial_1\partial_0)...(\partial_{n-2}...\partial_1\partial_0)\Big(\sum_{i\geq 0}(-x_n)^i\sum\widetilde{Q}_I(X_{n-1}^\vee)\Big)$$

where the sum is over all partitions $I \subset (n-1, n-2, ..., k+1, J)$ such that the diagram $(n-1, n-2, ..., k+1, J)/I$ is of weight $i$ and has at most one box in every row (use the linearity formula, i.e. Proposition 4.1). Each time we get two equal parts $p$ in a partition $I$ such that $\widetilde{Q}_I(X_{n-1}^\vee)$ appears in the expression, we factor out $\widetilde{Q}_{p,p}(X_{n-1}^\vee)$ by Proposition 4.3. The last sum can be rewritten in the form:

$$\sum_{i\geq 0}\sum_M(-x_n)^i\widetilde{Q}_M(X_{n-1}^\vee)f_M + \sum_{i\geq 0}\sum_N(-x_n)^i\widetilde{Q}_N(X_{n-1}^\vee)g_N,$$

where $M$ (resp. $N$) runs over the so-obtained partitions contained in the partition $(n-1, n-2, ..., k+1, J)$ where some box is removed from the first $n-k-1$ places (resp. no box is subtracted from the first $n-k-1$ places), and $f_M$ (resp. $g_N$) denotes the corresponding monomial in the elements $\widetilde{Q}_{p,p}(X_{n-1}^\vee)$ obtained by factoring out. By the first assertion (applied to $X_{n-1}^\vee$) we know that our operator annihilates the first sum. By the induction assumption we get (with $N = (n-1, n-2, ..., k+1, J')$ and $g_{J'} = g_N$)

$$(\partial_k...\partial_1\partial_0)...(\partial_{n-2}...\partial_1\partial_0)\Big(\sum_{i\geq 0}\sum_N(-x_n)^i\widetilde{Q}_N(X_{n-1}^\vee)g_N\Big)$$
$$= \sum_{i\geq 0}\sum_{J'}(-x_n)^i\widetilde{Q}_{J'}(X_{n-1}^\vee)g_{J'} = \widetilde{Q}_J(X_n^\vee)$$

by the factorization property and the linearity formula, now used backwards. $\square$

We now pass to a geometric interpretation of the proposition. The setup and the notation is the same as in the proof of Proposition 3.1: $V \to B$ - rank $2n$ vector bundle endowed with a nondegenerate symplectic form, $X = LG_nV$, $V_n$ denotes here the tautological subbundle on $X$ and $p : \mathcal{F} \to X$ is the composition (see Section 1):

$$LG_{n-k}(C^\perp/C) \xrightarrow{\pi_1} G_k(V_n) \xrightarrow{\pi_2} X,$$

where $C$ is the tautological rank $k$ bundle on $G_k(V_n)$. The tautological rank $n-k$ subbundle $S$ for $LG_{n-k}(C^\perp/C)$ is identified with $D/C_{\mathcal{F}}$ where $D$ is rank $n$ tautological subbundle on $\mathcal{F}$. Let $r_1, ..., r_n$ be the Chern roots of $V_n$ and $d_1, ..., d_n$ - the Chern roots of $D$. Since $C_{\mathcal{F}} \subset (V_n)_{\mathcal{F}}$ and $C_{\mathcal{F}} \subset D$, we can assume that $r_1 = d_1, ..., r_k = d_k$ are the Chern roots of $C$.

<u>Claim</u>: For any symmetric polynomial $f$ in $n$ variables,

$$(\pi_1)_* \left(f(d_{k+1}, ..., d_n, d_1, ..., d_k)\right) = (\partial_v f)(r_{k+1}, ..., r_n, r_1, ..., r_k),$$

where $v = (\overline{n-k}, \overline{n-k-1}, ..., \overline{1}, n-k+1, ..., n)$.

Indeed, for the Chern roots $d_{k+1}, ..., d_n$ of $S$ one has by Proposition 5.8,

$$\begin{aligned}(\pi_1)_* \left(f(d_{k+1}, ..., d_n, d_1, ..., d_k)\right) &= (\pi_1)_* \left(f(d_{k+1}, ..., d_n, r_1, ..., r_k)\right) \\ &= (\partial_v f)(d_{k+1}, ..., d_n, r_1, ..., r_k).\end{aligned}$$

We know by Proposition 5.9 that $\partial_v f$ is a polynomial symmetric in the squares of the first $n-k$ variables. By Lemma 1.1 we have

$$\begin{aligned}[S] + [S^\vee] &= [(C^\perp/C)_{\mathcal{F}}] = [V_{\mathcal{F}}] - [C_{\mathcal{F}}] - [C^\vee_{\mathcal{F}}] \\ &= [(V_n)_{\mathcal{F}}] + [(V_n^\vee)_{\mathcal{F}}] - [C_{\mathcal{F}}] - [C^\vee_{\mathcal{F}}] = [(V_n)_{\mathcal{F}}/C_{\mathcal{F}}] + [((V_n)_{\mathcal{F}}/C_{\mathcal{F}})^\vee].\end{aligned}$$

Hence, for the Chern roots $r_{k+1}, ..., r_n$ of $V_n/C$,

$$(\partial_v f)(d_{k+1}, ..., d_n, r_1, ..., r_k) = (\partial_v f)(r_{k+1}, ..., r_n, r_1, ..., r_k)$$

and the claim is established.

We are now in position to give

**Another proof of Proposition 3.1.**

By virtue of the previous proposition it suffices to show that for every symetric polynomial $f$ in $n$ variables $p_*\big(f(d_1, ..., d_n)\big) = \big(\partial_{w^{(k)}} f\big)(r_1, ..., r_n)$. For a polynomial $g$ symmetric in the first $n-k$ - and in the last $k$ variables, one has

$$(\pi_2)_* \left(g(r_{k+1}, ..., r_n, r_1, ..., r_k)\right) = (\partial_u g)(r_1, ..., r_n),$$

where $u = (k+1, ..., n, 1, 2, ..., k)$ (see [L2], [P2] and [Br]). (This can be proved using a reasoning similar to the one in the proof of Proposition 5.8 above.) Since $w^{(k)} = u \circ v$ and $l(w^{(k)}) = l(u) + l(v)$, we thus have, invoking the claim:

$$\begin{aligned}p_*\big(f(d_1, ..., d_n)\big) &= p_*\big(f(d_{k+1}, ..., d_n, d_1, ..., d_k)\big) = \pi_{2*}\Big(\pi_{1*}\big(f(d_{k+1}, ..., d_n, d_1, ..., d_k)\big)\Big) \\ &= \pi_{2*}\big((\partial_v f)(r_{k+1}, ..., r_n, r_1, ..., r_k)\big) = \big(\partial_u(\partial_v f)\big)(r_1, ..., r_n) \\ &= \big((\partial_u \circ \partial_v) f\big)(r_1, ..., r_n) = (\partial_{w^{(k)}} f)(r_1, ..., r_n)\end{aligned}$$

which is the desired assertion. $\square$

In the odd orthogonal case, by replacing $\widetilde{Q}$-polynomials by $\widetilde{P}$-polynomials and arguing in the same way as above, one proves the following proposition.

**Proposition 8.2.** *For a strict partition $I$ of length $\geqslant n-k$, $\partial_{w^{(k)}}\widetilde{P}_I(X_n^\vee) \neq 0$ iff $I \supset (n, n-1, \ldots, k+1)$. In this case, writing $I = (n, n-1, \ldots, k+1, j_1, \ldots, j_l)$, where $j_l > 0$ and $l \leqslant k$, one has in $\mathbb{Z}[X_n]$,*

$$\partial_{w^{(k)}}\widetilde{P}_I(X_n^\vee) = \widetilde{P}_{j_1,\ldots,j_l}(X_n^\vee).$$

Let $V \to B$ be a rank $2n+1$ vector bundle endowed with a nondegenerate orthogonal form, $X = OG_nV$ and $V_n$ denote the tautological subbundle on $X$. Then, by an appropriate interpretation of the Gysin map associated with the composition:

$$OG_{n-k}(C^\perp/C) \xrightarrow{\pi_1} G_k(V_n) \xrightarrow{\pi_2} X,$$

where $C$ is the tautological rank $k$ bundle on $G_k(V_n)$, one gets another proof of Proposition 3.4.

We refer the reader to [L-P-R] for another operator treatment of $\widetilde{Q}$- and $\widetilde{P}$-polynomials and their generalizations.

## 9. Main results in the generic case

Let $V$ be a rank $2n$ vector bundle over a smooth pure-dimensional scheme $X$ endowed with a nondegenerate symplectic form. Let $E$ and $F_\bullet : F_1 \subset F_2 \subset \ldots \subset F_n = F$ be Lagrangian subbundles of $V$ with *rank* $F_i = i$ and *rank* $E = n$. For a given sequence $a_\bullet = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$, we are interested in a locus

$$D(a_\bullet) := \Big\{\ x \in X |\ dim\big(E \cap F_{a_p}\big)_x \geqslant p,\ p = 1, \ldots, k\Big\}.$$

Let $G = LG_nV$ and let $R \subset V_G$ be the tautological rank $n$ subbundle on $G$. By a well known universality property of Grassmannians there exists a morphism $s : X \to G$ such that $E = s^*R$. Therefore (in the set-theoretic sense) we have:

$$D(a_\bullet) = s^{-1}(\Omega(a_\bullet; F_\bullet)),$$

where

$$\Omega(a_\bullet; F_\bullet) = \Big\{\ g \in G |\ dim(R \cap F_{a_p})_g \geqslant p,\ p = 1, \ldots, k\ \Big\}.$$

We take this equality as the definition of a scheme structure on $D(a_\bullet)$, i.e., $D(a_\bullet)$ is defined in $X$ by the inverse image ideal sheaf (see [Ha, p.163]): $s^{-1}\mathcal{I}(\Omega(a_\bullet; F_\bullet)) \cdot \mathcal{O}_X$ where $\mathcal{I}(\Omega(a_\bullet; F_\bullet)$ is the ideal sheaf defining $\Omega$ in $G$. It follows from the main theorem of [DC-L] that $\Omega(a_\bullet; F_\bullet)$ is a Cohen-Macaulay scheme. Hence, by [K-L, Lemma 9] we get $[D(a_\bullet)] = s^*[\Omega(a_\bullet; F_\bullet)]$ provided $D(a_\bullet)$ is either empty or of pure codimension equal to the codimension of $\Omega(a_\bullet; F_\bullet)$ in $G$. Therefore, having a formula for the fundamental class of $\Omega(a_\bullet; F_\bullet)$ given by a polynomial $P$ in $c.(R)$ and $c.(F_{a_p})_G$, $p = 1, \ldots, k$, the formula for $D(a_\bullet)$ becomes $P\big(c.(E), c.(F_{a_p})_{p=1,\ldots,k}\big)$. Moreover, by using the Chow groups for singular schemes and a technique from [F] one can prove the following refinement of the above. If $X$ is a pure-dimensional Cohen-Macaulay scheme and $D(a_\bullet)$ is either empty or of pure codimension equal to the codimension of $\Omega(a_\bullet; F_\bullet)$ in $G$ then the class of $D(a_\bullet)$ in the Chow group of $X$ equals $P\big(c.(E), c.(F_{a_p})_{p=1,\ldots,k}\big) \cap [X]$. This reasoning (with obvious modifications) also applies, word by word, to the case of rank $2n+1$ vector bundle endowed with a nondegenerate orthogonal form.

In particular, for $a_\bullet = (n-k+1, n-k+2, \ldots, n)$ we have by Proposition 3.2:

**Theorem 9.1.** *If $X$ is a pure-dimensional Cohen-Macaulay scheme and the subscheme*

$$D^k = \{x \in X | \ dim(E \cap F)_x \geqslant k\}$$

*is either empty or of pure codimension $k(k+1)/2$ in $X$, then the class of $D^k$ (endowed with the above scheme structure) in the Chow group of $X$ equals*

$$[D^k] = \Big(\sum \widetilde{Q}_I E^\vee \cdot \widetilde{Q}_{\rho_k \smallsetminus I} F^\vee\Big) \cap [X],$$

*where the sum is over all strict partitions $I \subset \rho_k$.*

**Example 9.2.** The expressions giving the classes for successive $k$ are:

k=1 $\widetilde{Q}_1 E^\vee + \widetilde{Q}_1 F^\vee$;

k=2 $\widetilde{Q}_{21} E^\vee + \widetilde{Q}_2 E^\vee \cdot \widetilde{Q}_1 F^\vee + \widetilde{Q}_1 E^\vee \cdot \widetilde{Q}_2 F^\vee + \widetilde{Q}_{21} F^\vee$;

k=3 $\widetilde{Q}_{321} E^\vee + \widetilde{Q}_{32} E^\vee \cdot \widetilde{Q}_1 F^\vee + \widetilde{Q}_{31} E^\vee \cdot \widetilde{Q}_2 F^\vee + \widetilde{Q}_{21} E^\vee \cdot \widetilde{Q}_3 F^\vee + \widetilde{Q}_3 E^\vee \cdot \widetilde{Q}_{21} F^\vee + \widetilde{Q}_2 E^\vee \cdot \widetilde{Q}_{31} F^\vee + \widetilde{Q}_1 E^\vee \cdot \widetilde{Q}_{32} F^\vee + \widetilde{Q}_{321} F^\vee$.

For $a_\bullet = (n+1-i)$ we get:

**Theorem 9.3.** *Let $X$ be a pure-dimensional Cohen-Macaulay scheme and assume that the subscheme $S^i = \{x \in X | \ dim(E \cap F_{n+1-i})_x \geqslant 1\}$ is either empty or of pure codimension $i$ in $X$. Then*

$$[S^i] = \Big(\sum_{p=0}^{i} c_p E^\vee \cdot s_{i-p} F^\vee_{n+1-i}\Big) \cap [X].$$

**Example 9.4.** The expressions giving the classes for successive $i$ are:

i=1 $c_1 E^\vee + s_1 F^\vee$;

i=2 $c_2 E^\vee + c_1 E^\vee s_1 F^\vee_{n-1} + s_2 F^\vee_{n-1}$;

i=3 $c_3 E^\vee + c_2 E^\vee s_1 F^\vee_{n-2} + c_1 E^\vee s_2 F^\vee_{n-2} + s_3 F^\vee_{n-2}$.

The theorem is a globalization to degeneracy loci of Proposition 6.1. Also other formulas from Sections 6 and 7 admit analogous globalizations. We concentrate ourselves on a solution to J. Harris' problem for Mumford-type degeneracy loci mentioned in the Introduction.

The odd orthogonal analog of Theorem 9.1 is a consequence of Proposition 3.4 and reads as follows:

**Theorem 9.5.** *Let $X$ be a pure-dimensional Cohen-Macaulay scheme over a field of characteristic different from 2. Suppose that $V$ is a rank $2n+1$ vector bundle endowed with a nondegenerate orthogonal form. Let $E$ and $F$ be two rank $n$ isotropic subbundles of $V$. If the subscheme*

$$D^k = \{x \in X | \ dim(E \cap F)_x \geqslant k\}$$

*is either empty or of pure codimension $k(k+1)/2$ in $X$, then the class of $D^k$ in the Chow group of $X$ equals*

$$\Big(\sum \widetilde{P}_I E^\vee \cdot \widetilde{P}_{\rho_k \smallsetminus I} F^\vee\Big) \ \cap \ [X],$$

*where the sum is over all strict partitions $I \subset \rho_k$.*[6]

Let now $V$ be a rank $2n$ vector bundle over a connected pure-dimensional scheme $X$ endowed with a nondegenerate orthogonal form. Let $E$ and $F_\bullet : F_1 \subset F_2 \subset \ldots \subset F_n = F$ be isotropic subbundles of $V$ with $rank\ F_i = i$ and $rank\ E = n$. One should be careful here with the definition of $D(a_\bullet)$. For a given sequence $a_\bullet = (1 \leqslant a_1 < \ldots < a_k \leqslant n)$, where $k$ is such that $dim(E \cap F)_x \equiv k (mod\ 2)$ if $a_k = n$, we are interested in the locus

$$D(a_\bullet) = \Big\{x \in X|\ dim(E \cap F_{a_p})_x \geqslant p\ ,\ p = 1, \ldots, k\Big\}.$$

There is a morphism $s = (s', s'') : X \to OG'_n V \cup OG''_n V$ such that $s^* R = E$ where $R$ is the tautological rank $n$ subbundle on $OG'_n V \cup OG''_n V$. We have (in the scheme – theoretic sense) that if $k \equiv n\ (mod\ 2)$ then

$$D(a_\bullet) = (s')^{-1}\Omega\big(a_\bullet; (F_\bullet)_{OG'_n V}\big);$$

and if $k \equiv n + 1 (mod\ 2)$ then

$$D(a_\bullet) = (s'')^{-1}\Omega\big(a_\bullet; (F_\bullet)_{OG''_n V}\big).$$

The even orthogonal analog of Theorem 9.1 reads as follows:

**Theorem 9.6.** *If $X$ is a connected pure-dimensional Cohen-Macaulay scheme over a field of characteristic different from 2 and the subscheme*

$$D^k = \big\{x \in X|\ dim(E \cap F)_x \geqslant k\big\},$$

*defined for $k$ such that $k \equiv dim(E \cap F)_x\ (mod\ 2)$ where $x \in X$, is either empty or is of pure codimension $k(k-1)/2$ in $X$, then the class of $D^k$ in the Chow group of $X$ equals*

$$\Big(\sum \widetilde{P}_I E^\vee \cdot \widetilde{P}_{\rho_{k-1} \smallsetminus I} F^\vee\Big) \cap [X],$$

*where the sum is over all strict partitions $I \subset \rho_{k-1}$.*

---

[6] Observe that though the $\widetilde{P}$-polynomials of a vector bundle are defined only when the Chern classes of the vector bundle are divisible by 2, the integrality property of the classes obtained in the theorem holds true. One argues as in the preceding footnote, taking into account that the base change argument [K-L, Lemma 9] preserves the integrality too. The same remark applies to the even orthogonal case (Theorem 9.6 below).

**Example 9.7.** The expressions giving the classes for successive $k$ are:

k=1 1;

k=2 $\widetilde{P}_1 E^\vee + \widetilde{P}_1 F^\vee$;

k=3 $\widetilde{P}_{21} E^\vee + \widetilde{P}_2 E^\vee \cdot \widetilde{P}_1 F^\vee + \widetilde{P}_1 E^\vee \cdot \widetilde{P}_2 F^\vee + \widetilde{P}_{21} F^\vee$;

k=4 $\widetilde{P}_{321} E^\vee + \widetilde{P}_{32} E^\vee \cdot \widetilde{P}_1 F^\vee + \widetilde{P}_{31} E^\vee \cdot \widetilde{P}_2 F^\vee + \widetilde{P}_{21} E^\vee \cdot \widetilde{P}_3 F^\vee + \widetilde{P}_3 E^\vee \cdot \widetilde{P}_{21} F^\vee +$
$\widetilde{P}_2 E^\vee \cdot \widetilde{P}_{31} F^\vee + \widetilde{P}_1 E^\vee \cdot \widetilde{P}_{32} F^\vee + \widetilde{P}_{321} F^\vee$.

**Remark 9.8.** All the formulas stated in this section in the Chow groups have their direct analogs in topology. Perhaps the simplest version is the following. Assume that $X$ is a compact complex manifold, the bundles $E$, $F_i$ are holomorphic and the morphism $s$ from $X$ to $LG_nV$ above is transverse to the smooth locus of the Schubert variety $\Omega(a_\bullet; F_\bullet)$. Then the cohomology fundamental classes of $D(a_\bullet)$ are evaluated by the corresponding (given above) expressions in the Chern classes of $E$ and $F_i$. The same applies to the orthogonal case.

## Appendix A : Quaternionic Schubert calculus

Let $\mathbb{H}$ denote the (skew) field of quaternions. Let $\mathbb{P}^n_{\mathbb{H}}$ be the projective space that is identified with $(\mathbb{H}^{n+1} \smallsetminus \{0\})/\sim$, where $(h_1, \ldots, h_{n+1}) \sim (h'_1, \ldots, h'_{n+1})$ iff there is $0 \neq h \in \mathbb{H}$ such that $h_i = h \cdot h'_i$ for every $i$. It is a compact, oriented manifold over $\mathbb{R}$ of dimension $4n$. Let us recall after Hirzebruch [H1], that, in general, this real manifold does not admit a structure of a complex analytic manifold.

Let $G_k(\mathbb{H}^n)$ be the set of all $k-$dimensional subspaces[7] of $\mathbb{H}^n$. $G_k(\mathbb{H}^n)$ has a natural structure of $4k(n-k)$-dimensional, compact, oriented manifold over $\mathbb{R}$. Of course $G_1(\mathbb{H}^{n+1}) = \mathbb{P}^n_{\mathbb{H}}$.

Let $Fl_{k_1,\ldots,k_r}(\mathbb{H}^n)$ be the set of all flags of subspaces of consecutive dimensions $(k_1, \ldots, k_r)$ over $\mathbb{H}$. It is also a compact, oriented manifold over $\mathbb{R}$. One has (see [B], [Sl]), $Fl_{k_1,\ldots,k_r}(\mathbb{H}^n) = Sp(n)/\prod_{i=0}^r Sp(k_{i+1} - k_i)$ (here, $k_0 = 0$ and $k_{r+1} = n$). Of course $Fl_{k_1}(\mathbb{H}^n) = G_{k_1}(\mathbb{H}^n)$.

**10.1.** ([B, 31.1 p.202]) *Let $y_1, \ldots, y_n$ be a sequence of independent variables with deg $y_i = 4$. Then*

$$H^*\Big(Fl_{k_1,\ldots,k_r}(\mathbb{H}^n), \mathbb{Z}\Big) \cong S\mathcal{P}(y_1, \ldots, y_n)/I_{k_1,\ldots,k_r},$$

*where $I_{k_1,\ldots,k_r}$ is the ideal generated by polynomials symmetric in each of the sets $\{y_{k_i+1}, \ldots, y_{k_{i+1}}\}$, $i = 0, 1, ..., r$, separately ($k_0 = 0$, $k_{r+1} = n$).*

For instance (all cohomology groups are taken with coefficients in $\mathbb{Z}$),

$$H^*\big(\mathbb{P}^n_{\mathbb{H}}\big) = \mathbb{Z}[y]/(y^{n+1}),\ deg\ y = 4;$$

[7] the word "(sub)space" means always a "left $\mathbb{H}$-(sub)space".

$$H^*\Big(G_k\big(\mathbb{H}^n\big)\Big) = S\mathcal{P}(y_1, \dots, y_n)/I_k, \ deg\ y_i = 4.$$

We see that these cohomology rings are double-degree isomorphic with the cohomology rings of their complex analogues.

Fix now a flag $V_\bullet \ : \ V_1 \subset V_2 \subset \ldots \subset V_n$ of subspaces of $\mathbb{H}^n$ with $dim_{\mathbb{H}} V_i = i$. For every partition $I \subset (n-k)^k$ we set

$$\mathring{\sigma}(I) = \Big\{ L \in G_k(\mathbb{H}^n) \mid dim_{\mathbb{H}}(L \cap V_{n-k+p-i_p}) = p \ , \ p = 1, \dots, k \Big\}.$$

The so defined $\mathring{\sigma}(I)$ $\big(I \subset (n-k)^k\big)$ give a cellular decomposition of $G_k(\mathbb{H}^n)$ and the codimension of $\mathring{\sigma}(I)$ is $4|I|$. Now define

$$\sigma(I) = \sigma(I, V_\bullet) = \Big\{ L \in G_k(\mathbb{H}^n) \mid dim_{\mathbb{H}}\big(L \cap V_{n-k+p-i_p}\big) \geqslant p \ , \ p = 1, \dots, k \Big\}.$$

The cohomology classes of $\sigma(I, V_\bullet)$, in fact, do not depend on the flag $V_\bullet$ chosen and will be denoted by the same symbol $\sigma(I)$. We record:

**10.2.** *(Pieri-type formula) In $H^*(G_k(\mathbb{H}^n)$ one has*

$$\sigma(I) \cdot \sigma(r) = \sum \sigma(J),$$

*where the sum is over $J$ such that $i_p \leqslant j_p \leqslant i_{p-1}$ and $|J| = |I| + r$.*

Not all proofs of the Pieri formula for Complex Grassmannians can be extended to the quaternionic case. However, the proof in [G-H, pp.198-204] has this advantage. As a matter of fact, $G_k(\mathbb{H}^n)$ is an oriented compact manifold and thus its cohomology ring is endowed with the Poincaré duality. Moreover, one checks by direct examination that

$$\sigma(I) \cdot \sigma\big(n-k-i_k, \dots, n-k-i_1\big) = \sigma\big((n-k)^k\big) = [pt].$$

Then the proof in loc.cit. goes through mutatis mutandis also in the quaternionic case.

We can restate this information about the multiplicative structure in $H^*\big(G_k(\mathbb{H}^n)\big)$ as follows:

**10.3.** *Let $Y = (y_1, \dots, y_k)$ be independent variables of degree 4. The assignment $s_I(y_1, \dots, y_k) \mapsto \sigma(I)$ for $I \subset (n-k)^k$, and 0 -otherwise, is a ring homomorphism, and allows one to identify $H^*\big(G_k(\mathbb{H}^n)\big)$ with a quotient of $S\mathcal{P}(Y)$ modulo the ideal $\oplus \mathbb{Z} s_I(Y)$, the sum over $I \not\subset (n-k)^k$.*

This result has a number of useful consequences. For example, it implies immediately that the signature of the Complex Grassmannian (see [H, p.163] and [H-S, Formula (23) p.336] is the same as the one of the Quaternionic Grassmannian - a result proved originally in [Sl] using different methods.

We now describe a certain fibration which makes the Quaternionic Grassmannians useful in study of the Grassmannians of non-maximal Lagrangian subspaces (which are not Hermitian symmetric spaces).

Let $V = \mathbb{C}^{2n}$ be endowed with a nondegenerate symplectic form $\Phi$ given by the matrix

$$A = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

where $I_n$ is the $(n \times n)$-identity matrix.

Having in mind the standard notation associated with $\mathbb{H}$ we endow $V$ with a structure of $\mathbb{H}$-space setting $\mathbf{j}{\cdot}v = A\overline{v}$, where " $^-$ " denotes the complex conjugation (note that $A^2 = -id_V$).

**10.4.** *If $U \subset V$ is $k-$dimensional Lagrangian $\mathbb{C}$-subspace of $V$ then $dim_{\mathbb{H}}(\mathbb{H} \cdot U) = k$. Moreover, the restriction of the symplectic form $\Phi$ to any $\mathbb{H}$-subspace of $V$, is nondegenerate.*

To show this consider the standard Hermitian scalar product $<,>$ on $V = \mathbb{C}^{2n}$. Now given $U$, we take a $\mathbb{C}-$basis $u_1, \ldots, u_k$ such that $< u_p, u_q >= \delta_{p,q}$. We claim that $u_1, \ldots, u_k, \mathbf{j}u_1, \ldots, \mathbf{j}u_k$ are linearly independent over $\mathbb{C}$ (which implies $dim_{\mathbb{H}}(\mathbb{H}{\cdot}U) = k$). This claim follows immediately from $\Phi(u_p, u_q) = 0 = \Phi(\mathbf{j}u_p, \mathbf{j}u_q)$ and $\Phi(u_p, \mathbf{j}u_q) = u_p^t A(A\overline{u}_q) = -< u_p, u_q >= -\delta_{p,q}$.

Suppose now a $\mathbb{H}-$subspace $W \subset V$ is given with $dim_{\mathbb{H}} W = k$, say. We can always find $\mathbb{C}-$linearly independent vectors $w_1, \ldots, w_k \in W$ such that $\Phi(w_p, w_q) = 0$ and $< w_p, w_q >= \delta_{p,q}$. Then $\mathbf{j}w_1, \ldots, \mathbf{j}w_k$ also belong to $W$. It follows from $\Phi(w_p, w_q) = 0 = \Phi(\mathbf{j}w_p, \mathbf{j}w_q)$ and $\Phi(w_p, \mathbf{j}w_q) = -\delta_{p,q}$ that $w_1, \ldots, w_k, \mathbf{j}w_1, \ldots, \mathbf{j}w_k$ form a $\mathbb{C}-$basis of $W$ and the form $\Phi$ restricted to $W$ is nondegenerate.

We infer from the above

**10.5.** *The assignment $U \mapsto \mathbb{H} \cdot U$, defines a locally trivial fibration of $LG_k\big(\mathbb{C}^{2n}\big)$ over $G_k\big(\mathbb{H}^n\big)$ with the fiber $LG_k\big(\mathbb{C}^{2k}\big)$.*

In other words, denoting by $S$ the tautological (sub)bundle over $G_k\big(\mathbb{H}^n\big)$, $rank_{\mathbb{H}} S = k$, we have an identification $LG_k\big(\mathbb{C}^{2n}\big) \cong LG_k(S)$, where the latter symbol denotes (the total space of) the corresponding Grassmannian bundle.

This identification can be used in reduction of some problems about Grassmannians of non-maximal Lagrangian subspaces to the problems about the Grassmannians of maximal ones. For example, we get from 10.5 the following identity of Poincaré series:

$$P_{LG_k(\mathbb{C}^{2n})}(t) = P_{G_k(\mathbb{H}^n)}(t) \cdot P_{LG_k(\mathbb{C}^{2k})}(t),$$

thus reproving the result from [P-R2, Corollary 1.7].

Similar fibrations exist for flag varieties. Let $LFl_{k_1,\ldots,k_r}(\mathbb{C}^{2n})$ be the variety parametrizing Lagrangian ( w.r.t. $\Phi$ ) flags of dimensions $(k_1, \ldots, k_r)$ in $\mathbb{C}^{2n}$.

**10.6.** *The assignment* $(dim_{\mathbb{C}} U_i = k_i,\ i = 1, \dots, r)$*:*

$$\Big(U_1 \subset U_2 \subset \ldots \subset U_r\Big) \mapsto \Big(\mathbb{H} \cdot U_1 \subset \mathbb{H} \cdot U_2 \subset \ldots \subset \mathbb{H} \cdot U_r\Big)$$

*is a locally trivial fibration of* $LFl_{k_1,\dots,k_r}(\mathbb{C}^{2n})$ *over* $Fl_{k_1,\dots,k_r}(\mathbb{H}^n)$*. If* $\mathbb{C}^{2k_1} \subset \mathbb{C}^{2k_2} \subset \ldots \subset \mathbb{C}^{2k_r}$ *is a (part of) the standard flag, then the fiber of this fibration is the variety parametrizing Lagrangian flags* $W_1 \subset W_2 \subset \ldots \subset W_r$ *such that* $W_i \subset \mathbb{C}^{2k_i}$ *and* $dim_{\mathbb{C}} W_i = k_i,\ i = 1, \dots, r$*.*

Therefore the fiber is a composition of Lagrangian Grassmannian bundles of maximal subspaces. In particular, we obtain the following formula for the Poincaré series of $LFl_{k_1,\dots,k_r}(\mathbb{C}^{2n})$:

$$P_{LFl_{k_1,\dots,k_r}(\mathbb{C}^{2n})}(t) = P_{Fl_{k_1,\dots,k_r}(\mathbb{H}^n)}(t) \cdot \prod_{i=1}^{r} P_{LG_{k_i - k_{i-1}}(\mathbb{C}^{2(k_i - k_{i-1})})}(t),$$

where $k_0 = 0$. Since explicit expressions for the factors on the right-hand side are known (see (10.1)), this gives an explicit formula for $P_{LFl_{k_1,\dots,k_r}(\mathbb{C}^{2n})}(t)$.

**10.7.** Finally, we show an algebro-topological interpretation (as well as another proof) of the identity:

$$s_I(x_1^2, \dots, x_n^2) \cdot s_{\rho_n}(x_1, \dots, x_n) = s_{2I+\rho_n}(x_1, \dots, x_n)$$

from Section 5. To this end we show two different ways of constructing $LFl := LFl\big(\mathbb{C}^{2n}\big)$. The first way is given by taking the total space of the flag bundle $Fl(R) \to LG_n\big(\mathbb{C}^{2n}\big)$ where $R$ is the tautological vector bundle on $LG_n\big(\mathbb{C}^{2n}\big)$. The second way relies on the following observation: $LFl$ can be interpreted as the variety of flags $W_1 \subset W_2 \subset \ldots \subset W_{2n}$ such that $dim_{\mathbb{C}} W_j = j$ and each $W_{2j}$ is a $\mathbb{H}$-subspace. This realization is given by the assignment:

$$\Big(V_1 \subset V_2 \subset \ldots \subset V_n\Big) \mapsto \Big(V_1 \subset \mathbb{H} \cdot V_1 \subset \mathbb{H} \cdot V_1 + V_2 \subset \mathbb{H} \cdot V_1 + \mathbb{H} \cdot V_2 \subset \ldots\Big)$$

Equivalently, using the tautological sequence $S_1 \subset S_2 \subset \ldots \subset S_n$, $rank_{\mathbb{H}} S_i = i$, on $Fl_{\mathbb{H}}$, this corresponds to taking the total space of the product of projective bundles

$$\mathbb{P} := \mathbb{P}(S_2/S_1) \times_{Fl_{\mathbb{H}}} \ldots \times_{Fl_{\mathbb{H}}} \mathbb{P}(S_n/S_{n-1}) \to Fl_{\mathbb{H}}$$

where $S_{i+1}/S_i$, $i = 1, \dots, n$, are considered as rank 2 complex bundles.

The same holds in the relative situation, i.e. given a rank $2n$ vector bundle $V \to X$ endowed with a symplectic form we get a commutative diagram

$$\begin{array}{ccccc}
\mathbb{P} & = & LFl(V) & = & Fl(R) \\
{\scriptstyle\tau_1}\downarrow & & & & {\scriptstyle\pi_1}\downarrow \\
Fl_{\mathbb{H}}(V) & \xrightarrow{\tau_2} & X & \xleftarrow{\pi_2} & LG_nV
\end{array}$$

where $Fl_{\mathbb{H}}(V)$ denotes the flag bundle parametrizing complete Quaternionic flags of $V$. Let $x_1, \dots, x_n$ be the sequence of the Chern roots of the tautological quotient bundle on $LG_nV$. By Corollary 5.6(i) we know that if there exists an even $i_p$, then

$$(\pi_2 \circ \pi_1)_*\big(x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}\big) = 0.$$

(Calculating the other way arround, this follows easily from the projection formula.) On the other hand, iff all $i_p$ are odd, then (see Proposition 5.5)

$$s_{\rho_n}(x_1, \dots, x_n) \cdot (\pi_2 \circ \pi_1)_*\big(x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}\big) = s_{I-\rho_{n-1}}(x_1, \dots, x_n).$$

Putting $i_p = 2j_p + 1$ and calculating the other way around, we get

$$\begin{aligned}
(\tau_2\tau_1)_*\Big(x_1^{2j_1+1}x_2^{2j_2+1}\ldots x_n^{2j_n+1}\Big) &= \\
&= (\tau_2)_*\Big((x_1^2)^{j_1} \cdot (x_2^2)^{j_2} \cdot \ldots \cdot (x_n^2)^{j_n}\Big) \\
&= s_{J-\rho_{n-1}}\big(x_1^2, \dots, x_n^2\big).
\end{aligned}$$

Indeed, recalling the notation from 10.1 we have $y_p = x_p^2,\ p = 1, \dots, n$ (see [B, 31.1]), and we use the fact that $(\tau_2)_*$ is induced by the Jacobi symmetrizer (recalled in the proof of Corollary 5.6(ii) and that of Lemma 5.7(ii) ) this time applied to $y_1, \dots, y_n$. The latter statement follows from 10.1 by exactly the same reasoning as that used in the proof of Lemma 2.4 in [P1]. Comparison of the results of both computations, yields the desired identity.

## Appendix B : Introduction to Schubert polynomials à la polonaise

We provide here a brief sketch of a theory of symplectic Schubert polynomials which has grown up from the present work. For details and further developments as well as for the orthogonal Schubert polynomials, we refer the reader to [L-P-R].

Let $(x_1, x_2, \dots)$ be a sequence of independent variables. Let $w_0$ be the longest element in the Weyl group $W_n$ of type $C_n$ . Define

$$\mathcal{C}_{w_0} := \mathcal{C}_{w_0}(x_1, \dots, x_n) := (-1)^{n(n-1)/2}\, x_1^{n-1}\, x_2^{n-2}\, \ldots\, x_{n-1}^1\, x_n^0\, \widetilde{Q}_{\rho_n}(x_1, \dots, x_n)\,,$$

and for an arbitrary $w \in W_n$ ,

$$\mathcal{C}_w := \mathcal{C}_w(x_1, \dots, x_n) := \partial'_{w^{-1}w_0}(\mathcal{C}_{w_0})\,.$$

Above, by $\partial'_w$ ($w \in W_n$) we understand the composition of the divided difference operators $\partial'_i$ defined by

$$\begin{aligned} &\partial'_0(f) = (f - s_0 f)/2x_1 \\ &\partial'_i(f) = (f - s_i f)/(x_{i+1} - x_i) \qquad i = 1, 2, \ldots, n-1\,, \end{aligned}$$

associated in a usual way with an arbitrary reduced decomposition of $w$ using $s_i$, $i = 0, 1, \ldots, n-1$.

These polynomials satisfy the following properties.

1. (Stability) Suppose that $m > n$. Let $W_n \hookrightarrow W_m$ be the embedding via the first $n$ components. Then, for any $w \in W_n$, the following equality holds:

$$\mathcal{C}_w(x_1, \ldots, x_m)|_{x_{n+1}=\ldots=x_m=0} = \mathcal{C}_w(x_1, \ldots, x_n)\,.$$

2. (the Grassmannian case) Let $I = (i_1 > \ldots > i_k > 0)$ be a strict partition contained in $\rho_n$. Set

$$w_I = (\overline{i_1}, \ldots, \overline{i_k}, j_1 < j_2 < \ldots < j_{n-k})\,,$$

where $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\}$. Then

$$\mathcal{C}_{w_I}(x_1, \ldots, x_n) = \widetilde{Q}_I(x_1, \ldots, x_n)\,.$$

As we know from Section 4, $\widetilde{Q}_I(x_1, \ldots, x_n)$ is a positive sum of monomials. The polynomial $\mathcal{C}_w$ has not this property. Also, it is in general neither negative nor positive sum of monomials.

The following is the list of symplectic Schubert polynomials for $n = 2$.

$$\begin{gathered} \mathcal{C}_{(\overline{1},\overline{2})} = -\,x_1^3 x_2 - x_1^2 x_2^2 \\ \mathcal{C}_{(1,\overline{2})} = -x_1^2 x_2 \quad , \quad \mathcal{C}_{(\overline{2},\overline{1})} = x_1^2 x_2 + x_1 x_2^2 \\ \mathcal{C}_{(\overline{2},1)} = x_1 x_2 \quad , \quad \mathcal{C}_{(2,\overline{1})} = x_2^2 \\ \mathcal{C}_{(2,1)} = x_2 \quad , \quad \mathcal{C}_{(\overline{1},2)} = x_1 + x_2 \\ \mathcal{C}_{(1,2)} = 1\,. \end{gathered}$$

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