Formulas for $\widetilde{Q}$-functions: a survey for Brian

Piotr Pragacz*
Institute of Mathematics of Polish Academy of Sciences
Śniadeckich 8, 00-956 Warszawa, Poland
P.Pragacz@impan.gov.pl

Abstract. We give a survey on $\widetilde{Q}$-functions and state some problems about them.

1. Introduction

The content of this note is a slightly rewritten version of a manuscript prepared for Brian Wybourne in October 2003. I wanted to discuss with him the so-called $\widetilde{Q}$-functions (and $\widetilde{Q}$-polynomials) which I invented with Jan Ratajski some time ago to study Lagrangian degeneracy loci. I believed that these functions could be of some interest to Brian, and, possibly, also to other physicists.

These polynomials though invented (in Algebraic Geometry) in our study of Lagrangian Schubert classes [14], [15], and Lagrangian degeneracy loci [16], provide an interesting family of symmetric functions. For example, they form an additive basis of the ring of symmetric functions. Do they also have some meaning in Physics ? We collect in the present note their basic properties as proved primarily in [16], [10], and [11].

We also discuss briefly some polynomials that are related to $\widetilde{Q}$-polynomials via divided differences as well as some formulas for them, coming from [16], [10], [11], [5], [6]. Some of these polynomials and identities are important for quantum cohomology of Grassmannians [5], [6], [17].

I thank Ron King for his comments on the preliminary draft of this note.

2. Definition of $\widetilde{Q}$-functions

Let $\mathbb{X}$ be an alphabet[1]. By $\mathbb{X}^2$ we shall denote the alphabet consisting of squares of elements of $\mathbb{X}$, and by $\mathbb{X}_n$ the alphabet consisting of the first n elements of $\mathbb{X}$. Given an alphabet of variables $\mathbb{X}$ by $\mathfrak{Sym}(\mathbb{X})$ we shall denote the ring of symmetric functions in $\mathbb{X}$.

---

*Supported by KBN grant 2P03A 024 23.

[1] By an alphabet we understand an ordered multiset of elements (possibly countable) in a commutative ring. In most cases these elements are variables over $\mathbf{Z}$.

The folowing definition of $\widetilde{Q}$-functions stems from [16] (this definition was inspired by Schur's Q-functions, cf., e.g., [13]). We set $\widetilde{Q}_i = \Lambda_i = \Lambda_i(\mathbb{X})$, the i-th elementary symmetric function in $\mathbb{X}$. Given two nonnegative integers $i \geq j$, we put

$$\widetilde{Q}_{i,j} = \widetilde{Q}_i \widetilde{Q}_j + 2 \sum_{p=1}^{j} (-1)^p \widetilde{Q}_{i+p} \widetilde{Q}_{j-p} \,. \tag{1}$$

For example, $\widetilde{Q}_{i,i}(\mathbb{X}) = \Lambda_i(\mathbb{X}^2)$. Given any partition $I = (i_1 \geq \cdots \geq i_k \geq 0)$, where we can assume k to be even, we set

$$\widetilde{Q}_I = \mathrm{Pfaffian}(M) \,, \tag{2}$$

where $M = (m_{p,q})$ is the $k \times k$ skew-symmetric matrix with $m_{p,q} = \widetilde{Q}_{i_p,i_q}$ for $1 \leq p < q \leq k$. Equivalently, $\widetilde{Q}_I$ is defined recursively on $\ell = \ell(I)$ by putting for odd ℓ,

$$\widetilde{Q}_I = \sum_{j=1}^{\ell} (-1)^{j-1} \widetilde{Q}_{i_j} \cdot \widetilde{Q}_{(i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_\ell)} \tag{3}$$

and for even ℓ,

$$\widetilde{Q}_I = \sum_{j=2}^{\ell} (-1)^{j} \widetilde{Q}_{i_1,i_j} \cdot \widetilde{Q}_{(i_2,\ldots,i_{j-1},i_{j+1},\ldots,i_\ell)} \,. \tag{4}$$

Invoking the raising operators R_{ij} [13], [4], the above definition is rewritten

$$\widetilde{Q}_I = \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}} \; \Lambda^I, \tag{5}$$

where Λ^I is the product of the elementary symmetric polynomials in $\mathbb{X}$ associated with the parts of I.

Let, after [9], $Q'_I(q)$ denote the Hall–Littlewood polynomial $Q_I(\mathbb{Y}; q)$, where the alphabet $\mathbb{Y}$ is equal to $\mathbb{X}/(1-q)$ (in the sense of λ-rings). Using the raising operators R_{ij}, we have (cf., e.g., [4])

$$Q'_I(q) = \prod_{i<j} (1 - qR_{ij})^{-1} S_I \,. \tag{6}$$

where S_I denotes the classical *Schur function* associated with the partition I, cf. [13], [8]. Specialize now $q = -1$ and invoke the Jacobi-Trudi formula

$$S_I = \prod_{i<j} (1 - R_{ij}) S^I \,, \tag{7}$$

where S^I is the product of complete homogeneous symmetric polynomials in $\mathbb{X}$ associated with the parts of I. We have

$$Q'_I(-1) = \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}} S^I \,. \tag{8}$$

Therefore, denoting by ω the ring authomorphism defined by $\omega(\Lambda_i) = S_i$ for $i = 1, 2, \ldots$, we get

$$\widetilde{Q}_I = \omega\big(Q'_I(-1)\big)\,. \tag{9}$$

3. First properties

In this section unless otherwise stated, the results stem from [16]. We start with a useful linearity-type formula for $\widetilde{Q}$-polynomials. Recall that a partition I is called *strict* if all its parts are distinct.

PROPOSITION 1. *For any strict partition I one has*

$$\widetilde{Q}_I(\mathbb{X}_n) = \sum_{j=0}^{\ell(I)} x_n^j \Big(\sum_{|I|-|J|=j} \widetilde{Q}_J(\mathbb{X}_{n-1}) \Big), \tag{10}$$

where the sum is over all (i.e. not necessary strict) partitions $J \subset I$ such that I/J has at most one box in every row. (Using the terminology of [13], this is equivalent to saying that I/J is a vertical strip; note that I/J is here also a horizontal strip.)

For some generalization to any partition I, but using compositions J on the RHS, cf. [5].

LEMMA 2. *For partitions $I = (i_1, \ldots, i_k)$, $I' = (i_1, i_2, \ldots, j, j, \ldots, i_{k-1}, i_k)$, the following equality holds:*

$$\widetilde{Q}_{I'} = \widetilde{Q}_{j,j} \widetilde{Q}_I \,. \tag{11}$$

LEMMA 3. *Let $I = (i_1, i_2, \ldots, i_k)$ be a partition. If $i_1 > n$, then $\widetilde{Q}_I(\mathbb{X}_n) = 0$.*

From Proposition 1 and Lemma 2 it follows

PROPOSITION 4. *Each $\widetilde{Q}$-function is a $\mathbf{Z}$-linear combination of monomials with nonnegative coefficients.*

It is, however not, true that a $\widetilde{Q}$-function is such a combination of S-functions.

EXAMPLE 5. *Using the computer program SYMMETRICA, we get*

$$\widetilde{Q}_{53} = S_{22211} \quad \widetilde{Q}_{542} = S_{33221} - S_{32222} \quad \widetilde{Q}_{532} = S_{33211} - S_{32221} + S_{22222}$$

$$\widetilde{Q}_{54321} = S_{54321} - S_{54222} - S_{53331} - S_{44421} + S_{43332} - 2S_{33333}\,.$$

(For the expansions for all strict partitions whose first part is ≤ 5, cf. [16].)

PROPOSITION 6. *The set* $\{\widetilde{Q}_I(\mathbb{X}_n)\}$ *indexed by all partitions such that* $i_1 \le n$ *forms an additive basis of the ring* $\mathfrak{Sym}(\mathbb{X}_n)$. *For a countable alphabet* $\mathbb{X}$, *the* $\widetilde{Q}$*-functions form an additive basis of* $\mathfrak{Sym}(\mathbb{X})$.

PROPOSITION 7. *Let* $I = (i_1, ..., i_k)$ *be a strict partition of length* k. *Then*

$$\widetilde{Q}_I \cdot \widetilde{Q}_r = \sum 2^{m(I,r;J)} \widetilde{Q}_J \,, \tag{12}$$

where the sum is over all partitions (i.e. not necessary strict) $J \supset I$ *such that* $|J| = |I| + r$ *and* J/I *is a horizontal strip. Moreover,*

$$m(I, r; J) = \mathrm{card}\{1 \le p \le k \ : \ j_{p+1} < i_p < j_p\} \tag{13}$$

or, equivalently, it is expressed as the number of connected components of the strip J/I *not meeting the first column.*

It is not true that the coefficients in the $\widetilde{Q}$-function expansion of the product of two $\widetilde{Q}$-functions are positive. For example, it is pointed out in [5] and [17] that $\widetilde{Q}_{4422}$ appears in $\widetilde{Q}^2_{321}$ with the coefficient -4. During the last "Arbeitstagung 2005" at the Max-Planck Institute in Bonn (June 10–16), Arun Ram (private communication) informed me that he knows now a "Littlewood-Richardson rule" for Hall-Littlewood polynomials. Because of Eq. (9), this makes it plausible to derive from this rule a "L-R rule" for $\widetilde{Q}$-functions.

4. Divided differences

In this section, unless otherwise stated, the results stem from [10] and [11].

Let n be a fixed positive integer. The symmetric group (i.e. the Weyl group of type A), $\mathfrak{S}_n$, is the group with generators $s_1, \ldots, s_{n-1}$ subject to the relations

$$s_i^2 = 1, \quad s_{i-1}\, s_i\, s_{i-1} = s_i s_{i-1}\, s_i\,, \quad s_i s_j = s_j s_i \quad \forall i, j : |i - j| > 1\,. \tag{14}$$

A presentation for the hyperoctahedral group $\mathfrak{C}_n$ (i.e. the Weyl group of type C) is obtained by adding a further generator s_0 to those of $\mathfrak{S}_n$, where the new generator satisfies the relations

$$s_0^2 = 1, \quad s_0\, s_1\, s_0\, s_1 = s_1\, s_0\, s_1\, s_0\,, \quad s_0\, s_i = s_i\, s_0 \quad \text{for} \quad i \ge 2\,. \tag{15}$$

A presentation for the Weyl group $\mathfrak{D}_n$ of type D is obtained by adding a further generator $s_\square$ to those of $\mathfrak{S}_n$, where the new generator satisfies the relations

$$s_\square^2 = 1, \quad s_1 s_\square = s_\square\, s_1, \quad s_\square\, s_2\, s_\square = s_2 s_\square\, s_2\,, \quad s_\square\, s_i = s_i\, s_\square \quad \text{for} \quad i > 2. \tag{16}$$

The groups $\mathfrak{S}_n$, $\mathfrak{C}_n$, and $\mathfrak{D}_n$ act on $\mathbf{Z}[\mathbb{X}_n]$:

$$s_i(x_i) = x_{i+1}, \ s_0(x_1) = -x_1, \ s_\square(x_1) = -x_2. \tag{17}$$

We shall also need divided differences associated with these groups. For a polynomial $f = f(x_1, x_2, \ldots, x_n)$, we set

$$\partial_i(f) = \big(f - f(\ldots, x_{i+1}, x_i, \ldots)\big)/(x_i - x_{i+1})\,, \tag{18}$$

$$\partial_0(f) = \big(f - f(-x_1, x_2, \ldots)\big)/2x_1\,, \tag{19}$$

$$\partial_\Box(f) = \big(f - f(-x_2, -x_1, x_3, \ldots)\big)/(x_1 + x_2)\,. \tag{20}$$

The ∂_i, ∂_0, $\partial_\Box$ satisfy the Moore-Coxeter relations, together with the relations

$$\partial_0^2 = \partial_\Box^2 = \partial_i^2 = 0 \quad \text{for} \quad 1 \le i < n\ . \tag{21}$$

Therefore, to any element w of the Weyl groups $\mathfrak{S}_n$, $\mathfrak{C}_n$, $\mathfrak{D}_n$, there corresponds a *divided difference* ∂_w. Any reduced decomposition $s_{i_1}\, s_{i_2}\, \cdots s_{i_\ell} = w$ gives rise to a factorization $\partial_{i_1}\, \partial_{i_2}\, \cdots \partial_{i_\ell}$ of ∂_w. The divided difference ∂_w acts naturally on $\mathbf{Z}[X]$ by a composition of simple divided differences.

To describe some properties of $\widetilde{Q}$-polynomials, we shall need the Schubert polynomials of Lascoux-Schützenberger [12], [8].

Let us fix a positive integer n. We shall index some objects by sequences in $\mathbf{N}^n$, denoted by $\alpha = [\alpha_1, \ldots, \alpha_n]$. For two sequences $\alpha = [\alpha_1, \ldots, \alpha_n]$ and $\beta = [\beta_1, \ldots, \beta_n]$ in $\mathbf{N}^n$, we shall write $\alpha \subseteq \beta$ if $\alpha_i \le \beta_i$ for $i = 1, \ldots, n$. Define the sequence $\rho = [n-1, \ldots, 1, 0]$.

One defines recursively *Schubert polynomials* Y_α, for any sequence $\alpha \in \mathbf{N}^n$, with $\alpha \subseteq \rho$, by

$$\partial_i(Y_\alpha) = Y_\beta\ , \quad \text{if} \ \ \alpha_i > \alpha_{i+1}\ \ , \tag{22}$$

where

$$\beta = [\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i - 1, \alpha_{i+2}, \ldots, \alpha_n]\,, \tag{23}$$

starting from $Y_\rho = x^\rho$. (For a sequence $\alpha = [\alpha_1, \ldots, \alpha_n]$, by x^α we mean the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.) If $\alpha \in \mathbf{N}^n$ is weakly decreasing, then Y_α is equal to the *monomial* x^α. If $\alpha_1 \le \cdots \le \alpha_k$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$, for some $k \le n$, then Y_α is the *Schur polynomial* $S_{(\alpha_k, \ldots, \alpha_1)}(\mathbb{X}_k)$.

The ring $\mathbf{Z}[\mathbb{X}_n]$ is a free $\mathfrak{Sym}(\mathbb{X}_n)$–module with a basis given by Schubert polynomials Y_α where $\alpha \in \mathbf{N}^n$ and $\alpha \subseteq \rho$. We set

$$\nabla = \partial_0(\partial_1\partial_0)\cdots(\partial_{n-1}\cdots\partial_1\partial_0)\,. \tag{24}$$

Denote by

$$\langle\ ,\ \rangle : \mathfrak{Sym}(\mathbb{X}_n) \times \mathfrak{Sym}(\mathbb{X}_n) \to \mathfrak{Sym}(\mathbb{X}_n^2)$$

the scalar product defined for $f, g \in \mathfrak{Sym}(\mathbb{X}_n)$ by

$$\langle f, g\rangle = \nabla(f \cdot g)\,. \tag{25}$$

Define a strict partition $\rho(k) = (k, k-1, \ldots, 1)$. We record the following orthogonality property of $\widetilde{Q}$-polynomials:

THEOREM 8. *For strict $I, J \subseteq \rho(n)$,*

$$\langle \widetilde{Q}_I(\mathbb{X}_n), \widetilde{Q}_{\rho(n)\smallsetminus J}(\mathbb{X}_n)\rangle = \pm\delta_{IJ}\,, \tag{26}$$

where $\rho(n) \smallsetminus J$ is the strict partition whose parts complement the parts of J in $\{n, n-1, \ldots, 1\}$.

(In this section, for simplicity, we shall often not give precise signs – they are given in the quoted papers. Also we shall often omit the argument "$\mathbb{X}_n$".)

For another alphabet $\mathbb{Y}_n$, we define

$$\widetilde{Q}(\mathbb{X}_n, \mathbb{Y}_n) = \sum \widetilde{Q}_I(\mathbb{X}_n)\,\widetilde{Q}_{\rho(n)\smallsetminus I}(\mathbb{Y}_n)\ , \tag{27}$$

where the summation is over all strict partitions $I \subseteq \rho(n)$. The polynomial $\widetilde{Q}(\mathbb{X}_n, \mathbb{Y}_n)$ is a *reproducing kernel* for $\langle\ ,\ \rangle$:

$$\langle f(\mathbb{X}_n)\,, \widetilde{Q}(\mathbb{X}_n, \mathbb{Y}_n)\rangle = \pm f(\mathbb{Y}_n)\,. \tag{28}$$

PROPOSITION 9. *Let $\alpha \in \mathbf{N}^n$ be a sequence such that $\alpha \subseteq \rho$.*
(i) We have

$$\nabla(Y_\alpha(x_1, \ldots, x_n)\,\widetilde{Q}_{\rho(n)}) = \pm Y_\alpha(x_n, x_{n-1}, \ldots, x_1)\,. \tag{29}$$

(ii) For a strict partition $I \subsetneq \rho(n)$, we have

$$\nabla(Y_\alpha\,\widetilde{Q}_I) = 0\,. \tag{30}$$

We now define two operators that will play a crucial role in the present note. For $k \le n$, we define

$$\nabla_k^C(n) = (\partial_{n-k}\cdots\partial_1\partial_0)\cdots(\partial_{n-1}\cdots\partial_1\partial_0)\,, \tag{31}$$

and for $k \le n/2$ we set

$$\nabla_k^D(n) = (\partial_{n-2k}\cdots\partial_2\partial_1\partial_{n-2k+1}\cdots\partial_2\partial_\square)\cdots(\partial_{n-2}\cdots\partial_2\partial_1\partial_{n-1}\cdots\partial_2\partial_\square)\,. \tag{32}$$

Given partitions I, (p), J, we denote by IpJ their juxtaposition sequence (in the indicated order). Suppose $k = 1$ and $n \ge p > 0$. Let $IpJ \subseteq \rho(n)$ be a strict partition. Then we have

$$\partial_{n-1}\cdots\partial_1\partial_0(x_1^{n-p}\widetilde{Q}_{IpJ}) = \pm\widetilde{Q}_{IJ}\,. \tag{33}$$

If $H \subseteq \rho(n)$ a strict partition not containing p, then

$$\partial_{n-1}\cdots\partial_1\partial_0(x_1^{n-p}\widetilde{Q}_H) = 0\,. \tag{34}$$

More generally, we have

THEOREM 10. *Let $k \le n$ and let $\alpha = [\alpha_1 \le \cdots \le \alpha_k] \in \mathbf{N}^k$ with $\alpha_k \le n-k$. Suppose that $I \subseteq \rho(n)$ is a strict partition. Then the image of $\widetilde{Q}_I\, Y_\alpha$ under $\nabla_k^C(n)$ is 0 unless $n-0-\alpha_1, \ldots, n-(k-1)-\alpha_k$ are parts of I. In this case, the image is $\pm\widetilde{Q}_J$, where J is the strict partition with parts*

$$\{i_1, \ldots, i_{\ell(I)}\} \setminus \{n-0-\alpha_1, \ldots, n-(k-1)-\alpha_k\}\,.$$

EXAMPLE 11. *For $n=7$ and $k=3$,*

$$\nabla_3^C(7)(\widetilde{Q}_{75431}\, Y_{[2,3,4]}) = \pm\widetilde{Q}_{74}\,.$$

There is a complement to the theorem on erasing the “zero part” :

EXAMPLE 12. *For $n=5$ and $k=1$,*

$$\partial_4\partial_3\partial_2\partial_1\partial_0(x_1^5\widetilde{Q}_{5321}) = \pm\widetilde{Q}_{5321} \quad \text{and} \quad \partial_4\partial_3\partial_2\partial_1\partial_0(x_1^5\widetilde{Q}_{521}) = 0.$$

A result, for any k, is the content of [11], Theorem 9.

Using the standard “barred permutation” notation[2], we associate now with every strict partition $I = (i_1, \ldots, i_\ell > 0)$ the following element of $\mathfrak{C}_n$:

$$v(I) = (i_1, \ldots, i_\ell, \overline{j_1}, \ldots, \overline{j_h})^{-1}\,, \tag{35}$$

where $j_1 < \cdots < j_h$ are complementary numbers to $i_1, \ldots, i_\ell$ in $\{1, \ldots, n\}$.

PROPOSITION 13. *For every strict partition $I \subseteq \rho(n)$,*

$$\partial_{v(I)}(x^\rho \widetilde{Q}_{\rho(n)}) = \pm\widetilde{Q}_I. \tag{36}$$

This leads to the following characterization of $\widetilde{Q}$-polynomials via divided differences.

COROLLARY 14. *For any strict partition I, let $w(I) = (\overline{i_1}, \ldots, \overline{i_\ell}, j_1, \ldots, j_h)$. Then $w = w(I)$ is the unique element of $\mathfrak{C}_n$ such that $\ell(w) = |I|$ and $\partial_w(\widetilde{Q}_I) \neq 0$. In fact, $\partial_{w(I)}(\widetilde{Q}_I) = \pm 1$.*

PROPOSITION 15. *For a strict partition $I = (i_1, i_2, \ldots)$,*

$$\partial_{i_1-1}\cdots\partial_1\partial_0(\widetilde{Q}_I) = \pm\widetilde{Q}_{(i_2,\ldots)}\,. \tag{37}$$

Formulas for type B are the same, cf. [11].

We now come to type D. Given a partition I, we set $\widetilde{P}_I = 2^{-\ell(I)}\widetilde{Q}_I$.

[2]See, e.g., [16] p.41.

THEOREM 16. *Let $k \le n/2$. Suppose that $I \subseteq \rho(n-1)$ is a strict partition. Let $\alpha = [\alpha_1 \le \alpha_2 \le \cdots \le \alpha_{2k}] \in \mathbf{N}^{2k}$ with $\alpha_{2k} \le n-2k$. Then the image of $\widetilde{P}_I\, Y_\alpha$ under $\nabla_k^D(n)$ is 0 unless all the integers: $n-1-\alpha_1, \ldots, n-2k-\alpha_{2k}$ belong to $\{i_1, \ldots, i_{\ell(I)}, 0\}$. In this case, the image is $\pm\widetilde{P}_J$, where J is the strict partition with parts*

$$\{i_1, \ldots, i_{\ell(I)}\} \setminus \{n-1-\alpha_1, \ldots, n-2k-\alpha_{2k}\}\,.$$

EXAMPLE 17. *For $n=7$ we have*

$$\nabla_1^D(7)(\widetilde{P}_{64321,0}\, Y_{[2,5]}) = \pm\widetilde{P}_{6321} \quad \text{and} \quad \nabla_2^D(7)(\widetilde{P}_{654321,0}\, Y_{[1,1,1,3]}) = \pm\widetilde{P}_{621}\,.$$

With a strict partition $I \subseteq \rho(n-1)$ with $\ell = \ell(I)$, we associate the following element $v(I) \in \mathfrak{D}_n$. If $n-\ell$ is even (resp. odd), we set

$$v(I) = (i_1+1, \ldots, i_\ell+1, \overline{j_1}, \ldots, \overline{j_h})^{-1}\ ,\ v(I) = (i_1+1, \ldots, i_\ell+1, 1, \overline{j_1}, \ldots, \overline{j_h})^{-1}\,. \tag{38}$$

PROPOSITION 18. *For a strict partition $I \subseteq \rho(n-1)$,*

$$\partial_{v(I)}(x^\rho \widetilde{P}_{\rho(n-1)}) = \pm\widetilde{P}_I\,. \tag{39}$$

COROLLARY 19. *For a strict partition $I \subseteq \rho(n-1)$, we set for even $\ell = \ell(I)$ (resp. odd ℓ)*

$$w(I) = (\overline{i_1+1}, \ldots, \overline{i_\ell+1}, j_1, \ldots, j_h) \quad , \quad w(I) = (\overline{i_1+1}, \ldots, \overline{i_\ell+1}, \overline{1}, j_1, \ldots, j_h)\,. \tag{40}$$

Then $w = w(I)$ is the unique element of $\mathfrak{D}_n$ such that $\ell(w) = |I|$ and $\partial_w(\widetilde{P}_I) \ne 0$; in fact $\partial_{w(I)}(\widetilde{P}_I) = \pm 1$.

PROPOSITION 20. *For a strict partition $I = (i_1, i_2, i_3, i_4, \ldots) \subseteq \rho(n-1)$,*

$$\partial_{i_2} \cdots \partial_2 \partial_1 \partial_{i_1} \cdots \partial_2 \partial_\square(\widetilde{P}_I) = \pm\widetilde{P}_{(i_3, i_4, \ldots)}. \tag{41}$$

Most results in this section were proved using *vertex operators* – a useful tool given to mathematicians by physicists, cf. [10], [11] and [1], [2], [3].

5. Related polynomials

Using $\widetilde{Q}$-polynomials (or $\widetilde{P}$-polynomials) and divided differences, one can produce some new interesting polynomials (which have applications, e.g., to Algebraic Geometry). Consider for any $w \in \mathfrak{C}_n$, a *symplectic Schubert polynomial* $\mathcal{X}_w(\mathbb{X}_n) = \partial_{w^{-1}w_0}(x^\rho \widetilde{Q}_{\rho(n)}(\mathbb{X}_n))$, where w_0 stands for the longest element in the group $\mathfrak{C}_n$ (cf. [16], [10]). So, $\mathcal{X}_{w_0} = x^\rho \widetilde{Q}_{\rho(n)}(\mathbb{X}_n)$. These Schubert polynomials have the *stability property*: we have for $w \in \mathfrak{C}_n \subset \mathfrak{C}_{n+1}$, $\mathcal{X}_w(\mathbb{X}_{n+1})|_{x_{n+1}=0} = \mathcal{X}_w(\mathbb{X}_n)$. Also, Proposition 13 tells us: $\mathcal{X}_{(\overline{i_1},\ldots,\overline{i_\ell},j_1,\ldots,j_h)} = \pm\widetilde{Q}_I$, a property that has some geometric meaning. Similarly one can define orthogonal Schubert polynomials; they enjoy analogous properties with respect to the orthogonal divided differences, cf. [11].

For a strict partition I, consider the symplectic Schubert polynomial $C_I = \partial_0(\widetilde{Q}_I(\mathbb{X}_n))$. In [5], Andrew Kresch and Harry Tamvakis gave the following identity for these polynomials: for any strict partition I of length $\ell \geq 3$, we have, with $r = [(\ell+1)/2]$,

$$\sum_{i=1}^{r-1} (-1)^{j-1} C_{i_j,j_r} C_{I\setminus\{i_j,i_r\}} = 0\,. \tag{42}$$

This identity was useful in describing the quantum cohomology ring of the Lagrangian Grassmannian (*loc.cit.*). Quantum cohomology is another concept coming to Mathematics from Physics. A similar identity for the polynomials $\partial_\Box(\widetilde{P}_I(\mathbb{X}_n))$ was useful, in turn, in describing the quantum cohomology of "maximal" orthogonal Grassmannians [6].

There exist also "double" Schubert polynomials, built using divided differences and the kernel $\widetilde{Q}(\mathbb{X}_n, \mathbb{Y}_n)$ (cf. (27)), as well as the analogous kernels involving $\widetilde{P}$-polynomials. They describe the fundamental classes of symplectic and orthogonal degeneracy loci, see [10], [11], and [7] for details.

6. Problems

1. Give a transparent non-inductive proof of the orthogonality property of $\widetilde{Q}$-polynomials from Theorem 8. The original proof in [16] used Proposition 7 and double induction.

2. Give a combinatorial description of a $\widetilde{Q}$-polynomial as a sum of monomials, cf. Proposition 4.

3. Work out more relations like Eq. (42).

4. Establish a "Littlewood-Richardson rule" for $\widetilde{Q}$-functions (see the discussion at the end of Section 1). Establish such a rule for symplectic and orthogonal Schubert polynomials.

5. Does there exist some "general structure" governing the divided difference and vertex operator computations in [10] and [11] (this question is not precise, but seems to be challenging).

6. Find applications of $\widetilde{Q}$-functions to Physics, if any.

Bibliography

1. E. Date, M. Kashiwara, T. Miwa, *Vertex operators and τ functions,* Proc. Japan Acad., Ser. A **57** (1981), 387–392.

2. E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *A new hierarchy of soliton equations of KP-type,* Physica D **4** (1982), 343–365.

3. E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Transformation groups for soliton equations* in: "Non-linear Integrable Systems – Classical Theory and Quantum Theory" (M. Jimbo, T. Miwa eds.), World Scientific (1983), 41-119.

4. J. Désarménien, B. Leclerc, J-Y. Thibon, *Hall-Littlewood functions and Kostka-Foulkes polynomials in representation theory,* Preprint – Institut Gaspard Monge vol. 94-13 (1994).

5. A. Kresch, H. Tamvakis, *Quantum cohomology of Lagrangian Grassmannian,* J. Alg. Geom. **12** (2003), 777–810.

6. A. Kresch, H. Tamvakis, *Quantum cohomology of orthogonal Grassmannians,* Compositio Math. **140** (2004), 482–500.

7. A. Kresch, H. Tamvakis, *Double Schubert polynomials and degeneracy loci for the classical groups,* Ann. Inst. Fourier **52** (2002), 1681–1727.

8. A. Lascoux, *Symmetric functions and combinatorial operators on polynomials*, CBMS/AMS Lectures Notes **99**, Providence (2003).

9. A. Lascoux, B. Leclerc, J-Y. Thibon, *Fonctions de Hall-Littlewood et polynômes de Kostka-Foulkes aux racines de l'unité,* C. R. Acad. Sc. Paris **316** (1993), 1–6.

10. A. Lascoux, P. Pragacz, *Operator calculus for $\widetilde{Q}$-polynomials and Schubert polynomials,* Adv. Math. **140** (1998), 1–43.

11. A. Lascoux, P. Pragacz, *Orthogonal divided differences and Schubert polynomials, $\widetilde{P}$-polynomials, and vertex operators,* Michigan Math. J. **48** in honor of William Fulton (2000), 417–441.

12. A. Lascoux, M-P. Schützenberger, *Polynômes de Schubert,* C. R. Acad. Sc. Paris **294** (1982), 447–450.

13. I.G. Macdonald, *S*ymmetric functions and Hall polynomials, 2nd ed., Oxford Univ. Press 1995.

14. P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials,* Séminaire d'Algèbre Dubreil–Malliavin 1989–1990 (M-P. Malliavin ed.), Springer Lecture Notes in Math. **1478** (1991), 130–191.

15. P. Pragacz, *Symmetric polynomials and divided differences in formulas of intersection theory,* in: "Parameter Spaces", Banach Center Publications **36** (1996), 125–177.

16. P. Pragacz, J. Ratajski, *Formulas for Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach,* Compositio Math. **107** (1997), 11–87.

17. H. Tamvakis, *Gromov-Witten invariants and quantum cohomology of Grassmannians,* in: "Topics in cohomological studies of algebraic varieties" (P. Pragacz ed.), Trends in Mathematics, Birkhäuser 2005, 271–297.