Positivity of Thom polynomials and Schubert calculus

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Abstract

We describe the positivity of Thom polynomials of singularities of maps, Lagrangian Thom polynomials and Legendrian Thom polynomials. We show that these positivities come from Schubert calculus.

1 Introduction

In the present paper, we discuss the issue of *positivity*. The positivity plays an important role in mathematics. For example, positivity in algebraic geometry is a subject of a vast monograph [36] of Lazarsfeld.

There are two questions related to positivity: 1. Are the numbers in question (mostly the coefficients of some polynomials) nonnegative? 2. If yes, what is a positive description of these numbers?

Positivity in Schubert calculus is an active area of the contemporary research, related mainly to the second question, see, e.g., the survey article [9]. Answers to question 1. in many important situations are known classically (many of them follow from the Bertini-Kleiman theorem). Since the author hopes that the present article will also be read by beginners, we discuss this issue briefly at the end of Section 3.

Our main goal here is to describe some positivities in the global geometry of singularities. These positivities come from Schubert calculus. The presented results are related mainly to the first question. The present knowledge about the second question in this area is rather restricted. We survey several recent positivity results about Thom polynomials. Some of them are obtained by the Bertini-Kleiman theorem and its variants; other are deduced using the Fulton-Lazarsfeld theorems on positive polynomials for ample vector bundles, or vector bundles generated by their global sections. We also discuss positivity of the restrictions of Schubert classes, and some other related positivities.

Here is a description of the content of the paper.

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After the preliminaries, in Section 3, we recall basic definitions and facts about Schubert classes in the cohomology rings of G/P. We put an emphasis on Poincaré duality. Then – to start the discussion on positivity – we recall two known positivities in the cohomology rings of flag manifolds and explain the positivity of restrictions of Schubert classes.

Thom polynomials came from algebraic topology and singularities. The classical ones are associated with singularities of maps. Nowadays, we also study Lagrangian Thom polynomials and Legendrian Thom polynomials. In Sections 5, 7 and 8, we give brief introductions to these three series of Thom polynomials. The computations of Thom polynomials are, in general, quite hard. There are basically two ways to compute the Thom polynomials of a singularity class Σ: 1. using desingularization of Σ, and push-forward formulas (one should mention here many names, making this article too long); 2. the interpolation method of Fehér and Rimányi: by restricting to singularities of smaller codimension than $\operatorname{codim}\Sigma$, and using symmetries of singularities (see [49]). It was the basis of monomials in Chern classes, which served at first to compute the Thom polynomials. About a decade ago, the basis of Schur functions started also to be used systematically for computations of Thom polynomials (see Section 5 for more details).

In 2006 Weber and the author proved the positivity of the Thom polynomials of stable singularity classes of maps in the basis of Schur functions [47]. The method relies on classifying spaces of singularities and on some global aspects of Schubert calculus. The Fulton-Lazarsfeld theory [17] of polynomials numerically positive for *ample* vector bundles is used. For details, see Sections 4 and 5. Thus methods of algebraic geometry appear to be useful to study Thom polynomials.

Section 6 presents a generalization, by the same two co-authors, of this positivity to Thom polynomials of invariant cones and, in particular, to the Thom polynomials of possibly nonstable singularity classes of maps [48].

In section 7, we describe the positivity of Lagrangian Thom polynomials in the basis of $\widetilde{Q}$-polynomials. This is a result of Mikosz, Weber and the author [37].

The positivity of Legendrian Thom polynomials is a subject of Section 8, where we report on results of the same three co-authors [38]. The argument is based on some variant of the Bertini-Kleiman theorem and the Schubert calculus for Lagrangian Grassmann bundle associated with a twisted skew-symmetric form. One constructs a basis in the cohomology ring of that Lagrangian Grassmann bundle such that any Legendrian Thom polynomial has, in this basis, an expansion with nonnegative coefficients. This leads to the construction of a one-parameter family of such bases in the ring of Legendrian characteristic classes.

This is a written account of the talk delivered by the author at the conference on Schubert calculus (July 23-27, 2012) at Osaka, where apart from interesting lectures he enjoyed beautiful Japanese gardens (cf. [50]). He wishes to thank the organizers of the conference for their devoted work.

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2 Preliminaries

General information about varieties, homology groups $H_*(-)$, cohomology groups $H^*(-)$ and Chow groups $A_*(-)$, $A^*(-)$ in the scope needed for this paper is contained in [19, App. A]. The multiplication in cohomology and that in Chow rings of nonsingular varieties will be denoted by "$\cdot$". For more detailed information concerning these matters, we refer the reader to [16]. We follow the notation for algebraic geometry from this book. We use the following variants of fundamental classes:

- Let X be a variety over a field k. Given a (closed) subscheme Z of X of pure dimension d, by $[Z] = \sum m_i[Z_i]$ we denote its fundamental class in the Chow group $A_d(X)$, where Z_i are irreducible components of Z and $m_i = l(\mathcal{O}_{Z,Z_i})$ are their geometric multiplicities.
- If $k = \mathbf{C}$, a (closed) subscheme Z of a compact variety X of pure dimension d determines in the same way a fundamental class in $H_{2d}(X, \mathbf{Z})$ denoted $[Z]$. If X is nonsingular, by Poincaré duality, we have the class $[Z]$ in $H^{2e}(X, \mathbf{Z}) = H_{2d}(X, \mathbf{Z})$, where e is the codimension of Z in X.
- If Z is a (closed) subscheme of a possibly *noncompact* complex manifold[1] X of pure codimension e, then we have a class $[Z] \in H^{2e}(X, \mathbf{Z})$. Indeed, Z has a fundamental class $[Z]$ in the Borel-Moore homology group $H_{2d}(X)$, $d = \dim(Z)$ (see [6]), and that group is naturally isomorphic to $H^{2e}(X, \mathbf{Z})$ (see [7, Thm 7.9]).

A cycle $\sum n_i[V_i]$ on a scheme X is *nonnegative* if each n_i is nonnegative.

Let E and F be vector bundles on a nonsingular variety X. We define two families of symmetric functions: $s_\lambda(E - F) \in A^{|\lambda|}(X)$ and $\widetilde{Q}_\mu(E) \in A^{|\mu|}(X)$. We follow the notation for partitions from [19]. Let $\{e\}$ and $\{f\}$ be the sets of Chern roots of E and F. We set

$$\sum s_i(E - F)z^i := \prod_f (1 - fz) / \prod_e (1 - ez) \,, \tag{1}$$

where z is an independent variable. We see that $s_i(E - F)$ interpolates between $s_i(E)$ – the i-th Segre class of E times $(-1)^i$ (cf. [16]) and $s_i(-F)$ – the i-th Chern class of F times $(-1)^i$ (*loc.cit.*). Given a partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0) \,,$$

we define

$$s_\lambda(E - F) := \left| s_{\lambda_i - i + j}(E - F) \right|_{1 \leq i,j \leq l} \,. \tag{2}$$

If $F = 0$, we obtain $s_\lambda(E)$, that is, the classical Schur function of E. In the following, the reader will find a formula, how, knowing the Chern classes of

[1] A manifold here is always *nonsingular*.

E, to get $s_\lambda(E)$. For more detailed information on the *supersymmetric* Schur functions $s_\lambda(E-F)$, see [19, Sect. 3, 4 and 5].

We now define the second family of functions. We set $\widetilde{Q}_i(E) = c_i(E)$. Given two nonnegative integers $i \geq j$, we define

$$\widetilde{Q}_{i,j}(E) := \widetilde{Q}_i(E)\widetilde{Q}_j(E) + 2\sum_{p=1}^{j}(-1)^p \widetilde{Q}_{i+p}(E)\widetilde{Q}_{j-p}(E)\,.$$

For a partition μ, $\widetilde{Q}_\mu(E)$ is defined recurrently on $l(\mu)$, by putting for odd $l(\mu)$,

$$\widetilde{Q}_\mu(E) = \sum_{p=1}^{l(\mu)}(-1)^{p-1}\widetilde{Q}_{\mu_p}(E)\ \widetilde{Q}_{\mu\smallsetminus\{\mu_p\}}(E),$$

and for even $l(\mu)$,

$$\widetilde{Q}_\mu(E) = \sum_{p=2}^{l(\mu)}(-1)^{p}\widetilde{Q}_{\mu_1,\mu_p}(E)\ \widetilde{Q}_{\mu\smallsetminus\{\mu_1,\mu_p\}}(E).$$

This family of functions is modeled on Schur Q-functions, and is useful in Schubert calculus of Lagrangian Grassmannians. The reader can find in [19, Sect. 3 and 7] more details concerning the polynomials $\widetilde{Q}_\mu(E)$.

3 Schubert varieties and Schubert classes

In this section, we collect basic information on the cohomology rings of the flag manifolds G/P. We begin by fixing some notation.

Let G be a semisimple group over an algebraically closed field k, and $B \subset G$ a Borel subgroup. Choose a maximal torus $T \subset B$ with Weyl group $W = N_G(T)/T$ of (G,T). This determines a root system R, simple roots Δ, positive roots R^+ etc. The group W is generated by simple reflections $\{s_\alpha : \alpha \in \Delta\}$ with length function $l(w)$ and longest word w_0: $l(w_0) = \mathrm{card}(R^+)$. The Chevalley-Bruhat decomposition $G = BWB$ provides a "cell-decomposition" of the flag manifold

$$G/B = \coprod_{w\in W} BwB/B\,.$$

Recall that the flag manifold G/B is nonsingular algebraic and projective of dimension $\mathrm{card}(R^+)$. Each subset Bw_0wB/B of G/B is isomorphic, as a k-variety, to the affine space $k^{l(w_0)-l(w)}$. Its closure $\overline{Bw_0wB/B}$ is called a *Schubert variety.* This is, in general, a singular algebraic variety of codimension $l(w)$ in G/B. We set in $A^{l(w)}(G/B)$, or in $H^{2l(w)}(G/B,\mathbf{Z})$

$$X^w := [\overline{Bw_0wB/B}]\,,$$

and call it a *Schubert class.*

The same applies to all partial flag manifolds G/P, where P is a parabolic subgroup of G. Let θ be a subset of Δ and let W_θ be the subgroup of W generated by $\{s_\alpha\}_{\alpha\in\theta}$. We set $P = P_\theta = BW_\theta B$, and $W_P = W_\theta$.

Consider the set

$$W^P = W^\theta := \{w \in W : \ l(ws_\alpha) = l(w) + 1 \quad \forall \alpha \in \theta\}.$$

This is the set of minimal length left coset representatives of W_P in W. The projection $G/B \to G/P$ induces an injection

$$A^*(G/P) \hookrightarrow A^*(G/B)\,,$$

which additively identifies $A^*(G/P)$ with $\bigoplus_{w\in W^P} \mathbf{Z}X^w$. In other words, the X^w, $w \in W^P$, form a $\mathbf{Z}$-basis for $A^*(G/P)$ [4, Thm 5.5]. Multiplicatively, $A^*(G/P)_{\mathbf{Q}}$ is identified with the ring of invariants ${A^*(G/B)_{\mathbf{Q}}}^{W_P}$ [4, Sect. 5].

If $k = \mathbf{C}$, since G/P admits a cell-decomposition, we have

$$H^{2i+1}(G/P, \mathbf{Z}) = 0 \quad \text{ and } \quad H^{2i}(G/P, \mathbf{Z}) = A^i(G/P) \tag{3}$$

(cf. [16, Ex. 19.1.11]).

Example 1. Let $G = SL_n$. We set $P = P_\theta$, where θ is obtained by omitting the root $\varepsilon_r - \varepsilon_{r+1}$ in the basis $\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n$ of the root system of type A_{n-1}:

$$\{\varepsilon_i - \varepsilon_j \ : \ i \neq j\} \subset \oplus_{i=1}^n \mathbf{R}\varepsilon_i\,.$$

We have an identification $SL_n/P = G_r(k^n)$, the Grassmannian parametrizing r-dimensional linear subspaces in k^n. It is an algebraic variety of dimension $r(n-r)$.

The Weyl group W is here the symmetric group S_n, and $W_P = S_r \times S_{n-r}$. The poset W^P is naturally identified with the poset of partitions λ contained in $((n-r)^r)$ (see, e.g., [24]) and the corresponding Schubert class X^λ is represented by the following locus in the Grassmannian. Consider a flag

$$V_0 \subset V_1 \subset \cdots \subset V_n = k^n$$

of vector spaces with $\dim(V_i) = i$. Consider the following locus:

$$\{L \in G_r(k^n) : \dim(L \cap V_{n-r+i-\lambda_i}) \geq i, \ 1 \leq i \leq r\}\,.$$

The class of this locus does not depend on the flag, and is equal to the Schubert class X^λ.

Theorem 2 (Giambelli formula)**.** *In $A^{|\lambda|}(G_r(k^n))$, we have*

$$X^\lambda = s_\lambda(R^*)\,,$$

where R is the tautological subbundle on the Grassmannian.

Example 3. Let V be a symplectic vector space over k of dimension $2n$, and let $G = Sp(V)$ be the symplectic group. We set $P = P_\theta$, where θ is obtained by omitting the root $2\varepsilon_n$ in the basis $\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$ of the root system of type C_n:

$$\{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \leq j \leq n\} \cap \{\pm 2\varepsilon_i : 1 \leq i \leq n\}\,.$$

We have an identification $Sp(V)/P = LG(V)$, the Lagrangian Grassmannian parametrizing all Lagrangian linear subspaces in V. It is an algebraic variety of dimension $n(n+1)/2$.

We set $\rho(n) = (n, n-1, \dots, 2, 1)$.

The Weyl group W is here the hyperoctahedral group that can be identified with group of signed permutations, and $W_P = S_n$. The poset W^P is naturally identified with the poset of strict partitions μ contained in $\rho(n)$ (see, e.g., [24]), and the corresponding Schubert class Y^μ, where

$$\mu = (n \geq \mu_1 > \cdots > \mu_r > 0)$$

is represented by the following variety. Consider a flag

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset V$$

of isotropic vector spaces with $\dim(V_i) = i$. Consider the following locus:

$$\{L \in LG(V) : \dim(L \cap V_{n+1-\mu_i}) \geq i, \ 1 \leq i \leq r\}.$$

The class of this locus does not depend on the flag, and is equal to the Schubert class Y^μ.

Theorem 4. *[41], [42] In $A^{|\mu|}(LG(V))$, we have*

$$Y^\mu = \widetilde{Q}_\mu(R^*),$$

where R is the tautological subbundle on the Lagrangian Grassmannian.

The original argument [42] made use of a comparison of Pieri formulas for Lagrangian Schubert classes and for Schur Q-functions (for references, see [42, Sect. 6]). There is also another proof in [35, p. 40], which uses the characteristic map for a Lagrangian Grassmannian, and relies on some divided difference and vertex operator computations.

We record the following "duality" result.

Theorem 5. *Let G be a semisimple group, and let $P \subset G$ be a parabolic subgroup. For any $w \in W^P$ there exists exactly one $w' \in W^P$ such that $\dim X^w + \dim X^{w'} = \dim G/P$, and in $A^*(G/P)$ we have $X^w \cdot X^{w'} \neq 0$. In fact, $X^w \cdot X^{w'} = 1$.*

We call $X^{w'}$ the *dual class* to X^w. Let us discuss the following three examples.

For $P = B$, we have $w' = w_0 \cdot w$. Indeed,

$$X^w \cdot X^{w'} = \delta_{w, w_0 w'} X^{w_0},$$

where X^{w_0} is the class of a point (see [8, p. 20]).

In the situation of Example 1, the dual class to X^λ is $X^{\lambda'}$ where $\lambda'_i = n - r - \lambda_{r+1-i}$ for $i = 1, \dots, r$. (See, e.g., [16, p. 271].)

In the situation of Example 3, the dual class to Y^μ is $Y^{\mu'}$ where the parts of the strict partition μ' complement the set of parts of μ in $\{1, 2, \dots, n\}$. (See [42, p. 178].)

In general, the Poincaré duality for a partial flag manifold G/P is described in terms of the Weyl group of G in [27, p. 197].

The following Bertini-Kleiman theorem is often used to show positivity.

Theorem 6. *[31] Suppose a connected algebraic group G acts transitively on a variety X (over an algebraically closed field k). Let Y, Z be subvarieties of X. Then, denoting by $g \cdot Y$ the translate of Y by $g \in G$, the following two statements hold.*
(1) There exists a nonempty open subset $U \subset G$ such that for all $g \in U$, $(g \cdot Y) \cap Z$ is either empty or of pure dimension

$$\dim(Y) + \dim(Z) - \dim(X) .$$

(2) If Y and Z are nonsingular, and char(k)=0, then there is a nonnempty open subset $U \subset G$ such that for all $g \in U$, $(g \cdot Y) \cap Z$ is nonsingular.

Corollary 7. *With the notation of the theorem, if* $\dim(Y) + \dim(Z) = \dim(X)$, *then $(g \cdot Y) \cap Z$ is either empty or a zero-dimensional subscheme. Under the assumptions of (2), all points in $(g \cdot Y) \cap Z$ are regular.*

We end this section with the following fact on positivity of Schubert classes. Assertions (i) and (ii) are classically known (also for G/P with similar proofs).

Proposition 8. *(i) Let Z be a subvariety of G/B. If in $A^*(G/B)$, we have*

$$[Z] = \sum_{w \in W} a_w X^w ,$$

where $a_w \in \mathbf{Z}$, then all the coefficients a_w are nonnegative.
(ii) If for $w, v \in W$, in $A^{l(w)+l(v)}(G/B)$, we have

$$X^w \cdot X^v = \sum_u c_{wv}^u X^u ,$$

then $c_{wv}^u \geq 0$.
(iii) Let $G \subset H$ be an inclusion of algebraic groups. Let $Q \subset H$ be a parabolic subgroup. Set $P = G \cap Q$, and let $i : G/P \to H/Q$ be the inclusion. If $Z \subset H/Q$ is a subvariety, and in $A^*(G/P)$ we have

$$i^*[Z] = \sum_{w \in W^P} a_w X^w ,$$

with $a_w \in \mathbf{Z}$, then all the coefficients a_w are nonnegative.

Let us show, for instance, (i) and (iii). As for (i): For any w, we have

$$a_w = \int_{G/B} [Z] \cdot X^{w'} ,$$

where $X^{w'}$ is the dual class to X^w. Let Y be a subvariety representing the class $X^{w'}$. We apply to Z and Y the Bertini-Kleiman theorem: for a general $g \in G$ we obtain a zero-dimensional scheme $(g \cdot Z) \cap Y$ and a_w is its length, hence $a_w \geq 0$.
As for (iii): We use the Bertini-Kleiman theorem for the subvarieties Z and G/P of H/Q: for a general $h \in H$, $h \cdot Z$ and G/P meet properly. Let

$$V = (h \cdot Z) \cap G/P \subset G/P ,$$

a schematic intersection. We now use Proposition 8(i) for the subvariety V of G/P. Alternatively, to conclude, we can use again the Bertini-Kleiman theorem, this time for $V \subset G/P$ and a subvariety representing the dual class to X^w. $\square$

Corollary 9. *Let V be symplectic vector space of dimension $2n$ and let $LG(V)$ be the Lagrangian Grassmannian. Denote by*

$$i : LG(V) \hookrightarrow G_n(V)$$

the inclusion. If in $A^*(LG(V))$ we have

$$i^*(X^\lambda) = \sum a_\mu Y^\mu \,,$$

where $a_\mu \in \mathbf{Z}$, then $a_\mu \geq 0$.

A combinatorial positive rule for the coefficients a_μ was given in [43, Prop. 2].

4 Ample vector bundles and positive polynomials

In this section, we work over an algebraically closed field k of arbitrary characteristic. Let X be a scheme, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *generated by its global sections* if there is a family of global sections $\{s_i\}_{i\in I}$, $s_i \in \Gamma(X, \mathcal{F})$, such that for each $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_x$-module.

Recall that a line bundle $\mathcal{O}(D)$ on a smooth curve of genus g is generated by its global sections if $\deg D \geq 2g$. It is ample iff $\deg D > 0$; so a sufficiently high power of an ample line bundle is generated by its global sections.

This is also the case of vector bundles of higher ranks on higher dimensional varieties. Given a vector bundle E, we denote by $S^p(E)$ its pth symmetric power. We say that a vector bundle E on a variety X is *ample* if for any coherent sheaf $\mathcal{F}$ there exists $p_0 \in \mathbf{N}$ such that for any $p \geq p_0$, the sheaf $S^p(E)\otimes\mathcal{F}$ is generated by its global sections. This is equivalent to say that the Grothendieck invertible sheaf $L_{E^*} = \mathcal{O}(1)$ on $\mathbf{P}(E^*)$, the projective bundle[2] of E^* is ample.

Let us mention two properties of ample vector bundles [23]:

- a direct sum of ample vector bundles is ample;
- for a partition λ, the *Schur bundle* $S^\lambda(E)$ (see [34], [19, p. 131]) of an ample vector bundle E is ample.

Perhaps this is a good moment to come back to positivity. Consider the following example. If E is a vector bundle, λ, μ partitions, then the integer coefficients a_ν in the expansion of the Schur polynomials of the Schur bundle $S^\lambda(E)$,

$$s_\mu(S^\lambda(E)) = \sum_\nu a_\nu s_\nu(E) \,,$$

in the basis of Schur functions $\{s_\nu(E)\}$ are *nonnegative* [44, Cor. 7.2] (see also [36, Ex. 8.3.13]). This is a consequence of the second property. This information is nontrivial even for Chern classes (i.e. for $\mu = (1, \ldots, 1)$); for some examples of explicit computations, see [44].

[2] i.e. the bundle of lines in the fibers

Let E be a vector bundle of rank n on a variety X, and C a subscheme of E. We say that $C \subset E$ is a *cone* if it is stable under the natural $\mathbf{G}_m$ action on E. If $C \subset E$ is a cone of pure dimension c, then one may intersect its cycle $[C]$ with the zero-section of the vector bundle:

$$z(C, E) = s_E^*([C]) \in A_{c-n}(X)\,, \tag{4}$$

where $s_E^* : A_c(E) \to A_{c-n}(X)$ is the Gysin map determined by the zero section $X \to E$. In fact, we can use any other section $X \to E$ and $z(C, E)$ is the unique cycle class on X such that

$$p^*(z(C, E)) = [C] \tag{5}$$

in $A_c(E)$ (see [17, (1.4)]).

Here is an example of a positivity result with a pretty simple proof.

Lemma 10. *Let E be a vector bundle on a variety X, and let C be an irreducible cone in E. If E is generated by its global sections, then $z(C, E)$ is represented by a nonnegative cycle.*

Proof. Restricting E to the support of C [3], we may assume that this support is equal to X. The inclusion $C \subset E$ gives rise to a subscheme $\mathbf{P}(C) \subset \mathbf{P}(E)$. If E is generated by its global sections, then $\mathcal{O}(1)$ on $\mathbf{P}(E)$ is generated by its global sections. By the Bertini theorem, a general hypersurface section on $\mathbf{P}(E)$ intersects $\mathbf{P}(C)$ properly or this intersection is empty. Hence a general section of E intersects C properly or the intersection is empty. Therefore $z(C, E)$ is represented by a nonnegative cycle. $\square$

For a projective variety X, there is well defined *degree*

$$\int_X : A_0(X) \to \mathbf{Z}$$

(see [16, Def. 1.4]). The following result of Fulton and Lazarsfeld is basic for applications to positivity.

Theorem 11. *[17] Let E be an ample vector bundle of rank n on a projective variety X. Let C be a cone in E of pure dimension n. Then we have*

$$\int_X z(C, E) > 0\,.$$

For a more extensive study of positivity in intersection theory, coming from ample vector bundles and vector bundles generated by their global sections, see [16, Thm 12.1].

Remark 12. Suppose $k = \mathbf{C}$. Under the assumptions of Theorem 11, we have in $H_0(X, \mathbf{Z})$ the homology analog of $z(C, E)$, denoted by the same symbol. We also have the homology degree map $\deg_X : H_0(X, \mathbf{Z}) \to \mathbf{Z}$. They are compatible with their Chow group counterparts via the cycle map: $A_0(X) \to H_0(X, \mathbf{Z})$ (cf. [16, Sect. 19]). Thus we have

$$\deg_X\big(z(C, E)\big) > 0\,. \tag{6}$$

[3] Cf. [16, B.5.3].

We record the following result.

Proposition 13. *[37] Let E be a vector bundle on a complete homogeneous variety X. Let C be a cone in E and let $Y \subset X$ be a subvariety of dimension $\dim(X) + \operatorname{rank}(E) - \dim C$. Assume that E is generated by its global sections. Then the intersection $[C] \cdot [Y]$ is nonnegative.*

Let $c_1, c_2, \ldots$ be commuting variables with $\deg(c_i) = i$. Fix $d, n \in \mathbf{N}$. Let $P(c_1, \ldots, c_n)$ be a weighted homogeneous polynomial of degree d. We say that P is *numerically positive for ample vector bundles*, or simply *positive*, if for every d-dimensional projective variety X and any ample vector bundle of rank n on X, we have

$$\int_X P(c_1(E), \ldots, c_n(E)) > 0 \, .$$

For example, Griffiths [21] who pioneered this subject, found the following positive polynomials: c_1, c_2, $c_1^2 - c_2$. Bloch-Gieseker [5] showed that c_d is positive for $d \leq n$.

Given a partition λ, with the conjugate partition μ, we set

$$s_\lambda = s_\lambda(c_1, c_2, \ldots) := |c_{\mu_i - i + j}|_{1 \leq i,j \leq l(\mu)} \, . \tag{7}$$

Kleiman [30] showed that positive polynomials for surfaces are nonnegative combinations of s_2 and $s_{1,1}$. Gieseker [20] proved that s_d (the d-th Segre class) is positive.

Fulton and Lazarsfeld gave the following characterization of positive polynomials. Let P be a weighted homogeneous polynomial of degree d in n variables. Write

$$P = \sum_\lambda a_\lambda s_\lambda \, , \tag{8}$$

where $a_\lambda \in \mathbf{Z}$.

Theorem 14. *[17] The polynomial P is positive iff P is not zero and all the coefficients a_λ in (8) are nonnegative.*

The proof of the theorem combines the hard Lefschetz theorem appropriately adapted to this subject by Bloch and Gieseker [5] and the Giambelli formula, which was recalled in Theorem 2.

Remark 15. We now discuss some results related to Theorems 11 and 14. The latter was generalized by Demailly, Peternell and Schneider to *nef* vector bundles in [11]. The former has a very simple proof due to Fulton and Lazarsfeld in [18] in the case when E is ample and generated by its global sections. Hacon [22] showed that these assumptions are not sufficient, for a positive polynomial P, to have $\int_X P(E) \geq P(n, \binom{n}{2}, \ldots, \binom{n}{d})$, as it was conjectured by Beltrametti, Schneider and Sommese in [2]. This last inequality is true for *very ample* bundles (*loc.cit.*). Consider a vector bundle E on a complex projective manifold. Griffiths [21] defined E to be *numerically positive* if for any analytic subvariety $W \subset M$, and any rank q quotient Q of $E_{|W}$, we have $\int_W P(c(Q)) > 0$ for any homogeneous polynomial of degree equal to $\dim(W)$ from the Griffiths cone associated to q (see also [17, App. A]). Griffiths speculated on the possibility that arbitrary ample bundles are numerically positive. This was proved, using Schubert calculus, by Usui and Tango [52] for bundles generated by their global sections. The numerical positivity of all ample bundles was proved in [17, App. A].

5 Thom polynomials for singularities of maps

Thom polynomials came from algebraic topology and singularities. They are tools to measure the complexity of singularities. In this section, we investigate Thom polynomials of singularities of maps. Let

$$f : M \to N$$

be a map of complex analytic manifolds; we say that $x \in M$ is a singularity of f if df_x fails to have the maximal rank.

We now follow the terminology from [49] for what concerns map germs $(\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$ and their stable versions. Two map germs $\kappa_1, \kappa_2 : (\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$ are said to be *right-left equivalent* if there exist germs of biholomorphisms ϕ of $(\mathbf{C}^m, 0)$ and ψ of $(\mathbf{C}^n, 0)$ such that $\psi \circ \kappa_1 \circ \phi^{-1} = \kappa_2$. A *suspension* of a germ map κ is its trivial unfolding: $(x, v) \mapsto (\kappa(x), v)$. Let us fix $l \in \mathbf{N}$. Consider the equivalence relation on stable map germs $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet + l}, 0)$ generated by right-left equivalence and suspension. An equivalence class of this relation is often called *singularity* and denoted by η.

According to Mather's classification [13], the finite dimensional (local) $\mathbf{C}$-algebras are in one-to-one correspondence with classes of contact equivalence classes of singularities (cf. [15]). For instance, A_i stands for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$; and $I_{a,b}$ for stable germs with local algebra $\mathbf{C}[[x, y]]/(xy, x^a + y^b)$, $b \geq a \geq 2$ (they also depend on l).

Following Thom, we look at the locus

$$\eta(f) := \{x \in M : \text{the singularity of } f \text{ at } x \text{ is } \eta\}$$

for a map $f : M \to N$, and try to compute its class in terms of the Chern classes of M and N. More precisely, we restrict ourselves only to *general* maps, i.e., the maps from some open subset in the space of all maps.

For example, let $f : M \to N$ be a general morphism of compact Riemann surfaces. Suppose that the singularity is A_1: $z \mapsto z^2$. Then $\eta(f)$ is the ramification divisor of f, and by the Riemann-Hurwitz formula the wanted class is $f^* c_1(N) - c_1(M)$. We refer the reader to [32, p. 300] for more details.

The space of germs of maps from $(\mathbf{C}^m, 0)$ to $(\mathbf{C}^n, 0)$ has infinite dimension, which is inconvenient from the point of view of algebraic geometry. To remedy this, we pass to the *spaces of jets* of germs of maps. Fix $m, n, p \in \mathbf{N}$. Consider the space $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$ of p-jets of analytic functions from $\mathbf{C}^m$ to $\mathbf{C}^n$, which map 0 to 0 (see [1, pp. 36-38]). This space will also be denoted by $\mathcal{J}(m, n)$ or simply by $\mathcal{J}$ to avoid too complicated notation.

Let Aut^p_n denote the group of p-jets of automorphisms of $(\mathbf{C}^n, 0)$.

Consider the natural right-left action of the group $\mathrm{Aut}^p_m \times \mathrm{Aut}^p_n$ on the space $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$. By a *singularity class* we mean a closed algebraic right-left invariant subset of $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$.

Given complex analytic manifolds M^m and N^n, a singularity class $\Sigma \subset \mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$ defines the following subset $\Sigma(M, N) \subset \mathcal{J}^p(M, N)$, where $\mathcal{J}^p(M, N)$ is the space of p-jets of maps from M to N: using the coordinate maps $M \cong \mathbf{C}^m$ and $N \cong \mathbf{C}^n$, we declare that a point belongs to $\Sigma(M, N)$ iff it belongs to Σ. If we change the coordinate maps, then the set $\Sigma(M, N)$ remains unchanged by virtue of right-left invariance of Σ.

Theorem 16. *Let $\Sigma \subset \mathcal{J}^p(\mathbf{C}_0^m, \mathbf{C}_0^n)$ be a singularity class. There exists a universal polynomial $\mathcal{T}^\Sigma$ over $\mathbf{Z}$ in $m+n$ variables $c_1, \ldots, c_m, c'_1, \ldots, c'_n$ which depends only on Σ, m and n such that for any manifolds M^m, N^n and for a general map $f : M \to N$, the class of*

$$\Sigma(f) := (j^p f)^{-1}(\Sigma(M, N))$$

is equal to

$$\mathcal{T}^\Sigma(c_1(M), \ldots, c_m(M), f^* c_1(N), \ldots, f^* c_n(N)),$$

where $j^p f : M \to \mathcal{J}^p(M, N)$ is the p-jet extension of f ([1, pp. 36-38]).

This is a theorem due to Thom, see [51]. The polynomial $\mathcal{T}^\Sigma$ is called the *Thom polynomial of* Σ.

Note that a singularity η corresponds here to the singularity class Σ being the closure of a single right-left orbit, and the locus $\eta(f)$ is generalized to $\Sigma(M, N)$. The key problem is to compute the classes of these varieties in terms of the Chern classes of the manifolds M and N.

Lemma 17. *Let $\Sigma \subset \mathcal{J}$ be a singularity class. Then Σ is a cone in the vector space $\mathcal{J}$.*

Proof. For a function $f \in \Sigma$ and a scalar $c \in \mathbf{C}^*$, we have $c \cdot f \in \Sigma$ because $\mathbf{G}_m \subset \mathrm{Aut}_n^p$, and the singularity class Σ is $\mathrm{Aut}_m^p \times \mathrm{Aut}_n^p$-invariant. □

We follow Kazarian's approach to Thom polynomials of singularities of maps [28]. We set

$$G := \mathrm{Aut}_m^p \times \mathrm{Aut}_n^p \,.$$

Consider the classifying principal G-bundle $EG \to BG$ [39] (see also [25, Sect. 7]). Here EG is a contractible space with a free action of the group G. This action extends to the diagonal action on the product space $EG \times \mathcal{J}$. Invoking [25, Def. 3.1] and its notation, we set

$$\widetilde{\mathcal{J}} := EG \times^G \mathcal{J} = (EG \times \mathcal{J})/G \,.$$

This space is often called the *classifying space of singularities*[4]. For a given singularity class $\Sigma \subset \mathcal{J}$, we define

$$\widetilde{\Sigma} := EG \times^G \Sigma \subset \widetilde{\mathcal{J}} \,.$$

We have $\mathrm{codim}(\widetilde{\Sigma}, \widetilde{\mathcal{J}}) = \mathrm{codim}(\Sigma, \mathcal{J})$. We denote by $\mathcal{T}^\Sigma \in H^{2\,\mathrm{codim}(\Sigma,\mathcal{J})}(\widetilde{\mathcal{J}}, \mathbf{Z})$ the dual class of $[\widetilde{\Sigma}]$. The classifying spaces BG, $\widetilde{\mathcal{J}}$, etc. have infinite dimensions and the notion of the "dual class" should be clarified, see [28, Rem. 1.6] and [47, footnote (4)].

The projection to the second factor $\widetilde{\mathcal{J}} \to BG$ is a bundle with fiber isomorphic to $\mathcal{J}$ and structure group G. Since $\mathcal{J}$ is contractible, and also G is contractible to the subgroup $GL_m \times GL_n$ of linear changes, we get

$$H^*(\widetilde{\mathcal{J}}, \mathbf{Z}) \cong H^*(BG, \mathbf{Z}) \cong H^*(BGL_m \times BGL_n, \mathbf{Z}) \,.$$

[4] Note that the same construction is used in the definition of Borel of equivariant cohomology for a G-space $\mathcal{J}$.

Hence $\mathcal{T}^\Sigma$ is identified with a polynomial in $c_1, \dots, c_m$ and $c'_1, \dots, c'_n$ which are the Chern classes of universal bundles R_m and R_n on BGL_m and BGL_n. This is the Thom polynomial $\mathcal{T}^\Sigma$.

We now explain what we mean by *stable* singularity class. The suspension

$$\mathcal{S} : \mathcal{J}(m,n) \hookrightarrow \mathcal{J}(m+1, n+1)$$

allows one to increase the dimension of the source and the target simultaneously: with the local coordinates $x_1, x_2, \dots$ for the source and a function $f = f(x_1, \dots, x_m)$, the jet $\mathcal{S}(f) \in \mathcal{J}(m+1, n+1)$ is defined by

$$\mathcal{S}(f)(x_1, \dots, x_m, x_{m+1}) := (f(x_1, \dots, x_m), x_{m+1}) .$$

Suppose that the singularity class Σ is *stable under suspension.* By this we mean that it is a member $\Sigma_0 = \Sigma$ of a family

$$\{\Sigma_r \subset \mathcal{J}(m+r, n+r)\}_{r \geq 0}$$

such that

$$\Sigma_{r+1} \cap \mathcal{J}(m+r, n+r) = \Sigma_r$$

and

$$\mathcal{T}^{\Sigma_{r+1}}{}_{|H^*(BGL_{m+r} \times BGL_{n+r}, \mathbf{Z})} = \mathcal{T}^{\Sigma_r} .$$

This means that if we specialize

$$c_{m+r+1} = c'_{n+r+1} = 0$$

in the polynomial $\mathcal{T}^{\Sigma_{r+1}}$, we obtain the polynomial $\mathcal{T}^{\Sigma_r}$. If Σ is closed under the *contact equivalence* (see [15]), then it is stable in our sense.

The theorem of Thom has the following refinement due to Damon [10] for singularity classes Σ which are stable under suspension: $\mathcal{T}^\Sigma$ is a polynomial in

$$c_i(R_m - R_n) , \quad \text{where} \quad i = 1, 2, \dots .$$

So, we can use the bases of monomials in the Chern classes in $R_m - R_n$ or $R_n - R_m$ or $R_m^* - R_n^*$ or $R_n^* - R_m^*$. We can also use the bases of (supersymmetric) Schur functions in $R_m - R_n$ or $R_n - R_m$ or $R_m^* - R_n^*$ or $R_n^* - R_m^*$. About a decade ago, calculations of the Thom polynomials using the bases of Schur functions were done independently by Fehér-Kömüves and Lascoux-Pragacz.

For Morin singularities A_i, there is a positivity conjecture of Rimányi (1998), asserting that the expansions of the Thom polynomials $\mathcal{T}^{A_i}$ in the basis of monomials in the Chern classes in $R_n - R_m$ have nonnegative coefficients. See [3] for a discussion of a link of this conjecture with the Green-Griffiths conjecture about holomorphic curves in nonsingular projective varieties.

Example 18. We display here three Thom polynomials for the Morin singularities between equal dimensional manifolds (so $l = 0$ in the notation from the beginning of this section):
A_3: $c_1^3 + 3c_1c_2 + 2c_3$;
A_4: $c_1^4 + 6c_1^2c_2 + 2c_2^2 + 9c_1c_3 + 6c_4$;
A_5: $c_1^5 + 10c_1^3c_2 + 25c_1^2c_3 + 10c_1c_2^2 + 38c_1c_4 + 12c_2c_3 + 24c_5$.

In general, the expansions of Thom polynomials of stable singularities in the basis of monomials in the Chern classes of $R_n - R_m$ can have negative coefficients.

Example 19. We give here three Thom polynomials for the singularities $I_{p,q}$ between equal dimensional manifolds (for the definition of these singularities, see the beginning of this section):
$I_{2,2}$: $c_2^2 - c_1c_3$;
$I_{2,3}$: $2c_1c_2^2 - 2c_1^2c_3 + 2c_2c_3 - 2c_1c_4$;
$I_{2,4}$: $2c_1^2c_2^2 + 3c_2^3 - 2c_1^3c_3 + 2c_1c_2c_3 - 3c_3^2 - 5c_1^2c_4 + 9c_2c_4 - 6c_1c_5$.

It is not obvious that $\mathcal{T}^\Sigma \neq 0$ for a nonempty stable singularity class Σ.

We now examine the expansions of Thom polynomials of stable singularities in the basis $\{s_\lambda(R_n - R_m)\}$ labeled by partitions λ. We refer the reader for a variety of examples to [47, p. 93-94], [15] and [40].[5]

Theorem 20. *[47] Let Σ be a nonempty stable singularity class. Then for any partition λ the coefficient a_λ in*

$$\mathcal{T}^\Sigma = \sum a_\lambda s_\lambda(R_n - R_m) \tag{9}$$

is nonnegative and $\sum a_\lambda > 0$.

This feature of Schur function expansions of Thom polynomials was pointed out in [45], conjectured for Thom-Boardman singularities by Fehér and Kömüves [14] (they computed the Schur function expansions of the Thom polynomials of $\Sigma^{i,j}[-i+1]$), and conjectured for all singularity classes in [46].

Note that each partition λ appearing in the RHS of (9) is contained in the (n,m)-hook (see [19, p. 35]).

To prove the theorem, we generalize the equation (9) for any pair of complex vector bundles (E, F) on any paracompact space X. To this end, we apply the techniques of fiber bundles. Apart from vector bundles, we also use principal G-bundles associated with finite collections of vector bundles[6] on a common base space (here $G = \prod_i GL_{n_i}$, where n_i are the ranks of the vector bundles). For principal bundles, we refer, e.g., to [33, Sect. I.5] or [25, Sect. 5].

Moreover, it is convenient to pass to the topological homotopy category, where any pair of vector bundles can be pulled back from the universal pair of vector bundles on $BGL_m \times BGL_n$.

We first pull back the bundle $\widetilde{\mathcal{J}}$ from BG to $BGL_m \times BGL_n$ via the embedding

$$GL_m \times GL_n \hookrightarrow \mathrm{Aut}_m \times \mathrm{Aut}_n\,.$$

Since $GL_m \times GL_n$ acts linearly on $\mathcal{J}$, the obtained pullback bundle is now the following vector bundle on $BGL_m \times BGL_n$:

$$\mathcal{J}(R_m, R_n) := \Big(\bigoplus_{i=1}^{p} S^i(R_m^*)\Big) \otimes R_n\,.$$

[5] In [47] and [40] the authors worked with the basis of Schur functions $\{s_\lambda(R_m^* - R_n^*)\}$, so the Schur functions given in the examples there are labeled by the conjugate partitions of those appearing in the present convention.

[6] The associated principal GL_n-bundle of a vector bundle E of rank n is often called the *frame bundle* of E (its fibers consist of all ordered bases of the fibers of E).

The bundle $\mathcal{J}(R_m, R_n)$ contains the preimage of $\widetilde{\Sigma}$, denoted by $\Sigma(R_m, R_n)$, whose class is

$$[\Sigma(R_m, R_n)] = \sum_\lambda a_\lambda s_\lambda(R_n - R_m)\,, \tag{10}$$

with the same coefficients a_λ as in (9).

Consider now a pair of vector bundles E and F of ranks m and n on a variety X. We set

$$\mathcal{J}(E, F) := \Big(\bigoplus_{i=1}^{p} S^i(E^*)\Big) \otimes F\,.$$

Let $P(E, F)$ be the principal $GL_m \times GL_n$-bundle associated with the pair of vector bundles (E, F). We have

$$\mathcal{J}(E, F) = P(E, F) \times^{GL_m \times GL_n} \mathcal{J}\,.$$

We set

$$\Sigma(E, F) := P(E, F) \times^{GL_m \times GL_n} \Sigma \subset \mathcal{J}(E, F)\,,$$

a locally trivial fibration with the fiber equal to Σ.

Lemma 21. *The variety $\Sigma(E, F)$ is a cone in the vector bundle $\mathcal{J}(E, F)$.*

Proof. The assertion follows from Lemma 17. □

Lemma 22. *The dual class of $[\Sigma(E, F)] \in H_{2\dim(\Sigma)}(\mathcal{J}(E, F), \mathbf{Z})$ in*

$$H^{2\,\mathrm{codim}(\Sigma,\mathcal{J})}(\mathcal{J}(E, F), \mathbf{Z}) = H^{2\,\mathrm{codim}(\Sigma,\mathcal{J})}(X, \mathbf{Z})$$

is equal to

$$\sum_\lambda a_\lambda s_\lambda(F - E)\,, \tag{11}$$

where the coefficients a_λ are the same as in (9) [7].

Proof. The pair of vector bundles (E,F) on a variety X can be pulled back from the universal pair $(R_m,\, R_n)$ on $BGL_m \times BGL_n$ using a C^∞ map. We get the assertion of the lemma by pulling back the equation (10). Consequently, the coefficients of $s_\lambda(F - E)$ in (11) are the same as the coefficients of $s_\lambda(R_n - R_m)$ in (10). □

Proof of Theorem 20.[8] Let $e = \mathrm{codim}(C, \mathcal{J})$. This means that for any partition λ appearing in (11) its weight $|\,\lambda\,|$ is equal to e.

The idea of the proof is to produce from (11) a numerically positive polynomial for ample vector bundles, which captures positivity information about all the a_λ's. Since (11) is a supersymmetric polynomial, and we want a usual symmetric polynomial, we wish to specialize E to be a trivial bundle. Since the singularity class Σ is stable, we can use a pair of vector bundles E and F on X of the corresponding ranks $m' = m + r$ and $n' = n + r$ for some $r \geq 0$, instead of m and n. So we can assume that $n' >> 0$. In particular, we may suppose that $n' \geq e$.

[7] The meaning of the "dual class of $[\Sigma(E, F)]$" for a *singular* X is explained in [47, Note 6].
[8] This is the same proof as that in [47], but with "mehr Licht".

We use a specialization argument: let X vary over projective varieties of dimension e, let F vary over ample vector bundles of rank n' on X, and let E be a trivial vector bundle $\mathbf{1}^{m'}$ of rank m' on X. By the theory of symmetric functions (see, e.g., [19, Sect. 3.2]), the Schur polynomials $s_\lambda(F-E)$ appearing in (11) are indexed by partitions λ of weight $|\lambda| = e$, which are contained in the (n,m)-hook. In general, such polynomials vanish under our specialization. But the assumption $n' \geq e$, or equivalently, $\operatorname{rank} F \geq |\lambda|$, guarantees that the partition corresponding to a summand $a_\lambda s_\lambda(F-E)$ appearing in (11) has at most n' parts, and thus this summand survives the specialization, giving $a_\lambda s_\lambda(F)$. After the specialization, the expression (11) becomes

$$\sum_\lambda a_\lambda s_\lambda(F)\,, \tag{12}$$

where the summation is as in (11). Consider the polynomial

$$P := \sum_\lambda a_\lambda s_\lambda\,,$$

with the s_λ's as in (7) and the summation as in (12). We want to show that P is positive. To this end, consider the cone $\Sigma(E,F)$ in $\mathcal{J}(E,F)$ (see Lemma 21) and its cone class $z(\Sigma(E,F),\mathcal{J}(E,F))$ (see (4) and Remark 12). Since $\dim\Sigma(E,F) = \operatorname{rank}\mathcal{J}(E,F)$, this cone class belongs to $H_0(X,\mathbf{Z})$. It follows from Lemma 22 that the dual class of $z(\Sigma(\mathbf{1}^{m'},F),\mathcal{J}(\mathbf{1}^{m'},F))$ is the element of $H^{2e}(X,\mathbf{Z})$ given by the expression (12).

Since a direct sum of ample vector bundles is ample (see [23, Prop. (2.2)]), and the vector bundle $\mathcal{J}(\mathbf{1}^{m'},F)$ is a direct sum of several copies of F, then $\mathcal{J}(\mathbf{1}^{m'},F)$ is ample. Therefore by Theorem 11 and the inequality (6), we have

$$\int_X P(F) = \deg_X(z(\Sigma(\mathbf{1}^{m'},F),\mathcal{J}(\mathbf{1}^{m'},F)) > 0\,,$$

and thus conclude that P is positive.

In turn, by Theorem 14 we get that P is nonzero, and all the coefficients a_λ are nonnegative; hence also $\sum_\lambda a_\lambda > 0$. □

Question. Does there exists a basis different (up to rescaling) from the basis $\{s_\lambda(R_n - R_m)\}$ with the property that any Thom polynomial of a stable singularity class has a positive expansion in that basis?

6 Thom polynomials for invariant cones

In the previous section, in the context of classical Thom polynomials, we have investigated the functor of p-jets:

$$(E,F) \mapsto \mathcal{J}^p(E,F) = \Big(\bigoplus_{i=1}^{p} S^i(E^*)\Big) \otimes F\,,$$

defined on pairs of vector bundles, where p is large enough.

We want to generalize this setting. Suppose that $(n_1,\ldots,n_l) \in \mathbf{N}^{*l}$ and that V is a representation of $G = \prod_{i=1}^{l} GL_{n_i}$. The representation V gives rise to a *functor* ϕ defined for a collection of bundles on a variety X:

$$E_1,\ldots,E_l \mapsto \phi(E_1,\ldots,E_l)\,,$$

with $\dim E_i = n_i$, $i = 1, \ldots, l$. By passing to the dual bundles, we may assume that the functor ϕ is covariant in each variable.

Let $P(E_\bullet) = P(E_1, \ldots, E_l)$ be the principal G-bundle associated with the vector bundles $E_1, \ldots, E_l$. We define a new vector bundle:

$$V(E_\bullet) = V(E_1, \ldots, E_l) := P(E_\bullet) \times^G V$$

with fiber equal to V.

Suppose now that a G-invariant cone $\Sigma \subset V$ is given. We set

$$\Sigma(E_\bullet) = \Sigma(E_1, \ldots, E_l) := P(E_\bullet) \times^G \Sigma \subset V(E_\bullet)\,,$$

a fibration with fiber equal to Σ.

Let $R^{(i)}$, $i = 1, \ldots, l$, be the pullback of the tautological vector bundle from BGL_{n_i} to

$$BG = \prod_{i=1}^{l} BGL_{n_i}\,.$$

We denote by

$$\mathcal{T}^\Sigma \in H^{2codim(\Sigma,V)}\big(V(R^{(1)}, \ldots, R^{(l)}), \mathbf{Z}\big) = H^{2codim(\Sigma,V)}(BG, \mathbf{Z})$$

the dual class[9] of $[\Sigma(R^{(1)}, \ldots, R^{(l)})]$, and call it the *Thom polynomial* of Σ.

Then, the so defined Thom polynomial $\mathcal{T}^\Sigma \in H^*(BG, \mathbf{Z})$ depends on the Chern classes of the universal bundles $R^{(i)}$'s. We write $\mathcal{T}^\Sigma(E_1, \ldots, E_l)$ for the Thom polynomial $\mathcal{T}^\Sigma$, with $c_j(R^{(i)})$ replaced by $c_j(E_i)$ for $i = 1, \ldots, l$.

Arguing like in Lemma 22, we know that for any vector bundles $E_1, \ldots, E_l$ on a variety X, the class $[\Sigma(E_\bullet)]$ in

$$H^{2\,\mathrm{codim}(\Sigma,\mathcal{J})}(V(E_\bullet), \mathbf{Z}) = H^{2\,\mathrm{codim}(\Sigma,\mathcal{J})}(X, \mathbf{Z})$$

is equal to $\mathcal{T}^\Sigma(E_1, \ldots, E_l)$.

Since the Schur functions form an additive basis of the ring of symmetric functions, the Thom polynomial $\mathcal{T}^\Sigma$ is uniquely written in the following form:

$$\mathcal{T}^\Sigma = \sum a_{\lambda^{(1)}, \ldots, \lambda^{(l)}}\ s_{\lambda^{(1)}}(R^{(1)}) \cdot \ldots \cdot s_{\lambda^{(l)}}(R^{(l)})\,, \tag{13}$$

where $a_{\lambda^{(1)}, \ldots, \lambda^{(l)}} \in \mathbf{Z}$

We say that the functor ϕ, associated with a representation V, *preserves spannedness* if for a collection of vector bundles $E_1, \ldots, E_l$ generated by their global sections, the bundle $\phi(E_1, \ldots, E_l)$ is generated by its global sections.

Examples of functors preserving spannedness over fields of characteristic zero are *polynomial functors*. They are, at the same time, quotient functors and subfunctors of the tensor power functors (cf. [23]).

Theorem 23. *[48] Suppose that the functor ϕ preserves spannedness. Then the coefficients $a_{\lambda_1, \ldots, \lambda_l}$ in (13) are nonnegative. Assume additionally that there exists a projective variety X of dimension greater than or equal to* $\mathrm{codim}(\Sigma, V)$, *and there exist vector bundles $E_1, \ldots, E_l$ on X such that the bundle $\phi(E_1, \ldots, E_l)$ is ample. Then at least one of the coefficients $a_{\lambda_1, \ldots, \lambda_l}$ is positive.*

[9] Here the "dual class" has the same meaning as in the approach to Thom polynomials via classyfying spaces in the previous section.

Consider now the Thom polynomial $\mathcal{T}^\Sigma$ associated with a nonempty, possibly nonstable singularity class Σ in the space of jets $\mathcal{J}(m,n)$. By the theory of symmetric functions (see, e.g., [19, Sect. 3]), there exist coefficients $b_{\lambda\mu} \in \mathbf{Z}$ such that

$$\mathcal{T}^\Sigma = \sum_{\lambda,\mu} b_{\lambda\mu} s_\lambda(R_n) \cdot s_\mu(R_m^*) \,. \tag{14}$$

The following result follows from Theorem 23.

Corollary 24. *For any pair of partitions λ, μ, we have $b_{\lambda\mu} \geq 0$ and $\sum b_{\lambda\mu} > 0$.*

Let now Σ be a stable singularity class. There exist coefficients $a_\lambda \in \mathbf{Z}$ such that

$$\mathcal{T}^\Sigma = \sum_{\lambda} a_\lambda s_\lambda(R_n - R_m) \,, \tag{15}$$

the sum is over partitions λ with $|\lambda| = \mathrm{codim}(\Sigma, \mathcal{J}(m,n))$.

Here is another proof of Theorem 20. By the theory of symmetric functions (*loc.cit.*), we have that the coefficient of $s_\lambda(R_n - R_m)$ in the RHS of (15) is equal to the coefficient of $s_\lambda(R_n)$ in the RHS of (14), that is, $a_\lambda = b_{\lambda,\emptyset}$ for any partition λ. The assertion now follows from Corollary 24.

Remark 25. Another proof of the *nonnegativity* assertions of Theorem 20 and Corollary 24 was communicated to the author by Klyachko and indendependently by Kazarian (private communications). For details, see [40, p. 452]. These proofs use the Bertini-Kleiman theorem. Coming back to the above proofs of Theorem 20, we see that the use of ample vector bundles and the Fulton-Lazarsfeld Theorem 11, apart from the nonnegativity of the considered coefficients, implies that at least one of them is strictly positive.

7 Lagrangian Thom polynomials

Lagrangian Thom polynomials were considered by Vassiliev [53] (see also [29]).

Let us fix a positive integer n. Suppose that W is a complex vector space, where $\dim W = n$. Let

$$V = W \oplus W^*$$

be a linear symplectic space, equipped with the symplectic form $\langle , \rangle$, defined by

$$\langle (w_1, f_1), (w_2, f_2) \rangle = f_1(w_2) - f_2(w_1)$$

for $w_i \in W$ and $f_i \in W^*$, $i = 1, 2$. We view V as a symplectic manifold. Writing $q = (q_1, \ldots, q_n)$ for the coordinates of W and $p = (p_1, \ldots, p_n)$ for the dual coordinates of W^*, the symplectic form on V is $\sum_{i=1}^n dp_i \wedge dq_i$.

Denote by $\varrho : V \to W$ the projection.

Any germ of a Lagrangian submanifold L of V through 0 such that $\varrho_{|L}$ is a submersion is a graph of a 1-form $\alpha : W \to W^*$. The condition that L is Lagrangian is equivalent to $d\alpha = 0$. Since we deal with germs, we can write $\alpha = df$ for some function $f : W \to \mathbf{C}$.

The space of germs of Lagrangian submanifolds $L \subset V$ passing through 0 has infinite dimension, which is inconvenient from the point of view of algebraic geometry. To remedy this, we pass to the space of jets of germs of Lagrangian submanifolds.

Let us fix, once for all, a nonnegative integer p. We identify two germs of Lagrangian submanifolds L_1, L_2 through 0 if the tangency order of L_1 to L_2 (see [12, Def. 2.6]) is greater than p. (See also [26, I.1], where the name "contact of order" is used.) The equivalence class is called a "p-jet of a submanifold". In this way we obtain the *space of p-jets of Lagrangian manifolds* denoted by $\mathcal{J}^p(V)$. This space is homogeneous with respect to the action of $\mathrm{Sympl}^p(V)$ - the *group of p-jet symplectomorphisms preserving* $0 \in V$: every p-jet of a Lagrangian submanifold can be obtained from the "distinguished" Lagrangian submanifold W by application of a symplectomorphism preserving 0.

The Lagrangian Grassmannian $LG(V)$ is embedded in $\mathcal{J}^p(V)$ in a natural way. On the other hand, we have the *Gauss map*

$$\pi : \mathcal{J}^p(V) \to LG(V)\,,$$

which is a retraction to $LG(V)$, defined for a Lagrangian submanifold L by $\pi(L) = T_0(L)$, the tangent space of L at $0 \in L$. Denote by $\{W\}$ the point of $LG(V)$ corresponding to the linear space W. The following fact and its proof stems from [37].

Lemma 26. *The fiber of π over $\{W\}$ is isomorphic to the linear space*

$$\bigoplus_{i=3}^{p+1} S^i(W^*)\,.$$

Proof. The fiber $\pi^{-1}\{W\}$ consists of those (jets of) Lagrangian submanifolds L such that $T_0(L) = W$. Every Lagrangian submanifold L such that $\varrho_{|L}$ is a submersion is the graph of the differential of a function $f : W \to \mathbf{C}$ (note that df acts from W to W^*). The condition: $0 \in L$ corresponds to the condition: $df(0) = 0$, and the condition: $T_0(L) = W$ corresponds to the vanishing of the second derivatives of f at 0. $\square$

Thus $\pi : \mathcal{J}^p(V) \to LG(V)$ is an affine fibration. (Note that π is not a vector bundle starting from $p = 3$, see [37, footnote on p. 68].)

Let H be the subgroup of $\mathrm{Sympl}^p(V)$ consisting of p-jets of holomorphic symplectomorphisms preserving the fibration $\varrho : V \to W$. Two Lagrangian p-jets are *Lagrangian equivalent* if they belong to the same orbit of H. A *Lagrange singularity class* is any closed pure dimensional algebraic subset of the manifold $\mathcal{J}^p(V)$, which is H-invariant.

A Lagrange singularity class $\Sigma \subset \mathcal{J}^p(V)$ defines the class $[\Sigma]$ in the cohomology groups

$$H^*(\mathcal{J}^p(V), \mathbf{Z}) \cong H^*(LG(V), \mathbf{Z})\,. \tag{16}$$

This cohomology class in $H^*(LG(V), \mathbf{Z})$ will be called the *(Lagrangian) Thom polynomial* of Σ, and denoted $\mathcal{T}^\Sigma$.

We now use Schubert calculus to investigate Lagrangian Thom polynomials, that is, we study the expansions of Lagrangian Thom polynomials in the basis of Lagrangian Schubert classes. (These are the classes of the closures of the cells of a cellular decomposition of $LG(V)$, and thus they form a basis of $H^*(LG(V), \mathbf{Z})$.) By Theorem 4, we have

$$\mathcal{T}^\Sigma = \sum_{\text{strict } \mu \subset \rho(n)} a_\mu \widetilde{Q}_\mu(R^*)\,,$$

where $a_\mu \in \mathbf{Z}$.

Lemma 27. *[37] We have the following expression for the normal bundle of $LG(V)$ in $\mathcal{J}^p(V)$:*

$$N_{LG(V)}\mathcal{J}^p(V) \cong \bigoplus_{i=3}^{p+1} S^i(R^*)\,.$$

In particular, $N_{LG(V)}\mathcal{J}^p(V)$ is generated by its global sections.

Proposition 28. *Let Z be a subvariety of $\mathcal{J}^p(V)$. If, using (16), we have*

$$[Z] = \sum a_\mu \widetilde{Q}_\mu(R^*)\,,$$

where $a_\mu \in \mathbf{Z}$, then all the coefficients a_μ are nonnegative.

Proof. Set $G = LG(V)$, $\mathcal{J} = \mathcal{J}^p(V)$ and $N = N_G\mathcal{J}$. Denote by $i : G \hookrightarrow \mathcal{J}$ the inclusion. We look at the coefficients a_μ in the expression

$$i^*[Z] = \sum a_\mu \; \widetilde{Q}_\mu(R^*) = \sum a_\mu Y^\mu\,,$$

where the last equality follows from Theorem 4. Let $Y^{\mu'}$ be the dual class to Y^μ (see Example 3). We have

$$a_\mu = i^*[Z] \cdot Y^{\mu'}\,.$$

Invoking (3), we may compute this last intersection number using the Chow groups of G. Let $C = C_{G\cap Z}Z \subset N$ be the *normal cone* of $G \cap Z$ in Z. Denote by $j : G \hookrightarrow N$ the zero-section inclusion. By deformation to the normal cone (see [16, Sect. 6.1 and 6.2]), we have

$$i^*[Z] = j^*[C] \qquad \text{(equality in } A^*(G))\,.$$

It follows that

$$a_\mu = [C] \cdot Y^{\mu'} \qquad \text{(intersection in } N)\,.$$

By virtue of Lemma 27, the assertion now follows from Proposition 13 for $X = G$, $E = N$, and $Y = Y^{\mu'}$. □

Theorem 29. *[37] For any Lagrange singularity class Σ, the Thom polynomial $\mathcal{T}^\Sigma$ is a nonnegative combination of $\widetilde{Q}$-functions.*

Question. Does there exists a basis different (up to rescaling) from $\{Y^\mu = \widetilde{Q}_\mu(R^*)\}$ with the property that any Lagrangian Thom polynomial has a positive expansion in that basis?

8 Legendrian Thom polynomials

Fix $n \in \mathbf{N}$. Let W be a complex vector space of dimension n, and let L be a one dimensional complex vector space. Consider

$$V := W \oplus (W^* \otimes L) \tag{17}$$

– a symplectic space equipped with the *twisted symplectic form* $\omega \in \Lambda^2 V^* \otimes L$.

Consider a *contact space*

$$V \oplus L = W \oplus (W^* \otimes L) \oplus L\,.$$

Let α be a contact form on $V \oplus L$ (cf. [1, Sect. 20.1]). Legendrian submanifolds of $V \oplus L$ are maximal integral submanifolds of the form α, i.e., the manifolds of dimension n with tangent spaces contained in $\mathrm{Ker}(\alpha)$.

To study Legendrian submanifolds (through 0) of $V \oplus L$, we use Lagrangian submanifolds (through 0) of V. Any Legendrian submanifold in $V \oplus L$ is determined by its Lagrangian projection to V and any Lagrangian submanifold in V lifts to $V \oplus L$.

Legendrian Thom polynomials were considered by Vassiliev [53] (see also [29]). In [38], the space $\mathcal{J}^p(W, L)$ was constructed (with the help of Kazarian) that can serve to address positivity questions about Legendrian Thom polynomials. This space is not a naive generalization of the space of Lagrangian p-jets from the previous section. Roughly speaking, one wants to parametrize the relative positions of two Lagrangian submanifolds. More precisely, we define $\mathcal{J}^p(W, L)$ to be the set of pairs of p-jets of Lagrangian submanifolds of V consisting of a linear space and a submanifold whose tangent space at 0 is W. For a motivation of this construction and more details, we refer the reader to [38, Sect. 2 and 3]. The projection to the first factor gives a map

$$\pi : \mathcal{J}^p(W, L) \to LG(V)\,,$$

which is a trivial vector bundle with the fiber equal to

$$\bigoplus_{i=3}^{p+1} S^i(W^*) \otimes L\,.$$

In fact, we need a relative version of this construction. Let X be a topological space, W a complex rank n vector bundle over X, and L a complex line bundle over X. Define a vector bundle V on X by (17). Let

$$\tau : LG(V) \to X$$

denote the induced Lagrange Grassmann *bundle*. We have a *relative* version of the map π

$$\pi : \mathcal{J}^p(W, L) \to LG(V)\,,$$

which is denoted by the same letter.

The space $\mathcal{J}^p(W, L)$ fibers over X. It is equal to the pull-back

$$\mathcal{J}^p(W, L) = \tau^* \left(\bigoplus_{i=3}^{p+1} S^i(W^*) \otimes L \right)\,.$$

By a *Legendre singularity class* we mean a closed algebraic subset $\Sigma \subset \mathcal{J}^p(\mathbf{C}^n, \mathbf{C})$ invariant with respect to holomorphic *contactomorphisms* of $\mathbf{C}^{2n+1}$. Additionally, we assume that Σ is stable with respect to enlarging the dimension of W. Since any changes of coordinates of W and L induce holomorphic contactomorphisms of $V \oplus L$, any Legendre singularity class Σ defines

$$\Sigma(W, L) \subset \mathcal{J}^p(W, L)\,.$$

The element $[\Sigma(W,L)]$ of $H^*(\mathcal{J}^p(W,L),\mathbf{Z})$ is called the *Legendrian Thom polynomial of* Σ.

In the following, we shall write $\mathcal{J}$ for the vector bundle $\mathcal{J}^p(W,L)$.

We now use Schubert calculus to study Legendrian Thom polynomials. Let $L, M_1, M_2, \ldots, M_n$ be one dimensional vector spaces, and let

$$W := \bigoplus_{i=1}^{n} M_i\,, \qquad V = W \oplus (W^* \otimes L)\,.$$

We have a symplectic form ω defined on V with values in L. The Lagrangian Grassmannian $LG(V)$ is a homogeneous space for the symplectic group $Sp(V) \subset \mathrm{End}(V)$. We fix two "opposite" isotropic flags E^+ and E^- in V:

$$E_j^+ := \bigoplus_{i=1}^{j} M_i\,, \qquad E_j^- := \bigoplus_{i=1}^{j} M_{n-i+1}^* \otimes L\,, \qquad (j = 1, 2, \ldots, n)\,.$$

Consider two Borel groups $B^\pm \subset Sp(V)$, preserving the flags $E^\pm$. The orbits of $B^\pm$ in $LG(V)$ form two cell decompositions $\{C^\mu(E^\pm, L)\}$ of the space $LG(V)$, labeled by strict partitions $\mu \subset \rho(n)$ (see [42] and Sect.3). The cells of the C^--decomposition are transverse to the cells of C^+-decomposition. Denote the class of the closure of $C^\mu(E^\pm, L)$ in $H^*(LG(V),\mathbf{Z})$ by $Y^\mu(E^\pm, L)$.

All these data behave functorially with respect to the automorphisms of the lines L and M_i's (they form a torus $(\mathbf{C}^*)^{n+1}$). Thus the construction of the cell decompositions can be repeated for bundles L and $\{M_i\}_{i=1}^n$ over any base X. We get a Lagrange Grassmann bundle

$$\tau : LG(V) \to X\,,$$

endowed with two (relative) stratifications

$$\{C^\mu(E^\pm, L) \to X\}_\mu\,.$$

Suppose that $X = G/P$ is a compact manifold, homogeneous with respect to an action of a linear group G. Then X admits a Chevalley-Bruhat cell decomposition $\{\sigma_\lambda\}$. The subsets

$$Z_{\mu\lambda}^- := \tau^{-1}(\sigma_\lambda) \cap C^\mu(E^-, L)$$

form an algebraic cell decomposition of $LG(V)$. Another cell decomposition of $LG(V)$ is given by the collection of subsets

$$Z_{\mu\lambda}^+ := \tau^{-1}(\sigma_\lambda) \cap C^\mu(E^+, L)\,.$$

Example 30. If $X = \mathbf{P}^1$, $W = \mathbf{1}$, $L = \mathcal{O}(d)$ (for $d > 0$), then $LG(V)$ is the Hirzebruch surface H_d which can be presented as the sum of the space of the line bundle L and the section at infinity, $H_d = L \cup \mathbf{P}^1_\infty$. Then $\mathbf{P}^1_0$, the zero section of the bundle L, is a stratum of the C^+-decomposition and the section at infinity $\mathbf{P}^1_\infty$ is a stratum of the C^--decomposition. For the cell decomposition of $X = \mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$, we obtain two cell decompositions of H_d. Two resulting bases of cohomology are mutually dual with respect to the intersection product. The closures of strata of Z^+-decomposition have the following property: any effective cycle has a nonnegative intersection number with them. This is not true for the closures of strata of Z^--decomposition: for example, the self-intersection of $\mathbf{P}^1_\infty$ is equal to $-d$.

Theorem 31. *[38] Fix a strict partition $\mu \subset \rho(n)$ and an index λ. Suppose that the vector bundle $\mathcal{J}$ is generated by its global sections. Then, in $\mathcal{J}$, the intersection of $\Sigma(W,L)$ with the closure of any $\pi^{-1}(Z^-_{\mu\lambda})$ is represented by a nonnegative cycle.*

The proof in [38] is based (apart from the Schubert calculus for $LG(V) \to X$) on some variant of the Bertini-Kleiman theorem.

We apply the theorem in the situation when all M_i are equal to the same line bundle M (and then $W = M^{\oplus n}$) and $M^{-m} \otimes L$ is generated by its global sections for $m \geq 3$.

Consider the following three cases: the base is always $X = \mathbf{P}^n$ and

- $L_1 = \mathcal{O}(-2), \ \ M_1 = \mathcal{O}(-1)$, or
- $L_2 = \mathcal{O}(1), \ \ M_2 = \mathbf{1}$, or
- $L_3 = \mathcal{O}(-3), \ \ M_3 = \mathcal{O}(-1)$.

We obtain the symplectic bundles $V_i = M_i^{\oplus n} \oplus (M_i^* \otimes L_i)^{\oplus n}$ with twisted symplectic forms ω_i for $i = 1, 2, 3$.

These three cases were crucial to discover and prove the forthcoming Theorem 32. Case 1 was the subject of [37, Rem. 14]. In Case 2, the integral cohomology $H^*(LG(V), \mathbf{Z})$ is isomorphic to the ring of Legendrian characteristic classes up to degree n; the Z^--decomposition of $LG(V)$ gives us another basis of cohomology. In Case 3, the cohomology of $LG(V)$ is isomorphic, up to degree n, to the ring of Legendrian characteristic classes, provided we invert the number 3 this time. The positivity property in Case 1 was known (*loc.cit.*), whereas in Cases 2 and 3, it was Kazarian who suggested the positivity.

To overlap all these three cases we consider $X := \mathbf{P}^n \times \mathbf{P}^n$ and set

$$W := p_1^* \mathcal{O}(-1)^{\oplus n}\,, \qquad L := p_1^* \mathcal{O}(-3) \otimes p_2^* \mathcal{O}(1)\,,$$

where $p_i : X \to \mathbf{P}^n$, $i = 1, 2$, are the projections. Restricting the bundles W and L to the diagonal or to the factors, we obtain the three cases considered above.

The space $LG(V)$ has a cell decomposition $Z^+_{\mu\lambda}$, where μ runs over strict partitions contained in $\rho(n)$, and $\lambda = (a, b)$ with a and b natural numbers smaller than or equal to n. The classes of closures of the cells of this decomposition give a basis of the cohomology of $LG(V)$.

Let v_1 and v_2 be the first Chern classes of $p_1^*(\mathcal{O}(1))$ and $p_2^*(\mathcal{O}(1))$. We have

$$[\overline{Z^+_{\mu,a,b}}] = Y^\mu(E^+, L)\ v_1^a v_2^b\,. \tag{18}$$

Theorem 32. *[38] Let Σ be a Legendre singularity class. Then $[\Sigma(W,L)]$ has nonnegative coefficients in the basis $\{[\overline{Z^+_{\mu,a,b}}]\}$.*

Example 33. Using the names of singularities from [29], we display some Legendrian Thom polynomials in the basis from the theorem. The bold terms give the Thom polynomials of the corresponding Lagrange singularities.

A_2: $\widetilde{\mathbf{Q}}_{\mathbf{1}}$
A_3: $\mathbf{3}\widetilde{\mathbf{Q}}_{\mathbf{2}} + v_2 \widetilde{Q}_1$

A_4: $\mathbf{12\widetilde{Q}_3 + 3\widetilde{Q}_{21}} + (3v_1 + 7v_2)\widetilde{Q}_2 + (v_1v_2 + v_2^2)\widetilde{Q}_1$
D_4: $\mathbf{\widetilde{Q}_{21}}$
P_8: $\mathbf{\widetilde{Q}_{3\underset{\sim}{2}1}}$.
A_5: $\mathbf{60\widetilde{Q}_4 + 27\widetilde{Q}_{31}} + (6v_1 + 16v_2)\widetilde{Q}_{21} + (39v_1 + 47v_2)\widetilde{Q}_3 +$
$(6v_1^2 + 22v_1v_2 + 12v_2^2)\widetilde{Q}_2 + (2v_1^2v_2 + 3v_1v_2^2 + v_2^3)\widetilde{Q}_1$
D_5: $\mathbf{6\widetilde{Q}_{\underset{\sim}{3}1}} + 4v_2\widetilde{Q}_{2\underset{\sim}{1}}$,
P_9: $\mathbf{12\widetilde{Q}_{4\underset{\sim}{2}1}} + 12v_2\widetilde{Q}_{3\underset{\sim}{2}1}$.

Using the theorem, one constructs, in the ring of Legendrian characteristic classes, a one-parameter family of bases such that any Legendrian Thom polynomial has, in any basis from the family, an expansion with nonnegative coefficients (see [38]).

Remark 34. Positive descriptions of the coefficients of Schur function expansions of Thom polynomials are known for several series of singularity classes of maps, see a survey article [40]. For those coefficients, which do not admit such descriptions, it is interesting to establish their bounds. For the Legendrian Thom polynomials, this issue is discussed in [38, Sect. 9 and 10]. In particular, one examines there how positivity of Thom polynomials of maps to curves implies some upper bounds on the coefficients of Legendrian Thom polynomials.

Remark 35. It is shown in [38, Sect. 10] that the Thom polynomials of nonempty stable Lagrangian and Legendrian singularity classes are nonzero.

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