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A GENERALIZATION OF THE MACDONALD–YOU FORMULA

Piotr Pragacz

Mathematical Institute of Polish Academy of Sciences,
Chopina 12,
PL 87-100 Toruń, Poland

To the memory of M.-P. Schützenberger

The goal of the present paper is to state and prove a new identity for Schur Q-functions.

We recall Schur's original definition of these functions, which appeared in [S]. Let $x = (x_1, x_2, \dots)$ be an infinite sequence of independent indeterminates. The following identity between formal power series in t

$$\prod_{i\geq 1} \frac{1 + x_i t}{1 - x_i t} = \sum_a Q_a(x) t^a$$

defines symmetric functions $Q_a = Q_a(x)$ in $x_1, x_2, \dots$ for all nonnegative integers a (with $Q_0 = 1$). Directly from the definition it follows that

$$(1) \qquad \sum_{a+b=c} (-1)^a Q_a \cdot Q_b = 0$$

for all $c \geq 1$.

For integers a_1, $a_2 \geq 0$, we define

$$(2) \qquad Q_{a_1 a_2} = Q_{a_1} \cdot Q_{a_2} + 2 \sum_{i=1}^{a_2} (-1)^i Q_{a_1+i} \cdot Q_{a_2-i}$$

By (1) we have $Q_{a_1 a_2} = -Q_{a_2 a_1}$; in particular $Q_{a_1 a_2} = 0$ when $a_1 = a_2$. Notice that $Q_{a\,0} = Q_a = -Q_{0\,a}$. Also, in general, it is sufficient to index Schur Q-functions by strict partitions, though for formal reasons it is convenient to extend the range of indices to the sequences of positive integers, or even to the sequences of nonnegative integers having at most one zero placed at the end.

Let $(a_1,\dots,a_m)\in(\mathbb{N}^*)^m$. We define, recurrently on m, the Schur Q-function $Q_{a_1\dots a_m}=Q_{(a_1,\dots,a_m)}$ as follows. If m is odd, then

(3) $$Q_{a_1\dots a_m}=Q_{a_1}\cdot Q_{a_2\dots a_m}-Q_{a_2}\cdot Q_{a_1a_3\dots a_m}+\dots+Q_{a_m}\cdot Q_{a_1\dots a_{m-1}}\,.$$

If m is even, then

(4) $$Q_{a_1\dots a_m}=Q_{a_1a_2}\cdot Q_{a_3\dots a_m}-Q_{a_1a_3}\cdot Q_{a_2a_4\dots a_m}+\dots+Q_{a_1a_m}\cdot Q_{a_2\dots a_{m-1}}\,.$$

Of course, this definition can be restated, more compactly, using appropriate Pfaffians. Namely, we have

$$Q_{a_1\dots a_m}=\operatorname{Pf}\big(Q_{a_ia_j}\big)_{1\le i,j\le m}$$

for m even, and

$$Q_{a_1\dots a_m}=\operatorname{Pf}\big(Q_{b_ib_j}\big)_{1\le i,j\le m+1},$$

for m odd, where $(b_1,\dots,b_m,b_{m+1})=(a_1,\dots,a_m,0)$. We then record the following property. For a permutation $\sigma\in S_m$, we have

(5) $$Q_{a_{\sigma(1)}\dots a_{\sigma(m)}}=\operatorname{sgn}(\sigma)Q_{a_1\dots a_m}\,.$$

Notice that Equation (3) also holds for m even, and Equation (4) also holds for m odd. This can be restated as follows. Letting, for $(a_1,\dots,a_m)\in(\mathbb{N}^*)^m$, $Q_{a_1\dots a_m0}=Q_{a_1\dots a_m}$, Equations (3) and (4) hold true if we formally set $a_m=0$.

The above algebraic properties will be sufficient for the purposes of the present paper. For more about Schur Q-functions and their algebraic applications, the reader is referred to [H-H] and especially to Schur's original paper [S]. For recent geometric applications of these functions to enumerative geometry of degeneracy loci and cohomology rings of isotropic Grassmannians, see [P1] and [P2].

We define Γ to be the polynomial algebra in $\{Q_a : a\ge 1\}$ over $\mathbb{Z}$ and call it the ring of Schur Q-functions. We consider Γ as a graded ring where $\deg Q_a=a$. We denote by Λ the ring of symmetric functions; recall that $\Lambda=\mathbb{Z}[e_1,e_2,\dots]$ where e_i is the i-th elementary symmetric function.

Let $\eta:\Lambda\to\Gamma$ be the ring homomorphism from the ring of symmetric functions to the ring of Schur Q-functions, defined by $\eta(e_i)=Q_i$. The goal of the present paper is to give an explicit expression for the image under η of an arbitrary S-function, $\eta(s_\lambda)$, as a quadratic polynomial in Schur Q-functions. This will be a consequence of a more general identity given in Theorem 4 which is the principal result of the present paper. The main results are illustrated by examples and accompanied by a geometric application (Proposition 8) which was, in fact, the main author's motivation of the present research.

To state our results we need some notation. For a sequence $A=(a_1,\dots,a_n)$ and a subsequence B of A, i.e., $B=(a_{i_1},\dots,a_{i_k})$ with $1\le i_1<\dots<i_k\le n$, we denote by $A\smallsetminus B$ the subsequence of A obtained by removing $\{a_{i_1},\dots,a_{i_k}\}$ from A and leaving the remaining a's in the original ordering. Given additionally a sequence $B=(b_1,\dots,b_n)$, we denote by $A\#B$ the sequence $(a_1,b_1,a_2,b_2,\dots,a_n,b_n)$. We will identify a subsequence $(a_{i_1},\dots,a_{i_k})$ of A with the subsequence of $A\#B$, which occupies the places numbered by $2i_1-1,\dots,2i_k-1$. Similarly, we will identify a subsequence $(b_{i_1},\dots,b_{i_k})$ of B with the subsequence of $A\#B$, which occupies the places numbered by $2i_1,\dots,2i_k$.

Theorem 1. *Suppose that* $\lambda = (\alpha_1, \dots, \alpha_n \mid \beta_1, \dots, \beta_n)$ *is a partition written in Frobenius notation. Then, for* $A = (a_1, \dots, a_n) := (\alpha_1 + 1, \dots, \alpha_n + 1)$ *and* $B := (\beta_1, \dots, \beta_n)$,

$$\mathrm{Det}\big(Q_{a_i\beta_j} + Q_{a_i} \cdot Q_{\beta_j}\big)_{1\le i,j\le n} = \sum Q_{(a_{i_1},\dots,a_{i_k})} \cdot Q_{A\#B\smallsetminus(a_{i_1},\dots,a_{i_k})},$$

where the sum is over all sequences $1 \le i_1 < \dots < i_k \le n$ *and* $k = 0, 1, \dots, n$.

Corollary 2. *In the notation of Theorem 1,*

$$\eta(s_\lambda) = \frac{1}{2^n} \sum Q_{(a_{i_1},\dots,a_{i_k})} \cdot Q_{A\#B\smallsetminus(a_{i_1},\dots,a_{i_k})},$$

the sum as in the theorem.

Indeed, using Giambelli's hook-formula for s_λ (see [G]), the equality

$$\eta\left(s_{(\alpha|\beta)}\right) = \frac{1}{2}\left(Q_{(\alpha+1,\beta)} + Q_{\alpha+1} \cdot Q_\beta\right)$$

(see, e.g., [J-P]), and Theorem 1, we get

$$\eta(s_\lambda) = \mathrm{Det}\Big(\frac{1}{2}(Q_{a_i\beta_j} + Q_{a_i} \cdot Q_{\beta_j})\Big)_{1\le i,j\le n}$$
$$= \frac{1}{2^n} \sum Q_{(a_{i_1},\dots,a_{i_k})} \cdot Q_{A\#B\smallsetminus(a_{i_1},\dots,a_{i_k})},$$

the sum as asserted.

Note that for $A = B$, Theorem 1 says

$$\mathrm{Det}\big(Q_{\beta_i\beta_j} + Q_{\beta_i} \cdot Q_{\beta_j}\big)_{1\le i,j\le n} = Q^2_{\beta_1\dots\beta_n}$$

and was proved originally by Macdonald, answering a conjecture of De Concini (see [J-P, Remark 2]), and by You [Y].

Example 3. a) With $A = (2,1)$ and $B = (3,1)$, we have

$$\eta(s_{2221}) = \frac{1}{2^2}\big(Q_{21} \cdot Q_{31} - Q_1 \cdot Q_{321}\big).$$

(Here, $n = 2$, $A\#B = (2,3,1,1)$.)

b) Note that $\eta(s_\lambda)$ is not always a positive combination of Q-functions. For instance, with $A = (5,4,3)$, $B = (6,2,1)$ and $A\#B = (5,6,4,2,3,1)$, the decomposition of $2^3\eta(s_{5553111})$ is equal to

$$Q_{564231} + Q_5 \cdot Q_{64231} + Q_4 \cdot Q_{56231} + Q_3 \cdot Q_{56421}$$
$$+ Q_{54} \cdot Q_{6231} + Q_{53} \cdot Q_{6421} + Q_{43} \cdot Q_{5621} + Q_{543} \cdot Q_{621}$$

$$= Q_{654321} - Q_5 \cdot Q_{64321} + Q_4 \cdot Q_{65321} - Q_3 \cdot Q_{65421} \\ - Q_{54} \cdot Q_{6321} + Q_{53} \cdot Q_{6421} - Q_{43} \cdot Q_{6521} + Q_{543} \cdot Q_{621}$$

and its decomposition into Q-functions contains the following negative combination of Q-functions:

$$-8Q_{9543} - 10Q_{95421} - 108Q_{85431} - 60Q_{75432}.$$

To show Theorem 1, we prove the following more general result. Let us fix a commutative ring with unity. Suppose that for every sequence $(a_1, \dots, a_m) \in (\mathbb{N}^*)^m$, where $m \in \mathbb{N}$, there is an associated element $[a_1 \dots a_m] = [(a_1, \dots, a_m)]$ of the ring. The empty bracket $[\,]$ is the unity of the ring. Moreover, assume that the family $\{[a_1 \dots a_m]\}$ satisfies the following three conditions:

For a permutation $\sigma \in S_m$,

$$[a_{\sigma(1)} \dots a_{\sigma(m)}] = \operatorname{sgn}(\sigma)[a_1 \dots a_m]. \tag{6}$$

If m is odd, then

$$[a_1 \dots a_m] = [a_1][a_2 \dots a_m] - [a_2][a_1 a_3 \dots a_m] + \dots + [a_m][a_1 \dots a_{m-1}]. \tag{7}$$

If m is even, then

$$[a_1 \dots a_m] = [a_1 a_2][a_3 \dots a_m] - [a_1 a_3][a_2 a_4 \dots a_m] + \dots + [a_1 a_m][a_2 \dots a_{m-1}]. \tag{8}$$

For instance, $[a_1 \dots a_m] = Q_{a_1 \dots a_m}$ satisfy (6)–(8) in the ring Γ of Schur Q-functions.

Theorem 4. *For $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n) \in (\mathbb{N}^*)^n$ one has*

$$\operatorname{Det}\big([a_i b_j] + [a_i]\,[b_j]\big)_{1 \le i,j \le n} = \sum [a_{i_1} \dots a_{i_k}]\big[A\#B \smallsetminus (a_{i_1}, \dots, a_{i_k})\big],$$

where the sum is over all sequences $1 \le i_1 < \dots < i_k \le n$ and $k = 0, 1, \dots, n$.

In the proof of the theorem, we will need some additional notation. Given a subset C of the set of the elements of a certain fixed sequence, we denote by (C) the subsequence of this sequence formed by the elements of C.

Moreover, in the proof of the theorem we will use the following

Lemma 5. *For any pair of different numbers $i, j = 1, \dots, n$ and a sequence $1 \le i_1 < \dots < i_k \le n$ where all i_p are different from j, the element*

$$(-1)^{i+j}[(A \smallsetminus a_j)\#(B \smallsetminus b_i) \smallsetminus (a_{i_1}, \dots, a_{i_k})]$$

is equal to

$$(-1)^c[A\#B \smallsetminus (\{a_j, b_i, a_{i_1}, \dots, a_{i_k}\})],$$

where $c = 1 + \operatorname{card}\{\ p\ :\ i < i_p < j\ \}$ for $i < j$ and $c = \operatorname{card}\{\ p\ :\ j < i_p \le i\ \}$ for $i > j$.

Proof. Assume first $k = 0$.

If $i < j$, we pass from $A\#B \smallsetminus (b_i, a_j)$, i.e.

$$(a_1, b_1, \dots, a_i, a_{i+1}, b_{i+1}, a_{i+2}, b_{i+2}, \dots, a_{j-1}, b_{j-1}, b_j, a_{j+1}, b_{j+1}, \dots, a_n, b_n)$$

to

$$(a_1, b_1, \dots, a_i, b_{i+1}, a_{i+1}, b_{i+2}, \dots, a_{j-2}, b_{j-1}, a_{j-1}, b_j, a_{j+1}, b_{j+1}, \dots, a_n, b_n)$$

i.e. $(A \smallsetminus a_j)\#(B \smallsetminus b_i)$ by transposing a_{i+1} and b_{i+1}, a_{i+2} and b_{i+2}, ... , a_{j-1} and b_{j-1}. The number of these transpositions is $j - i - 1$, and thus $c = 1$ in this case.

If $i > j$, we pass from $A\#B \smallsetminus (a_j, b_i)$, i.e.

$$(a_1, b_1, \dots, b_{j-1}, b_j, a_{j+1}, b_{j+1}, a_{j+2}, \dots, b_{i-2}, a_{i-1}, b_{i-1}, a_i, a_{i+1}, \dots, a_n, b_n)$$

to

$$(a_1, b_1, \dots, b_{j-1}, a_{j+1}, b_j, a_{j+2}, b_{j+1}, \dots, a_{i-1}, b_{i-2}, a_i, b_{i-1}, a_{i+1}, \dots, a_n, b_n)$$

i.e. $(A \smallsetminus a_j)\#(B \smallsetminus b_i)$ by transposing b_j and a_{j+1}, b_{j+1} and a_{j+2}, ... , b_{i-1} and a_i. The number of these transpositions is $i - j$ and thus $c = 0$ in this case.

If $k > 0$, we do not perform the transpositions involving a_{i_p} $(i < i_p < j)$ if $i < j$, and a_{i_p} $(j < i_p \le i)$ if $i > j$, and the assertion follows. □

Proof of Theorem 4. Define two $n \times n$ matrices

$$X = \big([a_i b_j]\big)_{1 \le i,j \le n} \quad \text{and} \quad Y = \big([a_i]\,[b_j]\big)_{1 \le i,j \le n},$$

so that we want to compute $\operatorname{Det}(X + Y)$. We claim that

$$\operatorname{Det}(X + Y) = \operatorname{Det}(X) + \operatorname{Tr}\big(\wedge^{n-1}(X) \cdot Y\big). \tag{9}$$

Indeed, this follows from a familiar identity

$$\wedge^n(X + Y) = \sum_{i=0}^{n} \operatorname{Tr}\big(\wedge^{n-i}(X) \cdot \wedge^i(Y)\big)$$

(see, e.g., [J-P]) by observing that the rank of Y is equal to 1.

We now claim that

(10)
$$\mathrm{Det}(X) = \sum [a_{i_1} \dots a_{i_k}] \big[A\#B \setminus (a_{i_1}, \dots, a_{i_k})\big] ,$$

where the sum is over all sequences $1 \le i_1 < \dots < i_k \le n$ with k even ($k = 0$ is included). To prove this, we use induction on n. For $n = 1$ the assertion is obviously true. Using the Laplace expansion along the first row of X and the induction assumption, $\mathrm{Det}(X)$ is equal to

(11)
$$\sum_{i=1}^{n} (-1)^{i+1} [a_1 b_i] \Big(\sum [a_{i_1} \dots a_{i_k}] \big[(A \setminus a_1)\#(B \setminus b_i) \setminus (a_{i_1}, \dots, a_{i_k})\big] \Big) ,$$

where the latter sum is over all sequences $2 \le i_1 < \dots < i_k \le n$ and $k = 0, 2, 4, \dots$.

Let us fix an arbitrary even $k = 0, 2, \dots$. By Lemma 5 the element

$$(-1)^{i+1} [(A \setminus a_1)\#(B \setminus b_i) \setminus (a_{i_1}, \dots, a_{i_k})]$$

is equal to

$$(-1)^{\mathrm{card}\{h : i_h < i\}} [A\#B \setminus (\{a_1, a_{i_1}, \dots, a_{i_k}, b_i\})].$$

Therefore, using (8), the contribution in (11) of the summands corresponding to k is equal to

(12)
$$\sum_{2 \le i_1 < \dots < i_k \le n} [a_{i_1} \dots a_{i_k}] [A\#B \setminus (a_{i_1}, \dots, a_{i_k})]$$
$$+ \sum_{2 \le i_1 < \dots < i_k \le n} \; \sum_{p \ne 1, i_1, \dots, i_k} \pm [a_1 a_p] [a_{i_1} \dots a_{i_k}] [A\#B \setminus (\{a_1, a_{i_1}, \dots, a_{i_k}, a_p\})] ,$$

where the sign "$\pm$" equals "$+$" (resp. "$-$") if a_p occupies an odd (resp. even) place in the sequence $A\#B \setminus (a_{i_1}, \dots, a_{i_k})$. For example, such a contribution for $k = 0$ equals

$$[A\#B] + \sum_{j \ge 2} [a_1 a_j] [A\#B \setminus (a_1, a_j)] .$$

Let us now fix a positive even k together with a sequence of integers $2 \le j_1 < \dots < j_{k+1} \le n$, and look at all the summands appearing in the second line of (12), for which

$$\{j_1, \dots, j_{k+1}\} = \{p, i_1, \dots, i_k\}.$$

Observe that the summand

$$\pm [a_1 a_{j_q}] [a_{j_1} \dots a_{j_{q-1}} a_{j_{q+1}} \dots a_{j_{k+1}}]$$

appears with sign "$+$" iff q is odd. Hence using (8):

$$[a_1 a_{j_1} \dots a_{j_{k+1}}] =$$
$$[a_1 a_{j_1}][a_{j_2} \dots a_{j_{k+1}}] - [a_1 a_{j_2}][a_{j_1} a_{j_3} \dots a_{j_{k+1}}] + \dots + [a_1 a_{j_{k+1}}][a_{j_1} \dots a_{j_k}],$$

the expression (12) is rewritten in the form

$$\sum_{2 \le i_1 < \dots < i_k \le n} [a_{i_1} \dots a_{i_k}][A \# B \setminus (a_{i_1}, \dots, a_{i_k})]$$
$$+ \sum_{2 \le j_1 < \dots < j_{k+1} \le n} [a_1 a_{j_1} \dots a_{j_{k+1}}][A \# B \setminus (a_1, a_{j_1}, \dots, a_{j_{k+1}})] .$$

Summing all these contributions for $k = 0, 2, 4, \dots$, Equation (10) follows. See also Example 6.

Next, we claim that

$$\text{(13)} \qquad \operatorname{Tr}\bigl(\wedge^{n-1}(X) \cdot Y\bigr) = \sum [a_{i_1} \dots a_{i_k}][A \# B \setminus (a_{i_1}, \dots, a_{i_k})] ,$$

where the sum is over all sequences $1 \le i_1 < \dots < i_k \le n$ with k odd. The (i, j)-th entry of $\wedge^{n-1}(X)$ computed with the help of (10) is

$$\text{(14)} \qquad (-1)^{i+j} \sum [a_{i_1} \dots a_{i_k}]\bigl[(A \setminus a_j) \# (B \setminus b_i) \setminus (a_{i_1}, \dots, a_{i_k})\bigr],$$

where the sum is over all sequences $1 \le i_1 < \dots < i_k \le n$ such that each i_p is different from j, and $k = 0, 2, 4, \dots$ is even. Then

(15)
$$\operatorname{Tr}\bigl(\wedge^{n-1}(X) \cdot Y\bigr)$$
$$= \sum_{i,j=1}^{n} (-1)^{i+j} [a_j][b_i] \left(\sum [a_{i_1} \dots a_{i_k}]\bigl[(A \setminus a_j) \# (B \setminus b_i) \setminus (a_{i_1}, \dots, a_{i_k})\bigr] \right) ,$$

where the latter sum is as in (14). Now let us fix $j = 1, \dots, n$ and then also fix an arbitrary even k together with a sequence $1 \le i_1 < \dots < i_k \le n$ such that each i_p is different from j. Consider only the summands corresponding to these fixed j and $(i_1, \dots, i_k)$:

$$\text{(16)} \qquad [a_j][a_{i_1} \dots a_{i_k}] \sum_{i=1}^{n} (-1)^{i+j} [b_i]\bigl[(A \setminus a_j) \# (B \setminus b_i) \setminus (a_{i_1}, \dots, a_{i_k})\bigr].$$

Using Lemma 5 and (7), the expression (16) is rewritten as

(17)
$$(-1)^{\operatorname{card}\{h : i_h < j\}} [a_j][a_{i_1} \dots a_{i_k}]\bigl[A \# B \setminus \bigl(\{a_j, a_{i_1}, \dots, a_{i_k}\}\bigr)\bigr]$$
$$+ \sum \pm [a_p][a_j][a_{i_1} \dots a_{i_k}]\bigl[A \# B \setminus \bigl(\{a_p, a_j, a_{i_1}, \dots, a_{i_k}\}\bigr)\bigr] ,$$

where the last sum is over $a_p \in A \setminus \{a_j, a_{i_1}, \dots, a_{i_k}\}$. Here, the sign "$\pm$" is $(-1)^{\operatorname{card}\{h : i_h < j\}}$ (resp. $-(-1)^{\operatorname{card}\{h : i_h < j\}}$) if a_p occupies an even (resp. odd) place in the sequence $A\#B\setminus(\{a_j, a_{i_1}, \dots, a_{i_k}\})$. Suppose that a_j (resp. a_p) occupies the s-th (resp. t-th) place in the sequence $A\#B\setminus(\{a_p, a_{i_1}, \dots, a_{i_k}\})$ (resp. $A\#B\setminus(\{a_j, a_{i_1}, \dots, a_{i_k}\})$). Then s and t are of different parity if the cardinality of the i_h's between p and j is even, and s and t are of the same parity if this cardinality is odd. This altogether implies that the sign "$\pm$" in the expressions (17) with fixed $(i_1, \dots, i_k)$, is an antisymmetric function of the pair (a_p, a_j).

Consequently, for fixed $(i_1, \dots, i_k)$, the sum over j (different from each i_p) of the expressions (17) becomes:

$$\sum (-1)^{\mathrm{card}\{h : i_h < j\}} [a_j][a_{i_1} \dots a_{i_k}]\big[A\#B\setminus(\{a_j, a_{i_1}, \dots, a_{i_k}\})\big], \tag{18}$$

the sum over j different from each i_p.

Now, keeping k fixed, let us make additionally $(i_1, \dots, i_k)$ vary. Fix a sequence $1 \le j_1 < \dots < j_{k+1} \le n$, and look at the summands appearing in the expressions (18), for which

$$\{j, i_1, \dots, i_k\} = \{j_1, \dots, j_{k+1}\}.$$

Observe that the summand

$$\pm[a_{j_q}][a_{j_1} \dots a_{j_{q-1}} a_{j_{q+1}} \dots a_{j_{k+1}}]$$

appears with sign "+" iff q is odd. Hence, using (7):

$$[a_{j_1} \dots a_{j_{k+1}}] =$$

$$[a_{j_1}][a_{j_2} \dots a_{j_{k+1}}] - [a_{j_2}][a_{j_1} a_{j_3} \dots a_{j_{k+1}}] + \dots + [a_{j_{k+1}}][a_{j_1} \dots a_{j_k}],$$

the sum of expressions (18) with k fixed, is equal to

$$\sum_{1 \le j_1 < \dots < j_{k+1} \le n} [a_{j_1} \dots a_{j_{k+1}}]\big[A\#B\setminus(a_{j_1}, \dots, a_{j_{k+1}})\big].$$

This is the contribution to $\mathrm{Tr}\big(\wedge^{n-1}(X) \cdot Y\big)$ of the summands in (15), associated with k.

Summing all these contributions for $k = 0, 2, 4, \dots$, we get

$$\sum_{1 \le i_1 < \dots < i_l \le n} [a_{i_1} \dots a_{i_l}]\big[A\#B\setminus(a_{i_1}, \dots, a_{i_l})\big],$$

the sum over all sequences $1 \le i_1 < \dots < i_l \le n$ where $l = 1, 3, \dots$ runs over odd positive integers, and Equation (13) follows. See also Example 7.

Equations (9), (10) and (13) imply the assertion of Theorem 4. $\square$

Of course, Theorem 4 implies Theorem 1 if $\beta_n \ne 0$. It implies Theorem 1 also if $\beta_n = 0$ because Equations (7) and (8) with $[a_1 \dots a_m] = Q_{a_1 \dots a_m}$ (i.e. Equations (3) and (4)) hold true if we formally put $a_m = 0$. Notice that we only use Equations (3) and (4) and we do not need Equation (2).

For the reader's convenience we provide the following two examples.

Example 6. We illustrate the proof of (10) by the example of $n = 4$. Using the Laplace expansion along the first row and the case $n = 3$, we have

$$\begin{vmatrix} [a_1b_1] & [a_1b_2] & [a_1b_3] & [a_1b_4] \\ [a_2b_1] & [a_2b_2] & [a_2b_3] & [a_2b_4] \\ [a_3b_1] & [a_3b_2] & [a_3b_3] & [a_3b_4] \\ [a_4b_1] & [a_4b_2] & [a_4b_3] & [a_4b_4] \end{vmatrix}$$

$$\begin{aligned} &= [a_1b_1]\big([a_2b_2a_3b_3a_4b_4] + [a_2a_3][b_2b_3a_4b_4] + [a_2a_4][b_2a_3b_3b_4] + [a_3a_4][a_2b_2b_3b_4]\big) \\ &- [a_1b_2]\big([a_2b_1a_3b_3a_4b_4] + [a_2a_3][b_1b_3a_4b_4] + [a_2a_4][b_1a_3b_3b_4] + [a_3a_4][a_2b_1b_3b_4]\big) \\ &+ [a_1b_3]\big([a_2b_1a_3b_2a_4b_4] + [a_2a_3][b_1b_2a_4b_4] + [a_2a_4][b_1a_3b_2b_4] + [a_3a_4][a_2b_1b_2b_4]\big) \\ &- [a_1b_4]\big([a_2b_1a_3b_2a_4b_3] + [a_2a_3][b_1b_2a_4b_3] + [a_2a_4][b_1a_3b_2b_3] + [a_3a_4][a_2b_1b_2b_3]\big)\,. \end{aligned}$$

The contribution coming from the first column in the brackets, using (8), is

$$\begin{aligned} &[a_1b_1a_2b_2a_3b_3a_4b_4] + [a_1a_2][b_1b_2a_3b_3a_4b_4]+ \\ &+ [a_1a_3][b_1a_2b_2b_3a_4b_4] + [a_1a_4][b_1a_2b_2a_3b_3b_4]\,. \end{aligned}$$

(This corresponds to the case $k = 0$ in the proof of (10).) The contribution coming from the second column in the brackets, using (8), is

$$[a_2a_3]\big([a_1b_1b_2b_3a_4b_4] + [a_1a_4][b_1b_2b_3b_4]\big)\,. \tag{19}$$

The contribution coming from the third column in the brackets, using (8), is

$$[a_2a_4]\big([a_1b_1b_2a_3b_3b_4] - [a_1a_3][b_1b_2b_3b_4]\big)\,. \tag{20}$$

The contribution coming from the fourth column in the brackets, using (8), is

$$[a_3a_4]\big([a_1b_2a_2b_2b_3b_4] + [a_1a_2][b_1b_2b_3b_4]\big)\,. \tag{21}$$

We use (8) once again to present the sum of the second summands in (19), (20) and (21) as

$$[a_1a_2a_3a_4][b_1b_2b_3b_4]\,. \tag{22}$$

(The contributions (19), (20), (21) and (22) correspond to the case $k = 2$ in the proof of (10).)

This ends the example.

Example 7. We illustrate the proof of (13) by the example of $n = 3$. Then $\wedge^2(X)$ is the matrix

$$\begin{bmatrix} \begin{array}{c}[a_2b_2a_3b_3]\\ +[a_2a_3][b_2b_3]\end{array} & \begin{array}{c}-[a_1b_2a_3b_3]\\ -[a_1a_3][b_2b_3]\end{array} & \begin{array}{c}[a_1b_2a_2b_3]\\ +[a_1a_2][b_2b_3]\end{array} \\ \\ \begin{array}{c}-[a_2b_1a_3b_3]\\ -[a_2a_3][b_1b_3]\end{array} & \begin{array}{c}[a_1b_1a_3b_3]\\ +[a_1a_3][b_1b_3]\end{array} & \begin{array}{c}-[a_1b_1a_2b_3]\\ -[a_1a_2][b_1b_3]\end{array} \\ \\ \begin{array}{c}[a_2b_1a_3b_2]\\ +[a_2a_3][b_1b_2]\end{array} & \begin{array}{c}-[a_1b_1a_3b_2]\\ -[a_1a_3][b_1b_2]\end{array} & \begin{array}{c}[a_1b_1a_2b_2]\\ +[a_1a_2][b_1b_2]\end{array} \end{bmatrix}$$

by (10). Therefore $\mathrm{Tr}(\wedge^2(X)\cdot Y)$, in this case, is equal to

$$\begin{aligned} &\big([a_2b_2a_3b_3] + [a_2a_3][b_2b_3]\big)[a_1][b_1] \\ -&\big([a_2b_1a_3b_3] + [a_2a_3][b_1b_3]\big)[a_1][b_2] \\ &\big([a_2b_1a_3b_2] + [a_2a_3][b_1b_2]\big)[a_1][b_3] \\ \\ -&\big([a_1b_2a_3b_3] + [a_1a_3][b_2b_3]\big)[a_2][b_1] \\ &\big([a_1b_1a_3b_3] + [a_1a_3][b_1b_3]\big)[a_2][b_2] \\ -&\big([a_1b_1a_3b_2] + [a_1a_3][b_1b_2]\big)[a_2][b_3] \\ \\ &\big([a_1b_2a_2b_3] + [a_1a_2][b_2b_3]\big)[a_3][b_1] \\ -&\big([a_1b_1a_2b_3] + [a_1a_2][b_1b_3]\big)[a_3][b_2] \\ &\big([a_1b_1a_2b_2] + [a_1a_2][b_1b_2]\big)[a_3][b_3]\,. \end{aligned}$$

Using (7), the contribution of the first column in the three displayed blocks is

$$[a_1][b_1a_2b_2a_3b_3] + [a_2][a_1b_1b_2a_3b_3] + [a_3][a_1b_1a_2b_2b_3]$$

(after the cancellation of three pairs of pairwise opposite elements:
from the first block: $[a_1]\big(-[a_2]\big)[b_1b_2a_3b_3]$, $[a_1]\big(-[a_3]\big)[b_1a_2b_2b_3]$;
from the second block: $[a_2][a_1][b_1b_2a_3b_3]$, $[a_2]\big(-[a_3]\big)[a_1b_1b_2b_3]$;
from the third block: $[a_3][a_1][b_1a_2b_2b_3]$, $[a_3][a_2][a_1b_1b_2b_3]$).
(This corresponds to the case $k = 0$ in the proof of (13).)

The contribution of the latter column in the first block is

$$[a_1][a_2a_3][b_1b_2b_3]\,. \tag{23}$$

The contribution of the latter column in the second block is

(24) $$-[a_2][a_1a_3][b_1b_2b_3]\,.$$

The contribution of the latter column in the third block is

(25) $$[a_3][a_1a_2][b_1b_2b_3]\,.$$

Thus, using (7), the latter columns in the blocks contribute in sum to

(26) $$[a_1a_2a_3][b_1b_2b_3]\,.$$

(The contributions (23),(24),(25) and (26) correspond to the case $k=2$ in the proof of (13). Here we have no further cancellations because of the absence of the a's in the latter brackets).

This ends the example.

We can restate Corollary 2 in geometric terms. Let V be a complex vector space of dimension $2n$ endowed with a nondegenerate symplectic form. Denote by G the Grassmannian of n-dimensional subspaces in V and by G' the Lagrangian Grassmannian of n-dimensional subspaces in V, which are isotropic w.r.t. the symplectic form.

Given a partition $\lambda \subset (n^n)$, we define the Schubert class σ_λ to be the Poincaré dual to the fundamental class of the cycle on G defined by

$$\{\, L \in G \,:\, \dim(L \cap V_{n+i-\lambda_i}) \geq i\,,\ i = 1, \dots, n\}\,,$$

where $0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{2n-1} \subset V_{2n} = V$ is a flag with $\dim(V_i) = i\,.$

Similarly, given a strict partition $\mu \subset (n, n-1, \dots, 1)$, we define the Schubert class σ'_μ to be the Poincaré dual to the fundamental class of the cycle on G' defined by

$$\{\, L \in G' \,:\, \dim(L \cap W_{n+1-\mu_i}) \geq i\,,\ i = 1, \dots, l(\mu)\}\,,$$

where $0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_n \subset V$ is an isotropic flag with $\dim(W_i) = i\,.$

To state the next proposition, we need the following notation. Given a sequence of different positive integers $C = (c_1, \dots, c_l)$, there is a permutation $\mu_C \in S_l$ such that $c_{\mu(1)} > \ldots > c_{\mu(l)} > 0$. Denote this last-mentioned strict partition by $<C>$.

Proposition 8. *Let $i : G' \hookrightarrow G$ be the inclusion and $i^* : H^*(G, \mathbb{Q}) \to H^*(G', \mathbb{Q})$ be the induced homomorphism of the cohomology rings. Then, for any partition λ, using the notation of Theorem 1, one has the equality*

$$i^*(\sigma_\lambda) = \frac{1}{2^n} \sum \sigma'_{(a_{i_1}, \dots, a_{i_k})} \cdot \mathrm{sgn}\big(\mu_{C(i_1, \dots, i_k)}\big) \cdot \sigma'_{<C(i_1, \dots, i_k)>}\,,$$

where the sum is over all sequences $1 \le i_1 < \ldots < i_k \le n$ *for which* $C(i_1, \ldots, i_k) := A\#B \smallsetminus (a_{i_1}, \ldots, a_{i_k})$ *is a sequence of different integers.*

The assertion of the proposition is a restatement of the formula for $\eta(s_\lambda)$ in Corollary 2. We invoke that $H^*(G)$ is a quotient of Λ and σ_λ is the image of s_λ (by the Giambelli formula of the Schubert Calculus). On the other hand, $H^*(G')$ is a quotient of Γ and σ'_μ is the image of Q_μ (by a result from [P2, Sect.6]). Using these identifications i^* is induced by η. This follows, e.g., by remarking that $\sigma_i = c_i(S^\vee)$ and $\sigma'_i = c_i(i^* S^\vee)$, where S is the tautological vector bundle on G. Hence $i^*(\sigma_i) = \sigma'_i$ and this equality corresponds to the algebraic equality $\eta(e_i) = Q_i$ defining $\eta : \Lambda \to \Gamma$.

Notice (see Example 3) that $i^*(\sigma_\lambda)$ is not, in general, a positive combination of the σ'_μ's.

By exchanging a_i for b_i and vice versa in Theorem 4, we get the following identity.

Corollary 9. *In the situation of Theorem 4, one also has:*

$$\mathrm{Det}\left([a_i b_j] + [a_i][b_j]\right)_{1 \le i,j \le n} = \sum [b_{i_1} \ldots b_{i_k}][A\#B \smallsetminus (b_{i_1}, \ldots, b_{i_k})],$$

where the sum is over sequences $1 \le i_1 < \ldots < i_k \le n$ *and* $k = 0, 1, \ldots, n$.

Observe that under this exchange, the LHS of (10) goes to $(-1)^n \,\mathrm{Det}(X)$ whereas its RHS goes to

$$\begin{aligned} &\sum [b_{i_1} \ldots b_{i_k}][B\#A \smallsetminus (b_{i_1}, \ldots, b_{i_k})] \\ &= (-1)^{n(n-k)} \sum [b_{i_1} \ldots b_{i_k}][A\#B \smallsetminus (b_{i_1}, \ldots, b_{i_k})], \end{aligned}$$

where the sum is over all sequences $1 \le i_1 < \cdots < i_k \le n$ with k even.

Under the same exchange, the LHS of (13) goes to $(-1)^{n(n-1)} \,\mathrm{Tr}\big(\wedge^{n-1}(X) \cdot Y\big)$ whereas its RHS goes to

$$(-1)^{n(n-k)} \sum [b_{i_1} \ldots b_{i_k}][A\#B \smallsetminus (b_{i_1}, \ldots, b_{i_k})],$$

where the sum is over all sequences $1 \le i_1 < \ldots < i_k \le n$ with k odd.

Since $n \equiv n(n-k) \pmod 2$ for k even, and $n(n-1) \equiv n(n-k) \pmod 2$ for k odd, the assertion of the corollary follows.

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2. B. Leclerc has informed me recently that he and S. Leidwanger have independently obtained a formula essentially equivalent to our Theorem 1 using the representation theory of affine Lie algebras – see their recent preprint [L-L2].

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