(to appear in Journal of Algebra)

ADDENDUM TO:

Piotr Pragacz: "A generalization of the Macdonald-You formula"
Journal of Algebra 204, 573–587 (1998); Article No. JA977342.

Introduction

The present note concerns mainly *restricting Schubert classes.* An example of this operation was discussed in Proposition 8 of the above paper (quoted in the following as [P]). This example is revisited in the present note, and a combinatorial interpretation of the restriction coefficients, based on Stembridge's results [St], is given.

Secondly, by comparing the formula appearing in the title of [P] with the mentioned results of Stembridge and some other combinatorial results, we deduce some new identities in Propositions 2–6. They concern (apart of the restrictions of Schubert classes to the cohomology of Lagrangian Grassmannians) also relations between Q-functions, Stembridge's coefficients, and various "hook numbers". Moreover, we provide some examples illustrating [P] and the formulas given in the present note.

In this note, any unexplained notation or quotation stems from [P]. However, in order to make the notation maximally compatible with that used in [St] (which is our prinicipal reference here), we label strict partitions by λ, and μ usually denotes an ordinary partition, contrariwise to [P].

But before passing to the proper content of this note, we correct some points in [P]. Namely, due to some bugs in the computer system SCHUR [Sch], [P, Example 3(b)] was miscalculated: the quadratic expression in Q-functions displayed there, written as a $\mathbb{Z}$-linear combination of Q-functions, contains *no* negative summands. Consequently the sentence on p.585, lines 6–7 from the bottom, is to be withdrawn from [P]. (These corrections *do not* affect other results of [P], in particular the main formulas.)

1. Nonnegativity of the restriction coefficients

In fact, if

$$i^*(\sigma_\mu) = \sum_\lambda c_{\lambda\mu}\sigma'_\lambda, \tag{1}$$

with $c_{\lambda\mu} \in \mathbb{Z}$, then all the coefficients $c_{\lambda\mu}$ are nonnegative. Perhaps the easiest way to see this, is the following. Let for $a \in H^*(G;\mathbb{Z})$, $\int_G a$ stand for the degree of the top codimensional component of a, and define similarly $\int_{G'} b$ for $b \in H^*(G';\mathbb{Z})$. Given a strict partition $\lambda \subset (n, n-1, \dots, 1)$, we denote by $\lambda^\vee$ the strict partition whose parts complement those of λ in $\{1, \dots, n\}$. We record the following property [P2]:

This note is a slightly rewritten version of `math.AG/9907092`, and it was supported by KBN grant No. 2P03A 05112.

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Lemma 1 (Duality). *The basis $\{\sigma'_\lambda\}$ of the group $H^{2p}(G';\mathbb{Z})$ and the basis $\{\sigma'_{\lambda^\vee}\}$ of the group $H^{n(n+1)-2p}(G';\mathbb{Z})$ are dual under the pairing $(a,b)\mapsto \int_{G'} a\cdot b$ of Poincaré duality.*

Now, if $i^*(\sigma_\mu) = \sum_\lambda c_{\lambda\mu}\sigma'_\lambda$, with $c_{\lambda\mu}\in\mathbb{Z}$, then it follows from the duality property that

$$c_{\lambda\mu} = \int_{G'} i^*(\sigma_\mu)\cdot\sigma'_{\lambda^\vee}. \tag{2}$$

Using the projection formula for i, this is rewritten as

$$c_{\lambda\mu} = \int_{G} \sigma_\mu\cdot i_*(\sigma'_{\lambda^\vee}). \tag{3}$$

Regard G as a homogeneous space $GL(V)/P$, where P is a suitable parabolic subgroup of $GL(V)$. Let $\Omega\subset G$ be a Schubert variety representing σ_μ and let $\Omega'\subset G'\subset G$ be a Schubert variety representing $\sigma'_{\lambda^\vee}$. Using e.g. Kleiman's theorem on a general translate [K], we can replace Ω by a translate by an element $g\in GL(V)$ such that $g\cdot\Omega$ and Ω' meet properly, and this intersection is represented as a nonnegative zero-cycle. This shows that $c_{\lambda\mu}\geq 0$.

A similar property holds in the following more general setting. Let now $G\supset P\supset B$ be a semisimple linear algebraic group, a parabolic subgroup, and a Borel subgroup. In a generalized flag variety G/P, one has Schubert varieties $\overline{BwP/P}$ and their Schubert classes in $H^*(G/P;\mathbb{Z})$ indexed by a corresponding subset of the Weyl group. These Schubert classes enjoy a similar duality property. In an analogous way, using a general translate argument, one shows that the fundamental class of any subscheme of G/P is a $\mathbb{Z}$-linear combination of the Schubert classes in $H^*(G/P;\mathbb{Z})$ with nonnegative coefficients. Combining this with a well-known fact about pulling back the class of a Cohen-Macaulay subscheme (see, e.g., Lemma on p.108 in [F-P]), we get the following result (also implying the nonnegativity of the above $c_{\lambda\mu}$):

Proposition 1. *Let $f: G/P\to Y$ be morphism to a nonsingular variety Y. Let Z be a pure-dimensional closed Cohen-Macaulay subscheme of Y. Then $f^*([Z])$ is a $\mathbb{Z}$-linear combination of the Schubert classes in $H^*(G/P;\mathbb{Z})$ with nonnegative coefficients.*

2. Stembridge's coefficients

We recall (see the discussion after [P, Proposition 8]) that the coefficients appearing in (1) and those appearing in:

$$\eta(s_\mu) = \sum_\lambda g_{\lambda\mu}Q_\lambda \tag{4}$$

satisfy $c_{\lambda\mu} = g_{\lambda\mu}$. Here, we take suficiently large Grassmannians $i: G'\hookrightarrow G$. To be more precise, this means that given μ, we take $n\geq|\mu|$ so that any strict partition λ with $|\lambda|=|\mu|$ is contained in $(n, n-1,\ldots,1)$. Consequently, all the coefficients $g_{\lambda\mu}$ are nonnegative. But this result, together with a combinatorial interpretation of the $g_{\lambda\mu}$'s, was already established by Stembridge in [St].[1] Indeed, the last displayed (unnumbered) equality before [St, Theorem 9.3]:

$$\text{“ } S_\mu = \sum_{\lambda\in DP_n} g_{\lambda\mu} Q_\lambda \text{ ”} \tag{5}$$

is identical with (4) because S_μ in the notation of [St] (and [M]) is equal to $\eta(s_\mu)$ in our notation.[2] In [St], (5) is a consequence of the equality

$$P_\lambda = \sum_{|\mu|=|\lambda|} g_{\lambda\mu} s_\mu, \tag{6}$$

where $P_\lambda = 2^{-l(\lambda)}Q_\lambda$, and comparison of the canonical scalar products on the ring of all symmetric functions with that on the ring of Q-functions. To the nonnegativity of $g_{\lambda\mu}$ is given, in loc. cit., several interpretations in representation theory, some of which go back to Morris and Stanley.

(Observe that (4) and (6) yield the following expression for $\eta(P_\lambda)$:

$$\eta(P_\lambda) = \sum_{|\mu|=|\lambda|} g_{\lambda\mu}\eta(s_\mu) = \sum_{|\mu|=|\lambda|}\sum_{|\nu|=|\lambda|} g_{\lambda\mu} g_{\nu\mu} Q_\nu.) \tag{7}$$

Stembridge [St] also established a combinatorial interpretation of the numbers $f^\lambda_{\mu\nu}$ appearing as coefficients in the expansion:

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu} P_\lambda, \tag{8}$$

where μ, ν, and λ denote now strict partitions. It will be convenient to set

$$e^\lambda_{\mu\nu} := 2^{l(\mu)+l(\nu)-l(\lambda)} f^\lambda_{\mu\nu}. \tag{9}$$

There exists a geometric analogue of (8): in the cohomology ring $H^*(G';\mathbb{Z})$ of a sufficiently large Lagrangian Grassmannian,

$$\sigma'_\mu \cdot \sigma'_\nu = \sum_\lambda e^\lambda_{\mu\nu} \sigma'_\lambda. \tag{10}$$

(See [P2, Sect.6].)

Stembridge's combinatorial description of the above $f^\lambda_{\mu\nu}$ and $g_{\lambda\mu}$ can be summarized by the following:

Theorem [St] (version from [M]). *(i) The coefficient $f^\lambda_{\mu\nu}$ is equal to the number of marked shifted tableaux T of shape λ/μ and weight ν such that:*
(a) The word $w(T)$ associated with T [M, p.258]) has the lattice property in the sense of loc.cit.;
(b) for each $k \geq 1$, the rightmost occurence of k' in $w(T)$ precedes the last occurence of k.
(ii) The coefficient $g_{\lambda\mu}$ is equal to the number of unshifted marked tableaux T of shape μ and weight λ satisfying (a) and (b) above.

For all unexplained here combinatorial notions, we refer the reader to [M, III.8 pp.255–259]; compare also [St, Sect.6 and 8] and [P2, Sect.4]. We make no attempt to make a complete survey here. Some examples of the coefficients $g_{\lambda\mu}$ will be given below.

Summarizing the content of this section, we record:

[1]The fact that this result was already established by Stembridge, has been learned by the author only in June 1999.

[2]Note that the map denoted in [P] and here by η, is denoted by φ in [M].

Proposition 2. *We have for a partition $\mu \subset (n^n)$*

$$i^*(\sigma_\mu) = \sum_\lambda g_{\lambda\mu}\ \sigma'_\lambda\,, \tag{11}$$

where λ runs over strict partitions contained in $(n, n-1, \ldots, 1)$, and $g_{\lambda\mu}$ is the Stembridge coefficient described in Theorem (ii).

3. Quadratic relations between Q-functions

We pass now to some applications of the generalized Macdonald-You formula ([L-L2], [P, Corollary 2]):

$$2^n \eta(s_\mu) = \sum Q_{(a_{i_1},\ldots,a_{i_k})} \cdot Q_{A\#B\smallsetminus(a_{i_1},\ldots,a_{i_k})}. \tag{12}$$

Recall that here, for $\mu = (\alpha_1, \ldots, \alpha_n|\beta_1, \ldots, \beta_n)$ in Frobenius notation,

$$A = (a_1, \ldots, a_n) := (\alpha_1 + 1, \ldots, \alpha_n + 1)\,, \qquad B := (\beta_1, \ldots, \beta_n)\,, \tag{13}$$

and the sum is over all sequences $1 \le i_1 < \cdots < i_k \le n$ and $k = 0, 1, \ldots, n$.

Since $\eta(e_i) = \eta(h_i)$, where h_i is the ith complete homogeneous symmetric function, we have for a partition μ

$$\eta(s_{\mu^\sim}) = \eta(s_\mu), \tag{14}$$

where $\mu^\sim = (\beta_1, \ldots, \beta_n|\alpha_1, \ldots, \alpha_n)$ is the conjugate partition of μ. We set in addition

$$C = (c_1, \ldots, c_n) := (\beta_1 + 1, \ldots, \beta_n + 1)\,, \qquad D := (\alpha_1, \ldots, \alpha_n). \tag{15}$$

Then (12) and (14) imply the following:

Proposition 3. *We have*

$$\sum Q_{(a_{i_1},\ldots,a_{i_k})} \cdot Q_{A\#B\smallsetminus(a_{i_1},\ldots,a_{i_k})} = \sum Q_{(c_{i_1},\ldots,c_{i_k})} \cdot Q_{C\#D\smallsetminus(c_{i_1},\ldots,c_{i_k})}, \tag{16}$$

where the sums are over all sequences $1 \le i_1 < \cdots < i_k \le n$ and $k = 0, 1, \ldots, n$.

The relations (16), regarded from the side of Q-functions, seem to be rather nontrivial. For instance, for $\mu = (5^3 31^3) = (432|621)$, so $A = (5, 4, 3)$, $B = (6, 2, 1)$, $C = (7, 3, 2)$, and $D = (4, 3, 2)$, we get the equation

$$\begin{aligned} &Q_{654321} - Q_5 \cdot Q_{64321} + Q_4 \cdot Q_{65321} - Q_3 \cdot Q_{65421} - Q_{54} \cdot Q_{6321} \\ &+ Q_{53} \cdot Q_{6421} - Q_{43} \cdot Q_{6521} + Q_{543} \cdot Q_{621} \\ &= Q_{32} \cdot Q_{7432} + Q_{732} \cdot Q_{432}. \end{aligned} \tag{17}$$

Using a (hopefully) debugged version of SCHUR, (16) is expressed as the following $\mathbb{Z}$-linear combination of the Q_λ's :

$$\begin{array}{lllll} 8Q_{11\ 64} & + 8Q_{11\ 631} & + 8Q_{11\ 541} & + 8Q_{11\ 532} & + 8Q_{10\ 74} \\ + 8Q_{10\ 731} & + 8Q_{10\ 65} & + 24Q_{10\ 641} & + 24Q_{10\ 632} & + 24Q_{10\ 542} \\ + 8Q_{10\ 5321} & + 8Q_{975} & + 16Q_{9741} & + 16Q_{9732} & + 16Q_{9651} \\ + 48Q_{9642} & + 16Q_{96321} & + 16Q_{9543} & + 16Q_{95421} & + 8Q_{8751} \\ + 24Q_{8742} & + 8Q_{87321} & + 24Q_{8652} & + 24Q_{8643} & + 24Q_{86421} \\ + 8Q_{85431} & + 8Q_{7653} & + 8Q_{76521} & + 8Q_{76431} & \end{array}$$

Consequently, taking sufficiently large Grassmannians $i : G' \hookrightarrow G$, we have

$$i^*(\sigma_{5^3 3 1^3}) =$$

$$\begin{array}{lllll}
\sigma'_{11\ 64} & +\ \sigma'_{11\ 631} & +\ \sigma'_{11\ 541} & +\ \sigma'_{11\ 532} & +\ \sigma'_{10\ 74} \\
+\ \sigma'_{10\ 731} & +\ \sigma'_{10\ 65} & +\ 3\sigma'_{10\ 641} & +\ 3\sigma'_{10\ 632} & +\ 3\sigma'_{10\ 542} \\
+\ \sigma'_{10\ 5321} & +\ \sigma'_{975} & +\ 2\sigma'_{9741} & +\ 2\sigma'_{9732} & +\ 2\sigma'_{9651} \\
+\ 6\sigma'_{9642} & +\ 2\sigma'_{96321} & +\ 2\sigma'_{9543} & +\ 2\sigma'_{95421} & +\ \sigma'_{8751} \\
+\ 3\sigma'_{8742} & +\ \sigma'_{87321} & +\ 3\sigma'_{8652} & +\ 3\sigma'_{8643} & +\ 3\sigma'_{86421} \\
+\ \sigma'_{85431} & +\ \sigma'_{7653} & +\ \sigma'_{76521} & +\ \sigma'_{76431} &
\end{array}$$

So e.g. we have: $g_{(5^3 3 1^3)\ (11\ 64)} = 1$, $g_{(5^3 3 1^3)\ (10\ 641)} = 3$, $g_{(5^3 3 1^3)\ (9741)} = 2$, and $g_{(5^3 3 1^3)\ (9642)} = 6$.

4. Linear relations between Stembridge's coefficients

Combining (4), (12), and (16), we have in the above notation, associated with a fixed μ

$$\begin{aligned}
&\sum Q_{(a_{i_1},\dots,a_{i_k})} \cdot Q_{A\#B\smallsetminus(a_{i_1},\dots,a_{i_k})} \\
&= \sum Q_{(c_{i_1},\dots,c_{i_k})} \cdot Q_{C\#D\smallsetminus(c_{i_1},\dots,c_{i_k})} = 2^n \sum_\lambda g_{\lambda\mu} Q_\lambda,
\end{aligned} \tag{18}$$

where the first two sums are over all sequences $1 \le i_1 < \cdots < i_k \le n$ and $k = 0, 1, \dots, n$.

The equalities (18) imply linear relations between the $e^\lambda_{\mu\nu}$'s and $g_{\lambda\mu}$'s. Given a sequence of different positive integers $K = (k_1, \dots, k_l)$, there is a permutation $w = w_K \in S_l$ such that $k_{w(1)} > \cdots > k_{w(l)} > 0$. Denote this last-mentioned strict partition by $< K >$. Then given strict partitions μ, λ and a sequence K as above, we set

$$e^\lambda_{\mu\ K} := \mathrm{sgn}(w_K)\ e^\lambda_{\mu\ <K>}. \tag{19}$$

From (18) and (16) we get the following result:

Proposition 4. *For a fixed partition μ and strict partition λ with $|\mu| = |\lambda|$, we have in the above notation associated with μ*

$$\begin{aligned}
2^n g_{\lambda\mu} &= \sum e^\lambda_{(a_{i_1},\dots,a_{i_k}),\ A\#B\smallsetminus(a_{i_1},\dots,a_{i_k})} \\
&= \sum e^\lambda_{(c_{i_1},\dots,c_{i_k}),\ C\#D\smallsetminus(c_{i_1},\dots,c_{i_k})},
\end{aligned} \tag{20}$$

where the sums are over all sequences $1 \le i_1 < \cdots < i_k \le n$ for which $A\#B \smallsetminus (a_{i_1}, \dots, a_{i_k})$ (resp. $C\#D \smallsetminus (c_{i_1}, \dots, c_{i_k})$) is a sequence of different integers, and $k = 0, 1, \dots, n$.

For instance, for any strict partition λ with $|\lambda| = 21$, and for $\mu = (5^3 3 1^3) = (432|621)$, we get the equations:

$$\begin{aligned}
2^3 g_{\lambda\ (5^3 3 1^3)} &= e^\lambda_{(654321)\ (\emptyset)} - e^\lambda_{(5)\ (64321)} + e^\lambda_{(4)\ (65321)} - e^\lambda_{(3)\ (65421)} - e^\lambda_{(54)\ (6321)} \\
&\quad + e^\lambda_{(53)\ (6421)} - e^\lambda_{(43)\ (6521)} + e^\lambda_{(543)\ (621)} \\
&= e^\lambda_{(32)\ (7432)} + e^\lambda_{(732)\ (432)}.
\end{aligned}$$

5. The class $i_*(\sigma'_\lambda)$ as a $\mathbb{Z}$-linear combination of the σ_μ's

By reasoning similarly as in §1, one shows that for any proper morphism $f : X \to G/P$ from a scheme X to a generalized flag variety G/P, and for any irreducible subscheme $Y \subset X$, $f_*([Y])$ is a $\mathbb{Z}$-linear combination of Schubert classes in $H^*(G/P;\mathbb{Z})$ with nonnegative coefficients.

The next proposition will give $i_*(\sigma'_\lambda)$ as an explicit $\mathbb{Z}$-linear combination of the σ_μ's. Given a partition $\mu \subset (n^n)$, we set $\mu^\star := (n-\mu_n, \dots, n-\mu_1)$. The following duality property is a well-known result of Schubert calculus [F]:

Lemma 2. *The basis $\{\sigma_\mu\}$ of the group $H^{2p}(G;\mathbb{Z})$ and the basis $\{\sigma_{\mu^\star}\}$ of the group $H^{2(n^2-p)}(G;\mathbb{Z})$ are dual under the pairing $(a,b) \mapsto \int_G a\cdot b$ of Poincaré duality.*

We now state:

Proposition 5. *For a fixed strict partition $\lambda \subset (n, n-1, \dots, 1)$, we have*

$$i_*(\sigma'_\lambda) = \sum_{|\mu|=|\lambda|+n(n-1)/2} g_{\lambda^\vee,\mu^\star}\ \sigma_\mu\,, \tag{21}$$

where μ runs over partitions contained in (n^n) and $g_{\lambda^\vee,\mu^\star}$ is the Stembridge coefficient described in Theorem (ii).

Indeed, if $i_*(\sigma'_\lambda) = \sum_\mu m_{\lambda\mu}\sigma_\mu$, with $m_{\lambda\mu} \in \mathbb{Z}$ (so that $|\mu| = |\lambda| + n(n-1)/2$), then it follows from Lemma 2 that

$$m_{\lambda\mu} = \int_G (i_*\sigma'_\lambda)\cdot\sigma_{\mu^\star}. \tag{22}$$

Using the projection formula for i, this is rewritten as

$$m_{\lambda\mu} = \int_{G'} \sigma'_\lambda \cdot i^*(\sigma_{\mu^\star}). \tag{23}$$

In turn, using the description of $i^*(\sigma_{\mu^\star})$ from Proposition 2, (23) is rewritten as

$$m_{\lambda\mu} = \int_{G'} \sigma'_\lambda \cdot (\sum_\nu g_{\nu\mu^\star}\ \sigma'_\nu) = \int_{G'} \sum_\tau \sum_\nu e^\tau_{\lambda\nu}\ g_{\nu\mu^\star}\ \sigma'_\tau = g_{\lambda^\vee,\mu^\star} \tag{24}$$

because only $\tau = (n, n-1, \dots, 1)$ and $\nu = \lambda^\vee$ give a nonzero contribution (note that for such τ and ν, we have $e^\tau_{\lambda\nu} = 1$).

6. Relations between the degrees of the ordinary and projective representations of the symmetric groups

For a partition μ, we set

$$\overline{f}^\mu := \prod_{x\in\mu} \frac{1}{h(x)}, \tag{25}$$

where $h(x)$ is the hook-length of μ at $x = (i,j)$ defined by $h(x) = h(i,j) = \mu_i + \mu^\sim_j - i - j + 1$. If $|\mu| = m$ then $f^\mu := m!\ \overline{f}^\mu$ is the degree of the irreducible representation of S_m corresponding to μ. Equivalently, f^μ is the number of standard tableaux of shape μ, obtained by labeling the squares of the diagram of μ with the numbers $1, 2, \ldots, m$. We refer to [F] for a detailed discussion of these facts.

For a strict partition λ, we set

$$\overline{g}^{\lambda} := \prod_{x \in S(\lambda)} \frac{1}{h(x)}, \tag{26}$$

where $S(\lambda)$ is the shifted diagram associated with λ [M, p.255], and for each square $x \in S(\lambda)$ the hook-length $h(x)$ is defined to be the hook-length at x in the "double diagram" $(\lambda_1, \lambda_2, \ldots | \lambda_1 - 1, \lambda_2 - 1, \ldots)$, containing $S(\lambda)$. If $|\lambda| = m$, $g^{\lambda} := m!\, \overline{g}^{\lambda}$ is the number of shifted standard tableaux of shape $S(\lambda)$, obtained by labeling the squares of $S(\lambda)$ with the numbers $1, 2, \ldots, m$ with strict increase along each row and down each column. The numbers g^{λ} also admit an interpretation as the degree of suitable projective representations of S_m. We refer to [H-H] for a detailed discussion of these results.

One has the following formulas, in terms of parts, for $\overline{f}^{\mu}$ [M, I.1 Example 1] and $\overline{g}^{\lambda}$ [M, III.8 Example 12]:

$$\overline{f}^{\mu} = \frac{\prod_{i<j}(\mu_i - \mu_j - i + j)}{\prod_{i \geq 1}(\mu_i + n - i)!}, \tag{27}$$

$$\overline{g}^{\lambda} = \frac{1}{\prod_{i \geq 1} \lambda_i!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}. \tag{28}$$

We now record:

Lemma 3. *(i) Under the specialization $e_i := \frac{1}{i!}$, s_{μ} becomes $\overline{f}^{\mu}$.*
(ii) Under the specialization $Q_i := \frac{1}{i!}$, Q_{λ} becomes $\overline{g}^{\lambda}$.

(For assertion (i), see [M, I.3 Example 5]. Assertion (ii) stems from [DC-P, Proposition 6].)

Given a partition μ, we want to apply formulas (12) and (16), so we adopt the notation of §3. Also, we follow the notation of §4 associated with a sequence K. For such a sequence, we set

$$\overline{g}^{K} := \operatorname{sgn}(w_K)\, \overline{g}^{<K>}. \tag{29}$$

From Lemma 3, (12), and (16), we get

Proposition 6. *For a fixed partition μ, we have*

$$\begin{aligned} 2^n \overline{f}^{\mu} &= \sum \overline{g}^{(a_{i_1}, \ldots, a_{i_k})}\, \overline{g}^{A\#B \setminus (a_{i_1}, \ldots, a_{i_k})} \\ &= \sum \overline{g}^{(c_{i_1}, \ldots, c_{i_k})}\, \overline{g}^{C\#D \setminus (c_{i_1}, \ldots, c_{i_k})}, \end{aligned} \tag{30}$$

where the sums are over all sequences $1 \leq i_1 < \cdots < i_k \leq n$ for which $A\#B \setminus (a_{i_1}, \ldots, a_{i_k})$ (resp. $C\#D \setminus (c_{i_1}, \ldots, c_{i_k})$) is a sequence of different integers, and $k = 0, 1, \ldots, n$.

For instance, for $\mu = (5^3 3 1^3) = (432|621)$, we get the equations:

$$\begin{aligned} 2^3 \overline{f}^{(5^3 3 1^3)} &= \overline{g}^{(654321)} - \overline{g}^{(5)}\, \overline{g}^{(64321)} + \overline{g}^{(4)}\, \overline{g}^{(65321)} - \overline{g}^{(3)}\, \overline{g}^{(65421)} - \overline{g}^{(54)}\, \overline{g}^{(6321)} \\ &\quad + \overline{g}^{(53)}\, \overline{g}^{(6421)} - \overline{g}^{(43)}\, \overline{g}^{(6521)} + \overline{g}^{(543)}\, \overline{g}^{(621)} \\ &= \overline{g}^{(32)}\, \overline{g}^{(7432)} + \overline{g}^{(732)}\, \overline{g}^{(432)}. \end{aligned}$$

ACKNOWLEDGEMENTS. The author would like to thank Brian Wybourne for debugging, on the author's request, the computer program SCHUR in June 1999. On the occasion of this debugging several examples were computed, which have helped the author to realize the link of [P] with Stembridge's results [St] on shifted tableaux.

We also thank Anders Thorup for his advice in preparing this note.

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Mathematical Institute of Polish Academy of Sciences, Chopina 12, PL 87100 Toruń, Poland