Tangency and regular separation

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in properly chosen regular parametrizations, those two curves have identical Taylor polynomials of degree k about x^0.

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Namely, the order of tangency allows one to define for example the **jets** of Lagrangian submanifolds.

The space of these jets is a fibration over the Lagrangian Grassmannian and leads to a positive decomposition of a Lagrangian Thom polynomial in the basis of Lagrangian Schubert cycles.

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when **there exist** a neighbourhood $U \ni u^0$ in $\mathbb{R}^p$ and parametrizations (diffeomorphisms onto their images)

$$q\colon \big(U, u^0\big) \to \big(M,\, x^0\big), \qquad \tilde{q}\colon \big(U, u^0\big) \to \big(\widetilde{M},\, x^0\big)$$

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means

$\lim_{u \to u_0} \frac{f(u)}{h(u)} = 0.$

“$f(u)$ is much smaller than $h(u)$ for u near u_0.”

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The condition (1) *is equivalent to*

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.(1) $\Rightarrow$ (2).

$$\mathrm{o}\Big(\big|u-u^0\big|^k\Big) = \tilde{q}(u) - q(u) \;=\; \big(\tilde{q}(u) - T^k_{u^0}(\tilde{q})(u-u^0)\big)$$
$$+\,\big(T^k_{u^0}(\tilde{q})(u-u^0) - T_{u^0}(q)(u-u^0)\big) + \big(T_{u^0}(q)(u-u^0) - q(u)\big)\,,$$

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where the first and last summands are $o\left(\left|u-u^0\right|^k\right)$ by Taylor.

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Lemma

Let $w \in \mathbb{R}[u_1,\ u_2, \ldots,\ u_p]$, $\deg w \leq k$, $w(u) = \mathrm{o}\big(|u|^k\big)$ when $u \to 0$ in $\mathbb{R}^p$. Then w is identically zero.

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The implication: Proposition $\Rightarrow$ (1) is easy.

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Let us assume additionally that

$$s < r\,. \tag{4}$$

When $r = \infty$, the condition (4) simply says that s is finite.

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Theorem

Under (4),

$$\min_{v}\Big(\max_{\gamma,\tilde{\gamma}}\big(\max\big\{l \in \{0\}\cup\mathbb{N} :\ |\gamma(t)-\tilde{\gamma}(t)| = \mathrm{o}\big(|t|^{l}\big) \text{ when } t \to 0\ \big\}\big)\Big) = s. \tag{5}$$

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The **minimum** *is taken over all* $0 \neq v \in T_{x^0}M = T_{x^0}\widetilde{M}$. *The* **outer maximum** *is taken over all pairs of* C^r *curves* $\gamma \subset M$, $\tilde\gamma \subset \widetilde{M}$ *such that* $\gamma(0) = x^0 = \tilde\gamma(0)$, *and – both non-zero! – velocities* $\dot\gamma(0)$, $\dot{\tilde\gamma}(0)$ *are both parallel to* v.

Attention. In this theorem the assumption (4) is essential; our proof would not work in the situation $s = r$.

It is quick to show that the LHS of (5) is at least s. Indeed, for every fixed vector v as above, $v = dq(u^0)\mathbf{u}$ (without loss of generality, $\mathbf{u} \in \mathbb{R}^p$, $|\mathbf{u}| = 1$). We now take $\delta(t) = q(u^0 + t\mathbf{u})$ and $\tilde{\delta}(t) = \tilde{q}(u^0 + t\mathbf{u})$.

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$$|\delta(t) - \tilde{\delta}(t)| \;=\; o\big(|t\mathbf{u}|^s\big) \;=\; o\big(|t|^s\big)$$

and so, in that equality,

$$\max_{\gamma,\tilde{\gamma}}\big(\max\big\{\, l\colon |\gamma(t) - \tilde{\gamma}(t)| = o\big(|t|^l\big) \text{ when } t \to 0\big\}\big) \;\geq\; s\,.$$

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In view of the arbitrariness in our choice of v, the same remains true after taking the minimum over all admissible v's on equality's LHS.

The opposite inequality is more involved. It is here where a delicate assumption $s \leq r - 1$ is needed. We skip the details.

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$$\mathcal{G}H(s) = dH(s)(T_s N)\,, \tag{6}$$

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Recall that $M, \widetilde{M} \subset \mathbb{R}^m$.

We use as previously the pair of parametrizations q and $\tilde{q}$. So we are now given the mappings

$$\mathcal{G}\, q : \; U \longrightarrow G_p\big(\mathbb{R}^m\big)\,, \qquad \mathcal{G}\, \tilde{q} : \; U \longrightarrow G_p\big(\mathbb{R}^m\big)\,.$$

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Upon putting $M^{(0)} = \mathbb{R}^m$, $\mathcal{G}^{(1)} = \mathcal{G}$, we get two sequences of recursively defined mappings. Namely, for $l \geq 1$,

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where, naturally, $M^{(l)} = \; G_p\big(M^{(l-1)}\big)$.

Theorem

C^r manifolds M and $\widetilde{M}$ have at x^0 the order of tangency at least k $(1 \le k \le r)$ iff

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Let now H be the graph of a C^1 mapping $h\colon \mathbb{R}^p \supset U \to \mathbb{R}^t$. That is, for $u \in U$, $H(u) = \big(u,\, h(u)\big) \in \mathbb{R}^{p+t} = \mathbb{R}^p \times \mathbb{R}^t$.

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$$\mathcal{G}H(u) = \Big(u,\ h(u)\,;\ d\big(u, h(u)\big)(u)\Big) = \Big(u,\ h(u);\ \mathrm{span}\{\partial_j + h_j(u)\}\Big) \tag{7}$$

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The h_j means the partial derivative of a vector mapping h w.r.t. u_j. Moreover, $\partial_j + h_j(u)$ is the partial derivative of $(\iota,\ h)\colon U \to \mathbb{R}^p\big(u_1, \ldots, u_p\big) \times \mathbb{R}^t$ w.r.t. u_j, where $\iota\colon U \hookrightarrow \mathbb{R}^p$ is the inclusion.

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We come back to the proof of the theorem.

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We shall show that the proposition (about Taylor series) implies the theorem.

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Attention. In this lemma we distinguish mixed derivatives taken in different orders.

Proof. $l = 1$. We note that

$$\mathcal{G}^{(1)}q\,(u) = \Big(u,\ f(u);\ \mathrm{span}\{\partial_j + f_j(u) \colon j = 1,\, 2, \ldots,\, p\}\Big)\,,$$

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This is nothing but

$$\big(u,\ f(u);\ f_{[1]}(u)\big)\ =\ \left(u,\ f(u);\ \binom{l}{1} \times f_{[1]}(u)\right).$$

The beginning of induction is done.

$l \Rightarrow l+1$, $l < k$. The mapping $\mathcal{G}^{(l)}q\colon U \to M^{(l)}$, evaluated at u, is already written down, in appropriate local chart assumed to exist in $M^{(l)}$, as

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These entries on the right hand side are to be juxtaposed with the **former** entries $\big(u,\ h(u)\big)$.

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The lemma is now proved by induction.

We now take $l = k$ in the lemma and get, for arbitrary $u \in U$, two similar expressions for $\mathcal{G}^{(k)}q(u)$ and $\mathcal{G}^{(k)}\tilde{q}(u)$.

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A natural question arises: What about branches of algebraic sets which often happen to be tangent one to another with various degrees of closeness?

It is well known that any pair of (closed) analytic subsets $X, Y \subset \mathbb{C}^m$ (of possibly different dimensions) satisfies so-called *Łojasiewicz regular separation property* at any point of $X \cap Y$:

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where ρ is a distance induced by any of the usual norms on $\mathbb{C}^m$.

If furthermore $x^0 \in \overline{X \setminus Y}$, then $\nu \geq 1$ and (10) is equivalent to

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This exponent is an interesting metric invariant of the pointed pair $(X, Y; x^0)$.

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Consider two curves N and Z in $\mathbb{C}^2(x, y)$ intersecting at $(0, 0)$:

$$N = \{y = 0\} \quad \text{and} \quad Z = \{y^d + yx^{d-1} + x^s = 0\},$$

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This is "$-x^{s-d+1}$" which dominates the computation.

Using $y(x)$, we compute the Łojasiewicz exponent. Here is a sketch. We discuss the inequality defining this exponent at $(0, 0)$.

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Let $A = (x, 0)$ be the points on N, $B = (x, y(x))$ be the points on Z, and let O be the point $(0,0)$.

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Using the function $y(x)$, the length AB is of order $|x|^{s-d+1}$. Since AO and BO are of order $|x|$, the triangle inequality:

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We get that the exponent is equal to $s - d + 1$. (The order of tangency is $s - d$.)

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Theorem

Let X and Y be analytic subsets in $\mathbb{C}^m$, and let $x^0 \in X \cap Y$ such that $\mathcal{L}(X, Y; x^0) \geq 1$.

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$$\mathcal{L}(X \cap H_0, Y \cap H_0; x^0) \leq \mathcal{L}(X, Y; x^0).$$

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then $\nu \geq \mathcal{L}(X, Y; 0) \geq 1$, and for some $c' > 0$ we have

$$\rho(x, Y) \geq c' \rho(x, X \cap Y)^{\nu} \tag{12}$$

By the proposition applied to $X \cap Y$, there is $c > 0$ such that for all $x \in H_0$ near 0 we have

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Combined with (12), this gives

$$\rho(x, Y \cap H_0) \geq \rho(x, Y) \geq c'\,\rho(x, X \cap Y)^{\nu} \geq \frac{c'}{c}\,\rho(x, X \cap Y \cap H_0)^{\nu}$$

for all $x \in X \cap H_0$ near 0, so that ν is a regular separation exponent for $X \cap H_0$ and $Y \cap H_0$ at 0 as desired.

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Let $\check{\mathbb{P}}^{m-1}$ denote the set of all hyperplanes of $\mathbb{C}^m$ through 0, with its usual structure of manifold. The distance between two elements $H, K \in \check{\mathbb{P}}^{m-1}$ is the angle $\sphericalangle(H, K)$ between them, that is,

$$\sphericalangle(H, K) := \arccos \frac{\langle v, w \rangle}{|v|\,|w|},$$

where v and w are normal vectors to the hyperplanes H and K respectively, considered with their underlying real structures, and where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^{2m}$.

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$$\mathcal{U} := \{(H, x) \in \check{\mathbb{P}}^{m-1} \times \mathbb{C}^m \mid \sphericalangle(H_0, H) < a \text{ and } |x| < b\}$$

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of a generic $(H_0, 0)$, the set $\mathcal{X}$ is *Lipschitz equisingular* over $\check{\mathbb{P}}^{m-1} \times \{0\}$.

That is, for any $(H,0) \in \mathcal{U} \cap (\check{\mathbb{P}}^{m-1} \times \{0\})$, there is a (germ of) Lipschitz homeomorphism

$$\varphi\colon (\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m, (H,0)) \to (\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m, (H,0))$$

such that $p \circ \varphi = p$ (where $p\colon \check{\mathbb{P}}^{m-1} \times \mathbb{C}^m \to \check{\mathbb{P}}^{m-1}$ is the standard projection) and $\varphi(\mathcal{X}) = \check{\mathbb{P}}^{m-1} \times (H \cap X)$ (as germs at $(H,0)$).

Actually, if $h = (h_1, \ldots, h_{m-1})$ are coordinates in $\check{\mathbb{P}}^{m-1}$ around H_0 such that

$$h_1(H_0) = \cdots = h_{m-1}(H_0) = 0\,,$$

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then locally near $(H_0, 0)$, the standard vector fields ∂_{h_j} $(1 \leq j \leq m-1)$ on $\check{\mathbb{P}}^{m-1} \times \{0\}$ can be lifted to Lipschitz vector fields v_j on $\check{\mathbb{P}}^{m-1} \times \mathbb{C}^m$

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such that the flows of v_j preserve $\mathcal{X}$.

So, in particular, v_j is a Lipschitz vector field of the form

$$v_j(h,x) = \partial_{h_j}|_{(h,x)} + \sum_{\ell=1}^{m} w_{j\ell}(h,x)\,\partial_{x_\ell}|_{(h,x)},$$

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Using the integral curves of these Lipschitz vector fields, we prove the proposition.

Corollary

Assume that $\dim(X) = \dim(Y)$. *If* $x^0 \in \overline{X \setminus Y}$, *then the tangency order* $s(X, Y; x^0) \leq \mathcal{L}(X, Y; x^0)$.

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Assume that $\dim(X) = \dim(Y)$*. If* $x^0 \in \overline{X \setminus Y}$*, then the tangency order* $s(X, Y; x^0) \leq \mathcal{L}(X, Y; x^0)$*.*

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Thus the corollary holds true in this case.

The general case (i.e., $\dim X \cap Y = n > 0$) follows from the 0-dimensional case and the last theorem.

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Let s_i (respectively, $\mathcal{L}_i$) denote the order of tangency (respectively, the Łojasiewicz exponent) of $X \cap H_1 \cap \cdots \cap H_i$ and $Y \cap H_1 \cap \cdots \cap H_i$ at x^0.

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Thus the corollary follows from the inequality $s_n \leq \mathcal{L}_n$ (0-dimensional case).

THE END