Order of tangency between manifolds

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Introduction

Two plane curves, both nonsingular at a point x^0, are said to have a contact of order at least k at x^0 if, in properly chosen regular parametrizations, those two curves have identical Taylor polynomials of degree k about x^0.

Why it is important to study the **"order of tangency"**? Let us discuss this notion for **Thom polynomials of singularities** (real or complex). Thom polynomials measure complexity of singularities and were studied by René Thom and many others.

An important property of Thom polynomials is their **positivity** closely related to Schubert calculus. Namely, the order of tangency allows one to define for example the **jets** of Lagrangian submanifolds. The space of these jets is a fibration over the Lagrangian Grassmannian and leads to a positive decomposition of a Lagrangian Thom polynomial in the basis of Lagrangian Schubert cycles.

Two manifolds M and $\widetilde{M}$ embedded in $\mathbb{R}^m$, both of class C^r, $r \geq 1$, and the same dimension p, intersecting at $x^0 \in M \cap \widetilde{M}$, for $k \leq r$, *have at* x^0 *the order of tangency at least* k, when **there exist** a neighbourhood $U \ni u^0$ in $\mathbb{R}^p$ and parametrizations (diffeomorphisms onto the image)

$$q\colon \big(U, u^0\big) \to \big(M,\, x^0\big), \qquad \tilde{q}\colon \big(U, u^0\big) \to \big(\widetilde{M},\, x^0\big)$$

of class C^r such that

$$\Big(\tilde{q} - q\Big)(u) \,=\, \mathrm{o}\Big(\big|u - u^0\big|^k\Big) \tag{1}$$

when $U \ni u \to u^0$. This definition does not depend on the choice of local parametrizations q and $\tilde{q}$.

$$f(u) = o(h(u)) \quad \text{when } u \to u_0$$

means

$\lim_{u \to u_0} \frac{f(u)}{h(u)} = 0.$

"$f(u)$ is much smaller than $h(u)$ for u near u_0."

Proposition

The condition (1) is equivalent to

$$T^k_{u^0}(q) \;=\; T^k_{u^0}(\tilde{q})\,, \tag{2}$$

where $T^k_{u^0}(\,\cdot\,)$ means the Taylor polynomial about u^0 of degree k.

.(1) $\Rightarrow$ (2).

$$\mathrm{o}\left(\left|u-u^0\right|^k\right) = \tilde{q}(u) - q(u) \;=\; \left(\tilde{q}(u) - \,T^k_{u^0}(\tilde{q})(u-u^0)\right)$$
$$+\left(T^k_{u^0}(\tilde{q})(u-u^0) - \,T_{u^0}(q)(u-u^0)\right)+\left(T_{u^0}(q)(u-u^0) - q(u)\right),$$

where the first and last summands are $o\left(\left|u-u^0\right|^k\right)$ by Taylor.

Under (1), so is the middle summand

$$T^k_{u^0}(\tilde{q})(u-u^0) - T^k_{u^0}(q)(u-u^0) \;=\; \mathrm{o}\Big(\big|u-u^0\big|^k\Big)$$

and (2) follows from the following general result.

Lemma

Let $w \in \mathbb{R}[u_1,\, u_2, \ldots,\, u_p]\,,\ \deg w \le k,\ w(u) = \mathrm{o}\big(|u|^k\big)$ when $u \to 0$ in $\mathbb{R}^p$. Then w is identically zero.

The implication: Proposition $\Rightarrow$ (1) is easy.

Consider the quantity

$$s\colon = \sup\{k \in \mathbb{N}\colon \text{the order of tangency } \geq k\}\,. \tag{3}$$

Note that an additional restriction here on k is $k \leq r$. If the class of smoothness $r = \infty$, then the condition (1) holds for all k if and only if $s = \infty$.

Let us assume additionally that

$$s < r\,. \tag{4}$$

When $r = \infty$, the condition (4) simply says that s is finite.

Our second approach uses *pairs of curves* lying, respectively, in M and $\widetilde{M}$. We assume that $T_{x^0}M = T_{x^0}\widetilde{M}$.

Theorem

Under (4),

$$\min_v \Big(\max_{\gamma,\tilde\gamma} \big(\max \big\{ l \in \{0\} \cup \mathbb{N} : \; |\gamma(t) - \tilde\gamma(t)| = \mathrm{o}\big(|t|^l\big) \text{ when } t \to 0 \,\big\}\big)\Big) = s. \tag{5}$$

The **minimum** *is taken over all* $0 \neq v \in T_{x^0}M = T_{x^0}\widetilde{M}$. *The* **outer maximum** *is taken over all pairs of* C^r *curves* $\gamma \subset M$, $\tilde\gamma \subset \widetilde{M}$ *such that* $\gamma(0) = x^0 = \tilde\gamma(0)$, *and – both non-zero! – velocities* $\dot\gamma(0)$, $\dot{\tilde\gamma}(0)$ *are both parallel to* v.

Attention. In this theorem the assumption (4) is essential; our proof would not work in the situation $s = r$.

It is quick to show that the LHS of (5) is at least s. Indeed, for every fixed vector v as above, $v = dq(u^0)\mathbf{u}$ (without loss of generality, $\mathbf{u} \in \mathbb{R}^p$, $|\mathbf{u}| = 1$). We now take $\delta(t) = q(u^0 + t\mathbf{u})$ and $\tilde{\delta}(t) = \tilde{q}(u^0 + t\mathbf{u})$. Then

$$|\delta(t) - \tilde{\delta}(t)| \;=\; o\big(|t\mathbf{u}|^s\big) \;=\; o\big(|t|^s\big)$$

and so, in that equality,

$$\max_{\gamma,\tilde{\gamma}}\big(\max\big\{\, l\colon |\gamma(t) - \tilde{\gamma}(t)| = o\big(|t|^l\big) \text{ when } t \to 0\big\}\big) \;\geq\; s\,.$$

In view of the arbitrariness in our choice of v, the same remains true after taking the minimum over all admissible v's on equality's LHS.

The opposite inequality is more involved. It is here where a delicate assumption $s \leq r - 1$ is needed. We skip the details.

Our third approach is based on a **tower of consecutive Grassmannians** attached to a local C^r parametrization q.

To every C^1 *immersion* $H : N \to N'$, N – an n-dimensional manifold, N' – an n'-dimensional manifold, we attach the so-called image map $\mathcal{G}H \colon N \to G_n(N')$ of the tangent map $d\,H$: for $s \in N$,

$$\mathcal{G}H(s) \;=\; dH(s)(T_sN)\,, \tag{6}$$

where $G_n(N')$ is the total space of the Grassmann bundle, with base N', of all n planes tangent to N' (often denoted $G_n(T_{N'})$).

Recall that $M, \widetilde{M} \subset \mathbb{R}^m$.

We use as previously the pair of parametrizations q and $\tilde{q}$. So we are now given the mappings

$$\mathcal{G}\, q:\ U \longrightarrow G_p\big(\mathbb{R}^m\big)\,, \qquad \mathcal{G}\, \tilde{q}:\ U \longrightarrow G_p\big(\mathbb{R}^m\big)\,.$$

Upon putting $M^{(0)} = \mathbb{R}^m$, $\mathcal{G}^{(1)} = \mathcal{G}$, we get two sequences of recursively defined mappings. Namely, for $l \geq 1$,

$$\mathcal{G}^{(l)} q:\ U \longrightarrow G_p\big(M^{(l-1)}\big)\,, \qquad \mathcal{G}^{(l+1)} q\ =\ \mathcal{G}\Big(\mathcal{G}^{(l)} q\Big)$$

and

$$\mathcal{G}^{(l)} \tilde{q}:\ U \longrightarrow G_p\big(M^{(l-1)}\big)\,, \qquad \mathcal{G}^{(l+1)} \tilde{q}\ =\ \mathcal{G}\Big(\mathcal{G}^{(l)} \tilde{q}\Big),$$

where, naturally, $M^{(l)} =\ G_p\big(M^{(l-1)}\big)$.

Theorem

C^r manifolds M and $\widetilde{M}$ have at x^0 the order of tangency at least k $(1 \leq k \leq r)$ iff for every parametrizations q and $\tilde{q}$ of the vicinities of x^0 in, respectively, M and $\widetilde{M}$, there holds

$$\mathcal{G}^{(k)}q\left(u^0\right) = \mathcal{G}^{(k)}\tilde{q}\left(u^0\right).$$

Let now H be the graph of a C^1 mapping $h\colon \mathbb{R}^p \supset U \to \mathbb{R}^t$. That is, for $u \in U$, $H(u) = \big(u,\ h(u)\big) \in \mathbb{R}^{p+t} = \mathbb{R}^p \times \mathbb{R}^t$. Then (6) assumes by far more precise form

$$\mathcal{G}H(u) = \Big(u,\ h(u)\,;\ d\big(u, h(u)\big)(u)\Big) = \Big(u,\ h(u);\ \mathrm{span}\{\partial_j + h_j(u)\}\Big) \tag{7}$$

Here $j = 1, 2, \ldots, p$. The symbol h_j means the partial derivative of a vector mapping h with respect to the variable u_j. Moreover, $\partial_j + h_j(u)$ denotes the partial derivative of the vector mapping $(\iota,\ h)\colon U \to \mathbb{R}^p\big(u_1, \ldots, u_p\big) \times \mathbb{R}^t$ with respect to u_j, where $\iota\colon U \hookrightarrow \mathbb{R}^p$ is the inclusion.

Now observe that the expression for $\mathcal{G}H(u)$ on the right hand side of (7) is still not quite useful. Yet there are charts in each newly appearing Grassmannian.

The chart in a typical fibre G_p over a point in the base $\mathbb{R}^{p+t}$, good for (7) consists of all the entries in the bottommost rows (indexed by numbers $p+1,\ p+2,\ \ldots,\ p+t$) in the $(p+t) \times p$ matrices

$$\left[\begin{array}{c|c|c|c} v_1 & v_2 & \ldots & v_p \end{array}\right]$$

with non-zero upper $p \times p$ minor, **after** multiplying the matrix on the right by the inverse of that upper $p \times p$ submatrix. That is to say, taking as the local coordinates all the entries in rows $(p+1)$-st,$\ldots,(p+t)$-th of the matrix

$$\left[v_j{}^i\right]_{\substack{1\le i\le p+t\\ 1\le j\le p}}\left(\left[v_j{}^i\right]_{\substack{1\le i\le p\\ 1\le j\le p}}\right)^{-1}.$$

That is, these coordinates are all $t\times p$ entries of the matrix

$$\left[v_j{}^i\right]_{\substack{p+1\le i\le p+t\\ 1\le j\le p}}\left(\left[v_j{}^i\right]_{\substack{1\le i\le p\\ 1\le j\le p}}\right)^{-1}.$$

In these coordinates, (7) becomes

$$\mathcal{G}H(u)=\left(u,\ h(u)\,;\ \frac{\partial h}{\partial u}(u)\right), \tag{8}$$

where, under the symbol $\frac{\partial h}{\partial u}(u)$ understood are all the entries of this *Jacobian* $(t\times p)$-matrix written in a row.

We come back to the theorem on Grassmann bundles. We assume without loss of generality that both M and $\widetilde{M}$ are, in the vicinities of x^0, just graphs of C^r mappings, and the parametrizations q and $\tilde{q}$ are the graphs of those mappings. That is,

$$q(u) = (u, f(u)), \text{ where } f\colon U \to \mathbb{R}^{m-p}(y_{p+1}, \ldots, y_m)$$

and

$$\tilde{q}(u) = (u, \tilde{f}(u)), \text{ where } \tilde{f}\colon U \to \mathbb{R}^{m-p}(y_{p+1}, \ldots, y_m).$$

We shall show that the proposition implies the theorem.

Lemma

For $1 \leq l \leq k$ there exists such a local chart on the Grassmannian $G_p\big(M^{(l-1)}\big)$ in which the mapping $\mathcal{G}^{(l)}q$ evaluated at u has the form

$$\left(u,\ f(u);\ \binom{l}{1} \times f_{[1]}(u),\ \binom{l}{2} \times f_{[2]}(u),\ \ldots,\ \binom{l}{l} \times f_{[l]}(u)\right),$$

where $f_{[\nu]}(u)$ is the aggregate of all the partials of the ν-th order at u, of all the components of f, which are in the number $p^{\nu}(m-p)$.

Attention. In this lemma we distinguish mixed derivatives taken in different orders, simply disregarding the Schwarz symmetricity discovery.

Proof. $l = 1$. We note that

$$\mathcal{G}^{(1)}q\,(u) = \Big(u,\ f(u);\ \mathrm{span}\{\partial_j + f_j(u) \colon j = 1, 2, \ldots, p\}\Big)\,,$$

in the relevant chart, is nothing but

$$\big(u,\ f(u);\ f_{[1]}(u)\big)\ =\ \left(u,\ f(u);\ \binom{I}{1} \times f_{[1]}(u)\right).$$

The beginning of induction is done.

$l \Rightarrow l+1$, $l < k$. The mapping $\mathcal{G}^{(l)}q\colon U \to M^{(l)}$, evaluated at u, is already written down, in appropriate local chart assumed to exist in $M^{(l)}$, as

$$\left(u,\, f(u),\, \binom{l}{1} \times f_{[1]}(u),\, \binom{l}{2} \times f_{[2]}(u), \ldots,\, \binom{l}{l} \times f_{[l]}(u)\right). \tag{9}$$

We work with $\mathcal{G}^{(l+1)}q = \mathcal{G}\left(\mathcal{G}^{(l)}q\right)$. Now, (9) being clearly of the form $H(u) = (u,\, h(u))$ in the previously introduced notation, the mapping h reads

$$h(u) = \left(f(u),\, \binom{l}{1} \times f_{[1]}(u),\, \binom{l}{2} \times f_{[2]}(u), \ldots,\, \binom{l}{l} \times f_{[l]}(u)\right).$$

In order to have $\mathcal{G}H(u)$ written down, in view of (8), one ought to write in row: u, then $h(u)$, and then all the entries of the Jacobian matrix $\frac{\partial h}{\partial u}(u)$, also written in row.

The **latter**, in our shorthand notation, are computed immediately. Namely

$$\frac{\partial h}{\partial u}(u) = \left(\binom{l}{0}\times f_{[1]}(u),\ \binom{l}{1}\times f_{[2]}(u),\ \binom{l}{2}\times f_{[3]}(u), \ldots,\ \binom{l}{l}\times f_{[l+1]}(u)\right)$$

These entries on the right hand side are to be juxtaposed with the **former** entries $\big(u,\ h(u)\big)$.

For better readability, we put together the groups of *same* partials.

In view of the elementary identities

$$\binom{l}{\nu-1}+\binom{l}{\nu}=\binom{l+1}{\nu},$$

we get in the outcome

$$\left(u,\ f(u),\ \binom{l+1}{1}\times f_{[1]}(u),\ \binom{l+1}{2}\times f_{[2]}(u),\ldots,\ \binom{l+1}{l+1}\times f_{[l+1]}(u)\right).$$

The lemma is now proved by induction.

We now take $l = k$ in the lemma and get, for arbitrary $u \in U$, two similar expressions for $\mathcal{G}^{(k)}q\,(u)$ and $\mathcal{G}^{(k)}\tilde{q}\,(u)$.

Suppose that the proposition holds for $u = u^0$. As a consequence, the theorem now follows.

Conversely, assuming this theorem, we get that the partial derivatives of q and $\tilde{q}$ at u^0 are mutually equal. This gives the proposition.

A natural question arises: What about branches of algebraic sets which often happen to be tangent one to another with various degrees of closeness?

Let M be a finite-dimensional real analytic manifold, d be a distance function on M induced by a Riemannian metric on M, and let $X, Y \subset M$ be closed subanalytic sets. The following important fact says that X and Y are regularly separated at any x_0:

Theorem (Łojasiewicz) For any $x_0 \in X \cap Y$ there exist $\nu > 0$ and $C > 0$ such for some neighbourhood $\Omega \subset M$ of x_0

$$d(x, X) + d(x, Y) \geq Cd(x, X \cap Y)^{\nu} ,$$

where $x \in \Omega$.

The exponent ν is called a *regular separation exponent* of X and Y at x_0. The infimum of such exponents is called the *Łojasiewicz exponent* and denoted $\mathcal{L}_{x_0}(X, Y)$.

Example Let $C = \{(x, y) \colon (y - x^2)^2 = x^5\}$,

The two branches of C issuing from the point $(0, 0)$,

$$C_- = \{y = x^2 - x^{5/2}\,,\ x \geq 0\} \text{ and } C_+ = \{y = x^2 + x^{5/2}\,,\ x \geq 0\}\,,$$

could be naturally extended to one-dimensional manifolds D_- and D_+, both of class C^2 – the graphs of functions

$$y_-(x) = x^2 - |x|^{5/2} \quad \text{and} \quad y_+(x) = x^2 + |x|^{5/2}\,,$$

respectively. The Taylor polynomials of degree 2 about $x = 0$ of y_- and y_+ coincide. Hence by Proposition 1, D_- and D_+ have at $(0, 0)$ the order of tangency 2.

This example suggests that, in the real algebraic geometry category, it would be suitable to use non-integer measures of closeness. For instance, for the above sets $y_-(x)$ and $y_+(x)$, we may take

$$\sup\{\alpha > 0 : \ y_+(x) - y_-(x) = o(|x|^\alpha) \text{ when } x \to 0\}\,.$$

This generalised order of tangency would be $5/2$ in the above example. This is the minimal regular separation exponent of the semialgebraic sets C_- and C_+. That quantity is also the Łojasiewicz exponent $\mathcal{L}_{(0,0)}\big(C_-,\ C_+\big)$.

This example generalises, for $(y - x^N)^2 = x^{2N+1}$, to a pair of C^N manifolds having the order of tangency N and the minimal separation exponent $\nu = N + \frac{1}{2}$.

Example Here we shall see that the order and exponent can be both integer and different. Consider two curves N and Z in $\mathbb{R}^2(x,y)$ intersecting at $(0,0)$:

$$N = \{y = 0\} \quad \text{and} \quad Z = \{y^d + yx^{d-1} + x^s = 0\},$$

where $1 < d < s$, and assume that d is odd. What is their minimal regular separation exponent at $(0,0)$? We want to present Z as the graph of some function $y(x)$.

Lemma

There is a locally unique function

$$y(x) = x^{s-d+1}z(x) - \ x^{s-d+1}$$

whose graph is Z, with a C^∞ function $z(x)$, $z(0) = 0$.

This is "$-x^{s-d+1}$" which dominates the computation.

Using $y(x)$, we compute the minimal reqular separation exponent. Here is a sketch.

We discuss the inequality defining the regular separation exponent at $(0,0)$. Let $A=(x,0)$ be the points on N, $B=(x,y(x))$ be the points on Z, and let O be the point $(0,0)$. Using the function $y(x)$, the length AB is of order $|x|^{s-d+1}$. Since AO and BO are of order $|x|$, the triangle inequality:

$$AB \leq AO + BO$$

implies the inequality from the Łojasiewicz theorem. We get that the exponent is equal to $s-d+1$.

(The order of tangency is $s-d$.)

Do we have tools to compute the exponents? Consider a singular plane curve

$$x^4 - y^3 + 6x^2y + 6y^2 - 2x^2 - 9y \;=\; 0\,.$$

This curve has two cusp-like 'return' points $P_\pm = (\pm 2,\, -1)$ and a self-intersection point $P_{\mathrm{self}} = (0,\, 3)$, all of them – critical points of the polynomial on the LHS. (The fourth critical point $(0,\, 1)$ lies off the curve.) From each of points $P_\pm$ there emerge a pair of branches. This curve admits a parametrization $x(t) = t^3 - 3t$, $y(t) = t^4 - 2t^2$. Its Taylor expansion about $t_0 = 1$ is

$$\begin{pmatrix} t^3 - 3t \\ t^4 - 2t^2 \end{pmatrix} \;=\; P_- + (t-1)^2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + (t-1)^3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + (t-1)^4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence the Euclidean distance of points of the curve for $t = 1 - \epsilon$ and $t = 1 + \epsilon$ is

$$2\sqrt{17}\,\epsilon^3 + O(\epsilon^4)\,,$$

while the distances of these points to the reference point P_- are asymptotically equal $5\,\epsilon^2$ when $\epsilon \to 0^+$. But

$$2\sqrt{17}\,\epsilon^3 + O(\epsilon^4) = O\big((5\epsilon^2)^{3/2}\big)\,.$$

So the minimal regular exponent is $3/2$. The two branches of the curve (semialgebraic sets!) from P_- are characterised by an inequality $4x - 3y + 5 \le 0$, or else $4x - 3y + 5 \ge 0$. A general theory gives a very loose upper bound

$$\nu \le \frac{(2\cdot 4 - 1)^{2+2} + 1}{2} = 1201.$$

Example (a) Consider a curve in $\mathbb{R}^2(x,y)$:

$$(xy)^2 = \frac{1}{4}\left(x^2+y^2\right)^3.$$

This set possesses a pair (even more than one such pair) of semialgebraic branches touching each other at the point $(0,0)$. Their minimal regular separation exponent is equal to 2.

(b) The set

$$\left(x^2+y^2-\frac{1}{2}x\right)^2 = \frac{1}{4}(x^2+y^2).$$

possesses as well a pair of semialgebraic branches $\{y \leq 0\}$ and $\{y \geq 0\}$ touching each other at the point $(0,0)$. Their minimal regular separation exponent is $3/2$.

A general theory gives unrealistically high estimates.

Thank you!