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A fundamental theorem of the geometry on algebraic surfaces and the splitting principle

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Joint research project with Camilla Felisetti (Trento) inspired by a classical paper of Beniamino Segre (1903-1977):

Beniamino Segre (1938)

Un teorema fondamentale della geometria sulle superficie algebriche ed il principio di spezzamento. Annali di Matematica Pura ed Applicata 17, 107–126 (1938).
https://link.springer.com/article/10.1007/BF02410697

Camilla Felisetti and Claudio Fontanari (2021)

On the Splitting Principle of Beniamino Segre. arXiv e-print (2021).
https://arxiv.org/abs/2105.00892

Today's program:

1. Motivation (p. 4)

2. History (p. 11)

3. Geometry (p. 30)

V smooth complex projective variety of dimension n

$\mathcal{O}_V$ sheaf of holomorphic functions on V

Ω_V sheaf of holomorphic differentials on V

$p_g(V) := h^0(V, \Omega_V^n) = h^0(V, K_V) = h^n(V, \mathcal{O}_V)$ (geometric genus)

$p_a(C) := (-1)^n(\chi(V, \mathcal{O}_V) - 1)$ (arithmetic genus)

C smooth complex projective curve

$$p_g(C) := h^0(C, \Omega_C) = h^0(C, K_C) = h^1(C, \mathcal{O}_C)$$

$$p_a(C) := 1 - \chi(C, \mathcal{O}_C) = 1 - h^0(C, \mathcal{O}_C) + h^1(C, \mathcal{O}_C)$$

$p_g(C) = p_a(C) =: g(C)$ (genus)

S smooth complex projective surface

$$p_g(S) := h^0(S, \Omega^2_S) = h^0(S, K_S) = h^2(S, \mathcal{O}_S)$$

$$\begin{aligned} p_a(S) &:= \chi(S, \mathcal{O}_S) - 1 = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) - 1 \\ &= -h^1(S, \mathcal{O}_S) + p_g(S) \end{aligned}$$

$p_g(S) - p_a(S) = h^1(S, \mathcal{O}_S) =: q(S)$ (irregularity)

Proposition. We have $q(S) := h^1(S, \mathcal{O}_S) = h^0(S, \Omega_S)$.

Proof. Indeed,

$$\begin{aligned} h^1(S, \mathcal{O}_S) = h^1(S, \Omega^0_S) \quad &=: \quad h^{0,1}(S), \\ h^0(S, \Omega_S) = h^0(S, \Omega^1_S) \quad &=: \quad h^{1,0}(S), \end{aligned}$$

and by Hodge symmetry we have $h^{0,1}(S) = h^{1,0}(S)$.

□

But without Hodge theory? (W. V. D. Hodge, The Theory and Applications of Harmonic Integrals, Cambridge University Press, Cambridge, 1941)

Oscar Zariski (1935)

Algebraic Surfaces. Springer-Verlag Berlin Heidelberg (1935).

(...) the quantitative specification that *an algebraic surface of irregularity q possesses complete continuous systems consisting of* ∞^q *distinct linear systems*, is a fundamental result of the theory of surfaces, due to the combined efforts of several geometers (Humbert, Castelnuovo, Enriques, Severi, Picard, Poincaré). (...)
A proof of the above fundamental result has been first proposed by Enriques (1904). Immediately after, another proof was proposed by Severi (1905). Severi himself has later pointed out that neither proof is entirely rigorous (1921). (...)
A rigorous proof of the existence on a surface of irregularity q of continuous systems consisting of ∞^q linear systems was given for the first time by Poincaré (1910). Poincaré's proof is analytic and has been subsequently simplified by Severi (1921) and by Lefschetz (1924).

Proposition. We have $q(S) := h^1(S, \mathcal{O}_S) = h^0(S, \Omega_S)$.

Proof. Indeed, from the above fundamental result it follows that

$$2q(S) = h^1(S, \mathbb{C}).$$

Since $H^{1,0}(S) \cap \overline{H^{1,0}(S)} = 0$ as subspaces of $H^1(S, \mathbb{C})$, we have $2h^0(S, \Omega_S) \leq h^1(S, \mathbb{C}) = 2q(S)$ and $h^0(S, \Omega_S) \leq q(S)$.

On the other hand, from the exact sequence

$$0 \to H^0(S, \Omega_S) \to H^1(S, \mathbb{C}) \to H^1(S, \mathcal{O}_S)$$

it follows that $h^0(S, \Omega_S) + h^1(S, \mathcal{O}_S) \geq h^1(S, \mathbb{C}) = 2q(S)$, hence $h^0(S, \Omega_S) \geq q(S)$.

□

Beniamino Segre (1938)

Un teorema fondamentale della geometria sulle superficie algebriche ed il principio di spezzamento. Annali di Matematica Pura ed Applicata 17, 107–126 (1938).
`https://link.springer.com/article/10.1007/BF02410697`

In the first part of this work we will give what we may call *the first algebro-geometric proof of the fundamental theorem*. Such proof moves from the construction of complete continuous systems in the form given by SEVERI in 1905 (...)
The new idea on which it relies – which provides the essential element of its success – consists in a suitable series of equivalence on a reducible curve, defined as the limit of a linear series over a varying irreducible curve, and in the fact that the dimension of the former cannot be less than the dimension of the latter.

The above proof yields at the same time in a natural way, and without leaving the field of algebraic geometry, to the following *splitting principle*, established in full generality in the second part of this work: (...)

This splitting principle provides a remarkable generalization to the degeneration principle due to ENRIQUES, according to which, if an irreducible algebraic curve E – varying continuously on an algebraic surface F – splits into two components, C and D, then there exists at least one connecting point between them. (...)

Here (in n. 4) we give another proof, purely algebro-geometric and of irreducible simplicity, inspired by the new idea mentioned before.

Proposition (*Degeneration principle*). Let $\{E\}$ be an algebraic system of nodal curves in $\mathbb{P}^k$. Suppose that the generic element E of $\{E\}$ is smooth and irreducible and a special element E_0 of $\{E\}$ splits as $E_0 = C + D$ with C, D irreducible smooth curves. Then $C \cap D \neq \emptyset$.

Proposition (*Principle of connectedness*). Let $\{X_t\}$ be a flat family of closed subschemes of $\mathbb{P}^k$ parameterized by an irreducible curve T. If X_t is connected for a general $t \in T$, then X_t is connected for all $t \in T$.

(R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1977, III, Exercise 11.4 on p. 281)

Segre's argument.

Suppose by contradiction that $E_0 = C + D$ with $C \cap D = \emptyset$. Let n_C, n_D, n_E be the degrees of respectively C, D and E. Also, let g_C, g_D, g_E be the genera of respectively C, D and E. Clearly $n_E = n_C + n_D$, moreover one can show that $g_E = g_C + g_D - 1$. Let $T = |\mathcal{O}_{\mathbb{P}^k}(t)|$ for $t \gg 0$. Then T determines over C, D, E linear series I^γ, I^δ, I^ϵ which, by Riemann-Roch, have dimension respectively

$$\gamma = n_C t - g_C$$
$$\delta = n_D t - g_D$$
$$\epsilon = n_E t - g_E$$

When E varying in $\mathcal{E}$ tends to the limit curve E_0, then l^ϵ tends to a linear series l on $E_0 = C + D$ contained in $l^\gamma + l^\delta$. By semicontinuity we have that

$$\epsilon \leq \dim l$$

and if

$$\epsilon \leq \dim l \leq \dim(l^\gamma + l^\delta) = \dim(l^\gamma) + \dim(l^\delta) = \gamma + \delta \tag{1}$$

then a direct computation would conclude the proof, since we have

$$\gamma + \delta = (n_C + n_D)t - (g_C + g_D) = n_E t - (g_E + 1) = \epsilon - 1,$$

contradicting (1).

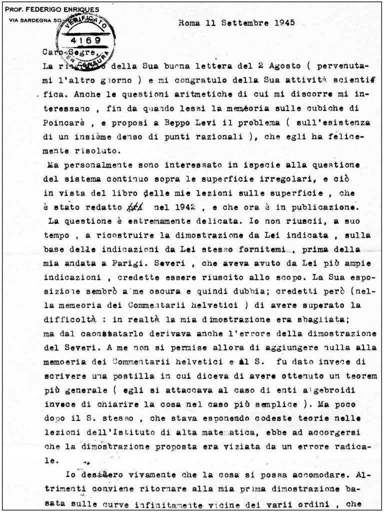
Notice however that the equality $\dim(I^\gamma + I^\delta) = \dim(I^\gamma) + \dim(I^\delta)$ is false; what is true is the corresponding equality between affine dimensions:

$$\dim(I^\gamma + I^\delta) + 1 = (\dim I^\gamma + 1) + (\dim I^\delta + 1),$$

which holds since C and D are supposed to be disjoint.

Federigo Enriques (1945)

Letter to Beniamino Segre (Roma, 11 Settembre 1945), stored in: Beniamino Segre Papers, Box 5, Institute Archives, California Institute of Technology, Pasadena, CA, reproduced and translated in: Donald Babbitt and Judith Goodstein, Federigo Enriques's Quest to Prove the "Completeness Theorem", Notices of the AMS 58, 240–249 (2011). `https://www.ams.org/notices/201102/rtx110200240p.pdf`



PROF. FEDERIGO ENRIQUES
VIA SARDEGNA 50

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Roma 11 Settembre 1945

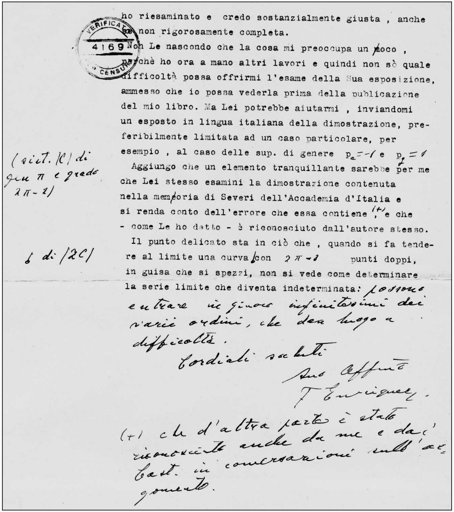
Caro Segre,

La ringrazio della Sua buona lettera del 2 Agosto (pervenutami l'altro giorno) e mi congratulo della Sua attività scientifica. Anche le questioni aritmetiche di cui mi discorre mi interessano , fin da quando lessi la memoria sulle cubiche di Poincarè , e proposi a Beppo Levi il problema (sull'esistenza di un insième denso di punti razionali), che egli ha felicemente risoluto.

Ma personalmente sono interessato in ispecie alla questione del sistema continuo sopra le superficie irregolari, e ciò in vista del libro delle mie lezioni sulle superficie , che è stato redatto nel 1942 , e che ora è in publicazione.

La questione è estremamente delicata. Io non riuscii, a suo tempo , a ricostruire la dimostrazione da Lei indicata , sulla base delle indicazioni da Lei stesso fornitemi , prima della mia andata a Parigi. Severi , che aveva avuto da Lei più ampie indicazioni , credette essere riuscito allo scopo. La Sua esposizione sembrò a me oscura e quindi dubbia; credetti però (nella memoria dei Commentarii helvetici) di avere superato la difficoltà : in realtà la mia dimostrazione era sbagliata; ma dal constatarlo derivava anche l'errore della dimostrazione del Severi. A me non si permise allora di aggiungere nulla alla memoria dei Commentarii helvetici e al S. fu dato invece di scrivere una postilla in cui diceva di avere ottenuto un teorem più generale (egli si attaccava al caso di enti algebroidi invece di chiarire la cosa nel caso più semplice). Ma poco dopo il S. stesso , che stava esponendo codeste teorie nelle lezioni dell'Istituto di alta matematica, ebbe ad accorgersi che la dimostrazione proposta era viziata da un errore radicale.

Io desidero vivamente che la cosa si possa accomodare. Altrimenti conviene ritornare alla mia prima dimostrazione basata sulle curve infinitamente vicine dei varii ordini , che



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ho riesaminato e credo sostanzialmente giusta , anche se non rigorosamente completa.

Non Le nascondo che la cosa mi preoccupa un poco , perchè ho ora a mano altri lavori e quindi non sò quale difficoltà possa offrirmi l'esame della Sua esposizione, ammesso che io possa vederla prima della publicazione del mio libro. Ma Lei potrebbe aiutarmi , inviandomi un esposto in lingua italiana della dimostrazione, preferibilmente limitata ad un caso particolare, per esempio , al caso delle sup. di genere $p_a = -1$ e $p_g = 1$

Aggiungo che un elemento tranquillante sarebbe per me che Lei stesso esamini la dimostrazione contenuta nella memoria di Severi dell'Accademia d'Italia e si renda conto dell'errore che essa contiene (+), e che - come Le ho detto - è riconosciuto dall'autore stesso.

Il punto delicato sta in ciò che , quando si fa tendere al limite una curva con $2\pi - 3$ punti doppi, in guisa che si spezzi, non si vede come determinare la serie limite che diventa indeterminata: possono entrare in giuoco infinitesimi dei varii ordini, che dan luogo a difficoltà.

(sist. |C| di gen π e grado 2π-2)

b di |2C|

Cordiali saluti

Suo Affmo

F. Enriques.

(+) che d'altra parte è stato riconosciuto anche da me e dai Cast. in conversazioni sull'argomento.

Appendix B. Text of a letter from F. Enriques to Beniamino Segre, 11 September 1945, recounting the recent history of his efforts to prove the “Completeness Theorem”. The war had officially ended. But the censor’s stamp in the upper left-hand corner reminds us that postal authorities still opened and read letters. ***Courtesy of California Institute of Technology.***

On September 11, 1945, writing from Rome, Enriques sent a letter to Beniamino Segre in response to a letter Segre had sent to Enriques from Manchester, U.K., on August 2nd. Most of Enriques's letter deals with his ongoing quest to vindicate his 1905 theorem.

Rome, September 11, 1945

Dear Segre, I thank you for your nice letter of August 2 (I received it the other day), and I congratulate you on your scientific activity. (...) *Personally, I am especially interested in the problem of a continuous system on irregular surfaces which was discussed in my 1942 lectures on surfaces, which is now in press. This question is extremely delicate. I was unable at that time to reconstruct the proof that you had indicated on the basis of the information that you had given me before my departure for Paris.*

Severi, with whom you have had more interaction, believed he had finally succeeded in giving a proof. His exposition seems obscure to me and therefore dubious; I believed (in the Commentarii Helvetici paper) to have overcome the difficulty: In reality, my proof was erroneous, but this realization also pointed out the error in Severi's proof. At that time, I was not allowed to add anything to my paper in the Commentarii Helvetici although Severi was allowed to write a note to my paper in which he says that he has obtained a more general theorem. (...)

But shortly afterward, Severi himself, who was expounding that theory in his lectures at the Institute of Higher Mathematics, realized that his proposed proof was flawed due to a radical error. I really wish that this thing could be settled. I have reexamined my earlier proof based on infinitely close curves of various orders and I believe it is substantially right, even if it is not rigorously complete.

In closing, Enriques asks Segre, whose August 2nd letter must have been written in English, a language he became proficient in while living in England during the war, to give him more details in Italian. In particular, he asks Segre to limit himself to a very specific type of surface and continuous system, where Enriques apparently expects there will be the need for infinitesimally close curves of higher order.

David Mumford (1966)

Lectures on curves on an algebraic surface. Princeton University Press, Princeton, New Jersey, 1966.

The goal of these lectures is a complete clarification of one "theorem" on Algebraic surfaces: the so-called completeness of the characteristic linear system of a good complete algebraic system of curves, on a surface F. If the characteristic is 0, this theorem was first proven by Poincaré in 1910 by analytic methods. Until 1960, no algebraic proof of this purely algebraic theorem was known. (Although an endless and depressing controversy obscured this fact). (...)
What was the key, the essential point which the Italians had overlooked? There is no doubt at all that it is the systematic use of nilpotent elements: in particular, a systematic analysis of all families of curves on a surface over a parameter space with only one point, but with non-trivial nilpotent structure sheaf.

David Mumford (2011)

Intuition and Rigor and Enriques's Quest. Notices of the AMS 58, 250–260 (2011).
https://www.ams.org/notices/201102/rtx110200250p.pdf

In the preceding article we have seen that Enriques and, indeed, the whole Italian school of algebraic geometry in the first half of the twentieth century were frustrated by one glaring gap in their theory of algebraic surfaces. (...)
In my own education, I had assumed they were irrevocably stuck, and it was not until I learned of Grothendieck's theory of schemes and his strong existence theorems for the Picard scheme that I saw that a purely algebro-geometric proof was indeed possible. I say here "algebro-geometric", not "geometric", because the first requirement in moving ahead had been the introduction of new algebraic tools into the subject first by Zariski and Weil and subsequently by Serre and Grothendieck. When Professors Babbitt and Goodstein wrote me about Enriques's work in the 1930s, I realized that the full story was more complex. As I see it now, Enriques must be credited with a nearly complete geometric proof *using, as did Grothendieck, higher order infinitesimal deformations.*

In other words, he anticipated Grothendieck in understanding that the key to unlocking the Fundamental Theorem was understanding and manipulating geometrically higher order deformations. Let's be careful: he certainly had the correct ideas about infinitesimal geometry, though he had no idea at all how to make precise definitions. If you compare his ideas here with, for example, the way Leibniz described his calculus, the level of rigor is about the same. To use a fashionable word, his "yoga" of infinitesimal neighborhoods was correct, but basic parts of it needed some nontrivial algebra before they could ever be made into a proper mathematical theory. (...)
In short, Enriques was a visionary. And, remarkably, his intuitions never seemed to fail him (unlike those of Severi, whose extrapolations of known theories were sometimes quite wrong). Mathematics needs such people – and perhaps, with string theory, we are again entering another age in which intuitions run ahead of precise theories.

Camilla Felisetti and Claudio Fontanari (2021)

On the Splitting Principle of Beniamino Segre. arXiv e-print (2021).
https://arxiv.org/abs/2105.00892

Finally we come back to Segre's splitting principle:

Theorem (*Splitting Principle*).
Let $\{E\}$ be an algebraic system of nodal curves on a smooth surface F of geometric genus p_g. Suppose that the general element E of $\{E\}$ is irreducible and that a special element E_0 of $\{E\}$ splits as $E_0 = C + D$ with C, D irreducible. Set $\Gamma := C \cap D$ and let c be the cardinality of Γ. Write $\Gamma = \Gamma_1 \sqcup \Gamma_2$ where Γ_1 is the set of points which are limits of nodes of curves in $\{E\}$ and Γ_2 are the connecting points between C and D originated by the splitting $E_0 = C + D$. Assume that $|D_{|D}| \neq \emptyset$ and that C is sufficiently general with respect to D, in particular that $|C(-\Gamma_1)|$ has no base points on D. If c_i is the cardinality of Γ_i then we have $c_2 \geq p_g + 1$, unless the points in Γ_2 are linearly dependent with respect to K_F.

The assumptions that all curves in $\{E\}$ are nodal and that C is general with respect to D are both missing from Segre's statement.

Note that the splitting of Γ into the disjoint union of Γ_1 and Γ_2 is well-defined only if E_0 is assumed to have double points as singularities (hence it is implicit in Segre's argument, see in particular *Un teorema fondamentale della geometria sulle superficie algebriche ed il principio di spezzamento*, §10, p. 122: *I punti di* Γ *si potranno allora distinguere in due categorie, secondochè provengono o meno come limiti da punti doppi di* E*, ossia rispettivamente a seconda che non risultano oppure risultano punti di collegamento tra C e D; denotiamo ordinatamente con* Γ_1*,* Γ_2 *i gruppi costuituiti dai punti del primo o del secondo tipo,* (...) *talchè sarà* $\Gamma = \Gamma_1 + \Gamma_2$).

For simplicity's sake, we focus on the case of algebraic systems with smooth general element, but essentially the same argument extends to the nodal case:

Theorem (*Splitting Principle*).
Let $\{E\}$ be an algebraic system of nodal curves on a smooth surface F of geometric genus p_g. Suppose that the general element E of $\{E\}$ is a smooth irreducible curve and that a special element E_0 of $\{E\}$ splits as $E_0 = C + D$ with C, D irreducible smooth curves. Assume that $|D_{|D}| \neq \emptyset$ and that C is sufficiently general with respect to D, in particular that $\mid C \mid$ has no base points on D. Set $\Gamma := C \cap D$ and let c be the cardinality of Γ. Then we have $c \geq p_g + 1$, unless the points in Γ are linearly dependent with respect to K_F.

Proof.

First of all, we claim that there is at least one point $P \in \Gamma$ which is not a base point of the complete linear series $|E_{0|D}|$ on D. Indeed, assume by contradiction that

$$h^0(D, E_{0|D}) = h^0(D, E_{0|D} - \Gamma). \tag{2}$$

On the other hand, since $E_0 = C + D$ we have $|E_{0|D}| \supseteq |C_{|D}| + |D_{|D}|$. Moreover, since C intersects D by the Principle of connectedness and $\mid C \mid$ has no base points on D by assumption, we have $h^0(D, C_{|D}) \geq 2$. Hence we deduce

$$\begin{aligned} h^0(D, E_{0|D}) &\geq h^0(D, C_{|D}) + h^0(D, D_{|D}) - 1 \geq h^0(D, D_{|D}) + 1 \\ &= h^0(D, E_{0|D} - C_{|D}) + 1 = h^0(D, E_{0|D} - \Gamma) + 1, \end{aligned}$$

contradicting (2), so the claim is established.

Now we follow Segre's approach. Let $d := h^1(D, D_{|D})$. We have two possibilities: (i) $c \geq d + 1$ or (ii) $d \geq c$.
(i) Suppose $c \geq d + 1$. Let $i := h^2(F, D)$. We first prove that

$$d \geq p_g - i. \tag{3}$$

Indeed, by adjunction $K_{F_{|D}} = K_D - D_{|D}$ and by Serre duality

$$d = h^1(D, D_{|D}) = h^0(D, K_D - D_{|D}) = h^0(D, K_{F_{|D}}).$$

The short exact sequence

$$0 \to K_F(-D) \to K_F \to K_{F|D} \to 0$$

yields a long exact sequence

$$0 \to H^0(K_F(-D)) \to H^0(K_F) \to H^0(K_{F_{|D}}) \to \dots$$

hence $p_g \leq i + d$.

If $i = 0$, we immediately get $c \geq d + 1 \geq p_g + 1$. If instead $i > 0$, then the points in Γ are dependent with respect to K_F, i.e. $h^0(F, K_F(-\Gamma)) > p_g - c$. Indeed, on the one hand by (3) we have

$$p_g - c \leq p_g - d - 1 \leq p_g - p_g + i - 1 = i - 1. \tag{4}$$

On the other hand, since any global section of K_F which vanishes on D vanishes in particular on Γ, we have

$$h^0(F, K_F(-\Gamma)) \geq h^0(F, K_F(-D)) = h^2(F, D) = i.$$

By (4) we conclude that $h^0(F, K_F(-\Gamma)) \geq i > p_g - c$.

(ii) Suppose $c \leq d$. Let $P \in \Gamma$ and set $\Gamma^* := \Gamma \setminus P$ (note that Γ is not empty by the degeneration principle above, but Γ^* might be). Observe first that the linear series

$$|D_{|D} + \Gamma^*|$$

on D is special. In fact, $h^1(D, D_{|D} + \Gamma^*) = h^0(D, K_D - D_{|D} - \Gamma^*) \geq h^0(D, K_D - D_{|D}) - c + 1 = h^1(D, D_{|D}) - c + 1 = d - c + 1 \geq 1$. In particular, by adjunction we have that
$H^0(D, K_D - D_{|D} - \Gamma^*) \cong H^0(D, K_{F_{|D}} - \Gamma^*)$ is nonzero.
We are going to prove that the natural inclusion

$$H^0(F, K_F - \Gamma) \subseteq H^0(F, K_F - \Gamma^*)$$

is an isomorphism for some choice of $P \in \Gamma$, i.e. that the points in Γ are dependent with respect to K_F.

Indeed, by the claim at the beginning of the proof, there exists at least one $P \in \Gamma$ such that the complete linear series

$$|E_{0|D}| = |D_{|D} + \Gamma|$$

does not admit P as a base point.
On the other hand, by the Riemann-Roch theorem

$$h^0(D_{|D} + \Gamma) = h^1(D_{|D} + \Gamma) + \deg(D_{|D} + \Gamma) + 1 - g(D)$$

$$h^0(D_{|D} + \Gamma^*) = h^1(D_{|D} + \Gamma^*) + \deg(D_{|D} + \Gamma) - 1 + 1 - g(D).$$

Since $h^0(D_{|D} + \Gamma^*) = h^0(D_{|D} + \Gamma) - 1$ then $h^1(D_{|D} + \Gamma) = h^1(D_{|D} + \Gamma^*)$ and by Serre duality

$$h^0(D, K_D - D_{|D} - \Gamma^*) = h^0(D, K_D - D_{|D} - \Gamma). \tag{5}$$

Suppose now by contradiction that the inclusion $H^0(F, K_F - \Gamma) \subseteq H^0(F, K_F - \Gamma^*)$ is strict, i.e. that there exists an effective divisor A in $\mathbb{P}H^0(F, K_F - \Gamma^*)$ not passing through P. Note that $A \cap D \neq D$, since $P \in D \setminus A$.
Now, if Γ^* is not empty, then $A \cap D \neq \emptyset$, since $\emptyset \neq \Gamma^* \subset A \cap D$, and $A_{|D}$ is a nontrivial effective divisor on D lying in $\mathbb{P}H^0(D, K_{F_{|D}} - \Gamma^*) \setminus \mathbb{P}H^0(D, K_{F_{|D}} - \Gamma)$, contradicting (5).
The same conclusion holds if Γ^* is empty but $A \cap D \neq \emptyset$. On the other hand, if Γ^* is empty and $A_{|D} = 0$, then $K_{F_{|D}} \cong \mathcal{O}_D$ and by adjunction we have $K_D = D_{|D}$. Hence (5) implies

$$1 = h^0(D, \mathcal{O}_D) = h^0(D, \mathcal{O}_D(-P)) = 0$$

and this contradiction ends the proof.

□

Empires perish, while Euclid theorems keep eternal youth

Vito Volterra