Higher dimensional Calabi-Yau varieties of Kummer type

Dominik Burek

Faculty of Mathematics and Computer Science of the Jagiellonian University

dominik.burek@uj.edu.pl

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Calabi-Yau manifolds

Definition

Calabi-Yau manifold is a complex, smooth, projective (kähler) d-fold X satisfying

1. $K_X = \mathcal{O}_X$,
2. $H^i\mathcal{O}_X = 0$ for $0 < i < d$.

Equivalently:

1. there are no global holomorphic i–forms on X,
2. there exists a nowhere vanishing holomorphic d–form on X.

Crepant resolution

Definition

Let X be a singular complex algebraic variety (with canonical line bundle) and $(\widetilde{X}, \pi)$ a resolution of X with the map $\pi\colon \widetilde{X} \to X$.
We say that that $\widetilde{X}$ is a crepant resolution of X if

$$\pi^*(K_X) = K_{\widetilde{X}}.$$

Theorem (Klein)

Let G be any finite subgroup of $SL_2(\mathbb{C})$. Then surface ${\mathbb{C}^2}/{G}$ admits a crepant resolution.

Surfaces ${\mathbb{C}^2}/{G}$ are called Kleinian singularities or Du Val surface singularities. There is a 1–1 correspondence between non-trivial finite subgroups $G \subset SL_2(\mathbb{C})$ and Dynkin diagrams of type $A_k (k \geq 1)$, $D_k (k \geq 4)$, E_6, E_7 and E_8.

Crepant resolution

The correspondence between Kleinian singularities ${}^{\mathbb{C}^2}/_{G}$, Dynkin diagrams and other areas of mathematics is known as the McKay correspondence.

Theorem (Roan)

Let G be a finite subgroup of $SL_3(\mathbb{C})$. Then ${}^{\mathbb{C}^3}/_{G}$ admits a crepant resolution.

For a subgroup $\{-1, +1\} \subseteq SL_4(\mathbb{C})$, variety ${}^{\mathbb{C}^4}/_{\{-1,+1\}}$ does not admit any crepant resolution!

Cynk-Hulek's Kummer type construction

Let $\mathbb{Z}_d$ be the cyclic group of order d.

Theorem (Cynk-Hulek)

Let E_d be an elliptic curve with an order d automorphism $\phi_d \colon E_d \mapsto E_d$, for $d = 2, 3, 4$. For any $n \in \mathbb{N}$, let

$$G_{d,n} := \{(m_1, m_2, \dots, m_n) \in \mathbb{Z}_d^n \colon m_1 + m_2 + \ldots + m_n = 0\} \simeq \mathbb{Z}_d^{n-1}$$

acts on E_d^n by $\phi_d^{m_i}$ on the i-th factor. There exists a crepant resolution

$$\widetilde{E_d^n / G_{d,n}} \to E_d^n / G_{d,n}.$$

Consequently, $X_{d,n} := \widetilde{E_d^n / G_{d,n}}$ is an n-dimensional Calabi-Yau manifold.

S. Cynk, K. Hulek, *Higher-dimensional modular Calabi-Yau manifolds*, Canad. Math. Bull. **50** (2007), 486–503.

Cynk-Hulek's Kummer type construction for $d = 6$

Let X_1, X_2 be two Calabi-Yau manifolds with automorphisms $\eta_i \colon X_i \to X_i$ (for $i = 1, 2$) of order 6 such that

$$\eta_1^*(\omega_{X_1}) = \zeta_6 \omega_{X_1} \quad \text{and} \quad \eta_2^*(\omega_{X_2}) = \zeta_6^5 \omega_{X_2},$$

where ω_{X_i} denotes a chosen generator of $H^{n,0}(X_i)$, for $i = 1, 2$.
Assume that:

1. the fixed point locus $\mathrm{Fix}(\eta_1)$ of η_1 is a disjoint union of smooth divisors, in particular η_1 has linearisation of the form $(\zeta_6, 1, 1, \ldots, 1)$ near any point of $\mathrm{Fix}(\eta_1)$,
2. $\mathrm{Fix}(\eta_2)$ is a disjoint union of submanifolds of codimension at most 3. In particular η_2 has linearisation of the form
   - $(\zeta_6^5, 1, 1, \ldots, 1)$ near a component of codimension one of $\mathrm{Fix}(\eta_2)$,
   - $(\zeta_6^4, \zeta_6, 1, 1, \ldots, 1)$ or $(\zeta_6^3, \zeta_6^2, 1, 1, \ldots, 1)$ near a component of codimension two of $\mathrm{Fix}(\eta_2)$,

Cynk-Hulek's Kummer type construction for $d = 6$

3. $\mathrm{Fix}(\eta_1^2) \setminus \mathrm{Fix}(\eta_1)$ is a disjoint union of smooth divisors in particular η_1^2 has linearisation $(\zeta_3, 1, 1, \ldots, 1)$ along any component of $\mathrm{Fix}(\eta_1^2) \setminus \mathrm{Fix}(\eta_1)$,
4. $\mathrm{Fix}(\eta_1^3) \setminus \mathrm{Fix}(\eta_1)$ is a disjoint union of smooth divisors in particular η_1^3 has linearisation $(-1, 1, 1, \ldots, 1)$ along any component of $\mathrm{Fix}(\eta_1^3) \setminus \mathrm{Fix}(\eta_1)$,
5. $\mathrm{Fix}(\eta_2^2) \setminus \mathrm{Fix}(\eta_2)$ is a disjoint union of smooth submanifolds of codimension at most 2, so η_2^2 has linearisation of the form $(\zeta_3^2, 1, 1, \ldots, 1)$ or $(\zeta_3, \zeta_3, 1, 1, \ldots, 1)$ along any component of $\mathrm{Fix}(\eta_2^2) \setminus \mathrm{Fix}(\eta_2)$,
6. $\mathrm{Fix}(\eta_2^3) \setminus \mathrm{Fix}(\eta_2)$ is a disjoint union of smooth divisors, so η_2^3 has linearisation of the form $(-1, 1, 1, \ldots, 1)$ along any component of $\mathrm{Fix}(\eta_2^3) \setminus \mathrm{Fix}(\eta_2)$,
7. the automorphism η_2 has a local linearisation of the form $(\zeta_6^2, \zeta_6^2, \zeta_6, 1, 1, \ldots, 1)$ along any codimensional 3 component of $\mathrm{Fix}(\eta_2)$.

Cynk-Hulek's Kummer type construction for $d = 6$

Proposition

Under the above assumptions the quotient ${X_1 \times X_2}/{\eta_1 \times \eta_2}$ admits a crepant resolution of singularities $\widetilde{{X_1 \times X_2}/{\eta_1 \times \eta_2}}$. Furthermore $\mathrm{id} \times \eta_2$ induces an automorphism of order 6 on $\widetilde{{X_1 \times X_2}/{\eta_1 \times \eta_2}}$ that satisfies all assumption we put on η_2.

Proof

The automorphism $\eta := \eta_1 \times \eta_2$ has a local linearisation around any fixed point of one of the following types:

1. $(\zeta_6, \zeta_6^5, 1, 1, \ldots, 1)$ which corresponds to singularity of type $\frac{1}{6}(1,5)$,
2. $(\zeta_6, \zeta_6, \zeta_6^4, 1, 1, \ldots, 1)$ which corresponds to singularity of type $\frac{1}{6}(1,1,4)$,
3. $(\zeta_6, \zeta_6^2, \zeta_6^3, 1, 1, \ldots, 1)$ which corresponds to singularity of type $\frac{1}{6}(1,2,3)$,
4. $(\zeta_6, \zeta_6, \zeta_6^2, \zeta_6^2, 1, 1, \ldots, 1)$ which corresponds to singularity of type $\frac{1}{6}(1,1,2,2)$.

Proof

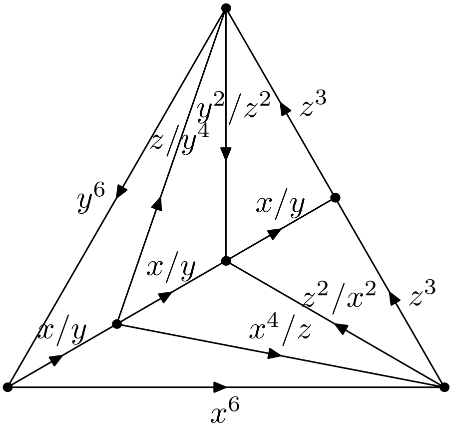
1. If η has a local linearisation given by $(\zeta_6, \zeta_6^5, 1, 1, \dots, 1)$ near $\mathrm{Fix}(\eta)$, then in local coordinates, the map from $X_1 \times X_2$ to the resolution is given in affine charts by

$$\left(x^6, \frac{y}{x^5}\right), \left(\frac{x^5}{y}, \frac{y^2}{x^4}\right), \left(\frac{x^4}{y^2}, \frac{y^3}{x^3}\right), \left(\frac{x^3}{y^3}, \frac{y^4}{x^2}\right), \left(\frac{x^2}{y^4}, \frac{y^5}{x}\right) \text{ or } \left(\frac{x}{y^5}, y^6\right).$$

The action of $\mathrm{id} \times \eta_2$ has a linearisation $(1, \zeta_6^5, 1, \dots, 1)$, so it lifts to the resolution as $(1, \zeta_6^5)$, (ζ_6, ζ_6^4), (ζ_6^2, ζ_6^3), (ζ_6^3, ζ_6^2), (ζ_6^4, ζ_6) and $(\zeta_6^5, 1)$, respectively.

Proof

2. If η has a local linearisation given by $(\zeta_6, \zeta_6, \zeta_6^4, 1, 1, \ldots, 1)$ near $\mathrm{Fix}(\eta)$, then we can use a toric resolution of $\frac{1}{6}(1,1,4)$ singularity.



A. Crew, M. Reid, *How to calculate A-Hilb* $\mathbb{C}^3$, Geometry of toric varieties, 129–154, Séminaires et Congrés 6, SMF, Paris, 2002.

Proof

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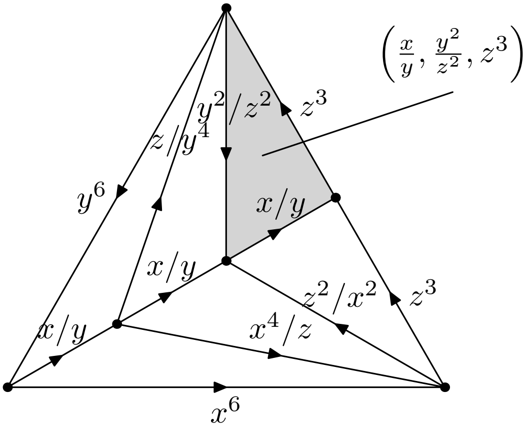

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Proof

Thus the map from $X_1 \times X_2$ to the resolution is given in affine charts as

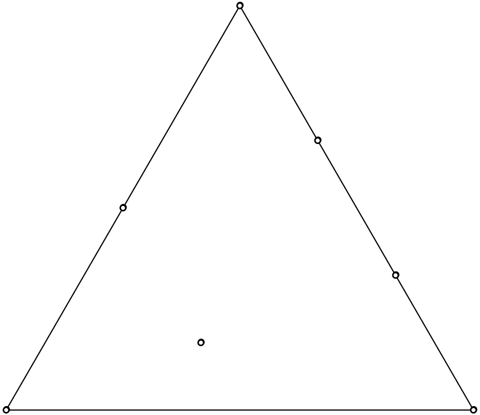
$$\left(x^6, \frac{y}{x}, \frac{z}{x^4}\right), \quad \left(\frac{x^4}{z}, \frac{y}{x}, \frac{z^2}{x^2}\right), \quad \left(\frac{x^2}{z^2}, \frac{y}{x}, z^3\right), \quad \left(\frac{x}{y}, y^6, \frac{z}{y^4}\right),$$

$$\left(\frac{x}{y}, \frac{y^4}{z}, \frac{z^2}{y^2}\right) \quad \text{or} \quad \left(\frac{x}{y}, \frac{y^2}{z^2}, z^3\right).$$

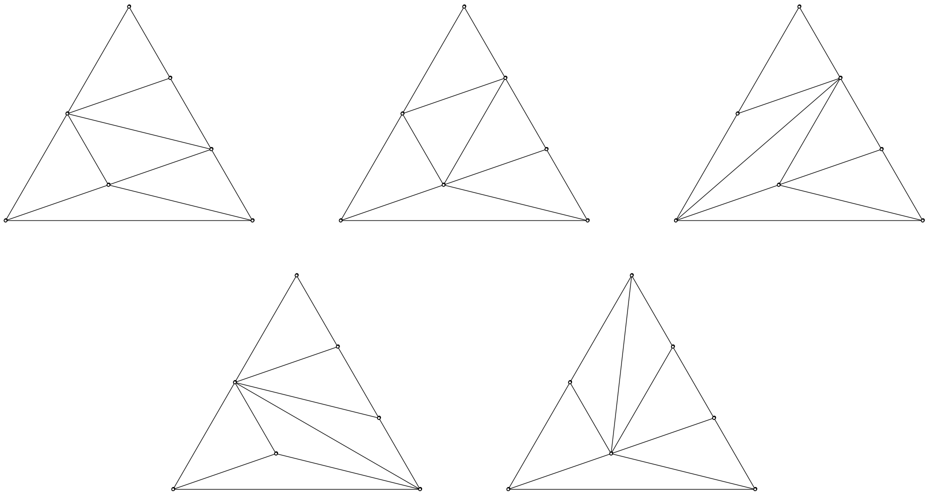
Therefore the action of $\mathrm{id} \times \eta_2$ lifts to the resolution as $(1, \zeta_6, \zeta_6^4)$, $(\zeta_6^2, \zeta_6, \zeta_6^2)$, $(\zeta_6^4, \zeta_6, 1)$, $(\zeta_6^5, 1, 1)$, $(\zeta_6^5, 1, 1)$, $(\zeta_6^5, 1, 1)$.

Proof

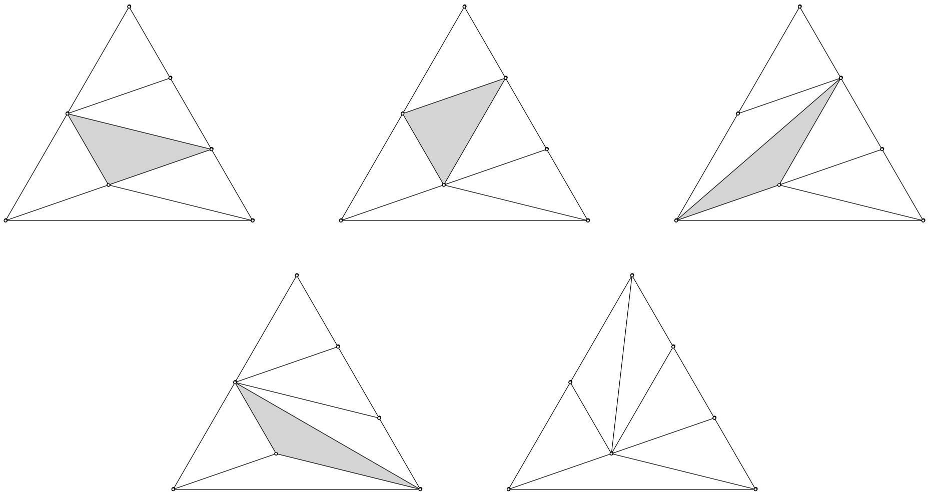
3. If η has a local linearisation given by $(\zeta_6, \zeta_6^2, \zeta_6^3, 1, 1, \dots, 1)$ near $\mathrm{Fix}(\eta)$, then we use again toric resolution of $\frac{1}{6}(1,2,3)$ singularity. There are five different decompositions of junior simplex which give a toric resolution.



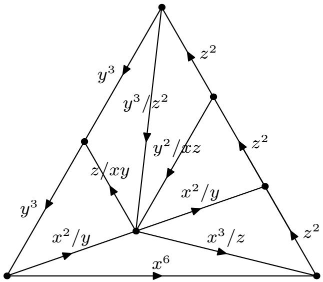
Proof



Proof



Proof



The map from $X_1 \times X_2$ to the resolution is given in affine charts as:

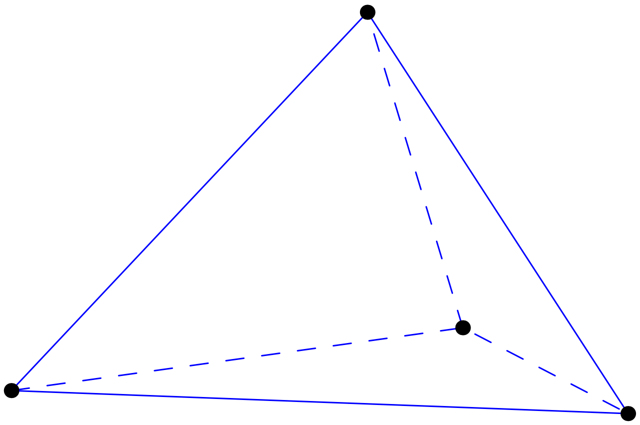
$$\left(x^6, \frac{z}{x^3}, \frac{y}{x^2}\right), \quad \left(\frac{x^3}{z}, z^2, \frac{y}{x^2}\right), \quad \left(\frac{x^2}{y}, z^2, \frac{y^2}{xz}\right), \quad \left(\frac{xz}{y^2}, z^2, \frac{y^3}{z^2}\right),$$

$$\left(\frac{z^2}{y^3}, y^3, \frac{xy}{z}\right) \quad \text{or} \quad \left(\frac{z}{xy}, y^3, \frac{x^2}{y}\right).$$

The action of $\mathrm{id} \times \eta_2$ has a local linearisation $(1, \zeta_6^2, \zeta_6^3, 1, \dots, 1)$, hence it lifts to the resolution as $(1, \zeta_6^3, \zeta_6^2)$, $(\zeta_6^3, 1, \zeta_6^2)$, $(\zeta_6^4, 1, \zeta_6)$, $(\zeta_6^5, 1, 1)$, $(1, 1, \zeta_6^5)$, $(\zeta_6, 1, \zeta_6^4)$, respectively.

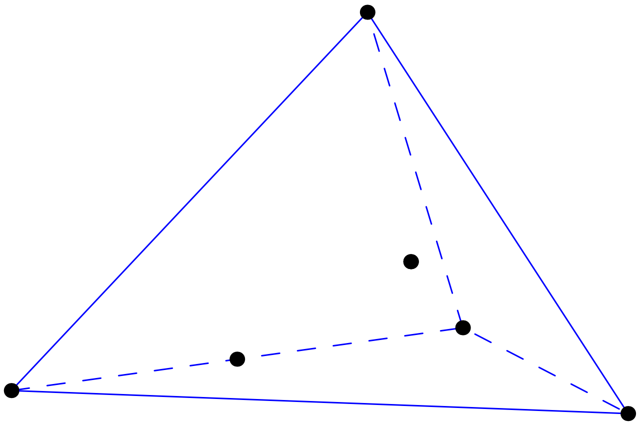
Proof

4. If η has a local linearisation given by $(\zeta_6, \zeta_6, \zeta_6^2, \zeta_6^2, 1, 1, \ldots, 1)$ near $\mathrm{Fix}(\eta_2)$.

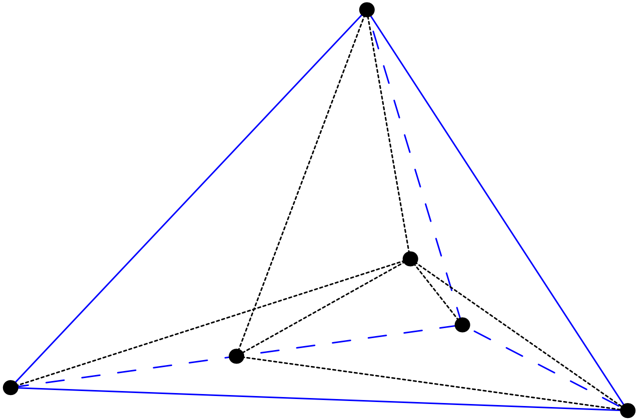


Proof

4. If η has a local linearisation given by $(\zeta_6, \zeta_6, \zeta_6^2, \zeta_6^2, 1, 1, \ldots, 1)$ near $\mathrm{Fix}(\eta_2)$.



4. If η has a local linearisation given by $(\zeta_6, \zeta_6, \zeta_6^2, \zeta_6^2, 1, 1, \ldots, 1)$ near $\mathrm{Fix}(\eta_2)$.



Proof

The map is given by

$$\left(x^6, \frac{y}{x}, \frac{z}{x^2}, \frac{t}{x^2}\right), \quad \left(\frac{x^2}{z}, \frac{y}{x}, z^3, \frac{t}{z}\right), \quad \left(\frac{x^2}{t}, \frac{y}{x}, t^3, \frac{z}{t}\right), \quad \left(\frac{x}{y}, y^6, \frac{z}{y^2}, \frac{t}{y^2}\right),$$

$$\left(\frac{x}{y}, \frac{y^2}{z}, z^3, \frac{t}{z}\right) \quad \text{or} \quad \left(\frac{x}{y}, \frac{y^2}{t}, \frac{z}{t}, t^3\right).$$

The action of $\mathrm{id} \times \eta_2$ has a local linearisation $(1, \zeta_6, \zeta_6^2, \zeta_6^2, 1, 1, \ldots, 1)$, hence it lifts to the resolution as $(1, \zeta_6, \zeta_6^2, \zeta_6^2)$, $(\zeta_6^4, \zeta_6, 1, 1)$, $(\zeta_6^4, \zeta_6, 1, 1)$, $(\zeta_6^5, 1, 1, 1)$, $(\zeta_6^5, 1, 1, 1)$, $(\zeta_6^5, 1, 1, 1)$.

Finally near the points of $\mathrm{Fix}(\eta^2) \setminus \mathrm{Fix}(\eta)$ and $\mathrm{Fix}(\eta^3) \setminus \mathrm{Fix}(\eta)$ we first consider the quotient $X_1 \times X_2 / \eta^2$ (resp. $X_1 \times X_2 / \eta^3$), we construct crepant resolutions of

$$\widetilde{\left(X_1 \times X_2 / \eta^2\right)} / \eta^3 \qquad \text{resp.} \qquad \widetilde{\left(X_1 \times X_2 / \eta^3\right)} / \eta^2.$$

□

Cynk-Hulek's Kummer type construction for $d = 6$

Let $X_1, X_2, \ldots, X_n$ be Calabi-Yau manifolds with automorphisms ϕ_i of order 6 such that

- $\phi_i^*(\omega_{X_i}) = \zeta_6 \omega_{X_i}$ where ω_{X_i} is a canonical form on X_i,
- ϕ_1 satisfies the assumptions we put on η_1 in proposition,
- ϕ_i^5 satisfies, for $i = 2, \ldots, n$, the assumptions we put on η_2 in proposition.

Proposition

The quotient of the product $X_1 \times X_2 \times \ldots \times X_n$ by the action of $G_{6,n}$ has a crepant resolution of singularities which is a Calabi-Yau manifold and such that the action of $\mathbb{Z}_6^n$ on $X_1 \times X_2 \times \ldots \times X_n$ lifts to a purely non-symplectic action of $\mathbb{Z}_6$ on this resolution.

Theorem

There exists a crepant resolution

$$\widetilde{E_6^n / G_{6,n}} \to E_6^n / G_{6,n}.$$

Borce-Voisin construction

Theorem

Let S_d be a $K3$ surface admitting a purely non-symplectic automorphism α_S of order $d = 2, 3, 4, 6$. Let E_d be an elliptic curve admitting an automorphism α_{E_d} of order d. Then ${S_d \times E_d}/{\alpha_{S_d} \times \alpha_{E_d}^{n-1}}$, is a singular variety which admits a crepant resolution of singularities $\widetilde{{S_d \times E_d}/{\alpha_{S_d} \times \alpha_{E_d}^{d-1}}}$. In particular $\widetilde{{S_d \times E_d}/{\alpha_{S_d} \times \alpha_{E_d}^{d-1}}}$ is a Calabi-Yau threefold.

C. Voisin, *Miroirs et involutions sur les surfaces $K3$*, Astérisque, (218):273–323, 1993. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992).

C. Borcea, *$K3$ surfaces with involution and mirror pairs of Calabi-Yau manifolds*, Mirror symmetry, II, 717–743, AMS/IP Stud. Adv. Math. 1, Amer. Math. Soc. Providence, RI, 1997.

A. Cattaneo, A. Garbagnati, *Calabi-Yau 3-folds of Borcea-Voisin type and elliptic fibrations*, Tohoku Math. J. **68** (2016), no. 4, 515–558.

Generalization of Borce-Voisin construction

Take $d \in \{2,3,4,6\}$ and let S_d be a $K3$-surface with non-symplectic automorphism γ_d of order d. Moreover, let E_d be elliptic curves admitting automorphisms α_d of order d. The following group

$$G_{d,n} := \{(m_1, m_2, \ldots, m_n) \in \mathbb{Z}_d^n \colon m_1 + m_2 + \ldots + m_n = 0\} \simeq \mathbb{Z}_d^{n-1}$$

acts on $S_d \times E_d^{n-1}$ by $(\gamma_d)^{m_1}$ on the first factor and $\alpha_d^{m_i}$ on the i-th factor, where $2 \leq i \leq n$. Moreover $G_{d,n}$ preserves canonical bundle of $S_d \times E_d^{n-1}$.

Theorem

There exists crepant resolution $Y_{d,n}$ of the quotient variety ${S_d \times E_d^{n-1}}/{G_{d,n}}$. In particular $Y_{d,n}$ is $(n+1)$-dimensional Calabi-Yau variety.

Orbifold's formula

Definition

For a variety ${}^{X}/_{G}$ define the Chen-Ruan cohomology by

$$H^{i,j}_{\mathrm{orb}}\left({}^{X}/_{G}\right) := \bigoplus_{[g]\in \mathrm{Conj}(G)} \left(\bigoplus_{U\in\Lambda(g)} H^{i-\mathrm{age}(g),\, j-\mathrm{age}(g)}(U) \right)^{\mathsf{C}(g)},$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G (we choose a representative g of each conjugacy class), $\mathsf{C}(g)$ is the centralizer of g, $\Lambda(g)$ denotes the set of irreducible connected components of the set fixed by $g \in G$ and $\mathrm{age}(g)$ is the age of the matrix of linearized action of g near a point of U.
The dimension of $H^{i,j}_{\mathrm{orb}}\left({}^{X}/_{G}\right)$ will be denoted by $h^{i,j}_{\mathsf{orb}}\left({}^{X}/_{G}\right)$ and it is called the orbifold Hodge number.

W. Chen, Y. Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248(1)**, 1–31, 2004.

Orbifold's formula

Definition

Consider $M \in GL_n(\mathbb{C})$ of finite order. Then M has eigenvalues $e^{2\pi i a_1}, e^{2\pi i a_2}, \ldots, e^{2\pi i a_m}$, where $a_1, a_2, \ldots, a_m \in [0,1) \cap \mathbb{Q}$ are uniquely defined up to order. The value of the sum $a_1 + a_2 + \ldots + a_m$ is called the age of M and is denoted by $\mathrm{age}(M)$.

Theorem (Yasuda)

Let G be a finite group acting on an algebraic smooth variety X. If there exists a crepant resolution $\widetilde{X/G}$ of variety X/G, then the following equality holds

$$h^{i,j}\left(\widetilde{X/G}\right) = h^{i,j}_{\mathrm{orb}}\left(X/G\right).$$

T. Yasuda, *Twisted jets, motivic measure and orbifold cohomology*, Compos. Math. **140** (2004), 396–422.

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Hodge numbers

Let X_i be a variety with automorphism $\phi_{i,d}\colon X_i \to X_i$ of order d for $i = 1, 2, \ldots, n$. Consider the following group

$$G_{d,n} := \{(m_1, m_2, \ldots, m_n) \in \mathbb{Z}_d^n \colon m_1 + m_2 + \ldots + m_n = 0\} \simeq \mathbb{Z}_d^{n-1}$$

which acts on $X_1 \times X_2 \times \ldots \times X_n$ by $\phi_{i,d}^{m_i}$ on the i-th factor. Suppose that there exists a crepant resolution $\widetilde{\mathcal{X}_{d,n}}$ of the quotient variety

$$\mathcal{X}_{d,n} := {X_1 \times X_2 \times \ldots \times X_n}/{\mathbb{Z}_d^{n-1}}.$$

Let

$$F_{X_i,k,j}(X,Y) := \text{Poincaré polynomial (in two variables } X, Y\text{) of } H^{**}\left(\operatorname{Fix}\left(\left(\phi_{i,d}\right)^k\right)_{\zeta_d^j}\right)$$

for $i \in \{1, 2, \ldots, n\}$ and $k, j \in \{0, 1, \ldots, d-1\}$.

Hodge numbers

Assume that $\mathrm{Fix}(\phi_{i,d}^{k})$ is a divisor. As

$$\mathrm{age}(\phi_{i,d}^{m_1} \times \phi_{i,d}^{m_2} \times \ldots \times \phi_{i,d}^{m_n}) = \frac{m_1 + m_2 + \ldots + m_n}{d}$$

we get the following

Theorem

$$h^{p,q}(\widetilde{\mathcal{X}_{d,n}}) = \sum_{j=0}^{d-1} \prod_{i=1}^{n} \left(\sum_{k=0}^{d-1} \sqrt[d]{(XY)^k} \cdot F_{X_i,k,j} \right) [X^p Y^q].$$

Hodge numbers

$k \backslash j$	0	1	$\cdots$	j	$\cdots$	$d-1$
0	$F_{X_i,0,0}$	$F_{X_i,0,1}$		$F_{X_i,0,j}$		$F_{X_i,0,d-1}$
1	$F_{X_i,1,0}$	$F_{X_i,1,1}$		$F_{X_i,1,j}$		$F_{X_i,1,d-1}$
2	$F_{X_i,2,0}$	$F_{X_i,2,1}$	$\cdots$	$F_{X_i,2,j}$	$\cdots$	$F_{X_i,2,d-1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$d-1$	$F_{X_i,d-1,0}$	$F_{X_i,d-1,1}$		$F_{X_i,d-1,j}$		$F_{X_i,d-1,d-1}$

Hodge numbers

$k \backslash j$	0	1	$\cdots$	j	$\cdots$	$d-1$
0	$F_{X_i,0,0}$	$F_{X_i,0,1}$		$F_{X_i,0,j}$		$F_{X_i,0,d-1}$
1	$F_{X_i,1,0}$	$F_{X_i,1,1}$		$F_{X_i,1,j}$		$F_{X_i,1,d-1}$
2	$F_{X_i,2,0}$	$F_{X_i,2,1}$	$\cdots$	$F_{X_i,2,j}$	$\cdots$	$F_{X_i,2,d-1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$d-1$	$F_{X_i,d-1,0}$	$F_{X_i,d-1,1}$		$F_{X_i,d-1,j}$		$F_{X_i,d-1,d-1}$

$$\sum_{k} F_{X_i,k,j} \;\stackrel{\cdot\cdot}{=}\; \upsilon_{X_i,j}$$

Hodge numbers

In order to compute $h^{p,q}(\widetilde{\mathcal{X}_{d,n}})$ we compute scalar product of vectors $v_{X_i,j}$ and

$$v_d := \left(1, \sqrt[d]{(XY)}, \sqrt[d]{(XY)^2}, \dots, \sqrt[d]{(XY)^{d-1}}\right)$$

for $1 \leq j \leq n$. Then we multiply all values of $v_{X_i,j} \circ v_d$ for $i \in \{1, 2, \dots, n\}$ and add all products for $j \in \{0, 1, \dots, d-1\}$.

Example – $X_{6,n}$

Let E_6 be an elliptic curve with the Weierstrass equation $y^2 = x^3 + 1$, and automorphism $\alpha_6(x, y) = (\zeta_6^2 x, -y)$, where ζ_6 denotes a fixed 6-th root of unity satisfying $\zeta_6^2 = \zeta_3$, then

$k \backslash j$	0	1	2	3	4	5
0	$1 + XY$	X	0	0	0	Y
1	1	0	0	0	0	0
2	2	0	0	1	0	0
3	2	0	1	0	1	0
4	2	0	0	1	0	0
5	1	0	0	0	0	0

Table: $F_{E_6,k,j}(X, Y)$

Example – $X_{6,n}$

$$h^{p,q}\left(\widetilde{E_6^n/G_{6,n}}\right)=$$

$$=\Big\{\left((1+XY)\cdot\sqrt[6]{(XY)^0}+1\cdot\sqrt[6]{XY}+2\cdot\sqrt[6]{(XY)^2}+2\cdot\sqrt[6]{(XY)^3}+2\cdot\sqrt[6]{(XY)^4}+1\cdot\sqrt[6]{(XY)^5}\right)^n+$$

$$+\left(X\cdot\sqrt[6]{(XY)^0}+0\cdot\sqrt[6]{XY}+0\cdot\sqrt[6]{(XY)^2}+0\cdot\sqrt[6]{(XY)^3}+0\cdot\sqrt[6]{(XY)^4}+0\cdot\sqrt[6]{(XY)^5}\right)^n+$$

$$+\left(0\cdot\sqrt[6]{(XY)^0}+0\cdot\sqrt[6]{XY}+0\cdot\sqrt[6]{(XY)^2}+1\cdot\sqrt[6]{(XY)^3}+0\cdot\sqrt[6]{(XY)^4}+0\cdot\sqrt[6]{(XY)^5}\right)^n+$$

$$+\left(0\cdot\sqrt[6]{(XY)^0}+0\cdot\sqrt[6]{XY}+1\cdot\sqrt[6]{(XY)^2}+0\cdot\sqrt[6]{(XY)^3}+1\cdot\sqrt[6]{(XY)^4}+0\cdot\sqrt[6]{(XY)^5}\right)^n+$$

$$+\left(0\cdot\sqrt[6]{(XY)^0}+0\cdot\sqrt[6]{XY}+0\cdot\sqrt[6]{(XY)^2}+1\cdot\sqrt[6]{(XY)^3}+0\cdot\sqrt[6]{(XY)^4}+0\cdot\sqrt[6]{(XY)^5}\right)^n+$$

$$+\left(Y\cdot\sqrt[6]{(XY)^0}+0\cdot\sqrt[6]{XY}+0\cdot\sqrt[6]{(XY)^2}+0\cdot\sqrt[6]{(XY)^3}+0\cdot\sqrt[6]{(XY)^4}+0\cdot\sqrt[6]{(XY)^5}\right)^n\Big\}[X^pY^q]=$$

$$=\Big\{X^n+Y^n+\left(1+XY+\sqrt[6]{XY}+2\sqrt[6]{(XY)^2}+2\sqrt[6]{(XY)^3}+2\sqrt[6]{(XY)^4}+\sqrt[6]{(XY)^5}\right)^n+$$

$$+2\cdot(XY)^{\frac{n}{2}}+\left(\sqrt[6]{(XY)^2}+\sqrt[6]{(XY)^4}\right)^n\Big\}[X^pY^q].$$

Hodge numbers of $X_{d,n}$

Theorem

The Hodge number $h^{p,q}(X_{d,n}) = \left\{F_{X_{d,n}}(X,Y)\right\}[X^pY^q]$ of the manifold $X_{d,n}$ is equal to

$$\begin{cases} \left\{(X+Y)^n + \left(XY + 4\sqrt{XY} + 1\right)^n\right\}[X^pY^q] & \text{if } d = 2, \\ \left\{X^n + Y^n + \left(1 + \sqrt[3]{XY}\right)^{3n}\right\}[X^pY^q] & \text{if } d = 3, \\ \left\{X^n + Y^n + \left(1 + XY + 2\sqrt[4]{XY} + 3\sqrt[4]{(XY)^2} + 2\sqrt[4]{(XY)^3}\right)^n + \left(\sqrt[4]{(XY)^2}\right)^n\right\}[X^pY^q] & \text{if } d = 4, \\ \left\{X^n + Y^n + \left(1 + XY + \sqrt[6]{XY} + 2\sqrt[6]{(XY)^2} + 2\sqrt[6]{(XY)^3} + 2\sqrt[6]{(XY)^4} + \sqrt[6]{(XY)^5}\right)^n + \right. & \\ \left. + 2\cdot(XY)^{\frac{n}{2}} + \left(\sqrt[6]{(XY)^2} + \sqrt[6]{(XY)^4}\right)^n\right\}[X^pY^q] & \text{if } d = 6. \end{cases}$$

D. Burek, *Higher-dimensional Calabi-Yau manifolds of Kummer type*, Math. Nach. **4** (2020), 638–650.

$X_{6,3}$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 84 & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & 84 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

$X_{6,4}$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 272 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 0 & & 1132 & & 0 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 272 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

$X_{6,5}$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 695 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 7645 & & 0 & & 0 & \\
1 & & 0 & & 0 & & 0 & & 0 & & 1 \\
 & 0 & & 0 & & 7645 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 695 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

Example – $Y_{3,n}$

- S_3 – $K3$ surfaces with a non-symplectic automorphism $\gamma_3\colon S_3 \to S_3$ of order 3,
- E_3 – elliptic curve with the Weierstrass equation $y^2 = x^3 + 1$, and automorphism α_3 is given by $\alpha_3(x,y) = (\zeta_3 x, y)$,
- $r = \dim H^2(S_3, \mathbb{C})^{\gamma_3}$,
- $m = \dim H^2(S_3, \mathbb{C})_{\zeta_3}$,
- $\mathrm{Fix}\,(\gamma_3) = \mathrm{Fix}\,(\gamma_3^2) = \{f_1, f_2, f_3\}$,
- $\mathrm{Fix}\,(\gamma_3) = L_1 \cup L_2 \cup \ldots \cup L_{k-1} \cup C \cup \{P_1, P_2, \ldots, P_h\}$, where
  - the set $\{L_1, L_2, \ldots L_{k-1}\} \cup \{C\}$ consists of curves which are fixed by γ_3 together with the curve C of maximal genus $g(C)$, in fact L_i are rational,
  - $\{P_1, P_2, \ldots, P_n\}$ is the set of points which are fixed by γ_3.

In this case

$$\mathrm{age}(\phi_{i,d}^{m_1} \times \phi_{i,d}^{m_2} \times \ldots \times \phi_{i,d}^{m_n}) = \frac{m_1 + m_2 + \ldots + m_n}{d}$$

except the case of $m_1 = 1$ and an isolated fixed point when

$$\mathrm{age}(\phi_{i,d}^{m_1} \times \phi_{i,d}^{m_2} \times \ldots \times \phi_{i,d}^{m_n}) = \frac{m_1 + m_2 + \ldots + m_n}{d} + 1$$

Example – $Y_{3,n}$

$k \backslash j$	0	1	2
0	$(XY)^2 + r \cdot XY + 1$	$X^2 + (m-1) \cdot XY$	$Y^2 + (m-1) \cdot XY$
1	$k + h \cdot XY + g(C) \cdot (X+Y) + k \cdot XY$	0	0
2	$k + h + g(C) \cdot (X+Y) + k \cdot XY$	0	0

Table: $F_{S_3,k,j}(X,Y)$ with correction

$k \backslash j$	0	1	2
0	$1 + XY$	X	Y
1	3	0	0
2	3	0	0

Table: $F_{E_3,k,j}(X,Y)$

Example – $Y_{3,n}$

$$h^{p,q}\left(\widetilde{S_3\times E_3^{n-1}\big/\mathbb{Z}_3^{n-1}}\right)=\Bigg\{\Bigg(\Big((XY)^2+r\cdot XY+1\Big)\cdot\sqrt[3]{(XY)^0}+$$

$$+(k+h\cdot XY+g(C)\cdot(X+Y)+k\cdot XY)\cdot\sqrt[3]{XY}+(k+h+g(C)\cdot(X+Y)+k\cdot XY)\cdot\sqrt[3]{(XY)^2}\Bigg)\times$$

$$\times\left((1+XY)\cdot\sqrt[3]{(XY)^0}+3\cdot\sqrt[3]{XY}+3\cdot\sqrt[3]{(XY)^2}\right)^{n-1}+$$

$$+\left((X^2+(m-1)\cdot XY)\cdot\sqrt[3]{(XY)^0}+0\cdot\sqrt[3]{XY}+0\cdot\sqrt[3]{(XY)^2}\right)\cdot\left(X\cdot\sqrt[3]{(XY)^0}+0\cdot\sqrt[3]{XY}+0\cdot\sqrt[3]{(XY)^2}\right)^{n-1}+$$

$$+\left((Y^2+(m-1)\cdot XY)\cdot\sqrt[3]{(XY)^0}+0\cdot\sqrt[3]{XY}+0\cdot\sqrt[3]{(XY)^2}\right)\cdot\left(Y\cdot\sqrt[3]{(XY)^0}+0\cdot\sqrt[3]{XY}+0\cdot\sqrt[3]{(XY)^2}\right)^{n-1}\Bigg\}[X^pY^q]=$$

$$=\Bigg\{\Bigg((XY)^2+r\cdot XY+1+\Big(k+h\cdot XY+g(C)\cdot(X+Y)+k\cdot XY\Big)\cdot\sqrt[3]{XY}+\Big(k+h+g(C)\cdot(X+Y)+$$

$$+k\cdot XY\Big)\cdot\sqrt[3]{(XY)^2}\Bigg)\cdot\Big(1+XY+3\sqrt[3]{XY}+3\sqrt[3]{(XY)^2}\Big)^{n-1}+\Big(X^2+(m-1)\cdot XY\Big)\cdot X^{n-1}+$$

$$+\Big(Y^2+(m-1)\cdot XY\Big)\cdot Y^{n-1}\Bigg\}[X^pY^q]$$

$Y_{3,2}$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

- $h^{1,1} = r + 3h + 6k + 1$
- $h^{1,2} = m - 1 + 6g(C)$

$Y_{3,3}$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & 0 & & h^{2,1} & & h^{1,2} & & 0 & \\
1 & & h^{3,1} & & h^{2,2} & & h^{1,3} & & 1 \\
 & 0 & & h^{3,2} & & h^{2,3} & & 0 & \\
 & & 0 & & h^{3,3} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

- $h^{1,1} = r + 6h + 21k + 20$
- $h^{2,2} = 2 + 42k + 30h + 20r$
- $h^{2,1} = 21g(C)$
- $h^{3,1} = m - 1$

$Y_{3,4}$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & h^{1,1} & & 0 & & & \\
 & & 0 & & h^{2,1} & & h^{1,2} & & 0 & & \\
 & 0 & & 0 & & h^{2,2} & & 0 & & 0 & \\
1 & & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,4} & & 1 \\
 & 0 & & 0 & & h^{3,3} & & 0 & & 0 & \\
 & & 0 & & h^{4,3} & & h^{3,4} & & 0 & & \\
 & & & 0 & & h^{4,4} & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

- $h^{1,1} = r + 9h + 45k + 84$
- $h^{2,2} = 85 + 297k + 162h + 84r$
- $h^{2,1} = 45g(C)$
- $h^{3,2} = 252g(C)$
- $h^{4,1} = m - 1$

Zeta functions

Let X_i be a variety with automorphism $\phi_{i,d}\colon X_i \to X_i$ of order d for $i = 1, 2, \ldots, n$. Consider the following group

$$G_{d,n} := \{(m_1, m_2, \ldots, m_n) \in \mathbb{Z}_d^n \colon m_1 + m_2 + \ldots + m_n = 0\} \simeq \mathbb{Z}_d^{n-1}$$

which acts on $X_1 \times X_2 \times \ldots \times X_n$ by $\phi_{i,d}^{m_i}$ on the i-th factor. Suppose that there exists a crepant resolution $\widetilde{\mathcal{X}_{d,n}}$ of the quotient variety

$$\mathcal{X}_{d,n} := {X_1 \times X_2 \times \ldots \times X_n}/{\mathbb{Z}_d^{n-1}}.$$

Let

$$Z_{X_i,k,j}(X,Y) := \prod_{\ell=0}^{2\dim X_i} \det\left(1 - \mathrm{Frob}_q^* t \mid H^{**}\left(\mathrm{Fix}(\phi_{i,d}^k)_{\zeta_d^j}\right)\right)^{(-1)^{\ell+1}}$$

for $i \in \{1, 2, \ldots, n\}$ and $k, j \in \{0, 1, \ldots, d-1\}$.

Assume that $\mathrm{Fix}(\phi_{i,d}^{k})$ is a divisor. Again using

$$\mathrm{age}(\phi_{i,d}^{m_1} \times \phi_{i,d}^{m_2} \times \ldots \times \phi_{i,d}^{m_n}) = \frac{m_1 + m_2 + \ldots + m_n}{d}$$

we get the following

Theorem

$$Z_q(\widetilde{\mathcal{X}_{d,n}}) = \left(\prod_{j=0}^{d-1} \bigotimes_{i=1}^{n} \left(\prod_{k=0}^{d-1} Z_{X_{i,k,j}} \left(q^{\frac{k}{d}} T \right) \right) \right)^{(-1)^{n+1}}$$

Zeta functions

$k \backslash j$	0	1	$\cdots$	j	$\cdots$	$d-1$
0	$Z_{X_i,0,0}$	$Z_{X_i,0,1}$		$Z_{X_i,0,j}$		$Z_{X_i,0,d-1}$
1	$Z_{X_i,1,0}$	$Z_{X_i,1,1}$		$Z_{X_i,1,j}$		$Z_{X_i,1,d-1}$
2	$Z_{X_i,2,0}$	$Z_{X_i,2,1}$	$\cdots$	$Z_{X_i,2,j}$	$\cdots$	$Z_{X_i,2,d-1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$d-1$	$Z_{X_i,d-1,0}$	$Z_{X_i,d-1,1}$		$Z_{X_i,d-1,j}$		$Z_{X_i,d-1,d-1}$

Zeta functions

$k \backslash j$	0	1	$\cdots$	j	$\cdots$	$d-1$
0	$Z_{X_i,0,0}$	$Z_{X_i,0,1}$		$Z_{X_i,0,j}$		$Z_{X_i,0,d-1}$
1	$Z_{X_i,1,0}$	$Z_{X_i,1,1}$		$Z_{X_i,1,j}$		$Z_{X_i,1,d-1}$
2	$Z_{X_i,2,0}$	$Z_{X_i,2,1}$	$\cdots$	$Z_{X_i,2,j}$	$\cdots$	$Z_{X_i,2,d-1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$d-1$	$Z_{X_i,d-1,0}$	$Z_{X_i,d-1,1}$		$Z_{X_i,d-1,j}$		$Z_{X_i,d-1,d-1}$

$$\upsilon_{X_i,j}$$

Zeta functions

In order to compute $Z^q(\widetilde{\mathcal{X}_{d,n}})$ we evaluate vector $v_{X_{i,j}}$ on

$$v_d := \left(T, \sqrt[d]{q}T, \sqrt[d]{q^2}T, \dots, \sqrt[d]{q^{d-1}}T\right)$$

and multiply all its terms. Then we take tensor product for all $i \in \{1, 2, \dots, n\}$ and take product over $j \in \{0, 1, \dots, d-1\}$. Finally we take $(-1)^{n+1}$ power of the result.

Example

Let S_6 be an elliptic $K3$ surface whose Weierstrass equation is

$$y^2 = x^3 + \lambda(z-1)^2 z^5$$

with the following ζ_6-action:

$$\alpha\colon (x, y, t) \to (\zeta_3^2 x, y, z).$$

Let E_6 be an elliptic curve E_6 with the Weierstrass equation $y^2 = x^3 + 1$ together with a non-symplectic automorphism of order 6.

Example

$k \backslash j$	0	1	2	3	4	5
0	$\frac{1}{(1-T)(1-qT)}$	$1-\alpha_q T$	1	1	1	$1-\overline{\alpha_q} T$
1	$\frac{1}{1-T}$	1	1	1	1	1
2	$\frac{1}{(1-T)^2}$	1	1	$\frac{1}{1-T}$	1	1
3	$\frac{1}{(1-T)^2}$	1	$\frac{1}{1-T}$	1	$\frac{1}{1-T}$	1
4	$\frac{1}{(1-T)^2}$	1	1	$\frac{1}{1-T}$	1	1
5	$\frac{1}{1-T}$	1	1	1	1	1

Table: $Z_{E_6,,j}(T)$

Example

$k \backslash j$	0	1	2	3	4	5
0	$\frac{1}{(1-T)(1-qT)^{19}(1-q^2T)}$	$\frac{1}{1-\beta_q T}$	1	$\frac{1}{1-c_q qT}$	1	$\frac{1}{1-\overline{\beta_q}T}$
1	$\frac{1}{(1-T)^3(1-qT)^{18}}$	1	1	1	1	1
2	$\frac{1}{(1-T)^6(1-qT)^{15}}$	1	1	1	1	1
3	$\frac{1}{(1-T)^{10}(1-qT)^{10}}$	1	$1-\delta_q T$	1	$1-\overline{\delta_q}T$	1
4	$\frac{1}{(1-T)^{15}(1-qT)^6}$	1	1	1	1	1
5	$\frac{1}{(1-T)^{18}(1-qT)^3}$	1	1	1	1	1

Table: $Z_{S_6,k,j}(T)$ with correction

Example

$$
\begin{aligned}
Z_q\left(\widetilde{S_6\times E_6/\mathbb{Z}_6}\right) = \Bigg[\Bigg(& \frac{1}{(1-T)(1-q\cdot T)}\cdot\frac{1}{(1-\sqrt[6]{q}\cdot T)}\cdot\frac{1}{(1-\sqrt[6]{q^2}\cdot T)^2}\cdot\frac{1}{(1-\sqrt[6]{q^3}\cdot T)^2}\cdot \\
&\cdot\frac{1}{(1-\sqrt[6]{q^4}\cdot T)^2}\cdot\frac{1}{(1-\sqrt[6]{q^5}\cdot T)}\Bigg)\otimes\Bigg(\frac{1}{(1-T)(1-q\cdot T)^{19}(1-q^2\cdot T)}\cdot\frac{1}{(1-\sqrt[6]{q}\cdot T)^3(1-\sqrt[6]{q}\cdot q\cdot T)^{18}}\cdot \\
&\cdot\frac{1}{(1-\sqrt[6]{q^2}\cdot T)^6(1-\sqrt[6]{q^2}\cdot q\cdot T)^{15}}\cdot\frac{1}{(1-\sqrt[6]{q^3}\cdot T)^{10}(1-\sqrt[6]{q^3}\cdot q\cdot T)^{10}}\cdot\frac{1}{(1-\sqrt[6]{q^4}\cdot T)^{15}(1-\sqrt[6]{q^4}\cdot q\cdot T)^6}\cdot \\
&\cdot\frac{1}{(1-\sqrt[6]{q^5}\cdot T)^{18}(1-\sqrt[6]{q^5}\cdot q\cdot T)^3}\Bigg)\Bigg]\times\left[(1-\alpha_q T)\otimes\left(\frac{1}{1-\beta_q T}\right)\right]\times\left[\left(\frac{1}{1-\sqrt[6]{q^3}\cdot T}\right)\otimes(1-\sqrt[6]{q^3}\cdot\delta_q\cdot T)\right]\times \\
&\times\left[\left(\frac{1}{1-\sqrt[6]{q^2}\cdot T}\cdot\frac{1}{1-\sqrt[6]{q^4}\cdot T}\right)\otimes\left(\frac{1}{1-c_q\cdot q\cdot T}\right)\right]\times\left[\left(\frac{1}{1-\sqrt[6]{q^3}\cdot T}\right)\otimes(1-\sqrt[6]{q^3}\cdot\overline{\delta_q}\cdot T)\right]\times \\
&\times\left[(1-\overline{\alpha_q}\cdot T)\otimes\left(\frac{1}{1-\overline{\beta_q}\cdot T}\right)\right]^{-1} = \frac{(1-\alpha_q\beta_q T)(1-\delta_q T)(1-\overline{\delta_q}T)(1-\overline{\alpha_q}\overline{\beta_q}T)}{(1-T)(1-qT)^{103}(1-q^2T)^{103}(1-q^3T)}.
\end{aligned}
$$

$Y_{6,2}$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 103 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 103 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Example

$$Z_q\left(\widetilde{S_6\times E_6^2/_{\mathbb{Z}_6}}\right)=\frac{1}{(1-T)(1-qT)^{340}\left(1-\overline{\alpha_q}^2\overline{\beta_q}T\right)(1-\alpha_q{}^2\beta_q T)(1-q^2T)^{1402}(1-q^3T)^{340}(1-q^2c_qT)^2(1-q^4T)}$$

$$Z_q\left(\widetilde{S_6\times E_6^3/_{\mathbb{Z}_6}}\right)=\frac{(1-\alpha_q{}^3\beta_q T)(1-q^2\delta_q T)\left(1-q^2\overline{\delta_q}T\right)\left(1-\overline{\alpha_q}^3\overline{\beta_q}T\right)}{(1-T)(1-qT)^{868}(1-q^2T)^{9548}(1-q^2c_qT)(1-q^3c_qT)(1-q^3T)^{9548}(1-q^4T)^{868}(1-q^5T)}$$

$Y_{6,3}$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 340 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 0 & & 1404 & & 0 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 340 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

$Y_{6,4}$

					1					
				0		0				
			0		868		0			
		0		0		0		0		
	0		0		9549		0		0	
1		0		1		1		0		1
	0		0		9549		0		0	
		0		0		0		0		
			0		868		0			
				0		0				
					1					

Zariski Calabi-Yau manifolds

Definition

An algebraic variety X of dimension n, over an algebraically closed field of characteristic p is called a *Zariski variety* if there exists a purely inseparable dominant rational map $\mathbb{P}^n \longrightarrow X$ of degree p.

T. Katsura and M. Schütt, *Zariski K3 surfaces*, Rev. Mat. Iberoam. **43** (2019), 869–894.

Let $E_{3,i}$ be the elliptic curve given by the equation $y_i^2 + y_i = x_i^3$, for $i \in \{1, 2, \dots, n\}$ with the ζ_3 action $\tau_3 \colon (x, y) \mapsto (\zeta_3 x, y)$ and consider groups

$$F_i := \left\langle (\tau_3, 1, \dots, 1, \tau_3^i),\ (1, \tau_3, 1, \dots, 1, \tau_3^i),\ \dots,\ (1, \dots, 1, \tau_3, \tau_3^i) \right\rangle \simeq \mathbb{Z}_3^{n-1} \simeq G_{3,n},$$

for $i = 1, 2$.

Lemma

The quotient variety $Z_{3,n} := {E_{3,1} \times E_{3,2} \times \dots \times E_{3,n}} / {F_1}$ is rational.

Proof

The monomial $x_1^{i_1} x_2^{i_2} \cdot \ldots \cdot x_n^{i_n} y_1^{j_1} y_2^{j_2} \cdot \ldots \cdot y_n^{j_n}$ is invariant under F_1 iff $3 \mid i_n + i_k$ for $1 \le k < n$, thus

$$\mathbb{C}[Z_{3,n}] \simeq \mathbb{C}[y_1, y_2, \ldots, y_n, x_1 x_2 \ldots x_{n-1} x_n^2, x_1^2 x_2^2 \ldots x_{n-1}^2 x_n].$$

Now let $z := \dfrac{x_1 x_2 \ldots x_{n-1}}{x_n}$ and observe that $\mathbb{C}(Z_{3,n}) = \mathbb{C}(y_1, y_2, \ldots, y_n, z)$, since

$$x_1 x_2 \ldots x_{n-1} x_n^2 = z(y_n^2 + y_n) \quad \text{and} \quad x_1^2 x_2^2 \ldots x_{n-1}^2 x_n = z^2(y_n^2 + y_n).$$

Moreover we have the following relation

$$(y_1^2 + y_1)(y_2^2 + y_2) \cdot \ldots \cdot (y_{n-1}^2 + y_{n-1}) = z^3 (y_n^2 + y_n).$$

Taking $\alpha := \dfrac{y_n}{y_{n-1}}$, we get the equation

$$(y_1^2 + y_1)(y_2^2 + y_2) \cdot \ldots \cdot (y_{n-2}^2 + y_{n-2})(y_{n-1} + 1) = z^3 \alpha(\alpha y_{n-1} + 1),$$

from which we can compute y_{n-1} and $y_n = \alpha y_{n-1}$ as rational functions in y_1, y_2, $\ldots$, y_{n-2}, z, α. Hence the variety $Z_{3,n}$ is rational. □

Now, consider a prime number $p \equiv 2 \pmod 3$ and the supersingular elliptic curve E_3 over a field k, such that $\zeta_3 \in k$ and $\operatorname{char} k = p$, defined by equation $y^2 + y = x^3$, and with the ζ_3 action $\tau_3 \colon (x, y) \mapsto (\zeta_3 x, y)$. The endomorphism ring of E_3 may be represented as

$$\operatorname{End}(E_3) = \mathbb{Z} \oplus \mathbb{Z}F \oplus \mathbb{Z}\tau_3 \oplus \mathbb{Z}\frac{(1+F)(2+\tau_3)}{3},$$

where F is a Frobenius morphism of E_3, with the relation $F\tau_3 = \tau_3^2 F$ (Katsura).

T. Katsura, *Generalized Kummer surfaces and their unirationality in characteristic p*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 34 (1987), 1–41.

The following diagram

$$\begin{array}{ccc} E_3^n & \xrightarrow{F_1} & E_3^n \\ {\scriptstyle 1\times\cdots\times 1\times F}\downarrow & \circlearrowleft & \downarrow{\scriptstyle 1\times\cdots\times 1\times F} \\ E_3^n & \xrightarrow[F_2]{} & E_3^n \end{array}$$

leads to purely inseparable rational map $E_3^n/_{F_1} \longrightarrow E_3^n/_{F_2}$ of degree p. Therefore

Theorem

The Calabi-Yau manifold $\widetilde{E_3^n/_{F_2}} = X_{3,n}$ is a Zariski manifold.

Zariski Calabi-Yau manifolds

Taking a supersingular elliptic curve E_4 defined by the equation $y^2 = x^3 - x$ with order 4 automorphism $\tau_4(x, y) = (-x, iy)$ and supersingular elliptic curve E_6 defined by the equation $y^2 + y = x^3$, with order 6 automorphism $\tau_6(x, y) = (\zeta_3 x, -y - 1)$ we have an analogous theorem

Theorem

The Calabi-Yau manifolds

$$\widetilde{E_4^n / \mathbb{Z}_4^{n-1}} = X_{4,n} \quad \text{and} \quad \widetilde{E_6^n / \mathbb{Z}_6^{n-1}} = X_{6,n}$$

are Zariski manifolds.

Corollary

In any odd characteristic $p \not\equiv 1 \pmod{12}$ there exists a unirational Calabi-Yau manifold of arbitrary dimension.