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Schubert Calculus and Its Applications in Combinatorics and Representation Theory

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Preface

With roots in enumerative geometry and Hilbert's 15th problem, modern Schubert Calculus studies classical and quantum intersection rings on spaces with symmetries, such as flag manifolds. The presence of symmetries leads to particularly rich structures, and it connects Schubert Calculus to many branches of mathematics, including algebraic geometry, combinatorics, representation theory, and theoretical physics. For instance, the study of the quantum cohomology ring of a Grassmann manifold combines all these areas in an organic way. The current volume showcases some of the newest developments in the subject, as presented at the "International Festival in Schubert Calculus", a conference held at Sun Yat-sen University in Guangzhou, China, during November 6–10, 2017.

The event included a 1-day mini-school and a 4-day international conference entitled "Trends in Schubert Calculus". There were over 80 participants, more than one half of which were international, from countries such as Australia, France, Germany, India, Japan, Korea, Poland, Russia, United Kingdom, and the U.S.A. This event continued the tradition of conferences in Schubert Calculus with large international participation; there were three such conferences in the past decade (Toronto 2010, Osaka 2012 and Bedlewo 2015).

The current volume includes 12 papers authored by some of the speakers, covering a large array of topics, including several high-quality surveys. Each of the papers was refereed by two anonymous experts in the field. This volume could not have existed without the combined efforts of the authors and referees, and we are grateful for everyone's contribution.

Problems with roots in classical Schubert Calculus attracted significant attention. The factorial Grothendieck polynomials, which investigate polynomials representing Schubert classes in K theory, were discussed in the paper by Matsumura and Sugimoto. The related problem of finding formulas for cohomology classes of various degeneracy loci is addressed in a paper by Darondeau and Pragacz. Yet another problem with roots in Schubert's classical work, that of finding formulas for the order of contact between manifolds, is investigated in the paper by Domitrz, Mormul, and Pragacz. Finally, Duan and Zhao address and survey Schubert's classical problem of characteristics, and find presentations for the integral cohomology rings of flag manifolds, including those of exceptional Lie types.

There are rich connections between Schubert Calculus and the combinatorics of symmetric functions and polynomials. A survey by Pechenik and Searles highlights the properties of some of the most important bases of polynomials relevant for geometry. Expanding on this, a topic of current high interest is to relate and apply Schubert Calculus methods to problems in (combinatorial and geometric) representation theory.

Three papers in the volume address such connections: Anderson and Nigro investigate the geometric Satake correspondence in relation to minuscule Schubert Calculus; McGlade, Ram, and Yang wrote a survey on the combinatorics and geometry of integrable representations of quantum affine algebras with a particular focus on level 0; this is motivated by Schubert Calculus on semi-infinite flag manifolds. Finally, Su and Zhong wrote a survey showcasing applications of Maulik and Okounkov's theory of stable envelopes on the cotangent bundle of a flag variety to various problems in geometry and representation theory. This recent direction, which one may call "Cotangent Schubert Calculus", is closely related to the study of characteristic classes of singular varieties; from this viewpoint, it is studied by Fehér, Rimányi, and Weber.

Methods and questions from Schubert Calculus can be adapted to varieties related to flag manifolds or to generalizations of the cohomology ring. A survey by Abe and Horiguchi is investigating the properties of the cohomology rings of Hessenberg varieties; these generalize the usual flag varieties, the Peterson variety, and the Springer fibre. In another direction, Hudson, Matsumura, and Perrin address the problem of defining stable Bott-Samelson classes in the algebraic cobordism; this is closely related to the outstanding problem of defining Schubert classes in more general oriented cohomology theories.

Finally, a paper by Kim, Oh, Ueda, and Yoshida gives an expository account of quasimap theory, and proves a generalization of toric residue mirror symmetry to Grassmannians.

These papers provide a broad overview of current interests in Schubert Calculus and related areas. We would like to thank again all the anonymous referees for their invaluable help, and the Springer editorial staff for the assistance with various technical parts. Finally, we are grateful to Sun Yat-sen University for generously providing funds for this conference and to Springer Nature for providing us the opportunity to publish these papers.

All papers in this volume have been refereed and are in a final form. No version of any of them will be submitted for publication elsewhere.

Guangzhou, China — Jianxun Hu
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May 2020

Contents

Factorial Flagged Grothendieck Polynomials

Tomoo Matsumura and Shogo Sugimoto

Abstract The *factorial flagged Grothendieck polynomials* are defined by flagged set-valued tableaux of Knutson–Miller–Yong [10]. We show that they can be expressed by a Jacobi–Trudi type determinant formula, generalizing the work of Hudson–Matsumura [8]. As an application, we obtain alternative proofs of the tableau and the determinant formulas of vexillary double Grothendieck polynomials, which were originally obtained by Knutson–Miller–Yong [10] and Hudson–Matsumura [8] respectively. Furthermore, we show that each factorial flagged Grothendieck polynomial can be obtained by applying K-theoretic divided difference operators to a product of linear polynomials.

Keywords Factorial grothendieck polynomials · Flagged partitions · Flagged set-valued tableaux · Vexillary permutations · Jacobi–Trudi formula · Double grothendieck polynomials

1 Introduction

The *double Grothendieck polynomials* introduced by Lascoux and Schützenberger [11, 12] represent the torus-equivariant K-theory classes of the structure sheaves of Schubert varieties in the flag varieties. Their combinatorial formula in terms of pipe dreams or rc graphs was obtained by Fomin–Kirillov [4, 5]. By restricting to Grassmannian elements, or more generally, vexillary permutations, Knutson–Miller–Yong [10] expressed the associated double Grothendieck polynomials as *factorial flagged Grothendieck polynomials* defined in terms of flagged set-valued tableaux. This can be regarded as a unification of the work of Wachs [17] and Chen–Li–Louck [3] on flagged tableaux and the work of Buch [2] and McNamara [16] on set-valued

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tableaux. On the other hand, the first author, in the joint work [7] with Hudson, Ikeda and Naruse, obtained a determinant formula of the double Grothendieck polynomials associated to Grassmannian elements (cf. [14]). Such an explicit closed formula was later generalized to the vexillary case in [1, 8]. Motivated by these results, the first author [15] studied (non-factorial, or single) flagged Grothendieck polynomials and showed their determinant formula in general, beyond the ones given by vexillary permutations.

In this paper, we extend the results in [15] to the factorial (or double) case. Let $x = (x_i)_{i\in\mathbb{Z}_{>0}}$ and $b = (b_i)_{\mathbb{Z}_{>0}}$ be infinite sequences of variables and β a formal variable. We denote $u \oplus v = u + v + \beta uv$ for variables u and v. Consider a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ with a *flagging* $f = (0 < f_1 \leq \cdots \leq f_r)$. Let $\mathcal{T}(\lambda, f)$ be the set of flagged set-valued tableaux of the flagged partition (λ, f). For each $T \in \mathcal{T}(\lambda, f)$, we denote

$$[x|b]^T = \prod_{e\in T} \left(x_{val(e)} \oplus b_{val(e)-r(e)+c(e)}\right),$$

where the product runs over all entries e of T, $val(e)$ denotes the numeric value of e, and $r(e)$ (resp. $c(e)$) denotes the row (resp. column) index of e. Following the work [9, 10] of Knutson–Miller–Yong, we define the *factorial flagged Grothendieck polynomial* associated to (λ, f) by

$$G_{\lambda,f}(x|b) := \sum_{T\in\mathcal{T}(\lambda,f)} \beta^{|T|-|\lambda|}[x|b]^T.$$

The following is the main result of this paper.

Main Theorem (Theorem 3.5). For a flagged partition (λ, f) of length r, we have

$$G_{\lambda,f}(x|b) = \det\left(\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s \mathcal{G}^{[f_i, f_i+\lambda_i-i]}_{\lambda_i+j-i+s}\right)_{1\leq i,j\leq r}$$

where $\mathcal{G}^{[p,q]}_m = \mathcal{G}^{[p,q]}_m(x|b)$, $m \in \mathbb{Z}$, $p, q \in \mathbb{Z}_{\geq 0}$, are polynomials in x and b with coefficients in $\mathbb{Z}[\beta]$, defined by the generating function

$$\sum_{m\in\mathbb{Z}} \mathcal{G}^{[p,q]}_m u^m = \frac{1}{1+\beta u^{-1}} \prod_{1\leq i\leq p} \frac{1+\beta x_i}{1-x_i u} \prod_{1\leq i\leq q} (1 + (u + \beta)b_i).$$

Our proof is a generalization of the ones in [15, 17] to the factorial (double) case. We prove that both the tableau and determinant expressions satisfy the same compatibility with divided difference operators, allowing us to show that they coincide by induction. The first author employed a similar argument in order to show a tableau formula for double Grothendieck polynomials associated to 321-avoiding permutations [13]. If we specialize our formula at $\beta = 0$, we recover the result by

Chen–Li–Louck [3] for flagged double Schur functions. Our proof is different from theirs, which is based on the lattice-path interpretation of the tableau expression. It is also worth stressing that our proof is completely self-contained. In particular, as a consequence of our main theorem, we obtain a purely algebraic and combinatorial proof of the formula

$$\mathcal{G}_m^{[p,p+m-1]}(x|b) = \sum_{T \in \mathcal{T}((m),(p))} [x|b]^T.$$

To the best of the authors' knowledge, the only proof of this formula available in the literature uses a geometric argument established in [7].

As an application, we obtain an alternative proof of the tableau and determinant formulas of vexillary double Grothendieck polynomials obtained by Knutson–Miller–Yong [10] and Hudson–Matsumura [8] respectively. We also generalize it to arbitrary factorial flagged Grothendieck polynomials: we show that each of them can be obtained by consecutively applying divided difference operators to a product of linear polynomials. The corresponding results for (non-factorial) flagged Schur and Grothendieck polynomials were obtained in [15, 17] respectively.

2 Flagged Grothendieck Polynomials

Let $x = (x_i)_{\mathbb{Z}_{>0}}$ and $b = (b_i)_{\mathbb{Z}_{>0}}$ be sets of infinitely many variables. Let $\mathbb{Z}[\beta]$ be the polynomial ring of a formal variable β where we set $\deg \beta = -1$. Let $\mathbb{Z}[\beta][x, b]$ and $\mathbb{Z}[\beta][[x, b]]$ be the rings of polynomials and of formal power series in x and b respectively. Throughout the paper, for each $f \in \mathbb{Z}[\beta][[x, b]]$, let $f^\star$ be the element obtained from f by increasing by 1 the index of all the x_i's. We use the generalized binomial coefficients $\binom{n}{i}$ given by $(1+x)^n = \sum_{i \geq 0} \binom{n}{i} x^i$ for $n \in \mathbb{Z}$ with the convention that $\binom{n}{i} = 0$ for all integers $i < 0$.

2.1 *Flagged Partitions and Their Tableaux*

A *partition* λ of length r is a weakly decreasing sequence of positive integers $(\lambda_1, \ldots, \lambda_r)$. We identify λ with its *Young diagram* $\{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq r\}$, depicted as a left-aligned array of boxes such that in the ith row from the top there are exactly λ_i boxes. Let $|\lambda|$ be the total number of boxes in the Young diagram of λ, *i.e.* $|\lambda| = \lambda_1 + \cdots + \lambda_r$. A *flagging* f of a partition of length r is a weakly increasing sequence of positive integers $(f_1, \ldots, f_r)$. We call the pair (λ, f) a *flagged partition*.

A *set-valued tableau* T of shape λ is an assignment of a finite subset of positive integers to each box of the Young diagram of λ, satisfying the requirement that

- weakly increasing in each row: $\max A \leq \min B$ if A fills the box (i, j) and B fills the box $(i, j+1)$ for $1 \leq i \leq r$ and $1 \leq j \leq \lambda_i - 1$.
- strictly increasing in each column: $\max A < \min B$ if A fills the box (i, j) and B fills the box $(i+1, j)$ for $1 \leq i \leq r-1$ and $1 \leq j \leq \lambda_{i+1}$.

An element e of a subset assigned to a box of T is called an *entry* of T and denoted by $e \in T$. The total number of entries in T is denoted by $|T|$. For each $e \in T$, let $val(e)$ be its numeric value, $r(e)$ its row index, and $c(e)$ its column index. A *flagged set-valued tableau* of a flagged partition (λ, f) is a set-value tableau of shape λ additionally satisfying that the sets assigned to the boxes of the i-th row are subsets of $\{1, \ldots, f_i\}$. Let $\mathcal{T}(\lambda, f)$ be the set of all flagged set-valued tableaux of (λ, f).

For each tableau $T \in \mathcal{T}(\lambda, f)$, we define

$$[x|b]^T := \prod_{e \in T} (x_{val(e)} \oplus b_{val(e)-r(e)+c(e)})$$

where we set $u \oplus v := u + v + \beta uv$. Following Knutson–Miller–Yong ([9, 10]), we define the *factorial flagged Grothendieck polynomial* $G_{\lambda,f}(x|b)$ as follows.

Definition 2.1 For a flagged partition (λ, f), we define

$$G_{\lambda,f} = G_{\lambda,f}(x|b) = \sum_{T \in \mathcal{T}(\lambda,f)} \beta^{|T|-|\lambda|}[x|b]^T.$$

If λ is an empty partition, we set $G_{\lambda,f} = 1$, and if $f_1 = 0$, we set $G_{\lambda,f} = 0$.

Example 2.2 Let $\lambda = (3, 1)$ and $f = (2, 4)$. Then $\mathcal{T}(\lambda, f)$ contains tableaux such as

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 12 \\ \hline 23 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 12 & 2 & 2 \\ \hline 34 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 12 & 2 \\ \hline 23 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\,.$$

If we change f to $f' = (2, 3)$, then $\mathcal{T}(\lambda, f')$ doesn't contain the second and forth tableaux.

If T is the first tableau in the above list, $|T| = 6$ and we have

$$[x|b]^T = (x_1 \oplus b_1)(x_1 \oplus b_2)(x_1 \oplus b_3)(x_2 \oplus b_4)(x_2 \oplus b_1)(x_3 \oplus b_2).$$

2.2 Divided Difference Operators and $\mathcal{G}_m^{[p,q]}$

Definition 2.3 For each positive integer i, define the *K-theoretic divided difference operator* π_i by

$$\pi_i(f) = \frac{(1 + \beta x_{i+1})f - (1 + \beta x_i)s_i(f)}{x_i - x_{i+1}}$$

for each $f \in \mathbb{Z}[\beta][[x, b]]$, where s_i permutes x_i and x_{i+1}.

The following properties of π_i are easy to check (cf. [15, Sect. 2.1]).

Lemma 2.4 *For each positive integer i and $f, g \in \mathbb{Z}[\beta][[x, b]]$, we have*

(1) $\pi_i(fg) = \pi_i(f)g + s_i(f)\pi_i(g) + \beta s_i(f)g.$
(2) If f is symmetric in x_i and x_{i+1}, then $\pi_i(f) = -\beta f$ and $\pi_i(fg) = f\pi_i(g)$.
(3) $\pi_i(f) = -\beta f$, then f is symmetric in x_i and x_{i+1}.

Definition 2.5 Define $\mathcal{G}_m^{[p,q]} = \mathcal{G}_m^{[p,q]}(x|b)$, $m \in \mathbb{Z}$, $p, q \in \mathbb{Z}_{\geq 0}$, by the generating function

$$\mathcal{G}_u^{[p,q]} = \sum_{m \in \mathbb{Z}} \mathcal{G}_m^{[p,q]} u^m = \frac{1}{1 + \beta u^{-1}} \prod_{1 \leq i \leq p} \frac{1 + \beta x_i}{1 - x_i u} \prod_{1 \leq i \leq q} (1 + (u + \beta)b_i).$$

If $q = 0$, then we denote $\mathcal{G}_m^{[p]} = \mathcal{G}_m^{[p,q]}$.

Remark 2.6 (1) Since the degree of u in $\mathcal{G}_u^{[0,q]}$ is at most q, we see that $\mathcal{G}_m^{[0,q]} = 0$ for $m > q$.

(2) We can show that $\mathcal{G}_m^{[p,q]} = (-\beta)^{-m}$ for $m \leq 0$ by a direct computation. In fact, by using $1 + (u + \beta)b_i = \frac{1 - u\bar{b}_i}{1 + \beta\bar{b}_i}$ where $\bar{b}_i = \frac{1 - ub_i}{1 + \beta b_i}$, we have

$$\begin{aligned}\mathcal{G}_u^{[p,q]} &= \frac{1}{1 + \beta u^{-1}} \prod_{1 \leq i \leq p} \frac{1 + \beta x_i}{1 - x_i u} \prod_{1 \leq i \leq q} \frac{1 - u\bar{b}_i}{1 + \beta\bar{b}_i} \\ &= \prod_{1 \leq i \leq p} (1 + \beta x_i) \prod_{1 \leq i \leq q} \frac{1}{1 + \beta\bar{b}_i} \sum_{m \in \mathbb{Z}} \left(\sum_{l \geq 0} \sum_{n \geq 0} h_{m+n-l}(x) e_l(-\bar{b})(-\beta)^n \right) u^m.\end{aligned}$$

Suppose $m \leq 0$. Then the coefficient of u^m in the summation is

$$\prod_{1 \leq i \leq p} \frac{1}{1 + \beta x_i} \prod_{1 \leq i \leq q} (1 + \beta\bar{b}_i)(-\beta)^{-m}.$$

Thus we have $\mathcal{G}_m^{[p,q]} = (-\beta)^{-m}$ for $m \leq 0$.

(3) Similarly to (2), we can also check that $\mathcal{G}_m^{[1]} = x_1^m$ for $m \geq 0$.

(4) We have $\mathcal{G}_{q+r}^{[1,q]} = x_1^r \mathcal{G}_q^{[1,q]}$ for any integer $r \geq 0$. Indeed, if we let

$$\prod_{i=1}^{q} (1 + (u + \beta)b_i) = \sum_{i=0}^{q} E_i u^i,$$

then, by (3), we can compute

$$\mathcal{G}_{q+r}^{[1,q]} = \sum_{i=0}^{d} \mathcal{G}_{i+r}^{[1]} E_{q-i} = x_1^r \sum_{i=0}^{q} \mathcal{G}_i^{[1]} E_{q-i} = x_1^r \mathcal{G}_q^{[1,q]}.$$

The following basic lemmas will be used in the next section.

Lemma 2.7 *For each $m \in \mathbb{Z}$, $p, q \in \mathbb{Z}_{\geq 0}$, we have* $\pi_i(\mathcal{G}_m^{[p,q]}) = \begin{cases} \mathcal{G}_{m-1}^{[p+1,q]} & (i = p), \\ -\beta\mathcal{G}_m^{[p,q]} & (i \neq p). \end{cases}$

Proof The claim follows from applying divided difference operators to the generating function of $\mathcal{G}_m^{[p,q]}$ (cf. [15, Lemma 1]). □

Lemma 2.8 *Let t and t_i $(i = 1, \ldots, n)$ be arbitrary positive integers and $t' := t + 1$. We have*

$$\pi_t\left(\prod_{i=1}^{n}(x_t \oplus b_{t_i})\right) = \sum_{v=0}^{n-1}\left(\prod_{i=1}^{v}(x_t \oplus b_{t_i})\prod_{i=v+2}^{n}(x_{t'} \oplus b_{t_i})\right) + \beta\sum_{v=1}^{n-1}\left(\prod_{i=1}^{v}(x_t \oplus b_{t_i})\prod_{i=v+1}^{n}(x_{t'} \oplus b_{t_i})\right).$$

In particular, $\pi_t(x_t \oplus b_{t_1}) = 1$.

Proof We prove the claim by induction on n. When $n = 1$, we can show that $\pi_t(x_t \oplus b_s) = 1$ by a direct computation. For a general n, we apply Lemma 2.4 (1) with $f = x_t \oplus b_{t_n}$ and g the rest:

$$\pi_t\left(\prod_{i=1}^{n}(x_t \oplus b_{t_i})\right) = \prod_{i=1}^{n-1}(x_t \oplus b_{t_i}) + (x_{t'} \oplus b_{t_n})\pi_t\left(\prod_{i=1}^{n-1}(x_t \oplus b_{t_i})\right) + \beta(x_{t'} \oplus b_{t_n})\prod_{i=1}^{n-1}(x_t \oplus b_{t_i}).$$

By the induction hypothesis, we have

$$\begin{aligned}\pi_t\left(\prod_{i=1}^{n}(x_t \oplus b_{t_i})\right) &= \prod_{i=1}^{n-1}(x_t \oplus b_{t_i}) + (x_{t'} \oplus b_{t_n})\sum_{v=0}^{n-2}\left(\prod_{i=1}^{v}(x_t \oplus b_{t_i})\prod_{i=v+2}^{n-1}(x_{t'} \oplus b_{t_i})\right) \\ &+\beta\left((x_{t'} \oplus b_{t_n})\sum_{v=1}^{n-2}\left(\prod_{i=1}^{v}(x_t \oplus b_{t_i})\prod_{i=v+1}^{n-1}(x_{t'} \oplus b_{t_i})\right) + (x_{t'} \oplus b_{t_n})\prod_{i=1}^{n-1}(x_t \oplus b_{t_i})\right).\end{aligned}$$

This is exactly the right hand side of the desired formula. □

Remark 2.9 Since the left hand side of the formula in Lemma 2.8 is symmetric in the variables $b_{t_1}, \ldots, b_{t_n}$, we can conclude that the right hand side is symmetric in x_t and x_{t+1}.

Lemma 2.10 ([cf. Lemma 3 [15]]) For any integers $m \in \mathbb{Z}$, $p \in \mathbb{Z}_{\geq 1}$ and $q \in \mathbb{Z}_{\geq 0}$, we have

$$\frac{1}{1 + x_1\beta}\left(\mathcal{G}_m^{[p,q]} - x_1\mathcal{G}_{m-1}^{[p,q]}\right) = \left(\mathcal{G}_m^{[p-1,q]}\right)^{\star}.$$

Proof It follows from the identity

$$\sum_{m\in\mathbb{Z}}\frac{1}{1 + x_1\beta}\left(\mathcal{G}_m^{[p,q]} - x_1\mathcal{G}_{m-1}^{[p,q]}\right)u^m = \left(\sum_{m\in\mathbb{Z}}\mathcal{G}_m^{[p-1,q]}u^m\right)^{\star},$$

which can be checked by a direct computation. □

3 Propositions and the Main Theorem

In this section, we prove the main theorem. First we show four propositions that will be used in the proof of the main theorem given at the end of this section. Throughout the section, we let (λ, f) to be a flagged partition of length r.

Definition 3.1 We denote the following determinantal expression by $\widetilde{G}_{\lambda,f} = \widetilde{G}_{\lambda,f}(x|b)$:

$$\widetilde{G}_{\lambda,f}(x|b) = \det\left(\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s \mathcal{G}_{\lambda_i-i+j+s}^{[f_i, f_i+\lambda_i-i]}\right)_{(1\le i,j\le r)}.$$

If $\lambda = \varnothing$, we set $\widetilde{G}_{\lambda,f} = 1$. If $\lambda \neq \varnothing$ and $f_1 = 0$, the first row of the determinant is identically zero by Remark 2.6 (1) so that we set $\widetilde{G}_{\lambda,f} = 0$.

Proposition 3.2 *For any integer $q \geq 0$, we have $\widetilde{G}_{(q),(1)}(x|b) = G_{(q),(1)}(x|b)$, or equivalently, $\mathcal{G}_q^{[1,q]} = \prod_{i=1}^{q}(x_1 \oplus b_i)$.*

Proof We prove the claim by induction on q. When $q = 0$, it is trivial. Suppose that $q > 0$. By definition of $\mathcal{G}_m^{[p,q]}$, we have

$$\sum_{m\in\mathbb{Z}} \mathcal{G}_m^{[1,q]}u^m = \left(\sum_{m\in\mathbb{Z}} \mathcal{G}_m^{[1,q-1]}u^m\right)(1+\beta b_q + b_q u),$$

so that

$$\mathcal{G}_q^{[1,q]} = \mathcal{G}_q^{[1,q-1]}(1+\beta b_q) + \mathcal{G}_{q-1}^{[1,q-1]}b_q.$$

Since $\mathcal{G}_q^{[1,q-1]} = x_1\mathcal{G}_{q-1}^{[1,q-1]}$ by Remark 2.6 (4), we obtain

$$\mathcal{G}_q^{[1,q]} = \mathcal{G}_{q-1}^{[1,q-1]}x_1(1+\beta b_q) + \mathcal{G}_{q-1}^{[1,q-1]}b_q = \mathcal{G}_{q-1}^{[1,q-1]}(x_1 \oplus b_q).$$

The induction hypothesis implies the desired formula. □

Proposition 3.3 *If $f_1 = 1$, then we have*

(1) $\widetilde{G}_{\lambda,f}(x|b) = \mathcal{G}_{\lambda_1}^{[1,\lambda_1]} \cdot \widetilde{G}_{\lambda',f'}(x|b)^{\star}$,
(2) $G_{\lambda,f}(x|b) = \mathcal{G}_{\lambda_1}^{[1,\lambda_1]} \cdot G_{\lambda',f'}(x|b)^{\star}$,

where $\lambda' = (\lambda_2, \lambda_3, \ldots, \lambda_r)$ and $f' = (f_2 - 1, f_3 - 1, \ldots, f_r - 1)$.

Proof (1) We show that the left hand side coincides with the right hand side by the column operation of subtracting the $(j-1)$-st column multiplied with $x_1(1+x_1\beta)^{-1}$ from the j-th column for $j = 2, \ldots, r$. By Remark 2.6 (4), the $(1, j)$ entry of $\widetilde{G}_{\lambda,f}(x|b)$ is

$$\sum_{s=0}^{\infty}\binom{1-j}{s}\beta^s\mathcal{G}^{[1,\lambda_1]}_{\lambda_1-1+j+s}=\sum_{s=0}^{\infty}\binom{1-j}{s}\beta^s x_1^{j-1+s}\mathcal{G}^{[1,\lambda_1]}_{\lambda_1}=(1+x_1\beta)^{1-j}x_1^{j-1}\mathcal{G}^{[1,\lambda_1]}_{\lambda_1}.$$

Thus after the above column operation, the first row of $\widetilde{G}_{\lambda,f}(x|b)$ becomes $(\mathcal{G}^{[1,\lambda_1]}_{\lambda_1},0,\dots,0)$. We compute the (i,j) entry of the resulting determinant for $i,j\geq 2$:

$$\begin{aligned}
&\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_1-i+j+s}-\frac{x_1}{1+x_1\beta}\sum_{s=0}^{\infty}\binom{i-j+1}{s}\beta^s\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_i-i+j-1+s}\\
&=\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_1-i+j+s}-\frac{x_1}{1+x_1\beta}\sum_{s=0}^{\infty}\left(\binom{i-j}{s}+\binom{i-j}{s-1}\right)\beta^s\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_i-i+j-1+s}\\
&=\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s\left(\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_1-i+j+s}-\frac{x_1}{1+x_1\beta}\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_1-i+j+s-1}\right)-\frac{x_1\beta}{1+x_1\beta}\sum_{s'=-1}^{\infty}\binom{i-j}{s'}\beta^{s'}\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_i-i+j+s'}\\
&=\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s\frac{1}{1+x_1\beta}\left(\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_i-i+j+s}-x_1\mathcal{G}^{[f_i,f_i+\lambda_i-i]}_{\lambda_i-i+j+s-1}\right)\\
&=\sum_{s=0}^{\infty}\binom{i-j}{s}\beta^s\left(\mathcal{G}^{[f_i-1,f_i+\lambda_i-i]}_{\lambda_i-1+j+s}\right)^{\star}.
\end{aligned}$$

Here we have used a well-known identity for binomial coefficients for the first equality, the fact that $\binom{i-j}{s'}=0$ for $s'=-1$ for the third equality, and Lemma 2.10 for the last equality. Finally the desired formula follows from the cofactor expansion with respect to the first row for the resulting determinant after the column operation.

(2) For any $T\in\mathcal{T}(\lambda,f)$, the entries on the first row of T are all 1 and all other entries are greater than 1. There is a bijection from $\mathcal{T}(\lambda,f)$ to $\mathcal{T}(\lambda',f')$ sending T to T' obtained from T by deleting its first row and decreasing the numeric values of the rest of the entries by 1. Under this bijection, we have

$$[x|b]^T=\left([x|b]^{T'}\right)^{\star}\cdot\prod_{j=1}^{\lambda_1}(x_1\oplus b_j).$$

Now the claim follows from Proposition 3.2. □

Proposition 3.4 *If $\lambda_1>\lambda_2$ and $f_1<f_2$, then we have*

(1) $\pi_{f_1}\widetilde{G}_{\lambda,f}(x|b)=\widetilde{G}_{\lambda',f'}(x|b)$,
(2) $\pi_{f_1}G_{\lambda,f}(x|b)=G_{\lambda',f'}(x|b)$,

where $\lambda'=(\lambda_1-1,\lambda_2,\dots,\lambda_r)$ and $f'=(f_1+1,f_2,\dots,f_r)$.

Proof (1) First observe that the entries of the determinant are symmetric in x_{f_1} and x_{f_1+1} except the ones on the first row, since $f_1<f_2$. We consider the cofactor expansion of $\widetilde{G}_{\lambda,f}(x|b)$ with respect to the first row: let $\Delta_{i,j}$ be the cofactor of the (i,j) entry and we have

$$\widetilde{G}_{\lambda,f} = \sum_{j=1}^{r}(-1)^{1+j}\Delta_{1,j}\sum_{s=0}^{\infty}\binom{1-j}{s}\beta^s \mathcal{G}^{[f_1,f_1+\lambda_1-1]}_{\lambda_1-1+j+s}.$$

Applying π_{f_1} to this expansion, then we obtain

$$\begin{aligned}\pi_{f_1}\widetilde{G}_{\lambda,f} &= \sum_{j=1}^{r}(-1)^{1+j}\Delta_{1,j}\sum_{s=0}^{\infty}\binom{1-j}{s}\beta^s \pi_{f_1}\left(\mathcal{G}^{[f_1,f_1+\lambda_1-1]}_{\lambda_1-1+j+s}\right)\\ &= \sum_{j=1}^{r}(-1)^{1+j}\Delta_{1,j}\sum_{s=0}^{\infty}\binom{1-j}{s}\beta^s \mathcal{G}^{[f_1+1,f_1+\lambda_1-1]}_{\lambda_1-2+j+s},\end{aligned}$$

in view of Lemma 2.4 (2) and Lemma 2.7. The last expression is the cofactor expansion of $\widetilde{G}_{\lambda',f'}$, and thus we obtain the desired formula.

(2) Let $t := f_1$ and $t' := f_1 + 1$. First we define an equivalence relation in $\mathcal{T}(\lambda, f)$ as follows. Two tableaux $T_1, T_2 \in \mathcal{T}(\lambda, f)$ are equivalent if the next two conditions are satisfied:

(i) the boxes containing either t or t' in T_1 coincide with those in T_2;
(ii) if each box $(1, \lambda_1)$ in T_1 and T_2 contains t, then both of them contain only t or both of them contain t along with other entries.

Let $\mathscr{A}$ be an equivalence class for $\mathcal{T}(\lambda, f)$, then the configuration of t and t' for the tableaux in $\mathscr{A}$ can be depicted as in Fig. 1. The one row rectangle A_1 on the first row consists of m_1 boxes with entries t. Each one-row rectangle A_i $(2 \le i \le k)$ with $*$ consists of m_i boxes and each box contains t or t' or both so that the total number of entries t and t' in A_i is m_i or $m_i + 1$. Each two-row rectangle B_j $(1 \le j \le k)$ consists of r_i columns with t on the first row and t' on the second. Note that m_i and r_i may be 0 so that the rectangles in Fig. 1 may be not connected.

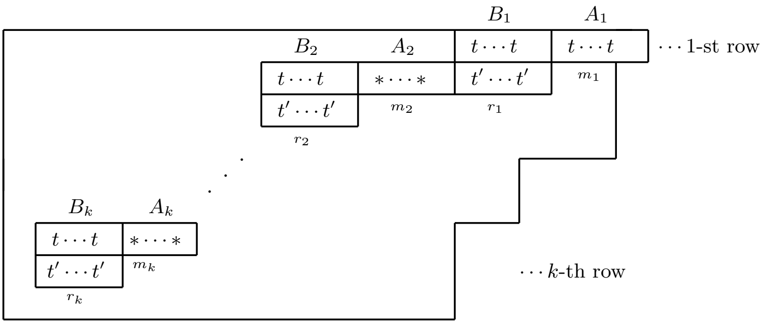

Fig. 1 Configuration of t and t' for $\mathscr{A}$

Let us write

$$G_{\mathscr{A}} := \sum_{T\in\mathscr{A}} \beta^{|T|-|\lambda|}[x|b]^T = R(A_1)R'(\mathscr{A}),\ R'(\mathscr{A}) := R(\mathscr{A})\left(\prod_{i=2}^{k} R(A_i)\right)\left(\prod_{j=1}^{k} R(B_j)\right),$$

where $R(\mathscr{A})$ is the polynomial contributed from the entries other than t and t' and $R(A_i)$ and $R(B_j)$ are the polynomials contributed from the entries t and t' in A_i and B_j respectively. It is obvious that $R(\mathscr{A})$ and $R(B_j)$ $(1 \le j \le k)$ are symmetric in x_t and $x_{t'}$. Moreover, in view of Remark 2.9, $R(A_i)$ $(2 \le i \le k)$ are also symmetric in x_t and $x_{t'}$. Thus $R'(\mathscr{A})$ is symmetric in x_t and $x_{t'}$.

Next we decompose $\mathcal{T}(\lambda, f)/\!\sim$ into the subsets $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ consisting of equivalence classes with configurations respectively satisfying conditions (1) $m_1 = 0$, (2) $m_1 = 1$ and the box $(1, \lambda_1)$ contains t along with other entries, (3) $m_1 \ge 1$ and the box $(1, \lambda_1)$ contains only t. There is a bijection from $\mathcal{H}_1$ to $\mathcal{H}_2$ sending $\mathscr{A}$ to $\mathscr{A}'$ whose configuration is obtained by inserting t in the box $(1, \lambda_1)$ of $\mathscr{A}$. This also defines a bijection from $\mathscr{A}$ to $\mathscr{A}'$ (say, it maps T to T'). Under this bijection, we have $G_{\mathscr{A}'} = \beta(x_t \oplus b_{t-1+\lambda_1})G_{\mathscr{A}}$. Since $G_{\mathscr{A}}$ is symmetric in x_t and $x_{t'}$, by Lemma 2.4 (2) and Lemma 2.8, we have $\pi_t(G_{\mathscr{A}'}) = \beta G_{\mathscr{A}}$. Thus

$$\pi_t\left(\sum_{\mathscr{A}\in\mathcal{H}_1\sqcup\mathcal{H}_2} G_{\mathscr{A}}\right) = \sum_{\mathscr{A}\in\mathcal{H}_1}\left(\pi_t\left(G_{\mathscr{A}}\right) + \pi_t\left(G_{\mathscr{A}'}\right)\right) = \sum_{\mathscr{A}\in\mathcal{H}_1}\left(-\beta G_{\mathscr{A}} + \beta G_{\mathscr{A}}\right) = 0.$$

As a consequence, we have

$$\pi_t\left(G_{\lambda,f}\right) = \pi_t\left(\sum_{\mathscr{A}\in\mathcal{H}_3} G_{\mathscr{A}}\right). \tag{1}$$

Now it remains to show that the right hand side of (1) coincides with $G_{\lambda',f'}$. Define an equivalence relation $\sim$ in $\mathcal{T}(\lambda', f')$ by the condition (i), that is, T'_1 and T'_2 in $\mathcal{T}(\lambda', f')$ are equivalent if the boxes containing either t or t' in T'_1 coincide with those in T'_2. For an arbitrary equivalence class $\mathscr{A}'$ for $\mathcal{T}(\lambda', f')$, we can describe its configuration of t and t' as in Fig. 2. Similarly to Fig. 1, A_i $(2 \le i \le k)$ is a rectangle consisting of m_i boxes with entries t, t' or both of them, B_j $(1 \le j \le k)$ is a two-row rectangle with r_j columns with t on the first row and t' on the second. The right-most rectangle A'_1 has m'_1 boxes with entries t, t' or both of them.

There is a bijection from $\mathcal{H}_3$ to $\mathcal{T}(\lambda', f')/\!\sim$ sending an equivalence class $\mathscr{A}$ to $\mathscr{A}'$ whose configuration of t and t' is obtained from the one for $\mathscr{A}$ by erasing the box $(1, \lambda_1)$ and replacing t with $*$ in the rectangle A_1. Under this bijection, $\pi_t\,(G_{\mathscr{A}}) = G_{\mathscr{A}'}$. Indeed, let $\mathscr{A} \in \mathcal{H}_3$ with the configuration as depicted in Fig. 1 and $\mathscr{A}' \in \mathcal{T}(\lambda', f')/\sim$ as in Fig. 2 where $m'_1 = m_1 - 1 \ge 0$. We have $R(A_1) = \prod_{i=1}^{m_1}(x_t \oplus b_{t-1+\lambda_1-m_1+i})$. By Lemma 2.4 (2) and Lemma 2.8, we have

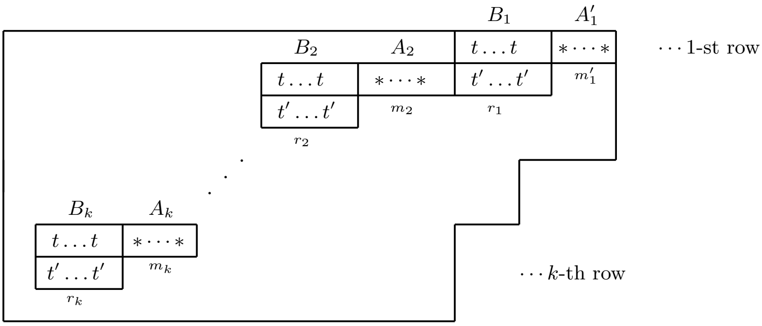

Fig. 2 Configuration of t and t' for $\mathscr{A}'$

$$\pi_t\,(G_{\mathscr{A}}) = \left\{ \sum_{v=0}^{m_1-1} \left(\prod_{i=1}^{v} (x_t \oplus b_{t-1+\lambda_1-m_1+i}) \prod_{i=v+2}^{m_1} (x_{t'} \oplus b_{t-1+\lambda_1-m_1+i}) \right) \right.$$
$$\left. + \beta \sum_{v=1}^{m_1-1} \left(\prod_{i=1}^{v} (x_t \oplus b_{t-1+\lambda_1-m_1+i}) \prod_{i=v+1}^{m_1} (x_{t'} \oplus b_{t-1+\lambda_1-m_1+i}) \right) \right\} R'(\mathscr{A}),$$

which is exactly $G_{\mathscr{A}'}$. Thus we have

$$\pi_t G_{\lambda,f} = \sum_{\mathscr{A}\in\mathcal{H}_3} \pi_t(G_{\mathscr{A}}) = \sum_{\mathscr{A}'\in\mathcal{T}(\lambda',f')/\sim} G_{\mathscr{A}'} = G_{\lambda',f'}.$$

This completes the proof. □

Theorem 3.5 *For a flagged partition* (λ, f)*, we have* $G_{\lambda,f}(x|b) = \widetilde{G}_{\lambda,f}(x|b)$.

Proof With the help of Proposition 3.2, 3.3, and 3.4, the proof is by induction on $|f| = f_1 + \cdots + f_r$, parallel to the one in [15]. If $|f| = 1$, then $\lambda = (\lambda_1)$ and $f = (1)$. Thus it follows from Proposition 3.2. Suppose that $|f| > 1$. We prove in two cases: $f_1 = 1$ or $f_1 > 1$. If $f_1 = 1$, then we can apply Proposition 3.3 to both $\widetilde{G}_{\lambda,f}$ and $G_{\lambda,f}$. The right hand sides of the resulting formulas coincide by the induction hypothesis, and thus the claim holds. If $f_1 > 1$, then Proposition 3.4 implies that $\pi_{g_1}\widetilde{G}_{\mu,g} = \widetilde{G}_{\lambda,f}$ and $\pi_{g_1}G_{\mu,g} = G_{\lambda,f}$ where $\mu = (\lambda_1 + 1, \lambda_2, \dots, \lambda_r)$ and $g = (f_1 - 1, f_2, \dots, f_r)$. The left hand sides of these equalities coincide by the induction hypothesis, and thus the claim holds. □

4 Vexillary Double Grothendieck Polynomials

In this section, we prove that the double Grothendieck polynomials associated to vexillary permutations are in fact factorial Grothendieck polynomials (Theorem 4.2), giving an alternative proof to the corresponding results in [8, 10]. Moreover we show that any factorial Grothendieck polynomial can be obtained from a product of certain linear polynomials by applying a sequence of divided difference operators (Theorem 4.3).

The *double Grothendieck polynomials* were introduced by Lascoux and Schützenberger [11, 12]. For any permutation $w \in S_n$, we define the associated double Grothendieck polynomial $\mathfrak{G}_w = \mathfrak{G}_w(x|b)$ as follows. Let w_0 be the longest element of the symmetric group S_n. We set

$$\mathfrak{G}_{w_0} = \prod_{i+j \leq n} (x_i \oplus b_j).$$

For an element $w \in S_n$ such that $\ell(w) < \ell(w_0)$, we can choose a simple reflection $s_i \in S_n$ such that $\ell(ws_i) = \ell(w) + 1$. Here $\ell(w)$ is the length of w. We then define

$$\mathfrak{G}_w = \pi_i(\mathfrak{G}_{ws_i}).$$

The polynomial $\mathfrak{G}_w$ is defined independently from the choice of such s_i, since the divided difference operators satisfy the Coxeter relations. From this point of view, we can write $\mathfrak{G}_w = \pi_v \mathfrak{G}_{w_0}$ with $v = w_0 w$ and $\pi_v = \pi_{i_k} \cdots \pi_{i_1}$ where $v = s_{i_1} \cdots s_{i_k}$ is a reduced expression.

A permutation $w \in S_n$ is called *vexillary* if it is 2143-avoiding, *i.e.* there is no $a < b < c < d$ such that $w(b) < w(a) < w(d) < w(c)$. We briefly recall how to obtain a flagged partition from a vexillary permutation. We follow [6, 10] (cf. [8]). For each $w \in S_n$, let r_w be the *rank function* of $w \in S_n$ defined by $r_w(p, q) := \sharp\{i \leq p |\ w(i) \leq q\}$ for $1 \leq p, q \leq n$. The *diagram* $D(w)$ of w is defined as

$$D(w) := \{(p, q) \in \{1, \dots, n\} \times \{1, \dots, n\} \mid w(p) > q, \text{ and } w^{-1}(q) > p\}.$$

The *essential set* $Ess(w)$ is the subset of $D(w)$ given by

$$Ess(w) := \{(p, q) \mid (p + 1, q), (p, q + 1) \notin D(w)\}.$$

If w is vexillary, we can choose a *flagging set* of w (cf. [8]), which is a subset $\{(p_i, q_i), i = 1, \dots, r\}$ of $\{1, \dots, n\} \times \{1, \dots, n\}$ containing $Ess(w)$ and satisfying

$$p_1 \leq p_2 \leq \cdots \leq p_r, \quad q_1 \geq q_2 \geq \cdots \geq q_r, \tag{2}$$
$$p_i - r_w(p_i, q_i) = i, \quad \forall i = 1, \dots, r. \tag{3}$$

An associated flagged partition $(\lambda(w), f(w))$ of length r is given by setting $f_i(w) := p_i$ and $\lambda(w)_i = q_i - p_i + i$ for $i = 1, \dots, r$. We remark that the set $\mathcal{T}(\lambda(w), f(w))$ depends only on w but not on the choice of a flagging set.

Example 4.1 A permutation $w \in S_n$ is *dominant* if $D(w)$ is the Young diagram of a partition and the values of r_w on $D(w)$ are zero. Such permutation is vexillary, and in this case, $\lambda(w)$ is the partition whose Young diagram is $D(w)$ and its flagging is $f(w) = (1, 2, \dots, r)$ where r is the length of $\lambda(w)$.

The following theorem was obtained in [8, 10]. We give an alternative proof.

Theorem 4.2 *If $w \in S_n$ is vexillary, then we have $\mathfrak{G}_w = G_{\lambda(w), f(w)}$.*

Proof We closely follow the proof of Theorem 2 in [15], which is by induction on $(n, \ell(w_0) - \ell(w))$ with the lexicographic order. For the longest element $w_0 \in S_n$, by definition, we have

$$G_{\lambda(w_0), f(w_0)} = \prod_{i+j \leq n} (x_i \oplus b_j) = \mathfrak{G}_{w_0}.$$

Consider $w \in S_n$ with $w \neq w_0$, id. Let d be the leftmost descent of w, *i.e.* d is the smallest number such that $w(d) > w(d+1)$. If $d > 1$ (**Case 1**), by Proposition 3.4, the proof is identical to the one in [15]. Suppose that $d = 1$. If $m := w(1) < n$ (**Case 2**), let $w' := s_m w$ and consider the dominant permutation

$$u = (m+1, m, m+2, m+3, \dots, n, m-1, m-2, \dots, 1),$$

as in [15]. The proof is identical to the one in [15], except that we have

$$\pi_1 \mathfrak{G}_u = (x_1 \oplus b_m)^{-1} \mathfrak{G}_u, \quad G_{\lambda(w), f(w)} = (x_1 \oplus b_m)^{-1} G_{\lambda(w'), f(w')}.$$

The former identity holds since we have $\mathfrak{G}_u = (x_1 \oplus b_m) R_u$ where R_u is symmetric in x_1 and x_2. The latter follows from the fact that there is a bijection from $\mathcal{T}(\lambda(w), f(w))$ to $\mathcal{T}(\lambda(w'), f(w'))$ sending T to T' which is obtained from T by adding a box in the first row with entry 1. Finally, if $w(1) = n$ (**Case 3**), we find a reduced expression $s_{i_1} \cdots s_{i_k}$ of $w_0 w$ where $i_1, \dots, i_k \geq 2$. Let $w' = (w(2), \dots, w(n)) \in S_{n-1}$. Then we have

$$\mathfrak{G}_w = \prod_{i=1}^{n-1} (x_1 \oplus b_i) \cdot \pi_{i_k} \pi_{i_{k-1}} \cdots \pi_{i_1} \left(\prod_{i+j \leq n-1} (x_{i+1} \oplus b_j) \right) = \prod_{i=1}^{n-1} (x_1 \oplus b_i) \cdot \left(\mathfrak{G}_{w'}\right)^{\star}.$$

By the induction hypothesis and Proposition 3.2, we have

$$\mathfrak{G}_w = \mathcal{G}_{n-1}^{[1,n-1]} \left(\mathfrak{G}_{\lambda(w'), f(w')}\right)^{\star}.$$

Since $f(w)_1 = 1$ and $(f(w')_1, \dots, f(w')_{n-1}) = (f(w')_2 - 1, \dots, f(w')_n - 1)$, Proposition 3.3 implies the claim. $\square$

Theorem 4.3 *Let (λ, f) be a flagged partition of length r. Then we have*

$$G_{\lambda, f} = \pi_w \left(\prod_{i=1}^{r} \prod_{j=1}^{a_i} (x_i \oplus b_j) \right),$$

where $a_i = \lambda_i + f_i - i$ *for* $1 \le i \le r$ *and* $w = (s_r s_{r+1} \cdots s_{f_r-1}) \cdots (s_2 s_3 \cdots s_{f_2-1})$ $(s_1 s_2 \cdots s_{f_1-1})$.

Proof The proof is by induction on $|f|$. If $|f| = 1$, we see that $G_{\lambda,f} = x_1 \oplus b_1$ and $w = \mathrm{id}$ so that the claim is trivial. Suppose that $|f| > 1$. If $f_1 = 1$, let $\lambda' = (\lambda_2, \dots, \lambda_r)$ and $f' = (f_2 - 1, \dots, f_r - 1)$. By Proposition 3.3, we have $G_{\lambda,f} = (G_{\lambda',f'})^\star \prod_{j=1}^{\lambda_1}(x_1 \oplus b_j)$. By the induction hypothesis, we can write $(G_{\lambda',f'})^\star = \pi_w \left(\prod_{i=2}^{r} \prod_{j=1}^{a_i}(x_i \oplus b_j)\right)$. Since s_1 doesn't appear in the reduced expression of w, we obtain the desired formula. If $f_1 > 1$, since $\lambda' = (\lambda_1 + 1, \lambda_2, \dots, \lambda_r)$, $f' = (f_1 - 1, f_2, \dots, f_r)$, Proposition 3.4 and the induction hypothesis imply the claim:

$$G_{\lambda,f} = \pi_{f_1-1} G_{\lambda',f'} = \pi_{f_1-1}\pi_{ws_{f_1-1}} \left(\prod_{i=1}^{r}\prod_{j=1}^{a_i}(x_i \oplus b_j)\right) = \pi_w \left(\prod_{i=1}^{r}\prod_{j=1}^{a_i}(x_i \oplus b_j)\right).$$

This completes the proof. □

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References

1. Anderson, D.: K-theoretic Chern class formulas for vexillary degeneracy loci. Adv. Math. **350**, 440–485 (2019)
2. Buch, A.S.: A Littlewood–Richardson rule for the K-theory of Grassmannians. Acta Math. **189**(1), 37–78 (2002)
3. Chen, W.Y.C., Li, B., Louck, J.D.: The flagged double Schur function. J. Algebraic Combin. **15**(1), 7–26 (2002)
4. Fomin, S., Kirillov, A. N.: The Yang–Baxter equation, symmetric functions, and Schubert polynomials. Proceedings of the 5th conference on formal power series and algebraic combinatorics (Florence, 1993). Discrete Math. **153**(1–3), 123–143 (1996)
5. Fomin, S., Kirillov, A.N.: Grothendieck polynomials and the Yang–Baxter equation. In: Formal Power Series and Algebraic Combinatorics/Séries Formelles et Combinatoire algébrique. DIMACS, Piscataway, NJ, pp. 183–189
6. Fulton, W.: Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. Duke Math. J. **65**(3), 381–420 (1992)
7. Hudson, T., Ikeda, T., Matsumura, T., Naruse, H.: Degeneracy loci classes in K-theory—determinantal and Pfaffian formula. Adv. Math. **320**, 115–156 (2017)
8. Hudson, T., Matsumura, T.: Vexillary degeneracy loci classes in K-theory and algebraic cobordism. European J. Combin. **70**, 190–201 (2018)
9. Knutson, A., Miller, E., Yong, A.: Tableau complexes. Israel J. Math. **163**, 317–343 (2008)
10. Knutson, A., Miller, E., Yong, A.: Gröbner geometry of vertex decompositions and of flagged tableaux. J. Reine Angew. Math. **630**, 1–31 (2009)
11. Lascoux, A.: Anneau de Grothendieck de la variété de drapeaux. In: The Grothendieck Festschrift, vol. III, vol. 88 of Progress in Mathematics. Birkhäuser Boston, Boston, MA, pp. 1–34 (1990)

12. Lascoux, A., Schützenberger, M.- P.: Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux. C. R. Acad. Sci. Paris Sér. I Math. **295**(11), 629–633 (1982)
13. Matsumura, T.: A tableau formula of double Grothendieck polynomials for 321-avoiding permutations. Ann. Comb. (2020). https://doi.org/10.1007/s00026-019-00481-4
14. Matsumura, T.: An algebraic proof of determinant formulas of Grothendieck polynomials. Proc. Jpn Acad. Ser. A Math. Sci. **93**(8), 82–85 (2017)
15. Matsumura, T.: Flagged Grothendieck polynomials. J. Algebraic Combin. **49**(3), 209–228 (2019)
16. McNamara, P.J.: Factorial Grothendieck polynomials, Research Paper 71 (electronic). Electron. J. Combin. **13**(1), 40 (2006)
17. Wachs, M.L.: Flagged Schur functions, Schubert polynomials, and symmetrizing operators. J. Combin. Theory Ser. A **40**(2), 276–289 (1985)

Flag Bundles, Segre Polynomials, and Push-Forwards

Lionel Darondeau and Piotr Pragacz

Abstract In this note, we give Gysin formulas for partial flag bundles for the classical groups. We then give Gysin formulas for Schubert varieties in Grassmann bundles, including isotropic ones. All these formulas are proved in a rather uniform way by using the step-by-step construction of flag bundles and the Gysin formula for a projective bundle. In this way we obtain a comprehensive list of new general formulas. The content of this paper was presented by Piotr Pragacz at the International Festival in Schubert Calculus in Guangzhou, November 6–10, 2017.

Keywords Push forward · Gysin maps · Segre polynomials · Classical flag bundles · Kempf-Laksov bundles · Schubert bundles

1 Introduction

Let $E \to X$ be a vector bundle of rank n on a variety X over an algebraically closed field. Let $\pi : \mathbf{F}(E) \to X$ be the bundle of flags of subspaces of dimensions $1, 2, \dots, n-1$ in the fibers of $E \to X$. The flag bundle $\mathbf{F}(E)$ is used, e.g., in splitting principle, a standard technique which reduces questions about vector bundles to the case of line bundles; namely the pullback bundle $\pi^* E$ decomposes as a direct sum of line bundles. One can construct $\mathbf{F}(E)$ inductively as a sequence of projective bundles, using the following iterative step, that decreases the rank by 1. Let $p_1 : \mathbf{P}(E) \to X$ denote the projective bundle of lines in E, and let $U_1 := \mathcal{O}_{\mathbf{P}(E)}(-1)$ denote the universal subbundle on $\mathbf{P}(E)$, then one has a universal exact sequence of vector bundles on $\mathbf{P}(E)$

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$$0 \to U_1 \longrightarrow p_1^* E \longrightarrow Q_{n-1} \to 0,$$

where Q_{n-1} (the universal quotient bundle on $\mathbf{P}(E)$) is a rank $n-1$ vector bundle. Replacing E by Q_{n-1}, one obtains a universal subbundle on $\mathbf{P}(Q_{n-1})$, together with a universal quotient bundle Q_{n-2}. Iterating this process until obtaining a quotient bundle Q_1 of rank one, one gets a sequence of projective bundles

$$\mathbf{F}(E) := \mathbf{P}(Q_2) \xrightarrow{p_{n-1}} \ldots \to \mathbf{P}(Q_{n-1}) \xrightarrow{p_2} \mathbf{P}(E) \xrightarrow{p_1} X, \tag{1}$$

a flag bundle filtration

$$0 \subsetneq (p_{n-1} \circ \cdots \circ p_2)^* U_1 \subsetneq (p_{n-1} \circ \cdots \circ p_3)^* U_2 \subsetneq \ldots \subsetneq U_{n-1} \subsetneq \pi^* E,$$

where $U_i \to \mathbf{P}(Q_{n+1-i})$ is the kernel bundle of the composition

$$(p_i \circ \cdots \circ p_1)^* E \twoheadrightarrow Q_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow Q_{n-i},$$

and universal exact sequences of vector bundles on $\mathbf{P}(Q_{n-i+1})$:

$$0 \to U_i / p_i^* U_{i-1} \to p_i^* Q_{n-i+1} \to Q_{n-i} \to 0. \tag{2}$$

In the Grothendieck group of $\mathbf{F}(E)$, one can write (droping the pullback notation)

$$\pi^* E = U_1 + U_2/U_1 + \cdots + U_{n-1}/U_{n-2} + Q_1,$$

as the sum of (the pullback of) the different line bundles appearing in (1).

Now we would like to outline how to obtain a Gysin formula for the flag bundle $\pi : \mathbf{F}(E) \to X$ (*cf.* Example 1), and introduce some notation.

We shall work in the framework of intersection theory of [3]. Recall that a proper morphism $g : Y \to X$ of nonsingular algebraic varieties over an algebraically closed field yields an additive map $g_* : A^\bullet Y \to A^\bullet X$ of Chow groups induced by pushforward cycles, called the Gysin map. The theory developed in [3] allows also one to work with singular varieties, or with cohomology. In this note, X will always be nonsingular.

For $E \to X$ a vector bundle, let $s(E)$ be the Segre class of E, that is the formal inverse of the Chern class $c(E)$ in the Chow ring of X. Let $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$; then $A^\bullet(\mathbf{P}(E))$ is generated algebraically by ξ over $A^\bullet X$—here we identify $A^\bullet X$ with a subring of $A^\bullet(\mathbf{P}(E))$—and

$$(p_1)_* \xi^i = s_{i-(n-1)}(E), \tag{3}$$

cf. [3]. To obtain a Gysin formula for the sequence of projective bundles (1), it suffices to appropriately iterate formula (3). The intermediate formulas involve the individual Segre classes of the universal quotient bundles, that can be eliminated using (2) and the Whitney sum formula. However, it seems rather difficult to obtain a universal formula in this way. A universal formula should hold for *any* polynomial in characteristic classes of universal vector bundles and depend explicitly on the Segre classes of the original bundle E. To obtain such a formula, we use the generating series of the Segre classes of the universal quotient bundles. A prototype is the reformulation of (3) in

$$(p_1)_*\xi^i = [t^{n-1}]\big(t^i s_{1/t}(E)\big), \tag{4}$$

where we consider the specialization in $x = 1/t$ of the Segre polynomial $s_x(E) = \sum_i s_i(E)x^i$ and where for a monomial m and a Laurent polynomial P, $[m](P)$ denotes the coefficient of m in P. Formula (4) and the projection formula imply that for any polynomial f in one variable with coefficients in $A^\bullet X$

$$(p_1)_* f(\xi) = [t^{n-1}]\big(f(t)s_{1/t}(E)\big). \tag{5}$$

In this formula, (i) one does not need to expand f into a combination of monomials; (ii) one uses the Segre polynomial that, like the total Segre class, is a group homomorphism from the Grothendieck group of X to the multiplicative group of units with degree zero term $= 1$ in $A^\bullet X$.

Iterating the Gysin formula (5) yields a closed universal Gysin formula for the flag bundle $\mathbf{F}(E) \to X$, as announced in Example 1.

It is clear that the outlined strategy of proof applies to more general step-by-step constructions than the construction (1) of the flag bundle $\mathbf{F}(E) \to X$. Considering the truncated composition $p_k \circ \cdots \circ p_1$ in (1) yields formulas for full flag bundles, i.e. bundles of flags of subspaces of dimensions $1, 2, \ldots, k$ in the fibers, for $k = 1, \ldots, n-1$. Then, using certain commutative diagrams (see [1, (5)]), one extends these formulas to arbitrary partial flag bundles.

One other interesting generalization is to restrict to the zero locus of a section of some vector bundle at each step of the sequence of projective bundles. In other words, one can impose some geometric conditions that the subspaces of the flag have to satisfy. An illustrative example is Theorem 2.3, in the orthogonal setting, obtained by considering at each step quadric bundles of isotropic lines in projective bundles of lines.

This method of step-by-step construction of generalized flag bundles leads to uniform short proofs of the different results announced in this note.

This note is organized as follows. In Sect. 2, we shall announce universal Gysin formulas for partial flag bundles for general linear groups, symplectic groups and orthogonal groups. The proofs of the results announced there can be found in [1].

In Sect. 3 we give Gysin formulas for Kempf–Laksov flag bundles. These generalized flag bundles are used to desingularize Schubert varieties in Grassmann bundles. Theorem 3.1 is established in [2]. Theorem 3.2 is announced for the first time in the present note.

2 Universal Gysin Formulas for Flag Bundles

In this section, the letter f denotes a polynomial in the indicated number of variables with coefficients in $A^\bullet X$. The appropriate symmetries that f has to satisfy to be in the Chow ring of the flag bundle under consideration are always implied. Here we consider non-singular varieties X. Note that the theory developed in Fulton's book [3] allows one to generalize the results to singular varieties over a field and their Chow groups; moreover, for complex varieties, one can also use the cohomology rings with integral coefficients.

We shall discuss separately the cases of general linear groups, symplectic groups and orthogonal groups.

2.1 General Linear Groups

Let $E \to X$ be a rank n vector bundle. Let $1 \le d_1 < \cdots < d_m = d \le n-1$ be a sequence of integers. We denote by $\pi : \mathbf{F}(d_1, \ldots, d_m)(E) \to X$ the bundle of flags of subspaces of dimensions $d_1, \ldots, d_m$ in E. On $\mathbf{F}(d_1, \ldots, d_m)(E)$, there is a universal flag $U_{d_1} \subsetneq \cdots \subsetneq U_{d_m}$ of subbundles of $\pi^* E$, where $\mathrm{rk}(U_{d_k}) = d_k$ (the fiber of U_{d_k} over the point $(V_{d_1} \subsetneq \cdots \subsetneq V_{d_m} \subset E(x))$, where $x \in X$, is equal to V_{d_k}). For a foundational account on flag bundles, see [4].

For $i = 1, \ldots, d$, set $\xi_i = -c_1(U_{d+1-i}/U_{d-i})$.

Theorem 2.1 *With the above notation, for $f(\xi_1, \ldots, \xi_d) \in A^\bullet(\mathbf{F}(d_1, \ldots, d_m)(E))$, one has*

$$\pi_* f(\xi_1, \ldots, \xi_d) = \left[t_1^{e_1} \ldots t_d^{e_d}\right]\left(f(t_1, \ldots, t_d) \prod_{1 \le i < j \le d} (t_i - t_j) \prod_{1 \le i \le d} s_{1/t_i}(E)\right),$$

where for $j = d - d_k + i$ with $i = 1, \ldots, d_k - d_{k-1}$, we denote $e_j = n - i$.

Example 1 For the complete flag bundle $\pi : \mathbf{F}(E) \to X$, one has

$$\pi_* f(\xi_1, \ldots, \xi_{n-1}) = \left[\prod_{i=1}^{n-1} t_i^{n-1}\right]\left(f(t_1, \ldots, t_{n-1}) \prod_{1 \le i < j \le n-1} (t_i - t_j) \prod_{i=1}^{n-1} s_{1/t_i}(E)\right);$$

and for the Grassmann bundle $\pi : \mathbf{F}(d)(E) \to X$, one has

$$\pi_* f(\xi_1, \ldots, \xi_d) = \left[\prod_{i=1}^{d} t_i^{n-i}\right]\left(f(t_1, \ldots, t_d) \prod_{1 \le i < j \le d} (t_i - t_j) \prod_{i=1}^{d} s_{1/t_i}(E)\right).$$

2.2 Symplectic Groups

Let $E \to X$ be a rank $2n$ vector bundle equipped with a non-degenerate symplectic form $\omega\colon E \otimes E \to L$ (with values in a certain line bundle $L \to X$). We say that a subbundle S of E is isotropic if S is a subbundle of its symplectic complement S^{ω}, where

$$S^{\omega} := \{w \in E \mid \forall v \in S\colon \omega(w, v) = 0\}.$$

Let $1 \le d_1 < \cdots < d_m \le n$ be a sequence of integers. We denote by $\pi : \mathbf{F}^{\omega}(d_1, d_m)(E) \to X$ the bundle of flags of isotropic subspaces of dimensions $d_1, \ldots, d_m$ in E. On $\mathbf{F}^{\omega}(d_1, \ldots, d_m)(E)$, there is a universal flag $U_{d_1} \subsetneq \cdots \subsetneq U_{d_m}$ of subbundles of π^*E, where $\mathrm{rk}(U_{d_k}) = d_k$.

For $i = 1, \ldots, d$, set $\xi_i = -c_1(U_{d+1-i}/U_{d-i})$.

Theorem 2.2 *With the above notation, for $f(\xi_1, \ldots, \xi_d) \in A^{\bullet}(\mathbf{F}^{\omega}(d_1, \ldots, d_m)(E))$, one has*

$$\pi_* f(\xi_1, \ldots, \xi_d) = \left[t_1{}^{e_1} \cdots t_d{}^{e_d}\right]\left(f(t_1, \ldots, t_d) \prod_{1 \le i < j \le d} (c_1(L) + t_i + t_j)(t_i - t_j) \prod_{1 \le i \le d} s_{1/t_i}(E)\right),$$

where for $j = d - d_k + i$ with $i = 1, \ldots, d_k - d_{k-1}$, we denote $e_j = 2n - i$.

Example 2 For the symplectic Grassmann bundle $\pi : \mathbf{F}^{\omega}(d)(E) \to X$, where ω has values in a trivial line bundle, one has

$$\pi_* f(\xi_1, \ldots, \xi_d) = \left[\prod_{i=1}^{d} t_i^{2n-i}\right]\left(f(t_1, \ldots, t_d) \prod_{1 \le i < j \le d} (t_i^2 - t_j^2) \prod_{i=1}^{d} s_{1/t_i}(E)\right).$$

2.3 Orthogonal Groups

Let $E \to X$ be a vector bundle of rank $2n$ or $2n+1$ equipped with a non-degenerate orthogonal form $Q\colon E \otimes E \to L$ (with values in a certain line bundle $L \to X$). We say that a subbundle S of E is isotropic if S is a subbundle of its orthogonal complement $S^{\perp}$, where

$$S^{\perp} := \{w \in E \mid \forall v \in S\colon Q(w, v) = 0\}.$$

Let $1 \le d_1 < \cdots < d_m \le n$ be a sequence of integers. We denote by $\pi : \mathbf{F}^{Q}(d_1, \ldots, d_m)(E) \to X$ the bundle of flags of isotropic subspaces of dimensions $d_1, \ldots, d_m$ in E. On $\mathbf{F}^{Q}(d_1, \ldots, d_m)(E)$, there is a universal flag $U_{d_1} \subsetneq \cdots \subsetneq U_{d_m}$ of subbundles of π^*E, where $\mathrm{rk}(U_{d_k}) = d_k$.

For $i = 1, \ldots, d$, set $\xi_i = -c_1(U_{d+1-i}/U_{d-i})$.

Theorem 2.3 *With the above notation, for $f(\xi_1, \ldots, \xi_d) \in A^\bullet(\mathbf{F}^Q(d_1, \ldots, d_m)(E))$, one has*

$$\pi_* f(\xi_1, \ldots, \xi_d) =$$
$$\left[t_1^{e_1} \cdots t_d^{e_d}\right]\Big(f(t_1, \ldots, t_d) \prod_{1 \le i \le d}(2t_i + c_1(L)) \prod_{1 \le i < j \le d}(c_1(L) + t_i + t_j)(t_i - t_j) \prod_{1 \le i \le d} s_{1/t_i}(E)\Big),$$

where for $j = d - d_k + i$ with $i = 1, \ldots, d_k - d_{k-1}$, we denote $e_j = \mathrm{rk}(E) - i$.

Note that, if the rank is $2n$ and $d = n$, we consider *both* of the two isomorphic connected components of the flag bundle. Thus, if one is interested in only one of the two components, the result should be divided by 2. When $c_1(L) = 0$, this makes appear the usual coefficient 2^{n-1}.

3 Universal Gysin Formulas for Kempf–Laksov Flag Bundles

In this section, we give Gysin formulas for Kempf–Laksov flag bundles, that are desingularizations of Schubert bundles in Grassmann bundles. We also extend the results to the symplectic setting. The orthogonal cases will be treated elsewhere.

3.1 General Linear Groups

Let $E \to X$ be a rank n vector bundle on a variety X with a reference flag of bundles $E_1 \subsetneq \cdots \subsetneq E_n = E$ on it, where $\mathrm{rk}(E_i) = i$. Let $\pi : \mathbf{G}_d(E) = \mathbf{F}(d)(E) \to X$ be the Grassmann bundle of subspaces of dimension d in the fibers of E. For any partition $\lambda \subseteq (n-d)^d$, there is the *Schubert bundle* $\varpi_\lambda : \Omega_\lambda(E_\bullet) \to X$ in $\mathbf{G}_d(E)$ given over the point $x \in X$ by

$$\Omega_\lambda(E_\bullet)(x) := \{V \in \mathbf{G}_d(E)(x) \colon \dim(V \cap E_{n-d-\lambda_i+i}(x)) \ge i, \text{ for } i = 1, \ldots, d\}. \tag{6}$$

We denote by

$$(\nu_1, \ldots, \nu_d) := (n - d - \lambda_d + d, \ldots, n - d - \lambda_1 + 1)$$

the dimensions of the spaces of the reference flag involved in the definition of $\Omega_\lambda(E_\bullet)$—in reverse order—. The partition ν is a strict partition, and furthermore, $n - i \le \nu_i \le \nu_1 = n - \lambda_d \le n$ for any i. Note that the above definition of $\Omega_\lambda(E_\bullet)$ can be restated using ν with the conditions

$$\dim(V \cap E_{\nu_i}(x)) \ge d + 1 - i, \text{ for } i = 1, \ldots, d. \tag{7}$$

For a strict partition $\mu \subseteq (n)^d$ with d parts, consider the flag bundle $\vartheta_\mu : F_\mu(E_\bullet) \to X$ given over the point $x \in X$ by

$$F_\mu(E_\bullet)(x) := \Big\{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d \in \mathbf{F}(1,\dots,d)(E)(x) \colon V_{d+1-i} \subseteq E_{\mu_i}(x), \text{ for } i = 1,\dots,d\Big\}. \tag{8}$$

We will call *Kempf–Laksov flag bundles* such bundles ϑ_μ introduced in [5].

These appear naturally as desingularizations of Schubert bundles. For a partition $\lambda \subseteq (n-d)^d$, defining ν as above, by (7) the forgetful map $\mathbf{F}(1,\dots,d)(E) \to \mathbf{G}_d(E)$ induces a birational morphism $F_\nu(E_\bullet) \to \Omega_\lambda(E_\bullet)$. On the *Schubert cell* given over the point $x \in X$ by

$$\mathring{\Omega}_\lambda(E_\bullet)(x) := \Big\{V \in \mathbf{G}_d(E)(x) \colon \dim(V \cap E_{\nu_i}(x)) = d+1-i, \text{ for } i = 1,\dots,d\Big\},$$

which is open dense in $\Omega_\lambda(E_\bullet)$, the inverse map is $V \mapsto (V \cap E_{\nu_d}(x), \dots, V \cap E_{\nu_1}(x))$. It establishes a desingularization of $\Omega_\lambda(E_\bullet)$ (see [5]).

We construct $F_\mu(E_\bullet)$ by induction on the length of flags. For $d = 1$, it is simply $\mathbf{P}(E_{\mu_d})$. Assume thus that for $d > 1$ we have constructed the variety $F' \subseteq \mathbf{F}(1,\dots,d-1)(E)$ parametrizing flags

$$\Big\{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{d-1} \in \mathbf{F}(1,\dots,d-1)(E)(x) \colon V_{d+1-i} \subseteq E_{\mu_i}(x), \text{ for } i = 2,\dots,d\Big\}.$$

Let U_{d-1} be the universal subbundle of rank $d-1$ on $\mathbf{F}(1,\dots,d-1)(E)$. Note that in restriction to F', the condition $V_{d-1} \subseteq E_{\mu_2}(x)$ yields: $U_{d-1} \subseteq E_{\mu_2} \subseteq E_{\mu_1}$; we can therefore consider the subvariety

$$\mathbf{P}((E_{\mu_1}|_{F'})/(U_{d-1}|_{F'})) \subseteq \mathbf{P}((E/U_{d-1})|_{F'}) \subseteq \mathbf{P}(E/U_{d-1}) = \mathbf{F}(1,\dots,d)(E).$$

Iterating this inductive step, we get a sequence of projective bundles

$$F_\mu(E_\bullet) = \mathbf{P}(E_{\mu_1}/U_{d-1}) \to \mathbf{P}(E_{\mu_2}/U_{d-2}) \to \dots \to \mathbf{P}(E_{\mu_{d-1}}/U_1) \to \mathbf{P}(E_{\mu_d}). \tag{9}$$

Set $\xi_i = -c_1(U_{d-i+1}/U_{d-i})$, $i = 1,\dots,d$, the first Chern class of the universal line bundle on $\mathbf{P}(E_{\mu_i}/U_{d-i})$ (we only imply the restriction of the universal subbundles to this subvariety).

Let f be a polynomial in d variables with coefficients in $A^\bullet(X)$.

Theorem 3.1 *With the above notation, one has*

$$(\vartheta_\mu)_* f(\xi_1,\dots,\xi_d) = \Big[t_1^{\mu_1-1}\cdots t_d^{\mu_d-1}\Big]\Bigg(f(t_1,\dots,t_d)\prod_{1\le i<j\le d}(t_i-t_j)\prod_{1\le i\le d}s_{1/t_i}(E_{\mu_i})\Bigg).$$

A proof of this theorem is based on (9) and (5).

3.2 Symplectic Groups

Let $E \to X$ be a rank $2n$ symplectic vector bundle endowed with the symplectic form $\omega\colon E \otimes E \to L$ with value in a line bundle $L \to X$, over a variety X. For $d \in \{1, \ldots, n\}$, let $\mathbf{G}^{\omega}_d(E) = \mathbf{F}^{\omega}(d)(E)$ be the Grassmannian bundle of isotropic d-planes in the fibers of E. Let

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E_n^{\omega} \subsetneq \cdots \subsetneq E_0^{\omega} = E$$

be a reference flag of isotropic subbundles of E and their symplectic complements, where $\mathrm{rk}(E_i) = i$. For $i = 1, \ldots, n$, we set $E_{n+i} := E^{\omega}_{n-i}$. For a partition $\lambda \subseteq (2n - d)^d$, there is the Schubert cell $\mathring{\Omega}_\lambda(E_\bullet)$ in $\mathbf{G}^{\omega}_d(E)$ given over the point $x \in X$ by the conditions

$$\mathring{\Omega}_\lambda(E_\bullet)(x) := \Big\{V \in \mathbf{G}^{\omega}_d(E)(x)\colon \dim\big(V \cap E_{2n-d+i-\lambda_i}(x)\big) = i,\ \text{for } i = 1, \ldots, d\Big\}.$$

Denote $\nu_{d+1-i} := 2n - d + i - \lambda_i$ the dimension of the reference space appearing in the ith condition. A partition indexing the Schubert cell $\mathring{\Omega}_\lambda$ must satisfy the conditions $\nu_i + \nu_j \neq 2n + 1$ (see [6, p. 174], where this is shown for $d = n$, and for arbitrary d the argument is the same). For such partitions one defines the Schubert bundle $\varpi_\lambda\colon \Omega_\lambda \to X$ as the Zariski-closure of $\mathring{\Omega}_\lambda$, given over a point $x \in X$ by the conditions

$$\Omega_\lambda(E_\bullet)(x) := \Big\{V \in \mathbf{G}^{\omega}_d(E)(x)\colon \dim\big(V \cap E_{2n-d+i-\lambda_i}(x)\big) \geq i,\ \text{for } i = 1, \ldots, d\Big\}.$$

For a strict partition $\mu \subseteq (2n)^d$ with d parts, such that $\mu_i + \mu_j \neq 2n + 1$ for all i, j, we introduce the *isotropic Kempf–Laksov bundle* $\vartheta_\mu\colon F_\mu(E_\bullet) \to X$ given over the point $x \in X$ by

$$F_\mu(E_\bullet)(x) := \Big\{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d \in \mathbf{F}^{\omega}(1, \ldots, d)(E)(x)\colon V_{d+1-i} \subseteq E_{\mu_i}(x)\Big\}.$$

Note that as in the previous section, $F_\nu(E_\bullet)$ is birational to $\Omega_\lambda(E_\bullet)$, but here it is not smooth in general.

Let U_i stands for the restriction to $F_\mu(E_\bullet)$ of the rank i universal bundle on $\mathbf{F}(1, \ldots, d)(E)$. Set $\xi_i = -c_1(U_{d-i+1}/U_{d-i})$, for $i = 1, \ldots, d$.

Let f be a polynomial in d variables with coefficients in $A^\bullet(X)$.

Theorem 3.2 *With the above notation, one has*

$$(\vartheta_\mu)_* f(\xi_1, \ldots, \xi_d) = \\ [t_1^{\mu_1-1} \cdots t_d^{\mu_d-1}]\Big(f(t_1, \ldots, t_d) \prod_{1\leq i<j\leq d}(t_i - t_j) \prod_{\substack{1\leq i<j\leq d\\ \mu_i+\mu_j>2n+1}}(c_1(L) + t_i + t_j) \prod_{1\leq j\leq d} s_{1/t_j}(E_{\mu_j})\Big).$$

A proof of this theorem will appear in a separate publication.

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References

1. Darondeau, L., Pragacz, P.: Universal Gysin formulas for flag bundles. Int. J. Math. **28**(11), 1750077 (2017)
2. Darondeau, L., Pragacz, P.: Gysin maps, duality, and Schubert classes. Fundam. Math. **244**, 191–208 (2019)
3. Fulton, W.: Intersection theory. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2. Springer, Berlin (1984)
4. Grothendieck, A.: Fibrés vectoriels: fibrés projectifs, fibrés en drapeaux. Seminaire H. Cartan 13e, vol. 1 (1960)
5. Kempf, G., Laksov, D.: The determinantal formula of Schubert calculus. Acta Math. **132**(1), 153–162 (1974)
6. Pragacz, P.: Algebro-geometric applications of Schur S– and Q– polynomials. Séminaire d'Algèbre Dubreil Malliavin 1989–1990. Springer Lecture Notes in Mathematics, vol. 1478, pp. 130–191 (1991)

Order of Tangency Between Manifolds

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Abstract We study the order of tangency between two manifolds of same dimension and give that notion three quite different geometric interpretations. Related aspects of the order of tangency, e.g., regular separation exponents, are also discussed.

Keywords Order of tangency · Order of contact · Taylor polynomial · Higher jets · Tower of Grassmannians · Regular separation exponent · Łojasiewicz exponent

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1 Introduction

In the present paper we discuss the order of tangency (or that of contact) between manifolds and its relation to enumerative geometry started with classical Schubert calculus.

Two plane curves, both sufficiently smooth and nonsingular at a point x^0, are said to have a contact of order at least k at x^0 if, in properly chosen regular parametriza-

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tions, those two curves have identical Taylor polynomials of degree k about the respective preimages of x^0.[1]

Alternatively, those curves have such contact when their minimal regular separation exponent at x^0, cf. [11], is not smaller than k or is not defined.

Formulas enumerating contacts have been widely investigated. For example in [3] the authors derive a formula for the number of contacts of order n between members of a specified-parameter family of plane curves and a generic plane curve of sufficiently high degree.

Contact problems of this sort have been of both old and new interests, particularly in the light of Hilbert's 15th problem to make rigorous the classical calculations of enumerative geometry, especially those undertaken by Schubert [16]. The situation regarding ordinary (i.e., first–order) contacts between families of varieties is now well understood thanks in large measure to the contact formula of Fulton, Kleiman and MacPherson [6]. The above mentioned formula in [3] generalizes that given by Schubert in [17] for the number of double contacts between a given plane curve and a specified 2-parameter family of curves. Schubert made his computations through the use of what has come to be known as "Schubert triangles". This theory has been made completely rigorous by Roberts and Speiser, see, e.g., [15], and independently by Collino and Fulton [2].

Apart from contact formulas, an important role is played by the "order of tangency". Let us discuss this notion for Thom polynomials. Among important properties of Thom polynomials we record their positivity closely related to Schubert calculus (see, e.g., [12] and also [14] for a survey). Namely, the order of tangency allows one to define the jets of Lagrangian submanifolds. The space of these jets is a fibration over the Lagrangian Grassmannian and leads to a positive decomposition of the Lagrangian Thom polynomial in the basis of Lagrangian Schubert cycles.

In this paper, we give three approaches to the order of tangency. The first one (in Sect. 2) is by the Taylor approximations of local parametrizations of manifolds. The second one (a min-max procedure in Sect. 3) makes use of curves sitting in the relevant manifolds. The third approach (in Sect. 4) is by Grassmann bundles. We show that these three approaches are equivalent. We basically work with manifolds over the reals (of various classes of smoothness), but the results carry over—in the holomorphic category—to complex manifolds.

In the last two sections, we discuss some issues related to the "closeness" of pairs of geometric objects: branches of algebraic sets and relations with contact geometry. In fact, in Sect. 5 discussed are the regular separation exponents of pairs of semialgebraic sets, sometimes called Łojasiewicz exponents (not to be mixed with the by now classical exponent in the renowned Łojasiewicz inequality for analytic functions). Then, in Sect. 6 we report on an unexpected application of a modification of tangency order in 3D which yields an elegant criterion for a rank-2 distribution on a 3-manifold to be contact.

[1] Some authors prefer to use at this place Taylor polynomials of degree $k-1$ instead, see for instance [3] and [4].

These concluding sections are not less important than the preceding ones. They show that the precise measurement of closeness is sometimes more demanding—and giving more—than merely bounding below tangency orders.

In the case of *singular* varieties different approaches to tangency orders lead to different notions. In this respect we refer the reader to [5] where compared were two discrete *symplectic* invariants of singular curves: the *Lagrangian tangency order* and *index of isotropy*.

2 By Taylor

One situation that is frequently encountered at the crossroads of geometry and analysis deals with pairs of manifolds which are the graphs of functions of the same number of variables. Such graphs can intersect, or touch each other, at a prescribed point, with various degrees of closeness.

Our departing point is a definition of such proximity going precisely in the spirit of a benchmark reference book [9], p. 18, although not formulated *expressis verbis* there.

Definition Two manifolds M and $\widetilde{M}$ embedded in $\mathbb{R}^m$, both of class C^r, $r \geq 1$, and the same dimension p, intersecting at $x^0 \in M \cap \widetilde{M}$, for $k \leq r$, *have at x^0 the order of tangency at least k*, when **there exist** a neighbourhood $U \ni u^0$ in $\mathbb{R}^p$ and parametrizations[2] (diffeomorphisms onto the image)

$$q\colon \big(U, u^0\big) \to \big(M,\, x^0\big), \qquad \tilde{q}\colon \big(U, u^0\big) \to \big(\widetilde{M},\, x^0\big)$$

of class C^r such that

$$\Big(\tilde{q} - q\Big)(u) \;=\; \mathrm{o}\Big(\big|u - u^0\big|^k\Big) \tag{1}$$

when $U \ni u \to u^0$.

(We underline the existence clause in this definition. Supposing having already such a couple of local parametrizations q and $\tilde{q}$, there is *an abundance* of other pairs of C^r parametrizations serving the vicinities of x^0 in M and $\widetilde{M}$, respectively, and *not* satisfying the condition (1). Note also that in this definition the order of tangency is automatically at least 0.)

Below in •• in Sect. 3, and also in Sect. 4 we restrict ourselves to parametrizations of very specific type—just the *graphs* of C^r mappings going from p dimensions to $m - p$ dimensions. This appears to be possible while *not* violating the key condition (1).

Naturally enough, the notion of the order of tangency not smaller than … is invariant under the local C^r diffeomorphisms of neighbourhoods in $\mathbb{R}^m$ of the tangency point x^0.

---

[2]The standard topology language adopted, among many other sources, in [13]

Attention. In the real C^∞ category it is possible for the order of tangency to be at least k for all $k \in \mathbb{N}$. In other words—be infinite even though $\{x^0\} = M \cap \widetilde{M}$. The rest of this paper is to be read with this remark in mind.

As a matter of record, basically the same definition is evoked in Proposition on page 4 in [8]. In [8] there is also proposed the following reformulation of (1).

Proposition 1 *The condition* (1) *is equivalent to*

$$T^k_{u^0}\big(q\big) \;=\; T^k_{u^0}\big(\tilde{q}\big)\,, \tag{2}$$

where $T^k_{u^0}\big(\,\cdot\,\big)$ *means the Taylor polynomial about* u^0 *of degree* k*.*

Implication (1) $\Rightarrow$ (2).

$$\begin{aligned} \mathrm{o}\Big(\big|u-u^0\big|^k\Big) = \tilde{q}(u) - q(u) \;&=\; \big(\tilde{q}(u) - T^k_{u^0}\big(\tilde{q}\big)(u-u^0)\big) \\ +\; \big(T^k_{u^0}\big(\tilde{q}\big)(u-u^0) - T^k_{u^0}\big(q\big)(u-u^0)\big) &+ \big(T^k_{u^0}\big(q\big)(u-u^0) - q(u)\big)\,, \end{aligned} \tag{3}$$

where the first and last summands on the right hand side are $\mathrm{o}\Big(\big|u-u^0\big|^k\Big)$ by Taylor. So is the middle summand

$$T^k_{u^0}\big(\tilde{q}\big)(u-u^0) - T^k_{u^0}\big(q\big)(u-u^0) \;=\; \mathrm{o}\Big(\big|u-u^0\big|^k\Big)$$

and (2) follows from the following general result.

Lemma 1 *Let* $w \in \mathbb{R}[u_1,\,u_2,\,\ldots,\,u_p]$, $\deg w \le k$, $w(u) = \mathrm{o}\big(|u|^k\big)$ *when* $u \to 0$ *in* $\mathbb{R}^p$*. Then* w *is identically zero.*

The proof goes by induction on $k \ge 0$, with an obvious start for $k = 0$. Then, assuming this for the polynomials of degrees smaller than $k \ge 1$ and taking a polynomial w of degree k as in the wording of the lemma, we can assume without loss of generality that w is *homogeneous* of degree k (the terms of lower degrees vanish altogether by the inductive assumption). Let $u \in \mathbb{R}^p$, $|u| = 1$, be otherwise arbitrary. Then

$$t^k w(u) = w(tu) = \mathrm{o}\big(|tu|^k\big) = \mathrm{o}\big(|t|^k\big) \quad \text{when } t \to 0\,.$$

Hence $w(u) = 0$ and the vanishing of w follows.

Implication (1) $\Leftarrow$ (2).

This implication is obvious, because now the middle term on the right hand side of (3) vanishes, so that the right hand side is automatically $\mathrm{o}\Big(\big|u-u^0\big|^k\Big)$. $\square$

3 By Curves

In the discussion in this section important will be the quantity

$$s: \; = \; \sup\{k \in \mathbb{N}\colon \text{the order of tangency } \geq k\}\,. \tag{4}$$

(Note that an additional restriction here on k is $k \leq r$, cf. Definition above.) If the class of smoothness $r = \infty$, then, by the very definition, the condition (1) holds for all k if and only if $s = \infty$.

Is it possible to ascertain something similar in the finite-order-of-tangency case?

With an answer to this question in view, we stick in the present section to the notation introduced in Sect. 2, but assume *additionally* that

$$s < r\,. \tag{5}$$

(Reiterating, the quantity s is defined in (4) above, and r is the assumed class of smoothness of the underlying manifolds, finite or infinite when the category is real. When $r = \infty$, the condition (5) simply says that s is finite.)

Our second approach uses *pairs of curves* lying, respectively, in M and $\widetilde{M}$.

We naturally assume that $T_{x^0}M = T_{x^0}\widetilde{M}$. Our actual objective is to show that

Theorem 1 *Under* (5)*,*

$$\min_{v}\Big(\max_{\gamma,\,\tilde\gamma}\big(\max\big\{l \in \{0\} \cup \mathbb{N}\colon\; |\gamma(t) - \tilde\gamma(t)| = \mathrm{o}\big(|t|^l\big) \text{ when } t \to 0\,\big\}\big)\Big) \;=\; s\,. \tag{6}$$

The **minimum** *is taken over all* $0 \neq v \in T_{x^0}M = T_{x^0}\widetilde{M}$*. The* **outer maximum** *is taken over all pairs of* C^r *curves* $\gamma \subset M$*,* $\tilde\gamma \subset \widetilde{M}$ *such that* $\gamma(0) = x^0 = \tilde\gamma(0)$*, and—both non-zero!—velocities* $\dot\gamma(0)$*,* $\dot{\tilde\gamma}(0)$ *are both parallel to* v*.*

Attention. In this theorem the assumption (5) is essential; our proof would not work in the situation $s = r$.

***Proof of Theorem* 1** • It is quick to show that the integer on the left hand side of equality (6) is at least s. Indeed, for every fixed vector v as above, $v = dq(u^0)u$ (without loss of generality, u is like in the proof of Lemma 1), one can take $\delta(t) = q(u^0 + tu)$ and $\tilde\delta(t) = \tilde q\big(u^0 + tu\big)$. Then

$$|\delta(t) - \tilde\delta(t)| \;=\; \mathrm{o}\big(|tu|^s\big) \;=\; \mathrm{o}\big(|t|^s\big)$$

and so

$$\max_{\gamma,\,\tilde\gamma}\big(\max\big\{\,l\colon |\gamma(t) - \tilde\gamma(t)| = \mathrm{o}\big(|t|^l\big) \text{ when } t \to 0\big\}\big) \;\geq\; s\,.$$

In view of the arbitrariness in our choice of v, the same remains true after taking the minimum over all admissible v's which is actually done on the left hand side of (6).

•• To show the opposite non-sharp inequality in (6) is more involved. It is precisely in this part that the additional assumption $s \le r-1$ is needed. We study the two manifolds in the vicinity of x^0 via an appropriate local C^r diffeomorphism of the ambient space, after which

$$\big(M,\, x^0\big) = \Big(\big\{x_{p+1} = x_{p+2} = \cdots = x_m = 0\big\},\, 0\Big)$$

and

$$(\widetilde{M},\, x^0) = \Big(\big\{x_j = F^j(x_1,\, x_2, \ldots,\, x_p)\,,\ j = p+1,\ p+2, \ldots,\, m\big\},\, 0\Big)$$

for some C^r functions F^j. Having the manifolds so neatly (graph-like) positioned, we take the most adapted parametrizations

$$q(u_1,\, u_2, \ldots,\, u_p) = \big(u_1,\, u_2, \ldots,\, u_p,\, 0,\, 0, \ldots,\, 0\big)\,,$$

$$\tilde{q}\big(u_1,\, u_2, \ldots,\, u_p\big) = \Big(u_1,\, u_2, \ldots,\, u_p,\, F\big(u_1,\, u_2, \ldots,\, u_p\big)\Big)\,,$$

where $F = \big(F^{p+1},\ F^{p+2}, \ldots,\ F^m\big)$. This—important—necessitates some extra technical work. Firstly the initial couple of parametrizations satisfying (1) is being straightened simultaneously with manifolds M and $\widetilde{M}$. Naturally enough, the resulting parametrizations keep satisfying (1), but are not yet of the above-desired form. So the parametrizations *and* manifolds are to be additionally slightly upgraded via another local C^r ambient diffeomorphism so as (a) to keep the simple description of manifolds and (b) to have the eventual parametrizations adapted as desired above.

Given the definition (4) of s, there hold

$$T^s_{u^0}(q) = T^s_{u^0}\big(\tilde{q}\big) \qquad \text{and} \qquad T^{s+1}_{u^0}(q) \neq T^{s+1}_{u^0}\big(\tilde{q}\big)\,,$$

that is,

$$T^s_{u^0}(F) = 0 \qquad \text{and} \qquad T^{s+1}_{u^0}(F) \neq 0\,.$$

It follows that there exist an integer $j \in \{p+1,\ p+2, \ldots,\ m\}$ and a vector $\mathbf{w} \in \mathbb{R}^p$ such that

$$T^s_{u^0}(F^j)(\mathbf{w}) \;=\; 0 \qquad \text{and} \qquad T^{s+1}_{u^0}\big(F^j\big)(\mathbf{w}) \neq 0\,. \tag{7}$$

Now let u and $\tilde{u}$ be two C^r curves in $\mathbb{R}^p$ passing at $t=0$ through u^0 and such that the vectors $\dot{u}(0)$ and $\dot{\tilde{u}}(0)$ are both non-zero and parallel to $\mathbf{w}$. These curves *in parameters* give rise to C^r curves $\delta(t) = q(u(t))$ and $\tilde{\delta}(t) = \tilde{q}(\tilde{u}(t))$ *in the manifolds*, both having at $t=0$ non-zero speeds parallel to the vector $\mathbf{v}\colon = dq(u^0)\mathbf{w} = d\tilde{q}(u^0)\mathbf{w}$. We will now estimate from above (by s) the left hand side of the equality (6) using, no wonder, $\mathbf{v}$, δ, and $\tilde{\delta}$:

$$|\delta(t) - \tilde{\delta}(t)| = \sqrt{|u(t) - \tilde{u}(t)|^2 + |F\big(\tilde{u}(t)\big)|^2}$$
$$\geq |F\big(\tilde{u}(t)\big)| \geq |F^j\big(\tilde{u}(t)\big)| \neq \mathrm{o}\big(|t|^{s+1}\big)\,, \qquad (8)$$

where the last inequality necessitates an explanation. In fact, by (7) and for every $c \neq 0$

$$T^{s+1}_{u^0}\big(F^j\big)(tc\mathbf{w}) = (ct)^{s+1} T^{s+1}_{u^0}\big(F^j\big)(\mathbf{w}) \neq \mathrm{o}\big(|t|^{s+1}\big) \quad \text{when} \;\; t \to 0\,.$$

But $\tilde{u}(t) - \tilde{u}(0) = ct\mathbf{w} + \mathrm{o}\big(|t|\big)$ for some non-zero c, hence

$$T^{s+1}_{u^0}\big(F^j\big)\big(\tilde{u}(t) - \tilde{u}(0)\big) \neq \mathrm{o}\big(|t|^{s+1}\big) \quad \text{when} \;\; t \to 0$$

as well. Also, just by Peano in the class of smoothness $s + 1 \leq r$, cf. (5),

$$F^j(u) = T^{s+1}_{u^0}\big(F^j\big)\big(u - u^0\big) + \mathrm{o}\Big(\big|u - u^0\big|^{s+1}\Big)$$

when $u \to u^0$ in $\mathbb{R}^p$. Therefore, $F^j\big(\tilde{u}(t)\big) \neq \mathrm{o}\big(|t|^{s+1}\big)$ as written in (8).

Now it is important to note that the pair of curves δ and $\tilde{\delta}$ produced by us above is completely general in the category C^r for that chosen vector $\mathbf{v}$. Hence it follows that—for this precise vector $\mathbf{v}$!—the quantity

$$\max_{\gamma,\,\tilde{\gamma}}\Big(\max\Big\{\, l \in \{0\} \cup \mathbb{N} : |\gamma(t) - \tilde{\gamma}(t)| = \mathrm{o}\big(|t|^l\big) \text{ when } t \to 0\Big\}\Big)$$

does not exceed s. Understandingly, so does the minimum of such quantities over all v's in $T_{x^0}M = T_{x^0}\widetilde{M}$. Theorem 1 is now proved. $\square$

4 By Grassmannians

Our third approach is based on the introductory pages of [8] where a natural tower of consecutive Grassmannians is being attached to every given local C^r parametrization q as used by us in the preceding sections. However, to allow for a recursive definition of tower's members, a more general framework is needed.

Namely, to every C^1 *immersion* $H : N \to N'$, N—an n-dimensional manifold, N'—an n'-dimensional manifold (manifolds not necessarily embedded in Euclidean spaces!), we attach the so-called image map $\mathcal{G}H\colon N \to G_n(N')$ of the tangent map dH: for $s \in N$,

$$\mathcal{G}H(s) \;=\; dH(s)(T_sN)\,, \qquad (9)$$

where $G_n(N')$ is the *total space* of the Grassmann bundle, with base N', of all n-planes tangent to N'. That is, $G_n(N')$ is a new manifold, much bigger than N' (whenever $n' > n$), of dimension $n' + n(n' - n)$.

We stick in the present section to the notation from Sect. 2 and invariably use the pair of parametrizations q and $\tilde{q}$. So we are given the mappings

$$\mathcal{G}q \,:\, U \longrightarrow G_p\big(\mathbb{R}^m\big)\,, \qquad \mathcal{G}\tilde{q} \,:\, U \longrightarrow G_p\big(\mathbb{R}^m\big)\,.$$

Upon putting $M^{(0)} = \mathbb{R}^m$, $\mathcal{G}^{(1)} = \mathcal{G}$, there emerge two sequences of recursively defined mappings. Namely, for $l \geq 1$,

$$\mathcal{G}^{(l)}q \,:\, U \longrightarrow G_p\big(M^{(l-1)}\big)\,, \qquad \mathcal{G}^{(l+1)}q \;=\; \mathcal{G}\Big(\mathcal{G}^{(l)}q\Big)$$

and

$$\mathcal{G}^{(l)}\tilde{q} \,:\, U \longrightarrow G_p\big(M^{(l-1)}\big)\,, \qquad \mathcal{G}^{(l+1)}\tilde{q} \;=\; \mathcal{G}\Big(\mathcal{G}^{(l)}\tilde{q}\Big),$$

where, naturally, $M^{(l)} = G_p\big(M^{(l-1)}\big)$. Now our objective is to show the following.

Theorem 2 C^r *manifolds* M *and* $\widetilde{M}$ *have at* x^0 *the order of tangency at least* k *(*$1 \leq k \leq r$*) iff there exist* C^r *parametrizations* q *and* $\tilde{q}$ *of the vicinities of* x^0 *in, respectively,* M *and* $\widetilde{M}$*, such that*

$$\mathcal{G}^{(k)}q\,(u^0) \;=\; \mathcal{G}^{(k)}\tilde{q}\,(u^0)\,. \tag{10}$$

(Observe that, in (10), there is clearly encoded that $q(u^0) = x^0 = \tilde{q}(u^0)$.)

4.1 *Proof of Theorem 2*

In what follows, of interest for us will be the situations when H in (9) above is locally (and all is local in tangency considerations!) the graph of a C^1 mapping $h\colon \mathbb{R}^p \supset U \to \mathbb{R}^t$. That is, for $u \in U$, $H(u) = \big(u,\, h(u)\big) \in \mathbb{R}^{p+t} = \mathbb{R}^p \times \mathbb{R}^t$. Then (9) assumes by far more precise form

$$\mathcal{G}H(u) = \Big(u,\, h(u)\,;\, d\big(u, h(u)\big)(u)\Big) = \Big(u,\, h(u);\, \text{span}\{\partial_j + h_j(u)\colon j = 1,\, 2, \ldots,\, p\}\Big) \tag{11}$$

where the symbol h_j means the partial derivative of a vector mapping h with respect to the variable u_j ($j = 1, \ldots,\, p$), and $\partial_j + h_j(u)$ denotes the partial derivative of the vector mapping $(\iota,\, h)\colon U \to \mathbb{R}^p\big(u_1, \ldots,\, u_p\big) \times \mathbb{R}^t$ with respect to u_j, where $\iota\colon U \hookrightarrow \mathbb{R}^p$ is the inclusion.

Now observe that the expression for $\mathcal{G}H(u)$ on the right hand side of (11) is still not quite useful. Yet there are charts in each newly appearing Grassmannian (see, for instance, [8] or p. 46 in [1])!

The chart in a typical fibre G_p over a point in the base $\mathbb{R}^{p+t}$, good for (11), consists of all the entries in the bottommost rows (indexed by numbers $p+1,\ p+2,\ \dots,\ p+t$) in the $(p+t)\times p$ matrices

$$\left[\, v_1 \mid v_2 \mid \dots \mid v_p \,\right]$$

with non-zero upper $p \times p$ minor, **after** multiplying the matrix on the right by the inverse of that upper $p \times p$ submatrix. That is to say, taking as the local coordinates all the entries in rows $(p+1)$-st,..., $(p+t)$th of the matrix

$$\left[v_j^{\,i}\right]_{\substack{1\le i\le p+t\\ 1\le j\le p}} \left(\left[v_j^{\,i}\right]_{\substack{1\le i\le p\\ 1\le j\le p}}\right)^{-1}.$$

That is, these coordinates are all $t \times p$ entries of the matrix

$$\left[v_j^{\,i}\right]_{\substack{p+1\le i\le p+t\\ 1\le j\le p}} \left(\left[v_j^{\,i}\right]_{\substack{1\le i\le p\\ 1\le j\le p}}\right)^{-1}.$$

In these, extremely useful, glasses the description (11) gets stenographed to

$$\mathcal{G}H(u) = \Big(u,\ h(u)\,;\ \frac{\partial h}{\partial u}(u)\Big), \tag{12}$$

where, under the symbol $\frac{\partial h}{\partial u}(u)$ understood are all the entries of this *Jacobian* $(t \times p)$-matrix written in row and separated by commas. This technical simplification is central for a proof that follows.

After this, basically algebraic, preparation we come back to Theorem 2.

The order of tangency between M and $\widetilde{M}$ at x^0 being at least k precisely means (Proposition 1) the existence of local C^r parametrizations q and $\tilde{q}$ satisfying (2). So we are just going to show that (10) $\Longleftrightarrow$ (2).

Moreover, we assume without loss of generality—much like it has been the case in the part •• of the proof of Theorem 1—that M and $\widetilde{M}$ are locally the graphs of parametrizations q and $\tilde{q}$, respectively. Which, at the same time, satisfy (2). So (2) holds for $q(u) = (u,\ f(u)),\ f\colon U \to \mathbb{R}^{m-p}\big(y_{p+1},\ \dots,\ y_m\big)$ and for $\tilde{q}(u) = (u,\ \tilde{f}(u)),\ \tilde{f}\colon U \to \mathbb{R}^{m-p}\big(y_{p+1},\ \dots,\ y_m\big),\ x^0 = \big(u^0,\ f(u^0)\big) = \big(u^0,\ \tilde{f}(u^0)\big)$.

Implication (2) $\Rightarrow$ (10).

We will derive such expressions for $\mathcal{G}^{(k)}q\ (u)$ and $\mathcal{G}^{(k)}\tilde{q}\ (u),\ u \in U$, that the use of the condition (2) will just prompt by itself. An added value of this derivation will be the control over the sets of natural local coordinates in the Grassmannians in question. (With this information at hand the opposite implication (2) $\Leftarrow$ (10) will follow in no time.) Our main technical tool for the $\Rightarrow$ implication is

Lemma 2 *For $1 \le l \le k$ there exists a local chart on the Grassmannian space $G_p\big(M^{(l-1)}\big)$ in which the mapping $\mathcal{G}^{(l)}q$ evaluated at u assumes the form*

$$\left(u,\ f(u);\ \binom{l}{1}\times f_{[1]}(u),\ \binom{l}{2}\times f_{[2]}(u),\ \ldots,\ \binom{l}{l}\times f_{[l]}(u)\right),$$

where $f_{[\nu]}(u)$ *is a shorthand notation for the aggregate of all the partials of the* ν*th order at* u*, of all the components of* f*, which are in the number* $(m-p)\times p^{\nu}$*, and the symbol* $N\times(*)$ *stands for the* N *copies going in row and separated by commas, of an object* $(*)$.

Attention. In this lemma we purposefully distinguish mixed derivatives taken in different orders, simply disregarding the Schwarz symmetricity discovery.

Proof $l=1$. We note that

$$\mathcal{G}^{(1)}q\,(u)=\Big(u,\ f(u);\ \operatorname{span}\{\partial_j+f_j(u)\colon j=1,\,2,\ldots,\,p\}\Big),$$

in the relevant Jensen–Borisenko–Nikolaevskii chart, is nothing but

$$\big(u,\ f(u);\ f_{[1]}(u)\big)\ =\ \left(u,\ f(u);\ \binom{l}{1}\times f_{[1]}(u)\right).$$

The beginning of induction is done.

$l\Rightarrow l+1, l<k$. The mapping $\mathcal{G}^{(l)}q\colon U\to M^{(l)}$, evaluated at u, is already written down, in appropriate local chart assumed to exist in $M^{(l)}$, as

$$\left(u,\ f(u),\ \binom{l}{1}\times f_{[1]}(u),\ \binom{l}{2}\times f_{[2]}(u),\ldots,\ \binom{l}{l}\times f_{[l]}(u)\right). \tag{13}$$

We work with $\mathcal{G}^{(l+1)}q=\mathcal{G}\Big(\mathcal{G}^{(l)}q\Big)$. Now, (13) being clearly of the form $H(u)=\big(u,\ h(u)\big)$ in the previously introduced notation, the mapping h reads

$$h(u)\ =\ \left(f(u),\ \binom{l}{1}\times f_{[1]}(u),\ \binom{l}{2}\times f_{[2]}(u),\ldots,\ \binom{l}{l}\times f_{[l]}(u)\right).$$

In order to have $\mathcal{G}H(u)$ written down, in view of (12), one ought to write in row: u, then $h(u)$, and then all the entries of the Jacobian matrix $\frac{\partial h}{\partial u}(u)$, also written in row and separated by commas. The **latter**, in our shorthand notation, are computed immediately. Namely

$$\frac{\partial h}{\partial u}(u)\ =\left(\binom{l}{0}\times f_{[1]}(u),\ \binom{l}{1}\times f_{[2]}(u),\ \binom{l}{2}\times f_{[3]}(u),\ldots,\ \binom{l}{l}\times f_{[l+1]}(u)\right).$$

These entries on the right hand side are to be juxtaposed with the **former** entries $\big(u,\ h(u)\big)$. For better readability, we put together the groups of *same* partials (a yet another permutation of Grassmann-type coordinates, cf. the wording of the lemma). In view of the elementary identities $\binom{l}{\nu-1}+\binom{l}{\nu}=\binom{l+1}{\nu}$, we get in the outcome

$$\left(u,\ f(u),\ \binom{l+1}{1}\times f_{[1]}(u),\ \binom{l+1}{2}\times f_{[2]}(u),\ldots,\ \binom{l+1}{l}\times f_{[l]}(u),\ \binom{l+1}{l+1}\times f_{[l+1]}(u)\right).$$

Lemma 2 is now proved by induction.

We now take $l = k$ in Lemma 2 and get, for arbitrary $u \in U$, two similar visualisations of $\mathcal{G}^{(k)}q\ (u)$ and $\mathcal{G}^{(k)}\tilde{q}\ (u)$. At that, the equality (2) holds true at $u = u^0$. As a consequence, (10) follows.

Implication (2) $\Leftarrow$ (10).

With the information on superpositions of the mappings $\mathcal{G}$, gathered in the course of proving the implication (2) $\Rightarrow$ (10), this opposite implication is clear. Theorem 2 is now proved. $\square$

5 Algebraic Geometry Examples and Regular Separation Exponents

In the present section, we shall work with the regular separation exponents of pairs of sets, a notion due to Łojasiewicz [11]. We shall compute these exponents in several natural examples, and compare the results with the information (when available) about the relevant orders of tangency. These examples deal with branches of algebraic sets which often happen to be tangent one to another, with various degrees of closeness.

Example 1 In the work [4] (Fig. 2 on page 37 there) analyzed is the following algebraic set in $\mathbb{R}^2(x, y)$

$$\mathcal{C} = \{(x, y)\colon (y - x^2)^2 = x^5\}. \tag{14}$$

The two branches of C issuing from the point $(0, 0)$,

$$C_- = \{y = x^2 - x^{5/2},\ x \geq 0\} \quad \text{and} \quad C_+ = \{y = x^2 + x^{5/2},\ x \geq 0\},$$

could be naturally extended to one-dimensional manifolds D_- and D_+, both of class C^2—the graphs of functions

$$y_-(x) = x^2 - |x|^{5/2} \quad \text{and} \quad y_+(x) = x^2 + |x|^{5/2},$$

respectively. The Taylor polynomials of degree 2 about $x = 0$ of y_- and y_+ coincide. Hence D_- and D_+ have at $(0, 0)$ the order of tangency at least 2 (cf. Sect. 2), and clearly *not* at least 3.

This example clearly suggests that, in real algebraic geometry, it would be pertinent to use non-integer measures of closeness. For instance, for the above sets $y_-(x)$ and $y_+(x)$, we may take

$$\sup\{\alpha > 0 : \; y_+(x) - y_-(x) = o(|x|^\alpha) \text{ when } x \to 0\}\,.$$

This kind of a *generalized* order of tangency would be 5/2 in the Colley–Kennedy example.

In the local analytic geometry there *is* a precise name for this notion—the minimal regular separation exponent for two (semialgebraic) branches, say X and Y, of an algebraic set. That is, here the minimal exponent for $X = C_-$ and $Y = C_+$. Some authors working after [11] used to call such a quantity the *Łojasiewicz exponent* of, in this case, C_-, C_+ at $(0, 0)$. And denoted it—when specialized to the present situation—by $\mathcal{L}_{(0,0)}\big(C_-,\ C_+\big)$.[3]

Remark 1 (i) Therefore, in Example 1, the rational number 5/2 is the minimal regular separation exponent of the semialgebraic sets C_- and C_+ which touch each other at $(0, 0)$.

(ii) Example 1 quickly generalizes, by means of the equation $(y - x^N)^2 = x^{2N+1}$ with N arbitrarily large, to yield a pair of C^N manifolds having the order of tangency at least N and not at least $N + 1$, and having the minimal regular separation exponent $N + \frac{1}{2}$.

It has not been difficult in Example 1 to discern the pair of branches C_- and C_+, initially slightly hidden in a synthetic equation (14). But it can happen considerably worse in this respect. Consider, for instance

Example 2 (a) an algebraic set in the plane $\mathbb{R}^2(x, y)$ defined by a single equation

$$(xy)^2 \;=\; \frac{1}{4}\Big(x^2 + y^2\Big)^3. \tag{15}$$

This set possesses a pair (even more than one such pair) of semialgebraic branches touching each other at the point $(0, 0)$. Yet it is not so immediate to ascertain their minimal regular separation exponent. Only after recognizing in (15) the classical *quatrefoil* $x = \cos(\varphi)\sin(2\varphi)$, $y = \sin(\varphi)\sin(2\varphi)$, it becomes quick to compute the relevant minimal regular separation exponent equal to 2.

(b) It is even more interestingly with another algebraic set in 2D given by the equation

$$\Big(x^2 + y^2 - \frac{1}{2}x\Big)^2 \;=\; \frac{1}{4}\big(x^2 + y^2\big). \tag{16}$$

This set possesses as well a pair of semialgebraic branches $\{y \le 0\}$ and $\{y \ge 0\}$ touching each other at the point $(0, 0)$. Yet it takes some time to find their minimal regular separation exponent. In fact, after discovering in (16) the classical *cardioid* $r = \frac{1}{2}(1 + \cos\varphi)$, that exponent turns out to be—one more time—a non-integer (3/2, in the occurrence).

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[3] However, this terminology is not yet definitely settled, as shown in a recent work [10]. The authors of the latter speak just descriptively about 'the Łojasiewicz exponent for the regular separation of closed semialgebraic sets'.

(c) It is worthy of note that for both explicitly defined algebraic sets (15) and (16) one can apply general type *bounds above*, for the minimal separation exponent, produced in [10]. Yet the estimations got in that way are unrealistically (by factors of thousands) high.

Returning to the notion of the order of tangency, in the realm of algebraic geometry the distinction between that order and minimal regular separation exponent sometimes happens to be fairly clear, with both discussed quantities effectively computable. An instructive instance of such a situation occurs in [18] (Example 3.5 there).

Example 3 The author of [18] deals there with a pair of one-dimensional algebraic manifolds N and Z in $\mathbb{R}^2(x, y)$ intersecting at $(0, 0)$. The manifold $N = \{y = 0\}$ is already utmostly simplified, whereas $Z = \{y^d + yx^{d-1} + x^s = 0\}$ depends on two integer parameters d and s, $1 < d < s$, d odd. What we are going to discuss here is a kind of reworking of Tworzewski's original approach, see also *Attention* below.

These manifolds have at $(0, 0)$ the order of tangency at least $s - d$, and not at least $s - d + 1$, while their minimal regular separation exponent at $(0, 0)$ is $s - d + 1$.

Indeed—to justify this one tries to present Z as the graph of a function $y = y(x)$. Clearly, $y(0) = 0$ and a function $y(x)$ could not be divisible by, for instance, x^{s+1}. So, with no loss of generality,

$$y(x) \;=\; x^k z(x) - \; x^{s-d+1}$$

for certain integer $k \geq 1$ and another function $z(x)$ such that $z(0) \neq 0$.

• The possibility $k < s - d + 1$ boils rather quickly down to $k = 1$, and then to the relation $\left(z - x^{s-d}\right)^d + z = 0$, impossible at $x = 0$, for $\left(z(0)\right)^d + z(0) \neq 0$, d being odd.

•• So $k \geq s - d + 1$ and now

$$y(x) \;=\; x^{s-d+1} z(x) - \; x^{s-d+1}$$

for a certain function $z(x)$. Upon substituting this $y(x)$ to the defining equation of Z and simplifying, $(z - 1)^d\, x^{(d-1)(s-d)} + z = 0$. Hence $z(0) = 0$. The Implicit Function Theorem is applicable here around $(0, 0)$, because

$$\frac{\partial}{\partial z}\Big((z - 1)^d x^{(d-1)(s-d)} + z\Big)\bigg|_{(0,0)} \;=\; 1\,.$$

One gets a locally unique C^∞ function $z(x)$, $z(0) = 0$, hence also a locally unique function $y(x) = x^{s-d+1} z(x) - \; x^{s-d+1}$ whose graph is Z. Because the function z vanishes at 0, the minuend in this expression for y is an 'o'of the subtrahend when $x \to 0$. So the statements about the order of tangency and minimal regular separation exponent follow immediately.

Attention. When $2|d$, the above-found resolving function $y(x)$ is *not* the only solution to the defining equation of Z. Namely, the necessary equality $\big(z(0)\big)^d + z(0) = 0$, $z(0) \neq 0$, is then possible with $z(0) = -1$ and

$$\frac{\partial}{\partial z}\Big(\big(z - x^{s-d}\big)^d + z\Big)\Big|_{(0,-1)} = d(-1)^{d-1} + 1 = 1 - d \neq 0\,.$$

Hence the Implicit Function Theorem gives this time a locally unique (for that $k = 1$) C^∞ function $\tilde{z}(x)$, $\tilde{z}(0) = -1$. Then the graph of

$$\tilde{y}(x) = x\,\tilde{z}(x) - x^{s-d+1} = -x + (\text{higher powers of } x)$$

is a second branch of Z passing through $(0, 0) \in \mathbb{R}^2$, transversal to N, in a stark distinction to the previously found, tangent to N, branch.

Remark 2 More generally, one could *not* hope to get a precise information regarding the minimal regular separation exponent for the pair of manifolds M, $\widetilde{M}$ on the sole basis of the assumptions in Theorem 1. That is, basically, under (5). Despite the inequality (8), that exponent need not necessarily be $s + 1$. Following Example 1 earlier in this section, one could just take the curves C_- and C_+ as the curves δ and $\tilde{\delta}$, respectively, in the proof of Theorem 1. That is, to take $M = C_- = \delta$ and $\widetilde{M} = C_+ = \tilde{\delta}$. Then, as the reader has lately seen, $r = 2$ and the quantity s defined in (4) is also 2, while the minimal regular separation exponent is but $s + \frac{1}{2}\,(= \frac{5}{2})$.

Even restricting oneself to a benchmark setting $\dim M = \dim \widetilde{M}$, an additional enormous complication could come from the fact that the intersection $M \cap \widetilde{M}$ might be a topologically highly nontrivial set (think about the C^∞ category). And it is precisely $M \cap \widetilde{M}$ which enters the definition of regular separation exponents for the pair M, $\widetilde{M}$.

6 Relation with Contact Topology

Unsurprisingly, the notion of order of contact proves useful not only in algebraic geometry (cf. Introduction), but also in geometry *tout court*. One not so obvious application in the *real* category deals with the real contact structures in three dimensions. Our summarising it here follows closely Sect. 1.6 in [7]. The author considers there a couple $\Sigma \subset M$, where M is a *contact* 3-dimensional manifold and Σ—a fixed embedded surface in it. Contact means M being endowed with a contact structure, say ξ, in TM.

When one approaches a given point $p \in \Sigma$ by points q staying within Σ, a natural question is about the order of smallness of the angle $\angle\big(T_q\Sigma,\ \xi_q\big)$. If that angle is an 'O' of the distance of q to p taken to power k (the distance measured in any chosen, and hence every, set of smooth local coordinates about p), then it is said that ξ has the order of contact at least k with Σ at p. (Therefore, what is discussed in this section differs a little from the notion of closeness of a pair of manifolds investigated in the preceding sections. Yet the added value is substantial.)

That is, to say that the *new* order of contact is at least 1 at a given point p is tantamount to saying that $\xi_p = T_p\Sigma$. And it is exactly 0 at p whenever $\xi_p \neq T_p\Sigma$.

So it comes as a not small surprise that this elementary notion allows one to characterise the contact structures as such! Namely, a theorem proved in [7] asserts that a rank-2 tangent distribution ξ on a 3-dimensional M is contact iff ξ has the new order of contact **at most** 1 with every surface Σ embedded in M, and this at every point of Σ.

The next natural question in this direction is whether it is possible to similarly characterise contact structures on $(2n+1)$-dimensional manifolds, $n \geq 2$. The author of [7] says nothing in this respect.

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References

1. Borisenko, A.A., Nikolaevskii, Y.A.: Grassmannian manifolds and Grassmann image of submanifolds. Russ. Math. Surv. **46**, 45–94 (1991) (English translation)
2. Collino, A., Fulton, W.: Intersection rings of spaces of triangles. Mem. Soc. Math. Fr. (N.S.) **38**, 75–117 (1989)
3. Colley, S.J., Kennedy, G.: A higher-order contact formula for plane curves. Commun. Algebra **19**(2), 479–508 (1991)
4. Colley, S.J., Kennedy, G.: Triple and quadruple contact of plane curves. Contemporary Mathematics, vol. 123, pp. 31–59. AMS, Providence (1991)
5. Domitrz, W., Trębska, Ż.: Symplectic T_7, T_8 singularities and Lagrangian tangency orders. Proc. Edinb. Math. Soc. **55**, 657–683 (2012)
6. Fulton, W., Kleiman, S.L., MacPherson, R.: About the enumeration of contacts. Lecture Notes in Mathematics, vol. 997, pp. 156–196. Springer, Berlin (1983)
7. Geiges, H.: An Introduction to Contact Topology. Cambridge University Press, Cambridge (2008)
8. Jensen, G.R.: Higher Order Contact of Submanifolds of Homogeneous Spaces. Lecture Notes in Mathematics, vol. 610. Springer, Berlin (1977)
9. Krasil'shchik, I.S., Lychagin, V.V., Vinogradov, A.M.: Geometry of Jet Spaces and Nonlinear Partial Differential Equations. Nauka, Moscow (1986) (in Russian)
10. Kurdyka, K., Spodzieja, S., Szlachcińska, A.: Metric properties of semialgebraic mappings. Discret. Comput. Geom. **55**, 786–800 (2016)
11. Łojasiewicz, S.: On semi-analytic and subanalytic geometry. Banach Cent. Publ. **34**, 89–104 (1995)
12. Mikosz, M., Pragacz, P., Weber, A.: Positivity of Thom polynomials II: the Lagrange singularities. Fundam. Math. **202**, 65–79 (2009)
13. Milnor, J.W.: Topology from the Differentiable Viewpoint. The University Press of Virginia, Charlottesville (1965)
14. Pragacz, P.: Positivity of Thom polynomials and Schubert calculus. Schubert Calculus–Osaka 2012. Advanced Studies in Pure Mathematics, vol. 71, pp. 419–451 (2016)

15. Roberts, J., Speiser, R.: Enumerative geometry of triangles I. Commun. Algebra **12**, 1213–1255 (1984)
16. Schubert, H.C.H.: Kalkül der abzählenden Geometrie. Teubner, Leipzig (1879); reprinted by Springer, Berlin (1979)
17. Schubert, H.C.H.: Anzahlgeometrische Behandlung des Dreiecks. Math. Ann. **17**, 153–212 (1880)
18. Tworzewski, P.: Isolated intersection multiplicity and regular separation of analytic sets. Ann. Pol. Math. **58**, 213–219 (1993)

On Schubert's Problem of Characteristics

Haibao Duan and Xuezhi Zhao

Abstract The Schubert varieties on a flag manifold G/P give rise to a cell decomposition on G/P whose Kronecker duals, known as the Schubert classes on G/P, form an additive base of the integral cohomology $H^*(G/P)$. The Schubert's problem of characteristics asks to express a monomial in the Schubert classes as a linear combination in the Schubert basis. We present a unified formula expressing the characteristics of a flag manifold G/P as polynomials in the Cartan numbers of the group G. As application we develop a direct approach to our recent works on the Schubert presentation of the cohomology rings of flag manifolds G/P.

Keywords Lie group · Flag manifold · Schubert variety · Cohomology

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1 The Problem

> *Schubert considered what he called the problem of characteristics to be the main theoretical problem of enumerative geometry.* —Kleimann [40, 1987]

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> *The existence of a finite basis for the homologies in every closed manifold implies furthermore the solvability of Schubert's "characteristics problems" in general.* —Van der Waerden [59, 1930][1]

Schubert calculus is the intersection theory of the 19th century, together with applications to enumerative geometry. Justifying this calculus was a major topic of the 20 century algebraic geometry, and was also the content of Hilbert's 15th problem "Rigorous foundation of Schubert's enumerative calculus" [33, 39, 50]. Thanks to the pioneer works [26, 59] of Van der Waerden and Ehresmann, the problem of characteristics [53], considered by Schubert as the fundamental problem of enumerative geometry, has had a concise statement by the 1950's.

Let G be compact connected Lie group with a maximal torus T. For an one parameter subgroup $\alpha : \mathbb{R} \to G$ its centralizer P is a parabolic subgroup on G, while the homogeneous space G/P is a projective variety, called a *flag manifold* of G. Let $W(P, G)$ be the set of left cosets of the Weyl group W of G by the Weyl group $W(P)$ of P with associated length function $l : W(P, G) \to \mathbb{Z}$. The following result was discovered by Ehresmann [26, 1934] for the Grassmannians $G_{n,k}$ of k-dimensional linear subspaces on $\mathbb{C}^n$, announced by Chevalley [13, 1958] for the complete flag manifolds G/T, and extended to all flag manifolds G/P by Bernstein–Gel'fand–Gel'fand [5, 1973].

Theorem 1.1 *The flag manifold G/P admits a decomposition into the cells indexed by the elements of $W(P, G)$,*

$$G/P = \bigcup_{w \in W(P,G)} X_w, \quad \dim X_w = 2l(w), \tag{1.1}$$

with each cell X_w the closure of an algebraic affine space, called the Schubert variety on G/P associated to w. □

Since only even dimensional cells are involved in the decomposition (1.1), the set $\{[X_w],\ w \in W(P, G)\}$ of fundamental classes forms an additive basis of the integral homology $H_*(G/P)$. The cocycle classes $s_w \in H^*(G/P)$ Kronecker dual to the basis (i.e. $\langle s_w, [X_u]\rangle = \delta_{w,u}, w, u \in W(P, G)$) gives rise to the *Schubert class* associated to $w \in W(P, G)$. Theorem 1.1 implies the following result, well known as *the basis theorem of Schubert calculus* [59, Sect. 8].

Theorem 1.2 *The set $\{s_w, w \in W(P, G)\}$ of Schubert classes forms an additive basis of the integral cohomology $H^*(G/P)$.* □

An immediate consequence is that any monomial $s_{w_1} \cdots s_{w_k}$ in the Schubert classes on G/P can be expressed as a linear combination of the basis elements

$$s_{w_1} \cdots s_{w_k} = \sum_{w \in W(P,G), l(w)=l(w_1)+\cdots+l(w_k),} a^w_{w_1,\ldots,w_k} \cdot s_w,\ a^w_{w_1,\ldots,w_k} \in \mathbb{Z}, \tag{1.2}$$

[1] This English translation of the profound discovery of van der Waerden in German [59] is due to Schappacher [56].

where the coefficients $a^w_{w_1,\ldots,w_k}$ are called *characteristics* by Schubert [26, 53, 59].

The problem of characteristics. Given a monomial $s_{w_1}\cdots s_{w_k}$ in the Schubert classes, determine the characteristics $a^w_{w_1,\ldots,w_k}$ for all $w \in W(P,G)$ with $l(w) = l(w_1) + \cdots + l(w_r)$. □

The characteristics are of particular importance in geometry, algebra and topology. They provide solutions to the problems of enumerative geometry [51–54]; were seen by Hilbert as "the degree of the final equations and the multiplicity of their solutions" of a system [33]; and are requested by describing the cohomology ring $H^*(G/P)$ in the Schubert basis [60, p. 331]. Notably, the degree of a Schubert variety [54] and the multiplicative rule of two Schubert classes [53] are two special cases of the problem which have received considerable attentions in literatures, see [15, 16] for accounts on the earlier relevant works.

This paper summaries and simplifies our series works [15, 16, 19–21] devoted to describe the integral cohomologies of flag manifolds by a minimal system of generators and relations in the Schubert classes. Precisely, based on a formula of the characteristics $a^w_{w_1,\ldots,w_k}$ stated in Sect. 2 and established in Sect. 3, we address in Sects. 4 and 5 a more direct approach to the Schubert presentations [20, 21] of the cohomology rings of flag manifolds G/P.

2 The Formula of the Characteristics $a^w_{w_1,\ldots,w_k}$

To investigate the topology of a flag manifold G/P we may assume that the Lie group G is 1-connected and simple. Resorting to the geometry of the Stiefel diagram of the Lie group G we present in Theorem 2.4 a formula that boils down the characteristics $a^w_{w_1,\ldots,w_k}$ to the Cartan matrix of the group G.

Fix a maximal torus T on G and set $n = \dim T$. Equip the Lie algebra $L(G)$ with an inner product $(,)$ so that the adjoint representation acts as isometries of $L(G)$. The *Cartan subalgebra* of G is the linear subspace $L(T) \subset L(G)$.

The restriction of the exponential map $\exp : L(G) \to G$ on $L(T)$ defines a set $S(G)$ of $\frac{1}{2}(\dim G - n)$ hyperplanes on $L(T)$, namely, the set of *singular hyperplanes* through the origin in $L(T)$ [11, p. 226]. These planes divide $L(T)$ into finitely many convex regions, each one is called a *Weyl chamber* of G. The reflections on $L(T)$ in these planes generate the Weyl group W of G [35, p. 49].

The map exp on $L(T)$ carries the normal line l (through the origin) of a hyperplane $L \in S(G)$ to a circle subgroup on T. Let $\pm\alpha \in l$ be the non-zero vectors with minimal length so that $\exp(\pm\alpha) = e$ (the group unit). The set $\Phi(G)$ consisting of all those vectors $\pm\alpha$ is called *the root system of* G. Fixing a regular point $x_0 \in L(T) - \cup_{L\in D(G)} L$ the set of *simple roots* relative to x_0 is

$$\Delta(x_0) = \{\beta \in \Phi(G) \mid (\beta, x_0) > 0\} \text{ [35, p. 47].}$$

In addition, for a simple root $\beta \in \Delta$ *the simple reflection* relative to β is the reflection σ_β in the plane $L_\beta \in S(G)$ perpendicular to β. If $\beta, \beta' \in \Delta$ *the Cartan number*

$$\beta \circ \beta' := 2(\beta, \beta')/(\beta', \beta')$$

is always an integer, and only $0, \pm 1, \pm 2, \pm 3$ can occur [35, p. 55].

Since the set of simple reflections $\{\sigma_\beta \mid \beta \in \Delta\}$ generates W [11, p. 193] every $w \in W$ admits a factorization of the form

$$w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m}, \beta_i \in \Delta. \tag{2.1}$$

Definition 2.1 The *length* $l(w)$ of an element $w \in W$ is the least number of factors in all decompositions of w in the form (2.1). The decomposition (2.1) is called *reduced* if $m = l(w)$.

For a reduced decomposition (2.1) of w the $m \times m$ (strictly upper triangular) matrix $A_w = (a_{i,j})$ with $a_{i,j} = 0$ if $i \geq j$ and $-\beta_j \circ \beta_i$ if $i < j$ is called *the Cartan matrix of* w relative to the decomposition (2.1). □

Example 2.2 In [57] Stembridge asked for an approach to find a reduced decomposition (2.1) for each $w \in W$. Resorting to the geometry of the Cartan subalgebra $L(T)$ this task can be implemented by the following method.

Picture W as the W-orbit $\{w(x_0) \in L(T) \mid w \in W\}$ through the regular point x_0. For a $w \in W$ let C_w be a line segment on $L(T)$ from the Weyl chamber containing x_0 to $w(x_0)$, that crosses the planes in $S(G)$ once at a time. Assume that they are met in the order $L_{\alpha_1}, \ldots, L_{\alpha_k}$, $\alpha_i \in \Phi(G)$. Then $l(w) = k$ and $w = \sigma_{\alpha_k} \circ \cdots \circ \sigma_{\alpha_1}$. Set

$$\beta_1 = \alpha_1, \quad \beta_2 = \sigma_{\alpha_1}(\alpha_2), \ldots, \quad \beta_k = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{k-1}}(\alpha_k).$$

Then, from $\beta_i \in \Delta$ and $\sigma_{\beta_i} = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{i-1}} \circ \sigma_{\alpha_i} \circ \sigma_{\alpha_{i-1}} \circ \cdots \circ \sigma_{\alpha_1}$ one sees that $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k}$, which is reduced because of $l(w) = k$. □

Let $\mathbb{Z}[x_1, \ldots, x_m] = \oplus_{n \geq 0} \mathbb{Z}[x_1, \ldots, x_m]^{(n)}$ be the ring of integral polynomials in $x_1, \ldots, x_m$, graded by $\deg x_i = 1$.

Definition 2.3 For a $m \times m$ strictly upper triangular integer matrix $A = (a_{i,j})$ *the triangular operator* T_A associated to A is the additive homomorphism $T_A : \mathbb{Z}[x_1, \ldots, x_m]^{(m)} \to \mathbb{Z}$ defined recursively by the following elimination rules:

(i) If $m = 1$ (consequently $A = (0)$) then $T_A(x_1) = 1$;
(ii) If $h \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m)}$ then $T_A(h) = 0$;
(iii) For any $h \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m-r)}$ with $r \geq 1$,
$$T_A(h \cdot x_m^r) = T_{A'}(h \cdot (a_{1,m}x_1 + \cdots + a_{m-1,m}x_{m-1})^{r-1}),$$

where A' is the $(m-1) \times (m-1)$ strictly upper triangular matrix obtained from A by deleting both of the mth column and row.

By additivity, T_A is defined for every $h \in \mathbb{Z}[x_1, \ldots, x_m]^{(m)}$ using the unique expansion $h = \sum_{0 \leq r \leq m} h_r \cdot x_m^r$ with $h_r \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m-r)}$. □

For a parabolic subgroup P of G the set $W(P, G)$ of left cosets of W by $W(P)$ can be identified with the subset of W

$$W(P, G) = \{w \in W \mid l(w) \leq l(ww'), w' \in W(P)\} \text{ (by [5, 5.1])},$$

where l is the length function on W. Assume that $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m}$, $\beta_i \in \Delta$, is a reduced decomposition of an element $w \in W(P, G)$ with associated Cartan matrix $A_w = (a_{i,j})_{m\times m}$. For a multi-index $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ we put $|I| := k$ and set

$$\sigma_I := \sigma_{\beta_{i_1}} \circ \cdots \circ \sigma_{\beta_{i_k}} \in W, x_I := x_{i_1} \cdots x_{i_k} \in \mathbb{Z}[x_1, \ldots, x_m].$$

Our promised formula for the characteristics is:

Theorem 2.4 *For every monomial $s_{w_1} \cdots s_{w_k}$ in the Schubert classes on G/P with $l(w) = l(w_1) + \cdots + l(w_k)$ we have*

$$a^w_{w_1,\ldots,w_k} = T_{A_w}\left(\prod_{i=1,\ldots,k}\left(\sum_{\sigma_I = w_i, |I| = l(w_i), I\subseteq\{1,\ldots,m\}} x_I\right)\right). \tag{2.2}$$

Remark 2.5 Formula (2.2) reduces the characteristic $a^w_{w_1,\ldots,w_k}$ to a polynomial in the Cartan numbers of the group G, hence applies uniformly to all flag manifolds G/P.

If $k = 2$ the characteristic $a^w_{w_1,w_2}$ is well known as *a Littlewood–Richardson coefficient*, and the formula (2.2) has been obtained by Duan in [16]. In [62] Willems generalizes the formula of $a^w_{w_1,w_2}$ to the more general context of flag varieties associated to Kac–Moody groups, and for the equivariant cohomologies. Recently, Bernstein and Richmond [10] obtained also a formula expressing $a^w_{w_1,w_2}$ in the Cartan numbers of G. □

3 Proof of the Characteristics Formula (2.2)

In this paper the homologies and cohomologies are over the ring $\mathbb{Z}$ of integers. If $f : X \to Y$ is a continuous map between two topological spaces X and Y, f_* (resp. f^*) denote the homology (resp. cohomology) homomorphism induced by f. For an oriented closed manifold M (resp. a connected projective variety) the notion $[M] \in H_{\dim M}(M)$ stands for the orientation class. In addition, the Kronecker pairing between cohomology and homology of a space X is written as $\langle , \rangle : H^*(X) \times H_*(X) \to \mathbb{Z}$. The proof of Theorem 2.4 makes use of the celebrated K-cycles on G/T constructed by Bott and Samelson in [8, 9]. We begin by recalling the construction of these cycles, as well as their basic properties developed in [8, 9, 15, 16].

For a simple Lie group G fix a regular point $x_0 \in L(T)$ and let Δ be the set of simple roots relative to x_0. For a $\beta \in \Delta$ let K_β be the centralizer of the subspace $\exp(L_\beta)$ on G, where $L_\beta \in S(G)$ is the plane perpendicular to β. Then $T \subset K_\beta$ and the quotient K_β/T is diffeomorphic to the 2-sphere [9, p. 996].

The 2-sphere K_β/T carries a natural orientation specified as follows. The Cartan decomposition of the Lie algebra $L(K_\beta)$ relative to the maximal torus $T \subset K_\beta$ takes the form $L(K_\beta) = L(T) \oplus \vartheta_\beta$, where $\vartheta_\beta \subset L(G)$ is the *root space* belonging to the root β [35, p. 35]. Taking a non-zero vector $v \in \vartheta_\beta$ and letting $v' \in \vartheta_\beta$ be such that $[v, v'] = \beta$, where [,] is the Lie bracket on $L(G)$, then the ordered base $\{v, v\prime\}$ furnishes ϑ_β with an orientation that is irrelevant to the choices of v. The tangent map of the quotient $\pi_\beta : K_\beta \to K_\beta/T$ at the group unit $e \in K_\beta$ maps the 2-plane ϑ_β isomorphically onto the tangent space to the sphere K_β/T at the point $\pi_\beta(e)$. In this manner the orientation $\{v, v\prime\}$ on ϑ_β furnishes the sphere K_β/T with the orientation $\omega_\beta = \{\pi_\beta(v), \pi_\beta(v\prime)\}$.

For an ordered sequence $\beta_1, \dots, \beta_m \in \Delta$ of m simple roots (repetitions like $\beta_i = \beta_j$ may occur) let $K(\beta_1, \dots, \beta_m)$ be the product group $K_{\beta_1} \times \cdots \times K_{\beta_m}$. With $T \subset K_{\beta_i}$ the product $T \times \cdots \times T$ (m-copies) acts on $K(\beta_1, \dots, \beta_m)$ by

$$(g_1, \dots, g_m)(t_1, \dots, t_m) = (g_1 t_1, t_1^{-1} g_2 t_2, \dots, t_{m-1}^{-1} g_m t_m).$$

Let $\Gamma(\beta_1, \dots, \beta_m)$ be the base manifold of this principal action that is oriented by the ω_{β_i}, $1 \le i \le m$. The point on $\Gamma(\beta_1, \dots, \beta_m)$ corresponding to the point $(g_1, \dots, g_m) \in K(\beta_1, \dots, \beta_m)$ is called $[g_1, \dots, g_m]$.

The integral cohomology of the oriented manifold $\Gamma(\beta_1, \dots, \beta_m)$ has been determined by Bott and Samelson in [8]. Let $\varphi_i : K_{\beta_i}/T \to \Gamma(\beta_1, \dots, \beta_m)$ be the embedding induced by the inclusion $K_{\beta_i} \to K(\beta_1, \dots, \beta_m)$ onto the ith factor group, and put

$$y_i = \varphi_{i_*}(\omega_{\beta_i}) \in H_2(\Gamma(\beta_1, \dots, \beta_m)), 1 \le i \le m.$$

Form the $m \times m$ strictly upper triangular matrix $A = (a_{i,j})_{m \times m}$ by setting

$$a_{i,j} = 0 \text{ if } i \ge j, \text{ but } a_{i,j} = -2(\beta_j, \beta_i)/(\beta_i, \beta_i') \text{ if } i < j.$$

It is easy to see from the construction that the set $\{y_1, \dots, y_m\}$ forms a basis of the second homology group $H_2(\Gamma(\beta_1, \dots, \beta_m))$.

Lemma 3.1 ([9]) *Let $x_1, \dots, x_m \in H^2(\Gamma(\beta_1, \dots, \beta_m))$ be the Kronecker duals of the cycle classes $y_1, \dots, y_m$ on $\Gamma(\beta_1, \dots, \beta_m)$. Then*

$$H^*(\Gamma(\beta_1, \dots, \beta_m)) = \mathbb{Z}[x_1, \dots, x_m]/J, \tag{3.1}$$

where J is the ideal generated by $x_j^2 - \sum_{i<j} a_{i,j} x_i x_j$, $1 \le j \le m$. □

In view of (3.1) the map $p_{\Gamma(\beta_1,\dots,\beta_m)}$ from the polynomial ring $\mathbb{Z}[x_1, \dots, x_m]$ onto its quotient $H^*(\Gamma(\beta_1, \dots, \beta_m))$ gives rise to the additive map

$$\int_{\Gamma(\beta_1,\dots,\beta_m)} : \mathbb{Z}[x_1, \dots, x_m]^{(m)} \to \mathbb{Z}$$

evaluated by $\int_{\Gamma(\beta_1,\dots,\beta_m)} h = \langle p_{\Gamma(\beta_1,\dots,\beta_m)}(h), [\Gamma(\beta_1, \dots, \beta_m)] \rangle$. The geometric implication of the triangular operator T_A in Definition 2.3 is shown by the following result.

Lemma 3.2 ([15, Proposition 2]) *We have*

$$\int_{\Gamma(\beta_1,\ldots,\beta_m)} = T_A : \mathbb{Z}[x_1,\ldots,x_m]^{(m)} \to \mathbb{Z}.$$

In particular, $\int_{\Gamma(\beta_1,\ldots,\beta_m)} x_1 \cdots x_m = 1.$ □

For a parabolic P on G we can assume, without loss of the generalities, that $T \subseteq P \subset G$. For a sequence $\beta_1,\ldots,\beta_m$ of simple roots *the associated Bott–Samelson's K-cycle* on G/P is the map

$$\varphi_{\beta_1,\ldots,\beta_m;P} : \Gamma(\beta_1,\ldots,\beta_m) \to G/P$$

defined by $\varphi_{\beta_1,\ldots,\beta_m;P}([g_1,\ldots,g_m]) = g_1 \cdots g_m P$. If $P = T$ Hansen [32] has shown that certain K-cycles are desingularizations of the Schubert varieties on G/T. The following more general result allows one to translate the calculation with Schubert classes on G/P to computing with monomials in the much simpler ring $H^*(\Gamma(\beta_1,\ldots,\beta_m))$.

Lemma 3.3 *With respect the Schubert basis on $H^*(G/P)$ the induced map of $\varphi_{\beta_1,\ldots,\beta_m;P}$ on the cohomologies is given by*

$$\varphi^*_{\beta_1,\ldots,\beta_m;P}(s_w) = (-1)^{l(w)} \sum_{\sigma_I = w, |I| = l(w), I \subseteq [1,\ldots,m]} x_I,\ w \in W(P;G). \tag{3.2}$$

Proof With $T \subseteq P \subset G$ the map $\varphi_{\beta_1,\ldots,\beta_m;P}$ factors through $\varphi_{\beta_1,\ldots,\beta_m;T}$ in the fashion

$$\begin{tikzcd} \Gamma(\beta_1,\ldots,\beta_m) \arrow[r, "\varphi_{\beta_1,\ldots,\beta_m;T}"] \arrow[dr, "\varphi_{\beta_1,\ldots,\beta_m;P}"'] & G/T \arrow[d, "\pi"] \\ & G/P \end{tikzcd},$$

where the map π is the fibration with fiber P/T. By [16, Lemma 5.1] formula (3.2) holds for the case $P = T$. According to [5, Sect. 5] the induced map $\pi^* : H^*(G/P) \to H^*(G/T)$ is given by $\pi^*(s_w) = s_w,\ w \in W(P;G)$, showing formula (3.2) for the general case $T \subset P$. □

Proof of Theorem 2.4 For a monomial $s_{w_1} \cdots s_{w_k}$ in the Schubert classes of G/P assume as in (1.2) that

$$s_{w_1} \cdots s_{w_k} = \sum_{w \in W(P;G), l(w) = m} a^w_{w_1,\ldots,w_k} \cdot s_w,\ a^w_{w_1,\ldots,w_k} \in \mathbb{Z}, \tag{3.3}$$

where $m = l(w_1) + \cdots + l(w_k)$. For an element $w_0 \in W(P;G)$ with a reduced decomposition $w_0 = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m}$, $\beta_i \in \Delta$, let $A_{w_0} = (a_{i,j})_{m \times m}$ be the relative Cartan matrix. Applying the ring map $\varphi^*_{\beta_1,\ldots,\beta_m;P}$ to the Eq. (3.3) on $H^*(G/P)$ we obtain by (3.2) the equality on the group $H^{2m}(\Gamma(\beta_1,\ldots,\beta_m))$

$$(-1)^{l(w_1)+\cdots+l(w_k)} \prod_{1\le i\le k}\left(\sum_{\sigma_I=w_i,|I|=l(w_i),I\subseteq\{1,\ldots,m\}} x_I\right) = (-1)^m a^{w_0}_{w_1,\ldots,w_k}\cdot x_1\cdots x_m.$$

Applying $\int_{\Gamma(\beta_1,\ldots,\beta_m)}$ to both sides we get by Lemma 3.2 that

$$(-1)^{l(w_1)+\cdots+l(w_k)}\cdot T_{A_w}\left(\prod_{1\le i\le k}\left(\sum_{\sigma_I=w_i,|I|=l(w_i),I\subseteq\{1,\ldots,m\}} x_I\right)\right) = (-1)^m\cdot a^{w_0}_{w_1,\ldots,w_k}.$$

This is identical to (2.2) because of $m = l(w_1)+\cdots+l(w_k)$. □

4 The Cohomology of Flag Manifolds G/P

The classical Schubert calculus amounts to the determination of the intersection rings on Grassmann varieties and on the so called "flag manifolds" of projective geometry. —Weil [60, p. 331]

A classical problem of topology is to express the integral cohomology ring $H^*(G/H)$ of a homogeneous space G/H by a minimal system of explicit generators and relations. The traditional approach due to H. Cartan, Borel, Baum, Toda utilize various spectral sequence techniques [2, 3, 37, 58, 64], and the calculation encounters the same difficulties when applied to a Lie group G with torsion elements in its integral cohomology, in particular, when G is one of the five exceptional Lie groups [38, 58, 61].

However, if $P \subset G$ is parabolic, Schubert calculus makes the structure of the ring $H^*(G/P)$ appearing in a new light. Given a set $\{y_1,\ldots,y_k\}$ of k elements let $\mathbb{Z}[y_1,\ldots,y_k]$ be the ring of polynomials in $y_1,\ldots,y_k$ with integer coefficients. For a subset $\{r_1,\ldots,r_m\}\subset\mathbb{Z}[y_1,\ldots,y_k]$ of homogenous polynomials denote by $\langle r_1,\ldots,r_m\rangle$ the ideal generated by $r_1,\ldots,r_m$.

Theorem 4.1 *For each flag manifold G/P there exist a set $\{y_1,\ldots,y_k\}$ of Schubert classes on G/P, and a set $\{r_1,\ldots,r_m\}\subset\mathbb{Z}[y_1,\ldots,y_k]$ of polynomials, so that the inclusion $\{y_1,\ldots,y_k\}\subset H^*(G/P)$ induces a ring isomorphism*

$$H^*(G/P) = \mathbb{Z}[y_1,\ldots,y_k]/\langle r_1,\ldots,r_m\rangle, \tag{4.1}$$

where both the numbers k and m are minimal subject to this presentation. □

Proof Let $D(H^*(G/P))\subset H^*(G/P)$ be the ideal of the decomposable elements. Since the ring $H^*(G/P)$ is torsion free and has a basis consisting of Schubert classes, there is a set $\{y_1,\ldots,y_k\}$ of Schubert classes on G/P that corresponds to a basis of the quotient group $H^*(G/P)/D(H^*(G/P))$. In particular, the inclusion $\{y_1,\ldots,y_k\}\subset H^*(G/P)$ induces a surjective ring map

$$f : \mathbb{Z}[y_1, \ldots, y_k] \to H^*(G/P).$$

Since $\ker f$ is an ideal the Hilbert basis theorem implies that there exists a finite subset $\{r_1, \ldots, r_m\} \subset \mathbb{Z}[y_1, \ldots, y_n]$ so that $\ker f = \langle r_1, \ldots, r_m \rangle$. We can of course assume that the number m is minimal subject to this constraint.

As the cardinality of a basis of the quotient group $H^*(G/P)/D(H^*(G/P))$ the number k is an invariant of G/P. In addition, if one changes the generators $y_1, \ldots, y_k$ to $y_1', \ldots, y_k'$, then each old generator y_i can be expressed as a polynomial g_i in the new ones $y_1', \ldots, y_k'$, and the invariance of the number m is shown by the presentation

$$H^*(G/P) = \mathbb{Z}[y_1', \ldots, y_k']/\langle r_1', \ldots, r_m' \rangle,$$

where r_j' is obtained from r_j by substituting g_i for y_i, $1 \leq j \leq m$. □

A presentation of the ring $H^*(G/P)$ in the form of (4.1) will be called a *Schubert presentation of the cohomology of* G/P, while the set $\{y_1, \ldots, y_k\}$ of generators will be called *a set of special Schubert classes* on G/P. Based on the characteristic formula (2.2) we develop in this section algebraic and computational machineries implementing Schubert presentation of the ring $H^*(G/P)$. To be precise the following conventions will be adopted throughout the remaining part of this section.

(i) G is a 1-connected simple Lie group with Weyl group W, and a fixed maximal torus T;
(ii) A set $\Delta = \{\beta_1, \ldots, \beta_n\}$ of simple roots of G is given and ordered as the vertex of the Dykin diagram of G pictured on [32, p. 58];
(iii) For each simple root $\beta_i \in \Delta$ write σ_i instead of $\sigma_{\beta_i} \in W$; ω_i in place of the Schubert class $s_{\sigma_{\beta_i}} \in H^2(G/T)$.

Note that Theorem 1.2 implies that the set $\{\omega_1, \ldots, \omega_n\}$ is the Schubert basis of the second cohomology $H^2(G/T)$, whose elements is identical to *the fundamental dominant weights* of G in the context of Borel and Hirzebruch [4, 17].

4.1 Decomposition

By convention (iii) each $w \in W$ admits a factorization of the form

$$w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}, 1 \leq i_1, \ldots, i_k \leq n, l(w) = k.$$

hence can be written as $w = \sigma[I]$ with $I = (i_1, \ldots, i_k)$. Such expressions of w may not be unique, but the ambiguity can be dispelled by employing the following notion. Furnish the set of all reduced decompositions of w

$$D(w) := \{w = \sigma[I] \mid I = (i_1, \ldots, i_k), l(w) = k\}.$$

with the order $\leq$ given by the lexicographical order on the multi-indexes I. Call a decomposition $w = \sigma[I]$ *minimized* if I is the minimal one with respect to the order. As result every $w \in W$ possesses a unique minimized decomposition.

For a subset $K \subset \{1, \ldots, n\}$ let $P_K \subset G$ be the centralizer of the 1-parameter subgroup $\{\exp(tb) \in G \mid t \in \mathbb{R}\}$ on G, where $b \in L(T)$ is a vector that satisfies

$$(\beta_i, b) > 0 \text{ if } i \in K;\ (\beta_i, b) = 0 \text{ if } i \notin K.$$

Then every parabolic subgroup P is conjugate in G to some P_K with $K \subset \{1, \ldots, n\}$, while the Weyl group $W(P) \subset W$ is generated by the simple reflections σ_j with $j \notin K$. Resorting to the length function l on W we embed the set $W(P; G)$ as the subset of W (as in Sect. 2)

$$W(P; G) = \{w \in W \mid l(w_1) \geq l(w), w_1 \in wW(P)\},$$

and put $W^r(P; G) := \{w \in W(P; G) \mid l(w) = r\}$. Since every $w \in W^r(P; G)$ has a unique minimized decomposition as $w = \sigma[I]$, the set $W^r(P; G)$ is also ordered by the lexicographical order on the multi-index I's, hence can be expressed as

$$W^r(P; G) = \{w_{r,i} \mid 1 \leq i \leq \beta(r)\}, \beta(r) := \left|W^r(P; G)\right|, \tag{4.2}$$

where $w_{r,i}$ is the ith element in the ordered set $W^r(P; G)$. In [19] a program entitled "*Decomposition*" is composed, whose function is summarized below.

Algorithm 4.2 *Decomposition.*

Input: *The Cartan matrix $A = (a_{ij})_{n\times n}$ of G, and a subset $K \subset \{1, \ldots, n\}$.*

Output: *The set $W(P_K; G)$ being presented by the minimized decompositions of its elements, together with the index system (4.2) imposed by the order $\leq$.*

For examples of the results coming from *Decomposition* we refer to [22, 1.1–7.1].

4.2 Factorization of the Ring $H^*(G/T)$ Using Fibration

The cardinality of the Schubert basis of G/T agrees with the order of the Weyl group W, which in general is very large. To reduce the computation costs we may take a proper subset $K \subset \{1, \ldots, n\}$ and let $P := P_K$ be the corresponding parabolic subgroup. The inclusion $T \subset P \subset G$ then induces the fibration

$$P/T \overset{i}{\hookrightarrow} G/T \overset{\pi}{\to} G/P, \tag{4.3}$$

where the induced maps π^* and i^* behave well with respect to the Schubert bases of the three flag manifolds P/T, G/P and G/T in the following sense:

(i) With respect to the inclusion $W(P) \subset W$ the map i^* carries the subset $\{s_w\}_{w \in W(P) \subset W}$ of the Schubert basis of $H^*(G/T)$ onto the Schubert basis of $H^*(P/T)$.
(ii) With respect to the inclusion $W(P; G) \subset W$ the map π^* identifies the Schubert basis $\{s_w\}_{w \in W(P;G)}$ of $H^*(G/P)$ with a subset of the Schubert basis of $H^*(G/T)$.

For these reasons we can make no difference in notation between an element in $H^*(G/P)$ and its π^* image in $H^*(G/T)$, and between a Schubert class on P/T and its i^* pre-image on G/T.

Assume now that $\{y_1, \ldots, y_{n_1}\}$ and $\{x_1, \ldots, x_{n_2}\}$ are respectively special Schubert classes on P/T and G/P, and with respect to them one has the Schubert presentations

$$H^*(P/T) = \frac{\mathbb{Z}[y_i]_{1 \le i \le n_1}}{\langle h_s \rangle_{1 \le s \le m_1}};\ H^*(G/P) = \frac{\mathbb{Z}[x_j]_{1 \le j \le n_2}}{\langle r_t \rangle_{1 \le t \le m_2}}, \tag{4.4}$$

where $h_s \in \mathbb{Z}[y_i]_{1 \le i \le n_1}$, $r_t \in \mathbb{Z}[x_j]_{1 \le j \le n_2}$. The following result allows one to formulate the ring $H^*(G/T)$ by the simpler ones $H^*(P/T)$ and $H^*(G/P)$.

Theorem 4.3 *The inclusions $y_i, x_j \in H^*(G/T)$ induces a surjective ring map*

$$\varphi : \mathbb{Z}[y_i, x_j]_{1 \le i \le n_1, 1 \le j \le n_2} \to H^*(G/T).$$

Furthermore, if $\{\rho_s\}_{1 \le s \le m_1} \subset \mathbb{Z}[y_i, x_j]$ is a system satisfying

$$\rho_s \in \ker \varphi \text{ and } \rho_s \mid_{x_j = 0} = h_s, \tag{4.5}$$

then φ induces a ring isomorphism

$$H^*(G/T) = \mathbb{Z}[y_i, x_i]_{1 \le i \le n_1, 1 \le j \le n_2} / \langle \rho_s, r_t \rangle_{1 \le s \le m_1, 1 \le t \le m_2}. \tag{4.6}$$

Proof By the property (i) above the bundle (4.3) has the Leray–Hirsch property. That is, the cohomology $H^*(G/T)$ is a free module over the ring $H^*(G/P)$ with the basis $\{1, s_w\}_{w \in W(P)}$:

$$H^*(G/T) = H^*(G/P)\{1, s_w\}_{w \in W(P)}([36, p.231]), \tag{4.7}$$

implying that φ surjects. It remains to show that for any $g \in \ker \varphi$ one has

$$g \in \langle \rho_s, r_t \rangle_{1 \le s \le m_1, 1 \le t \le m_2}.$$

To this end we notice by (4.5) and (4.7) that

$$g \equiv \sum\nolimits_{w \in W(P_K)} g_w \cdot s_w \bmod \langle \rho_s \rangle_{1 \le s \le m_1} \text{ with } g_w \in \mathbb{Z}[x_j]_{1 \le j \le n_2}.$$

Thus $\varphi(g) = 0$ implies $\varphi(g_w) = 0$, showing $g_w \in \langle r_t \rangle_{1 \le t \le m_2}$ by (4.4). $\square$

4.3 The Generalized Grassmannians

For a topological space X we set

$$H^{\text{even}}(X) := \oplus_{r\geq 0} H^{2r}(X), \quad H^{\text{odd}}(X) := \oplus_{r\geq 0} H^{2r+1}(X).$$

Then $H^{\text{even}}(X)$ is a subring of $H^*(X)$, while $H^{\text{odd}}(X)$ is a module over the ring $H^{\text{even}}(X)$.

If P is a parabolic subgroup that corresponds to a singleton $K = \{i\}$, the flag manifold G/P is called *generalized Grassmannians of* G *corresponding to the weight* ω_i [20]. With $W^1(P; G) = \{\sigma_i\}$ consisting of a single element the basis theorem implies that $H^2(G/P) = \mathbb{Z}$ is generated by ω_i. Furthermore, letting P^s be the semi-simple part of P, then the projection $p : G/P^s \to G/P$ is an oriented circle bundle on G/P with Euler class ω_i. With $H^{\text{odd}}(G/P) = 0$ by the basis theorem the Gysin sequence [48, p. 143]

$$\cdots \to H^r(G/P) \xrightarrow{p^*} H^r(G/P^s) \xrightarrow{\beta} H^{r-1}(G/P) \xrightarrow{\omega\cup} H^{r+1}(G/P) \xrightarrow{p^*} \cdots .$$

of p breaks into the short exact sequences

$$0 \to \omega_i \cup H^{2r-2}(G/P) \to H^{2r}(G/P) \xrightarrow{p^*} H^{2r}(G/P^s) \to 0 \tag{4.8}$$

as well as the isomorphisms

$$\beta : H^{2r-1}(G/P^s) \cong \ker\{H^{2r-2}(G/P) \xrightarrow{\omega_i\cup} H^{2r}(G/P)\}, \tag{4.9}$$

where $\omega_i\cup$ means taking cup product with ω_i. In particular, formula (4.8) implies that

Lemma 4.4 *If* $S = \{y_1, \ldots, y_m\} \subset H^*(G/P)$ *is a subset so that* $p^*S = \{p^*(y_1), \ldots, p^*(y_m)\}$ *is a minimal set of generators of the ring* $H^{even}(G/P^s)$*, then* $S' = \{\omega_i, y_1, \ldots, y_m\}$ *is a minimal set of generators of* $H^*(G/P)$. □

By Lemma 4.4 the inclusions $\{\omega_i\} \cup S \subset H^*(G/P)$, $p^*S \subset H^*(G/P^s)$ extend to the surjective maps π and $\overline{\pi}$ that fit in the commutative diagram

$$\begin{array}{ccccc} & \mathbb{Z}[\omega_i, y_1, \ldots, y_m]^{(2r)} & \xrightarrow{\varphi} & \mathbb{Z}[y_1, \ldots, y_m]^{(2r)} & \\ & \pi\downarrow & & \overline{\pi}\downarrow & \\ H^{2r-2}(G/P) \xrightarrow{\omega_i\cup} & H^{2r}(G/P) & \xrightarrow{p^*} & H^{2r}(G/P^s) & \to 0 \end{array} \tag{4.10}$$

where $\mathbb{Z}[\omega_i, y_1, \ldots, y_m]$ is graded by $\deg \omega_i = 2$, $\deg y_i$, and where

$$\varphi(\omega_i) = 0, \varphi(y_i) = y_i; \overline{\pi}(y_i) = p^*(y_i).$$

The next result showing in [20, Lemma 8] enables us to formulate a presentation of the ring $H^*(G/P)$ in term of $H^*(G/P^s)$.

Theorem 4.5 *Assume that* $\{h_1, \ldots, h_d\} \subset \mathbb{Z}[y_1, \ldots, y_m]$ *is a subset so that*

$$H^{even}(G/P^s) = \mathbb{Z}[p^*(y_1), \ldots, p^*(y_m)]/\left\langle p^*(h_1), \ldots, p^*(h_d)\right\rangle, \tag{4.11}$$

and that $\{d_1, \ldots, d_t\}$ *is a basis of the module* $H^{odd}(G/P^s)$ *over* $H^{even}(G/P^s)$. *Then*

$$H^*(G/P) = \mathbb{Z}[\omega_i, y_1, \ldots, y_m]/\left\langle r_1, \ldots, r_d; \omega_i g_1, \ldots, \omega_i g_t\right\rangle, \tag{4.12}$$

where $\{r_1, \ldots, r_d\}$, $\{g_1, \ldots, g_t\} \subset \mathbb{Z}[\omega_i, y_1, \ldots, y_m]$ *are two sets of polynomials that satisfy respectively the following "initial constraints"*

(i) $r_k \in \ker \pi$ *with* $r_k\mid_{\omega_i=0} = h_k, \quad 1 \leq k \leq d$;
(ii) $\pi(g_j) = \beta(d_j)$, $1 \leq j \leq t$. □

4.4 The Characteristics

To make Theorems 4.3 and 4.5 applicable in practical computation we develop in this section a series of three algorithms, entitled *Characteristics*, *Null-space*, *Giambelli polynomials*, all of them are based on the characteristic formula (2.2).

4.4.1 The Characteristics

For a $w \in W(P;G)$ with the minimized decomposition $w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}$, $1 \leq i_1, \ldots, i_m \leq n$, $l(w) = m$, we observe in formula (2.2) that

(i) The Cartan matrix A_w of w can be read directly from the Cartan matrix [35, p. 59] of the Lie group G;
(ii) For a $u \in W(P;G)$ with $l(u) = r < m$ the solutions in the multi-indices $I = \{j_1, \ldots, j_r\} \subseteq \{i_1, \ldots, i_m\}$ to the equation $\sigma_I = u$ in $W(P;G)$ agree with the solutions to the linear system $\sigma_I(x_0) = u(x_0)$ on the vector space $L(T)$, where x_0 is the fixed regular point;
(iii) The evaluation the operator T_{A_w} on a polynomial have been programmed using different methods in [19, 65].

Summarizing, granted with *Decomposition*, formula (2.2) indicates an effective algorithm to implement a parallel program whose function is briefed below.

Algorithm 4.6 : *Characteristics.*

Input: *The Cartan matrix* $A = (a_{ij})_{n\times n}$ *of* G, *and a monomial* $s_{w_1} \cdots s_{w_k}$ *in Schubert classes on* G/P.

Output: *The expansion* (1.2) *of* $s_{w_1} \cdots s_{w_k}$ *in the Schubert basis.* □

4.4.2 The Null-Space

Let $\mathbb{Z}[y_1, \dots, y_k]$ be the ring of polynomials in $y_1, \dots, y_k$ graded by $\deg y_i > 0$, and let $\mathbb{Z}[y_1, \dots, y_k]^{(m)}$ be the $\mathbb{Z}$-module consisting of all the homogeneous polynomials with degree m. Denote by $\mathbb{N}^k$ the set of all k-tuples $\alpha = (b_1, \dots, b_k)$ of non-negative integers. Then *the set of monomials basis* of $\mathbb{Z}[y_1, \dots, y_k]^{(m)}$ is

$$B(m) = \{y^\alpha = y_1^{b_1} \cdots y_k^{b_k} \mid \alpha = (b_1, \dots, b_k) \in \mathbb{N}^k,\ \deg y^\alpha = m\}. \tag{4.13}$$

It will be considered as an ordered set with respect to the lexicographical order on $\mathbb{N}^k$, whose cardinality is called $b(m)$.

Let $S = \{y_1, \dots, y_k\}$ be a set of Schubert classes on G/P that generates the ring $H^*(G/P)$ multiplicatively. Then the inclusion $S \subset H^*(G/P)$ induces a surjective ring map $f : \mathbb{Z}[y_1, \dots, y_k] \to H^*(G/P)$ whose restriction to degree $2m$ is

$$f_m : \mathbb{Z}[y_1, \dots, y_k]^{(2m)} \to H^{2m}(G/P).$$

Combining the *Characteristics* with the function "*Null-space*" in *Mathematica*, a basis of $\ker f_m$ can be explicitly exhibited.

Let $s_{m,i}$ be the Schubert class corresponding to the element $w_{m,i} \in W(P; G)$. With respect to the Schubert basis $\{s_{m,i} \mid 1 \le i \le \beta(m)\}$ on $H^{2m}(G/P)$ every monomial $y^\alpha \in B(2m)$ has the unique expansion

$$\pi_m(y^\alpha) = c_{\alpha,1} \cdot s_{m,1} + \cdots + c_{\alpha,\beta(m)} \cdot s_{m,\beta(m)},\ c_{\alpha,i} \in \mathbb{Z},$$

where the coefficients $c_{\alpha,i}$ can be evaluated by the *Characteristics*. The matrix $M(f_m) = (c_{\alpha,i})_{b(2m)\times\beta(m)}$ so obtained is called *the structure matrix of* f_m. The built-in function *Null-space* in *Mathematica* transforms $M(f_m)$ to another matrix $N(f_m)$ in the fashion

*In:=*Null-space$[M(f_m)]$

Out:= a matrix $N(f_m) = (b_{j,\alpha})_{(b(2m)-\beta(m))\times b(2m)}$,

whose significance is shown by the following fact.

Lemma 4.7 *The set* $\kappa_i = \sum_{y^\alpha \in B(2m)} b_{i,\alpha} \cdot y^\alpha$, $1 \le i \le b(2m) - \beta(m)$, *of polynomials is a basis of the* $\mathbb{Z}$ *module* $\ker f_m$. □

4.4.3 The Giambelli polynomials (i.e. The Schubert Polynomials [6])

For the unitary group $G = U(n)$ of rank n with parabolic subgroup $P = U(k) \times U(n-k)$ the flag manifold $G_{n,k} = G/P$ is the *Grassmannian* of k-planes through the origin on $\mathbb{C}^n$. Let $1 + c_1 + \cdots + c_k$ be the total Chern class of the canonical k-bundle on $G_{n,k}$. Then the c_i's can be identified with appropriate Schubert classes on $G_{n,k}$ (i.e. *the special Schubert class* on $G_{n,k}$), and one has *the classical Schubert presentation*

$$H^*(G_{n,k}) = \mathbb{Z}[c_1, \dots, c_k]/\left\langle r_{n-k+1}, \dots, r_n \right\rangle,$$

where r_j is the component of the formal inverse of $1 + c_1 + \cdots + c_k$ in degree j. It follows that every Schubert class s_w on $G_{n,k}$ can be written as a polynomial $\mathcal{G}_w(c_1, \dots, c_k)$ in the special ones, and such an expression is afforded by the classical *Giambelli formula* [34, p. 112].

In general, assume that G/P is a flag variety, and that a Schubert presentation (4.1) of the ring $H^*(G/P)$ has been specified. Then each Schubert class s_w of G/P can be expressed as a polynomial $\mathcal{G}_w(y_1, \dots, y_k)$ in these special ones, and such an expression will be called a *Giambelli polynomial* of the class s_w. Based on the *Characteristics* a program implementing $\mathcal{G}_w(y_1, \dots, y_k)$ has been compiled, whose function is summarized below.

Algorithm 4.8 : *Giambelli polynomials*

Input: *A set* $\{y_1, \dots, y_k\}$ *of special Schubert classes on* G/P.

Output: *Giambelli polynomials* $\mathcal{G}_w(y_1, \dots, y_k)$ *for all* $w \in W(P; G)$.

We clarify the details in this program. By (4.13) we can write the ordered monomial basis $B(2m)$ of $\mathbb{Z}[y_1, \dots, y_k]^{(2m)}$ as $\{y^{\alpha_1}, \dots, y^{\alpha_{b(2m)}}\}$. The corresponding structure matrix $M(f_m)$ in degree $2m$ then satisfies

$$\begin{pmatrix} y^{\alpha_1} \\ \vdots \\ y^{\alpha_{b(2m)}} \end{pmatrix} = M(f_m) \begin{pmatrix} s_{m,1} \\ \vdots \\ s_{m,\beta(m)} \end{pmatrix}.$$

Since f_m surjects the matrix $M(f_m)$ has a $\beta(m) \times \beta(m)$ minor equal to ± 1. The standard integral row and column operation diagonalizing $M(f_m)$ [55, p. 162–164] provides us with two unique invertible matrices $P = P_{b(2m)\times b(2m)}$ and $Q = Q_{\beta(m)\times\beta(m)}$ that satisfy

$$P \cdot M(f_m) \cdot Q = \begin{pmatrix} I_{\beta(m)} \\ C \end{pmatrix}_{b(2m)\times\beta(m)}, \tag{4.14}$$

where $I_{\beta(m)}$ is the identity matrix of rank $\beta(m)$. The *Giambelli polynomials* is realized by the procedure below.

Step 1. Compute $M(f_m)$ using the *Characteristics*;

Step 2. Diagonalize $M(f_m)$ to get the matrices P and Q;

Step 3. Set $\begin{pmatrix} \mathcal{G}_{m,1} \\ \vdots \\ \mathcal{G}_{m,\beta(m)} \end{pmatrix} = Q \cdot [P] \begin{pmatrix} y^{\alpha_1} \\ \vdots \\ y^{\alpha_{b(2m)}} \end{pmatrix}$, where $[P]$ is formed by the first $\beta(m)$ rows of P. Obviously, the polynomial $\mathcal{G}_{m,j}$ so obtained depends only on the special Schubert classes $\{y_1, \ldots, y_k\}$ on G/P, and is a Giambelli polynomial of $s_{m,j}$, $1 \le j \le \beta(m)$.

5 Application to the Flag Manifolds G/T

A calculus, or science of calculation, is one which has organized processes by which passage is made, mechanically, from one result to another. —De Morgan.

Among all the flag manifolds G/P associated to a Lie group G it is the complete flag manifold G/T that is of crucial importance, since the inclusion $T \subseteq P \subset G$ of subgroups induces the fibration

$$P/T \hookrightarrow G/T \xrightarrow{\pi} G/P$$

in which the induced map π^* embeds the ring $H^*(G/P)$ as a subring of $H^*(G/T)$, see the proof of Theorem 4.3. Further, according to E. Cartan [63, p. 674] all the 1-connected simple Lie groups consist of the three infinite families $SU(n)$, $Sp(n)$, $Spin(n)$ of the classical groups, as well as the five exceptional ones: G_2, F_4, E_6, E_7, E_8, while for any compact connected Lie group G with a maximal torus T one has a diffeomorphism

$$G/T = G_1/T_1 \times \cdots \times G_k/T_k$$

with each G_i an 1-connected simple Lie group and $T_i \subset G_i$ a maximal torus. Thus, the problem of finding Schubert presentations of flag manifolds may be reduced to the special cases G/T where G is 1-connected and simple.

In this section we determine the Schubert presentation of the ring $H^*(G/T)$ in accordance to G is classical or exceptional. Recall that for a simple Lie group G with rank n the fundamental dominant weights $\{\omega_1, \ldots, \omega_n\}$ of G [4] is precisely the Schubert basis of $H^2(G/T)$ [17, Lemma 2.4].

5.1 The Ring $H^*(G/T)$ Classical Lie Group

If $G = SU(n+1)$ or $Sp(n)$ Borel [3] has shown that

$$H^*(G/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n] / \langle \mathbb{Z}[\omega_1, \ldots, \omega_n]^{+,W} \rangle,$$

where $\mathbb{Z}[\omega_1, \ldots, \omega_n]^{+,W}$ is the ring of the integral Weyl invariants of G in positive degrees. It follows that if we let $c_r(G) \in H^{2r}(G/T)$ be respectively the rth element symmetric polynomial in the sets

$$\{\omega_1, \omega_k - \omega_{k-1}, -\omega_n \mid 2 \le k \le n\} \text{ or } \{\pm\omega_1, \pm(\omega_k - \omega_{k-1}) \mid 2 \le k \le n\},$$

then we have

Theorem 5.1 *For $G = SU(n)$ or $Sp(n)$ Schubert presentation of $H^*(G/T)$ is*

$$H^*(SU(n)/T) = \mathbb{Z}\left[\omega_1, \dots, \omega_{n-1}\right] / \langle c_2, \dots, c_n\rangle, c_r = c_r(SU(n)); \tag{5.1}$$

$$H^*(Sp(n)/T) = \mathbb{Z}\left[\omega_1, \dots, \omega_n\right] / \langle c_2, \dots, c_{2n}\rangle, c_{2r} = c_{2r}(Sp(n)). \tag{5.2}$$

□

Turning to the group $G = Spin(2n)$ let y_k be the Schubert class on $Spin(2n)/T$ associated to the Weyl group element

$$w_k = \sigma[n-k, \dots, n-2, n-1], 2 \le k \le n-1$$

(in the notation of Sect. 4.1). According to Marlin [46, Proposition 3]

$$H^*(Spin(2n)/T) = \mathbb{Z}[\omega_1, \dots, \omega_n, y_2, \dots, y_{n-1}]/\left\langle \delta_i, \xi_j, \mu_k \right\rangle \tag{5.3}$$

where

$$\begin{aligned}
&\delta_i := 2y_i - c_i(\omega_1, \dots, \omega_n), 1 \le i \le n-1,\\
&\xi_j := y_{2j} + (-1)^j y_j^2 + 2\sum_{1\le r\le j-1} (-1)^r y_r y_{2j-r}, 1 \le j \le \left[\tfrac{n-1}{2}\right],\\
&\mu_k := (-1)^k y_k^2 + 2\sum_{2k-n+1\le r\le k-1} (-1)^r y_r y_{2k-r}, \left[\tfrac{n}{2}\right] \le k \le n-1,
\end{aligned}$$

and where $c_i(\omega_1, \dots, \omega_n)$ is the ith elementary symmetric function on set

$$\{\omega_n, \omega_i - \omega_{i-1}, \omega_{n-1} + \omega_n - \omega_{n-2}, \omega_{n-1} - \omega_n \mid 2 \le i \le n-2\}.$$

Since each Schubert class y_{2j} with $1 \le j \le \left[\frac{n-1}{2}\right]$ can be expressed as a polynomial in the y_{2i+1}'s by the relations of the type ξ_k, we obtain that

Theorem 5.2 *The Schubert presentation of $H^*(Spin(2n)/T)$ is*

$$H^*(Spin(2n)/T) = \mathbb{Z}[\omega_1, \dots, \omega_n, y_3, y_5, \dots, y_{2[\frac{n-1}{2}]-1}]/\langle r_i, h_k\rangle, \tag{5.4}$$

where r_i and h_k are the polynomials obtained respectively from δ_i and μ_k by replacing the classes y_{2r} with the polynomials (by the relation ξ_r)

$$(-1)^{r-1} y_r^2 + 2\sum_{1\le k\le r-1} (-1)^{k-1} y_k y_{2r-k}.$$

□

Similarly, if $G = Spin(2n+1)$ one can deduce a Schubert presentation of the ring $H^*(G/T)$ from Marlin's formula [46, Proposition 2].

Remark 5.3 For the classical Lie groups G the Giambelli polynomials (i.e. *Schubert polynomials*) of the Schubert classes on G/T have been determined by Billey and Haiman [6].

In comparison with Marlin's formula (5.3) the presentation (5.4) is more concise for involving fewer generators and relations. A basic requirement of topology is to present the cohomology of a space X by a minimal system of generators and relations, so that the (rational) minimal model and κ -invariants of the Postnikov tower of X [30] can be formulated accordingly. □

5.2 The Ring $H^*(G/T)$ for an Exceptional Lie Group

Having clarified the Schubert presentation of the ring $H^*(G/T)$ for the classical G we proceed to the exceptional cases $G = F_4, E_6$ or E_7 (the result for the case E_8 comes from the same calculation, only the presentation [21, Theorem 5.1] is slightly lengthy). In these cases the dimension $s = \dim G/T$ and the number t of the Schubert classes on G/T are

$$(s, t) = (48, 1152), (72, 51840) \text{ or } (126, 2903040),$$

respectively. Instead of describing the ring $H^*(G/T)$ using the totality of t^3 Littlewood–Richardson coefficients $c_{u,v}^w$ (with $c_{u,v}^w = 0$ for $l(w) \neq l(u) + l(v)$ being understood) the idea of Schubert presentation brings us the following concise and explicit formulae of the ring $H^*(G/T)$.

Theorem 5.4 *For $G = F_4, E_6$ and E_7 the Schubert presentations of the cohomologies $H^*(G/T)$ are*

$$H^*(F_4/T) = \mathbb{Z}[\omega_1, \omega_2, \omega_3, \omega_4, y_3, y_4] / \langle \rho_2, \rho_4, r_3, r_6, r_8, r_{12} \rangle, \textit{ where} \tag{5.5}$$

$\rho_2 = c_2 - 4\omega_1^2;$
$\rho_4 = 3y_4 + 2\omega_1 y_3 - c_4;$
$r_3 = 2y_3 - \omega_1^3;$
$r_6 = y_3^2 + 2c_6 - 3\omega_1^2 y_4;$
$r_8 = 3y_4^2 - \omega_1^2 c_6;$
$r_{12} = y_4^3 - c_6^2.$

$$H^*(E_6/T) = \mathbb{Z}[\omega_1, \ldots, \omega_6, y_3, y_4] / \langle \rho_2, \rho_3, \rho_4, \rho_5, r_6, r_8, r_9, r_{12} \rangle, \textit{ where} \tag{5.6}$$

$\rho_2 = 4\omega_2^2 - c_2;$
$\rho_3 = 2y_3 + 2\omega_2^3 - c_3;$

$\rho_4 = 3y_4 + \omega_2^4 - c_4;$
$\rho_5 = 2\omega_2^2 y_3 - \omega_2 c_4 + c_5;$
$r_6 = y_3^2 - \omega_2 c_5 + 2c_6;$
$r_8 = 3y_4^2 - 2c_5 y_3 - \omega_2^2 c_6 + \omega_2^3 c_5;$
$r_9 = 2y_3 c_6 - \omega_2^3 c_6;$
$r_{12} = y_4^3 - c_6^2.$

$$H^*(E_7/T) = \mathbb{Z}[\omega_1, \ldots, \omega_7, y_3, y_4, y_5, y_9]/\langle \rho_i, r_j \rangle, \textit{ where} \tag{5.7}$$

$\rho_2 = 4\omega_2^2 - c_2;$
$\rho_3 = 2y_3 + 2\omega_2^3 - c_3;$
$\rho_4 = 3y_4 + \omega_2^4 - c_4;$
$\rho_5 = 2y_5 - 2\omega_2^2 y_3 + \omega_2 c_4 - c_5;$
$r_6 = y_3^2 - \omega_2 c_5 + 2c_6;$
$r_8 = 3y_4^2 + 2y_3 y_5 - 2y_3 c_5 + 2\omega_2 c_7 - \omega_2^2 c_6 + \omega_2^3 c_5;$
$r_9 = 2y_9 + 2y_4 y_5 - 2y_3 c_6 - \omega_2^2 c_7 + \omega_2^3 c_6;$
$r_{10} = y_5^2 - 2y_3 c_7 + \omega_2^3 c_7;$
$r_{12} = y_4^3 - 4y_5 c_7 - c_6^2 - 2y_3 y_9 - 2y_3 y_4 y_5 + 2\omega_2 y_5 c_6 + 3\omega_2 y_4 c_7 + c_5 c_7;$
$r_{14} = c_7^2 - 2y_5 y_9 + 2y_3 y_4 c_7 - \omega_2^3 y_4 c_7;$
$r_{18} = y_9^2 + 2y_5 c_6 c_7 - y_4 c_7^2 - 2y_4 y_5 y_9 + 2y_3 y_5^3 - 5\omega_2 y_5^2 c_7,$

where the c_r's are the polynomials $c_r(P)$ *in the weights* $\omega_1, \ldots, \omega_n$ defined in (5.17); and where the y_i's are the Schubert classes on G/T associated to the Weyl group elements tabulated below:

y_i	$F_4/T(F/P_{\{1\}})$	$E_6/T(E_6/P_{\{2\}})$	$E_7/T(E_7/P_{\{2\}})$
y_3	$\sigma_{[3,2,1]}$	$\sigma_{[5,4,2]}$	$\sigma_{[5,4,2]}$
y_4	$\sigma_{[4,3,2,1]}$	$\sigma_{[6,5,4,2]}$	$\sigma_{[6,5,4,2]}$
y_5			$\sigma_{[7,6,5,4,2]}$
y_6	$\sigma_{[3,2,4,3,2,1]}$	$\sigma_{[1,3,6,5,4,2]}$	$\sigma_{[1,3,6,5,4,2]}$
y_7			$\sigma_{[1,3,7,6,5,4,2]}$
y_9			$\sigma_{[1,5,4,3,7,6,5,4,2]}$.

(5.8)

To reduce the computational complexity of deriving Theorem 5.4 we choose for each $G = F_4, E_6$ or E_7 a parabolic subgroup P associated to a singleton $K = \{i\}$, where the index i, as well as the isomorphism types of P and its simple part P^s, is stated in the table below

G	F_4	E_6	E_7
i	1	2	2
$P;\ P^s$	$Sp(3) \cdot S^1;\ Sp(3)$	$SU(6) \cdot S^1;\ SU(6)$	$SU(7) \cdot S^1;\ SU(7)$

. (5.9)

In view of the circle bundle associated to P (see in Sect. 4.3)

$$S^1 \hookrightarrow G/P^s \xrightarrow{p} G/P \tag{5.10}$$

the calculation will be divided into three steps, in accordance to cohomologies of the three homogeneous spaces G/P^s, G/P and G/T.

Step 1. The cohomologies $H^*(G/P^s)$**.** By the formulae (4.8) and (4.9) the additive structure of $H^*(G/P^s)$ is determined by the homomorphisms

$$H^{2r-2}(G/P) \overset{\cup\omega}{\to} H^{2r}(G/P).$$

Explicitly, with respect to the Schubert basis $\{s_{r,1}, \ldots, s_{r,\beta(r)}\}$ of $H^{2r}(G/P)$, $\beta(r) = |W^r(P;G)|$, the expansions

$$\omega \cup s_{r-1,i} = \sum a_{i,j} \cdot s_{r,j}$$

give rise to a $\beta(r-1) \times \beta(r)$ matrix A_r that satisfies the linear system

$$\begin{pmatrix} \omega \cup s_{r-1,1} \\ \omega \cup s_{r-1,2} \\ \vdots \\ \omega \cup s_{r-1,\beta(r-1)} \end{pmatrix} = A_r \begin{pmatrix} s_{r,1} \\ s_{r,2} \\ \vdots \\ s_{r,\beta(r)} \end{pmatrix}.$$

Since $\omega \cup s_{r-1,i}$ is a monomial in Schubert classes, the *Characteristics* is applicable to evaluate the entries of A_r. Diagonalizing A_r by the integral row and column reductions [50, p. 162–166] one obtains the non-trivial groups $H^r(G/P^s)$, together with their basis, as that tabulated below, where

(1) $\overline{y}_i := p^*(y_i)$ with y_i the Schubert classes in table (5.8);
(2) For simplicity the non-trivial groups $H^r(G/P^s)$ are printed only up to the stage where all the generators and relations of the ring $H^{even}(G/P^s)$ emerge.

Step 2. The cohomologies $H^*(G/P)$**.** Summarizing the contents of Table 1 we find that

$$H^{even}(F_4/P^s) = \mathbb{Z}[\overline{y}_3, \overline{y}_4, \overline{y}_6]/\left\langle p^*(h_3), p^*(h_6), p^*(h_8), p^*(h_{12})\right\rangle,$$

where

$$h_3 = 2y_3, h_6 = 2y_6 + y_3^2, h_8 = 3y_4^2, h_{12} = y_6^2 - y_4^3,$$

and that $H^{odd}(F_4/P^s)$ has the $H^{even}(F_4/P^s)$ -module basis $\{d_{23}\}$. By Theorem 4.3 we obtain the partial presentation

$$H^*(F_4/P) = \mathbb{Z}[\omega_1, y_3, y_4, y_6]/\left\langle r_3, r_6, r_8, r_{12}, r_{12}'\right\rangle,$$

Table 1 Non-trivial cohomologies of F_4/P^s

Nontrivial H^k	Basis elements	Relations
$H^6 \cong \mathbb{Z}_2$	$\bar{s}_{3,1}(=\overline{y}_3)$	$2\overline{y}_3 = 0$
$H^8 \cong \mathbb{Z}$	$\bar{s}_{4,2}(=\overline{y}_4)$	
$H^{12} \cong \mathbb{Z}_4$	$\bar{s}_{6,2}(=\overline{y}_6)$	$-2\overline{y}_6 = \overline{y}_3^2$
$H^{14} \cong \mathbb{Z}_2$	$\overline{y}_3\overline{y}_4$	
$H^{16} \cong \mathbb{Z}_3$	$\overline{y}_4^2$	$3\overline{y}_4^2 = 0$
$H^{18} \cong \mathbb{Z}_2$	$\overline{y}_3\overline{y}_6$	
$H^{20} \cong \mathbb{Z}_4$	$\overline{y}_4\overline{y}_6$	
$H^{26} \cong \mathbb{Z}_2$	$\overline{y}_3\overline{y}_4\overline{y}_6$	
$H^{23} \cong \mathbb{Z}$	$d_{23} = \beta^{-1}(2s_{11,1} - s_{11,2})$	

Table 2 Non-trivial cohomologies of E_6/P^s

Nontrivial H^k	Basis elements	Relations
$H^6 \cong \mathbb{Z}$	$\bar{s}_{3,2}(=\overline{y}_3)$	
$H^8 \cong \mathbb{Z}$	$\bar{s}_{4,3}(=\overline{y}_4)$	
$H^{12} \cong \mathbb{Z}$	$\bar{s}_{6,1}(=\overline{y}_6)$	$-2\overline{y}_6 = \overline{y}_3^2$
$H^{14} \cong \mathbb{Z}$	$\overline{y}_3\overline{y}_4$	
$H^{16} \cong \mathbb{Z}_3$	$\overline{y}_4^2$	$3\overline{y}_4^2 = 0$
$H^{18} \cong \mathbb{Z}_2$	$\overline{y}_3\overline{y}_6$	$2\overline{y}_3\overline{y}_6 = 0$
$H^{20} \cong \mathbb{Z}$	$\overline{y}_4\overline{y}_6$	
$H^{22} \cong \mathbb{Z}_3$	$\overline{y}_3\overline{y}_4^2$	
$H^{26} \cong \mathbb{Z}_2$	$\overline{y}_3\overline{y}_4\overline{y}_6$	
$H^{28} \cong \mathbb{Z}_3$	$\overline{y}_4^2\overline{y}_6$	
$H^{23} \cong \mathbb{Z}$	$d_{23}= \beta^{-1}(s_{11,1}-s_{11,2}-s_{11,3}+s_{11,4}-s_{11,5}+s_{11,6})$	
$H^{29} \cong \mathbb{Z}$	$d_{29}= \beta^{-1}(-s_{14,1}+s_{14,2}+s_{14,4}- s_{14,5})$	$2d_{29} = \pm\overline{y}_3 d_{23}$

indicating that the inclusion $\{\omega_1, y_3, y_4, y_6\} \subset H^*(F_4/P)$ induces the surjective ring map $f : \mathbb{Z}[\omega_1, y_3, y_4, y_6] \to H^*(F_4/P)$. Further, according to Lemma 4.7, computing with the *Null-space* $N(f_m)$ in the order $m = 3, 6, 8$ and 12 suffices to decide the generators of the ideal $\ker f$ to yields the Schubert presentation

$$H^*(F_4/P) = \mathbb{Z}[\omega_1, y_3, y_4, y_6]/\langle r_3, r_6, r_8, r_{12}\rangle \text{ , where} \tag{5.11}$$

$r_3 = 2y_3 - \omega_1^3;$
$r_6 = 2y_6 + y_3^2 - 3\omega_1^2 y_4;$
$r_8 = 3y_4^2 - \omega_1^2 y_6;$
$r_{12} = y_6^2 - y_4^3.$

Similarly, combining *the Null-space* with the contents of Tables 2 and 3 one gets the Schubert presentations of the ring $H^*(G/P)$ for $G = E_6$ and E_7 as

Table 3 Non-trivial cohomologies of E_7/P^s

Nontrivial H^k	Basis elements
$H^6 \cong \mathbb{Z}$	$s_{3,2} = \bar{y}_3$
$H^8 \cong \mathbb{Z}$	$s_{4,3} = \bar{y}_4$
$H^{10} \cong \mathbb{Z}$	$s_{5,4} = \bar{y}_5$
$H^{12} \cong \mathbb{Z}$	$s_{6,5} = \bar{y}_3^2 + \bar{y}_6$
$H^{14} \cong \mathbb{Z} \oplus \mathbb{Z}$	$s_{7,6} = -\bar{y}_7$; $\bar{y}_3\bar{y}_4$
$H^{16} \cong \mathbb{Z}$	$\bar{y}_3\bar{y}_5 - 2\bar{y}_4^2$
$H^{18} \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\bar{y}_3^3 + \bar{y}_3\bar{y}_6 + \bar{y}_4\bar{y}_5 + \bar{y}_9$; $-\bar{y}_4\bar{y}_5$; $\bar{y}_3^3 + \bar{y}_9$
$H^{20} \cong \mathbb{Z} \oplus \mathbb{Z}$	$\bar{y}_3\bar{y}_7 - \bar{y}_5^2$; $\bar{y}_3\bar{y}_7 + \bar{y}_4\bar{y}_6$
$H^{22} \cong \mathbb{Z} \oplus \mathbb{Z}$	$\bar{y}_3^2\bar{y}_5 - \bar{y}_3\bar{y}_4^2 + \bar{y}_4\bar{y}_7$; $-2\bar{y}_4\bar{y}_7 + \bar{y}_5\bar{y}_6$
$H^{24} \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\bar{y}_3^4 + \bar{y}_3^2\bar{y}_6 + \bar{y}_3\bar{y}_4\bar{y}_5 + \bar{y}_3\bar{y}_9$ $\bar{y}_3^2\bar{y}_6 + \bar{y}_3\bar{y}_4\bar{y}_5 - \bar{y}_4^3$; $-\bar{y}_4^3 + \bar{y}_5\bar{y}_7 + \bar{y}_6^2$
$H^{26} \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\bar{y}_3^3\bar{y}_4 + \bar{y}_3\bar{y}_4\bar{y}_6 + \bar{y}_4^2\bar{y}_5 + \bar{y}_4\bar{y}_9$ $\bar{y}_3\bar{y}_4\bar{y}_6 + \bar{y}_3\bar{y}_5^2 - 3\bar{y}_4^2\bar{y}_5 + \bar{y}_6\bar{y}_7$ $\bar{y}_3^3\bar{y}_4 + \bar{y}_3^2\bar{y}_7 + 2\bar{y}_4^2\bar{y}_5 + \bar{y}_4\bar{y}_9$
$H^{28} \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\bar{y}_3^3\bar{y}_5 + \bar{y}_3\bar{y}_5\bar{y}_6 + \bar{y}_4\bar{y}_5^2 + \bar{y}_5\bar{y}_9$ $3\bar{y}_3^2\bar{y}_4^2 + 5\bar{y}_4\bar{y}_5^2 + \bar{y}_5\bar{y}_9$ $\bar{y}_3^3\bar{y}_5 + 2\bar{y}_3^2\bar{y}_4^2 + \bar{y}_3\bar{y}_4\bar{y}_7 + \bar{y}_3\bar{y}_5\bar{y}_6 + 4\bar{y}_4\bar{y}_5^2 +$ $\bar{y}_5\bar{y}_9$
$H^{30} \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$	$-\bar{y}_3^5 - \bar{y}_3\bar{y}_4^3 + \bar{y}_4\bar{y}_5\bar{y}_6 + \bar{y}_5^3 + \bar{y}_6\bar{y}_9$ $-\bar{y}_3^5 + \bar{y}_3^2\bar{y}_4\bar{y}_5 - 3\bar{y}_3\bar{y}_4^3 + \bar{y}_3\bar{y}_5\bar{y}_7 - \bar{y}_4^2\bar{y}_7 + \bar{y}_5^3$ $\bar{y}_3^2\bar{y}_4\bar{y}_5 + \bar{y}_3\bar{y}_5\bar{y}_7 + \bar{y}_4^2\bar{y}_7 + \bar{y}_4\bar{y}_5\bar{y}_6$
$H^{32} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$	$\bar{y}_3^4\bar{y}_4 + \bar{y}_3^2\bar{y}_4\bar{y}_6 + \bar{y}_3\bar{y}_4^2\bar{y}_5 + \bar{y}_3\bar{y}_4\bar{y}_9$ $\bar{y}_3^2\bar{y}_5^2 + \bar{y}_3\bar{y}_6\bar{y}_7 + \bar{y}_4\bar{y}_5\bar{y}_7 + \bar{y}_5^2\bar{y}_6 + \bar{y}_7\bar{y}_9$ $\bar{y}_3^4\bar{y}_4 + \bar{y}_3\bar{y}_4^2\bar{y}_5 + \bar{y}_3\bar{y}_4\bar{y}_9 + \bar{y}_3\bar{y}_6\bar{y}_7 + \bar{y}_4^4$ $\bar{y}_3^2\bar{y}_4\bar{y}_6 + \bar{y}_3\bar{y}_4^2\bar{y}_5 - \bar{y}_4^4 + \bar{y}_4\bar{y}_5\bar{y}_7 + \bar{y}_5^2\bar{y}_6$
$H^{34} \cong \mathbb{Z}_{38} \oplus \mathbb{Z}$	$\bar{y}_3^4\bar{y}_5 + \bar{y}_3\bar{y}_4\bar{y}_5^2 + \bar{y}_3\bar{y}_5\bar{y}_9 + 2\bar{y}_4^3\bar{y}_5$ $\bar{y}_3^4\bar{y}_5 + 2\bar{y}_3^3\bar{y}_4^2 + \bar{y}_3^2\bar{y}_4\bar{y}_7 + 9\bar{y}_4^3\bar{y}_5 + \bar{y}_4\bar{y}_6\bar{y}_7$
$H^{36} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}$	$\bar{y}_3^3\bar{y}_4\bar{y}_5 + \bar{y}_3\bar{y}_4\bar{y}_5\bar{y}_6 + \bar{y}_4^2\bar{y}_5^2 + \bar{y}_4\bar{y}_5\bar{y}_9$ $13\bar{y}_3^6 + 30\bar{y}_3^2\bar{y}_4^3 + \bar{y}_3\bar{y}_6\bar{y}_9 + \bar{y}_4^2\bar{y}_5^2 + \bar{y}_4\bar{y}_5\bar{y}_9 +$ $\bar{y}_6^3$ $-4\bar{y}_3^6 + \bar{y}_3^3\bar{y}_4\bar{y}_5 - 11\bar{y}_3^2\bar{y}_4^3 + \bar{y}_3\bar{y}_6\bar{y}_9 + \bar{y}_5\bar{y}_6\bar{y}_7$ $13\bar{y}_3^6 + \bar{y}_3^3\bar{y}_4\bar{y}_5 + 28\bar{y}_3^2\bar{y}_4^3 - \bar{y}_4^2\bar{y}_5^2 + \bar{y}_4\bar{y}_5\bar{y}_9$

$$H^*(E_6/P) = \mathbb{Z}[\omega_2, y_3, y_4, y_6]/\langle r_6, r_8, r_9, r_{12}\rangle, \text{ where} \tag{5.12}$$

$r_6 = 2y_6 + y_3^2 - 3\omega_2^2 y_4 + 2\omega_2^3 y_3 - \omega_2^6$;
$r_8 = 3y_4^2 - 6\omega_2 y_3 y_4 + \omega_2^2 y_6 + 5\omega_2^2 y_3^2 - 2\omega_2^5 y_3$;
$r_9 = 2y_3 y_6 - \omega_2^3 y_6$;
$r_{12} = y_6^2 - y_4^3$.

$$H^*(E_7/P) = \mathbb{Z}[\omega_2, y_3, y_4, y_5, y_6, y_7, y_9]/\ \langle r_j \rangle_{j\in\Lambda} \text{ where } \Lambda = \{6, 8, 9, 10, 12, 14, 18\}, \tag{5.13}$$

$$\begin{aligned}
&r_6 = 2y_6 + y_3^2 + 2\omega_2 y_5 - 3\omega_2^2 y_4 + 2\omega_2^3 y_3 - \omega_2^6;\\
&r_8 = 3y_4^2 - 2y_3 y_5 + 2\omega_2 y_7 - 6\omega_2 y_3 y_4 + \omega_2^2 y_6 + 5\omega_2^2 y_3^2 + 2\omega_2^3 y_5 - 2\omega_2^5 y_3;\\
&r_9 = 2y_9 + 2y_4 y_5 - 2y_3 y_6 - 4\omega_2 y_3 y_5 - \omega_2^2 y_7 + \omega_2^3 y_6 + 2\omega_2^4 y_5;\\
&r_{10} = y_5^2 - 2y_3 y_7 + \omega_2^3 y_7;\\
&r_{12} = y_6^2 + 2y_5 y_7 - y_4^3 + 2y_3 y_9 + 2y_3 y_4 y_5 + 2\omega_2 y_5 y_6 - 6\omega_2 y_4 y_7 + \omega_2^2 y_5^2;\\
&r_{14} = y_7^2 - 2y_5 y_9 + y_4 y_5^2;\\
&r_{18} = y_9^2 + 2y_5 y_6 y_7 - y_4 y_7^2 - 2y_4 y_5 y_9 + 2y_3 y_5^3 - \omega_2 y_5^2 y_7.
\end{aligned}$$

Step 3. Computing with the Weyl invariants. In addition to (5.10) the parabolic subgroup P on G specified by table (5.9) induces also the fibration

$$P/T \overset{i}{\hookrightarrow} G/T \overset{\pi}{\to} G/P (i.e.(4.3)), \tag{5.14}$$

where Schubert presentation of the cohomology of the base space G/P has been decided by (5.11), (5.12) and (5.13). On the other hand, with

$$P/T = Sp(3)/T^3,\ SU(6)/T^5 \text{ or } SU(7)/T^6 \text{ for } G = F_4,\ E_6 \text{ or } E_7$$

the cohomology of the fiber space P/T is given by Theorem 5.1 as

$$H^*(P/T) = \begin{cases} \frac{\mathbb{Z}[\omega_2,\omega_3,\omega_4]}{\langle c_2,c_4,c_6\rangle} \ if\, G = F_4 \\ \frac{\mathbb{Z}[\omega_1,\omega_3,\ldots,\omega_n]}{\langle c_r, 2\le r\le n\rangle} \ if\, G = E_n\, with\, n = 6, 7. \end{cases} \tag{5.15}$$

Thus, Theorem 4.3 is applicable to fashion the ring $H^*(G/T)$ in question from the known ones $H^*(P/T)$ and $H^*(G/P)$. To this end we need only to specify a system $\{\rho_r\}$ satisfying the constraints (4.5). The invariant theory of Weyl groups serves this purpose.

Recall that the Weyl group W of G can be identified with the subgroup of $Aut(H^2(G/T))$ generated by the automorphisms σ_i, $1 \le i \le n$, whose action on the Schubert basis $\{\omega_1, \ldots, \omega_n\}$ of $H^2(G/T)$ is given by the Cartan matrix $(a_{ij})_{n\times n}$ of G as

$$\sigma_i(\omega_k) = \begin{cases} \omega_i \text{ if } k \neq i; \\ \omega_i - \sum_{1\le j\le n} a_{ij}\omega_j \text{ if } k = i \end{cases}. \tag{5.16}$$

Introduce for each $G = F_4$, E_6 and E_7 the polynomials $c_r(P)$ in $\omega_1,\ldots,\omega_n$ by the formula

$$c_r(P) := \begin{cases} e_r(o(\omega_4, W(P))), 1 \le r \le 4 \text{ if } G = F_4; \\ e_r(o(\omega_n, W(P)), 1 \le r \le n \text{ if } G = E_n, n = 6, 7 \end{cases}, \tag{5.17}$$

where $o(\omega, W(P)) \subset H^2(G/T)$ denotes the $W(P)$-orbit through $\omega \in H^2(G/T)$, and where $e_r(o(\omega, W(P))) \in H^{2r}(G/T)$ is the rth elementary symmetric function on the set $o(\omega, W(P))$. For instance if $G = F_4, E_6$ we have by (5.16) that

$$\begin{aligned} o(\omega_4, W(P)) &= \left\{\omega_4, \omega_3 - \omega_4, \omega_2 - \omega_3, \omega_1 - \omega_2 + \omega_3, \omega_1 - \omega_3 + \omega_4, \omega_1 - \omega_4\right\}, \\ o(\omega_6, W(P)) &= \{\omega_6, \omega_5 - \omega_6, \omega_4 - \omega_5, \omega_2 + \omega_3 - \omega_4, \omega_1 + \omega_2 - \omega_3, \omega_2 - \omega_1\}. \end{aligned}$$

On the other hand, according to Bernstein–Gel'fand–Gel'fand [5, Proposition 5.1] the induced map π^* in (4.3) injects, and satisfies the relation

$$\mathrm{Im}\pi^* = H^*(G/T)^{W(P)} = H^*(G/P),$$

implying $c_r(P) \in H^*(G/P)$. Since $c_r(P)$ is an explicit polynomial in the Schubert classes ω_i the *Giambelli polynomials* is functional to express it as a polynomial g_r in the special Schubert classes on $H^*(G/P)$ given by table (5.8):

G	F_4	E_6	E_7
g_2	$4\omega_1^2$	$4\omega_2^2$	$4\omega_2^2$
g_3		$2y_3 + 2\omega_2^3$	$2y_3 + 2\omega_2^3$
g_4	$3y_4 + 2\omega_1 y_3$	$3y_4 + \omega_2^4$	$3y_4 + \omega_2^4$
g_5		$3\omega_2 y_4 - 2\omega_2^2 y_3 + \omega_2^5$	$2y_5 + 3\omega_2 y_4 - 2\omega_2^2 y_3 + \omega_2^5$
g_6	y_6	y_6	$y_6 + 2\omega_2 y_5$
g_7			y_7

. (5.18)

Up to now we have accumulated sufficient information to show Theorem 5.4.

Proof of Theorem 5.4 For each $G = F_4, E_6$ or E_7 Schubert presentations for the cohomologies of the base G/P and of the fiber P/T have been determined by (5.11)–(5.13) and (5.15), respectively, while a system $\{\rho_r\}$ satisfying the relation (4.5) is seen to be $\rho_r := c_r(P) - g_r$. Therefore, Theorem 4.3 is directly applicable to formulate a presentation of the ring $H^*(G/T)$. The results can be further simplified to yield the desired formulae (5.5)–(5.7) by the following observations:

(a) Certain Schubert classes y_k from the base space G/P can be eliminated against appropriate relations of the type ρ_k, e.g. if $G = E_7$ the generators y_6, y_7 and the relations ρ_6, ρ_7 can be excluded by the formulae of g_6 and g_7, which implies that $y_6 = c_6 - 2\omega_2 y_5$ and $y_7 = c_7$, respectively;
(b) Without altering the ideal, higher degree relations of the type r_i may be simplified modulo the lower degree ones by the following fact. For two ordered sequences $\{f_i\}_{1\le i\le n}$ and $\{h_i\}_{1\le i\le n}$ of a graded polynomial ring with

$$\deg f_1 < \cdots < \deg f_n \text{ and } \deg h_1 < \cdots < \deg h_n$$

write $\{h_i\}_{1\leq i\leq n} \sim \{f_i\}_{1\leq i\leq n}$ to denote the statements that $\deg h_i = \deg f_i$ and that $(f_i - h_i) \in \langle f_j \rangle_{1\leq j<i}$. Then $\{f_i\}_{1\leq i\leq n} \sim \{h_i\}_{1\leq i\leq n}$ implies that $\langle h_1, \ldots, h_n\rangle = \langle f_1, \ldots, f_n\rangle$. □

5.3 A Type Free Characterization of the Ring $H^*(G/T)$

For an 1-connected simple Lie group G with rank n denote by $D(G) \subset H^*(G/T)$ the ideal of decomposable elements. Let $h(G)$ be the cardinality of a basis of the quotient group $H^*(G/T)/D(G)$ and set $m = h(G) - n - 1$. The results of Theorems 5.1, 5.2 and 5.4 can be summarized into one formula, without referring to the types of the group G (see [21, Theorems 1.2 and 1.3]).

Theorem 5.5 *For each simple Lie group G there exist a set $\{y_1, \ldots, y_m\}$ of m Schubert classes on G/T with $2 < \deg y_1 < \cdots < \deg y_m$, so that the inclusion $\{\omega_1, \ldots, \omega_n, y_1, \ldots, y_m\} \in H^*(G/T)$ induces the Schubert presentation*

$$H^*(G/T) = \mathbb{Z}[\omega_1, \ldots, \omega_n, y_1, \ldots, y_m]/\langle e_i, f_j, g_j\rangle_{1\leq i\leq k; 1\leq j\leq m}, \tag{5.19}$$

where

(i) $k = n - m$ for all $G \neq E_8$ but $k = n - m + 2$ for $G = E_8$;

(ii) $e_i \in \langle \omega_1, \ldots, \omega_n\rangle$, $1 \leq i \leq k$;

(iii) the pair (f_j, g_j) of polynomials is related to the Schubert class y_j in the fashion

$$f_j = p_j \cdot y_j + \alpha_j, g_j = y_j^{k_j} + \beta_j, 1 \leq j \leq m,$$

where $p_j \in \{2, 3, 5\}$ and $\alpha_j, \beta_j \in \langle \omega_1, \ldots, \omega_n\rangle$;

(iv) ignoring the ordering, the sequence $\{\deg e_i, \deg g_j\}_{1\leq i\leq k; 1\leq j\leq m}$ of integers agrees with the degree sequence of the basic Weyl invariants of the group G (over the field of rationals). □

Concerning assertion (iv) we remark that for each simple Lie group G the degree sequence, as well as explicit formulae, of the basic Weyl invariants $P_1, \ldots, P_n$ of G has been determined by Chevalley and Mehta [12, 47].

6 Further Remarks on the Characteristics

6.1. Certain parameter spaces of the geometric figures concerned by Schubert [51, Chap. 4] may fail to be flag manifolds, but can be constructed by performing finite number of blow-ups on flag manifolds along the centers again in flag manifolds, see the examples in Fulton [27, Example 14.7.12], or in [18] for the construction of the parameter spaces of the complete conics and quadrics on the 3-space $\mathbb{P}^3$. As results the relevant characteristics can be computed from those of flag manifolds via strict transformations (e.g. [18, Examples 5.11; 5.12]).

6.2. As the intersection multiplicities of certain Schubert varieties on G/P, the characteristics $a^w_{w_1,\ldots,w_k}$ are always non-negative by Van der Waerden [59]. Due to the importance of these numbers in geometry their effective computability (rather than positivity) had been the top priority in the classical approach [51, 52, 54], see also Fulton [27, 14.7].

Motivated by the *Littlewood–Richardson rule* [45] for the structure constants of the Grassmannian $G_{n,k}$ a remarkable development of Schubert calculus has taken place in algebraic combinatorics since 1970's, where the main concern is to find combinatorial descriptions of characteristics by which the positivity become transparent. This idea has inspired beautiful results on the enumerations of Yong tableaux, Mondrian tableaux, Chains in the Bruhat order, and puzzles by Buch, Graham, Coskun, Knutson and Tao [7, 14, 28, 43, 44], greatly enriched the classical Schubert calculus.

6.3. According to Van der Waerden [59] and Weil [60, p. 331] Hilbert's 15th problem has been solved satisfactorily. In particular, in the context of modern intersection theory (e.g. [27, 29]) rigorous treatment of the major enumerative results of Schubert [51] had been completed independently by many authors (e.g. [1, 41, 42, 49])[2]; granted with the basis theorem the characteristics of flag manifolds can be evaluated uniformly by the formula (2.2), while the Schubert presentations of the cohomology rings of flag manifolds have also been available (e.g. [3, 6, 20, 21, 46]).

However, Schubert calculus remains a vital and powerful tool in constructing the cohomologies of much broad spaces, such as the homogeneous spaces G/H associated to Lie groups G. In contrast to the basis theorem (i.e. Theorem 1.2) the cohomologies of such spaces may be nontrivial in odd degrees, and may contain torsion elements. Nevertheless, inputting the formula (5.19) into the Koszul complex

$$E_2^{*,*}(G) = H^*(G/T) \otimes H^*(T)$$

associated to the fibration $G \to G/T$ a unified construction of the integral cohomology rings of all the 1-connected simple Lie groups G has been carried out by Duan and Zhao in [23]. In addition, the formula (2.2) of the characteristics has been extended to evaluate the Steenrod operators on the mod p cohomologies of flag manifolds [24], and of the exceptional Lie groups [25].

6.4. As is of today Schubert calculus has entered the intersection of several rapidly developing fields of mathematics, and has been generalized to the studies of other generalized cohomology theories, such as equivariant, quantum cohomology, K-theory, and cobordism, all of them are different deformations of the ordinary cohomology [31]. In this regard the present paper is by no means a comprehensive survey on the contemporary Schubert calculus. It illustrates a passage from the Cartan matrices of

[2]The enumerative results in Schubert [51] were mutually verifiable with the results of other geometers (e.g. Salmon, Clebsch, Chasles and Zeuthen) of the same period, hence were already known to be correct at that time.

Lie groups to the cohomology of homogeneous spaces, where Schubert's characteristics play a central role.

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References

1. Aluffi, P.: The enumerative geometry of plane cubics. II. Nodal and cuspidal cubics. Math. Ann. **289**(4), 543–572 (1991)
2. Baum, P.: On the cohomology of homogeneous spaces. Topology **7**, 15–38 (1968)
3. Borel, A.: Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts. Ann. Math. **57**, 115–207 (1953)
4. Borel, A., Hirzebruch, F.: Characteristic classes and homogeneous spaces (I). Am. J. Math. **80**, 458–538 (1958)
5. Bernstein, I.N., Gel'fand, I.M., Gel'fand, S.I.: Schubert cells and cohomology of the spaces G/P. Russ. Math. Surv. **28**(3), 1–26 (1973)
6. Billey, S., Haiman, M.: Schubert polynomials for the classical groups. J. AMS **8**, 443–482 (1995)
7. Buch, A.S.: Mutations of puzzles and equivariant cohomology of two-step flag varieties. Ann. Math. (2) **182**(1), 173–220 (2015)
8. Bott, R., Samelson, H.: The cohomology ring of G/T. Proc. Natl. Acad. Sci. USA **41**(7), 490–493 (1955)
9. Bott, R., Samelson, H.: Application of the theory of Morse to symmetric spaces. Am. J. Math. **LXXX**(4), 964–1029 (1958)
10. Berenstein, A., Richmond, E.: Littlewood-Richardson coefficients for reflection groups. Adv. Math. **284**, 54–111 (2015)
11. Bröcker, T., Tom Diek, T.: Representations of Compact Lie Groups. Graduate Texts in Mathematics, vol. 98. Springer, New York (1985)
12. Chevalley, C.: Invariants of finite groups generated by reflections. Am. J. Math. **77**, 778–782 (1955)
13. Chevalley, C.: Sur les Décompositions Celluaires des Espaces G/B. Proceedings of Symposia in Pure Mathematics, vol. 56 (part 1), pp. 1–26 (1994)
14. Coskun, I.: A Littlewood-Richardson rule for two-step flag varieties. Invent. Math. **176**, 325–395 (2009)
15. Duan, H.: The degree of a Schubert variety. Adv. Math. **180**, 112–133 (2003)
16. Duan, H.: Multiplicative rule of Schubert classes. Invent. Math. **159**(2), 407–436 (2005); **177**(3), 683–684 (2009) (with X. Zhao)
17. Duan, H.: On the Borel transgression in the fibration $G \to G/T$. Homol. Homotopy Appl. **20**(1), 79–86 (2018)
18. Duan, H., Li, B.: Topology of blow-ups and enumerative geometry. arXiv:0906.4152
19. Duan, H., Zhao, X.: Algorithm for multiplying Schubert classes. Int. J. Algebra Comput. **16**(6), 1197–1210 (2006)
20. Duan, H., Zhao, X.: The Chow rings of generalized Grassmannians. Found. Comput. Math. **10**(3), 245–274 (2010)
21. Duan, H., Zhao, X.: Schubert presentation of the cohomology ring of flag manifolds G/T. LMS J. Comput. Math. **18**(1), 489–506 (2015)
22. Duan, H., Zhao, X.: Appendix to the Chow rings of generalized Grassmannians. arXiv:math.AG/0510085
23. Duan, H., Zhao, X.: Schubert calculus and cohomology of Lie groups. Part I. 1-connected Lie groups. arXiv:0711.2541

24. Duan, H., Zhao, X.: A unified formula for Steenrod operations in flag manifolds. Compos. Math. **143**(1), 257–270 (2007)
25. Duan, H., Zhao, X.: Schubert calculus and the Hopf algebra structures of exceptional Lie groups. Forum Math. **26**(1), 113–140 (2014)
26. Ehresmann, C.: Sur la topologie de certains espaces homogenes. Ann. Math. (2) **35**, 396–443 (1934)
27. Fulton, W.: Intersection Theory. Springer, Berlin (1998)
28. Graham, W.: Positivity in equivariant Schubert calculus. Duke Math. J. **109**, 599–614 (2001)
29. Griffith, P., Harris, J.: Principles of Algebraic Geometry. Wiley, New York (1978)
30. Griffith, P., Morgan, J.: Rational Homotopy Theory and Differential Forms. Birkräuser, Boston (1981)
31. Gorbounov, V., Petrov, V.: Schubert calculus and singularity theory. J. Geom. Phys. **62**, 352–360 (2012)
32. Hansen, H.C.: On cycles in flag manifolds. Math. Scand. **33**, 269–274 (1973)
33. Hilbert, D.: Mathematische Probleme. Bull. AMS **8**, 437–479 (1902)
34. Hiller, H.: Geometry of Coxeter Groups. Research Notes in Mathematics, vol. 54. Pitman Advanced Publishing Program, Boston (1982)
35. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics, vol. 9. Springer, New York (1972)
36. Husemoller, D.: Fibre Bundles. Graduate Texts in Mathematics, vol. 20. Springer, New York (1975)
37. Husemoller, D., Moore, J., Stasheff, J.: Differential homological algebra and homogeneous spaces. J. Pure Appl. Algebra **5**, 113–185 (1974)
38. Ishitoya, K., Toda, H.: On the cohomology of irreducible symmetric spaces of exceptional type. J. Math. Kyoto Univ. **17**, 225–243 (1977)
39. Kleiman, S.: Problem 15: rigorous foundation of Schubert's enumerative calculus. Proceedings of Symposia in Pure Mathematics, vol. 28, pp. 445–482. American Mathematical Society, Providence (1976)
40. Kleiman, S.: Book review on "Intersection Theory by W. Fulton". Bull. AMS **12**(1), 137–143 (1985)
41. Kleiman, S.: Intersection theory and enumerative geometry: a decade in review. Proceedings of Symposia in Pure Mathematics, vol. 46, Part 2, pp. 321–370. American Mathematical Society, Providence (1987)
42. Kleiman, S., Str∅me, S., Xambó, S.: Sketch of a verification of Schubert's number 5819539783680 of twisted cubics. LNM, vol. 1266, pp. 156–180 (1987)
43. Knutson, A., Tao, T.: Puzzles and (equivariant) cohomology of Grassmannians. Duke Math. J. **119**, 221–260 (2003)
44. Knutson, A., Tao, T., Woodward, C.: The honeycomb model of $GL_n(\mathbb{C})$ tensor products, II. Puzzles determine facets of the Littlewood-Richardson cone. J. Am. Math. Soc. **17**, 19–48 (2004)
45. Littlewood, D.E., Richardson, A.R.: Group characters and algebra. Philos. Trans. R. Soc. Lond. **233**, 99–141 (1934)
46. Marlin, R.: Anneaux de Chow des groupes algériques $SU(n)$, $Sp(n)$, $SO(n)$, $Spin(n)$, G_2, F_4. Comptes Rendus Acad. Sci. Paris A **279**, 119–122 (1974)
47. Mehta, M.L.: Basic sets of invariant polynomials for finite reflection groups. Commun. Algebra **16**(5), 1083–1098 (1988)
48. Milnor, J., Stasheff, J.: Characteristic classes. Ann. Math. Stud. (Princeton University Press) **76** (1975)
49. Miret, J.M., Pujolàs, J., Saurav, K., Xambó-Descamps, S.: Computing some fundamental numbers of the variety of nodal cubics in P^3. J. Symb. Comput. **44**(10), 1425–1447 (2009)
50. Scottile, F.: Four entries for Kluwer encyclopaedia of mathematics (2001). arXiv:math.AG/0102047
51. Schubert, H.: Kalkül der abzählenden Geometrie. Springer, New York (1979)

52. Schubert, H.: Anzahl-Bestimmungen für lineare Räume beliebiger beliebiger Dimension. Acta Math. **8**, 97–118 (1886)
53. Schubert, H.: a Lösung des Characteristiken-Problems für lineare Räume beliebiger Dimension. Mitt. Math. Ges. Hambg. **1**, 134–155 (1886)
54. Schubert, H.: Über die Incidenz zweier linearer Räume beliebiger Dimensionen. Math. Ann. 206–221 (1903)
55. Scherk, J.: Algebra. A Computational Introduction. Studies in Advanced Mathematics
56. Schappacher, N.: A historical sketch of B. L. van der Waerden's work in algebraic geometry: 1926–1946. Episodes in the History of Modern Algebra (1800–1950). History of Mathematics, vol. 32, pp. 245–283. American Mathematical Society, Providence (2007)
57. Stembridge, J.: Computational aspects of root systems, coxeter groups, and Weyl characters. Interaction of Combinatorics and Representation Theory. MSJ Memoirs, vol. 11, pp. 1–38. Mathematical Society of Japan, Tokyo (2001)
58. Toda, H.: On the cohomology ring of some homogeneous spaces. J. Math. Kyoto Univ. **15**, 185–199 (1975)
59. van der Waerden, B.L.: Topologische Begründung des Kalkü ls der abzählenden Geometrie. Math. Ann. **102**(1), 337–362 (1930)
60. Weil, A.: Foundations of Algebraic Geometry. American Mathematical Society, Providence (1962)
61. Watanabe, T.: Cohomology of the homogeneous space $E_6/T^1 \cdot SU(6)$. Proceedings of Symposia in Pure Mathematics, vol. 63. American Mathematical Society, Providence (1998)
62. Willems, M.: Cohomologie équivariante des tours de Bott et calcul de Schubert équivariant. J. Inst. Math. Jussieu **5**(1), 125–159 (2006)
63. Whitehead, G.W.: Elements of Homotopy Theory. Graduate Texts in Mathematics, vol. 61. Springer, Berlin (1978)
64. Wolf, J.: The cohomology of homogeneous spaces. Am. J. Math. **99**(2), 312–340 (1977)
65. Zhang, K., Zhang, J., Duan, H., Li, J.: Effective algorithms for computing triangular operator in Schubert calculus. Front. Math. China **10**(1), 221–237 (2015)

Asymmetric Function Theory

Oliver Pechenik and Dominic Searles

Abstract The classical theory of symmetric functions has a central position in algebraic combinatorics, bridging aspects of representation theory, combinatorics, and enumerative geometry. More recently, this theory has been fruitfully extended to the larger ring of quasisymmetric functions, with corresponding applications. Here, we survey recent work extending this theory further to general asymmetric polynomials.

Keywords Symmetric function · Quasisymmetric function · Schubert polynomial · Slide polynomial · Demazure character · Demazure atom · Quasikey polynomial

1 The Three Worlds: Symmetric, Quasisymmetric, and General Polynomials

One of the gems of 20th-century mathematics is theory of *symmetric functions* and *symmetric polynomials*, as expounded in the classic textbooks [58, 59, 78]. In addition to its intrinsic beauty, this theory has important applications in representation theory, algebraic geometry, and combinatorics.

First, we will review aspects of this symmetric function theory. Then, we discuss the more general theory of *quasisymmetric functions and polynomials*, a very active area of contemporary research. Finally, we turn to the combinatorial theory of general asymmetric polynomials. While this seems naively like a very simple object, the polynomial ring turns out to have a rich and beautiful combinatorial structure analogous to that of the symmetric and quasisymmetric worlds, but far less explored.

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While the theory of symmetric polynomials is a textbook subject and that of quasisymmetric polynomials has recently been nicely expounded in [61], we believe this article is the first survey of corresponding developments in the asymmetric world.

In each world, we will consider a variety of additive bases for the algebra in question. The power of the combinatorial theory comes from the bases having the following three characteristics:

(1) positive combinatorial rules for change of basis,
(2) positive combinatorial multiplication rules in various bases, and
(3) algebraic/geometric interpretations of the basis elements.

We will attempt to elucidate all three properties in each case, to the extent that such properties are known to hold. In some cases, our descriptions of geometric interpretations require background that is beyond the scope of this article to recall; readers without such background knowledge may safely skip such episodes.

2 The Symmetric World

Consider the $\mathbb{Z}$-algebra $\mathrm{Poly}_n := \mathbb{Z}[x_1, \ldots, x_n]$ of integral multivariate polynomials. It carries a natural action of the symmetric group $\mathcal{S}_n$ on n letters, where the simple transposition $(i, i+1)$ acts on $f \in \mathrm{Poly}_n$ by swapping the variables x_i and x_{i+1}. Let $\mathrm{Sym}_n := \mathrm{Poly}_n^{\mathcal{S}_n}$, the $\mathcal{S}_n$-invariants. It is easy to see that Sym_n is a subring of Poly_n; we call it the **ring of symmetric polynomials in n variables**. Sym_n is naturally a graded ring, inheriting the grading by degree from Poly_n. We denote the degree m homogeneous piece of a graded ring R by $R^{(m)}$.

For $m \leq n$, we can map Sym_n onto Sym_m by setting the last $n - m$ variables equal to 0. The inverse limit of the $\{\mathrm{Sym}_n\}$ with respect to these restriction maps is called the **ring of symmetric functions** Sym, although its elements are not functions, but rather formal power series in infinitely-many variables. Classically, one generally prefers to study Sym; however, we will usually prefer the essentially equivalent theory of Sym_n, as it extends more naturally to the asymmetric setting that is our focus.

We will consider four of the most important additive bases of Sym_n: the monomial, elementary, homogeneous, and Schur bases. Given bases A, B of a free $\mathbb{Z}$-module, we say that A **refines** B if every element of B can be written as a sum of elements of A with nonnegative coefficients. For a much deeper exposition of symmetric function theory than we provide, see any of the textbooks [58, 59, 78], which also provide proofs that we omit. The relations among these four bases are illustrated in Fig. 1.

For a **weak composition** a (i.e., an infinite sequence of nonnegative integers with finite sum), define a monomial

$$\mathbf{x}^a := x_1^{a_1} x_2^{a_2} \cdots .$$

For a **partition** λ (i.e., a weakly decreasing weak composition), let

$$m_\lambda := \sum_a \mathbf{x}^a,$$

where the sum is over all distinct weak compositions that can be obtained by rearranging the parts of λ. If $\mathbf{x}^a$ is a monomial of the symmetric function f, then necessarily $\mathbf{x}^b$ is also a monomial of f for every b that can be obtained by rearranging the parts of a. Thus f can be written uniquely as a finite sum of the **monomial symmetric functions** m_λ. Therefore $\{m_\lambda\}$ is a $\mathbb{Z}$-linear basis of Sym and the dimension of $\mathrm{Sym}^{(m)}$ is the number of partitions of m.

We may consider a second action of $\mathcal{S}_n$ on Poly_n where a permutation acts by permuting variables and then multiplying by the sign of the permutation. Note that this merely amounts to twisting the original action by tensoring with the 1-dimensional sign representation of $\mathcal{S}_n$. The invariants of this twisted action are the **alternating polynomials in** n **variables**, $v_n\mathrm{Sym}_n$. These are precisely the polynomials where setting any two variables equal yields 0. The sum of two alternating polynomials is alternating, but the product is generally not. Hence $v_n\mathrm{Sym}_n$ is not a subring of Poly_n, although it is a module over Sym_n. As with Sym_n, $v_n\mathrm{Sym}_n$ is graded by degree, although for technical reasons one might prefer to shift the degree by $\binom{n}{2}$.

Let $v_n := \prod_{1\le i<j\le n}(x_i - x_j)$ be the **Vandermonde determinant**. This is an alternating polynomial and moreover divides every other alternating polynomial. The quotients are necessarily symmetric. Hence every alternating polynomial can be written as v_n times a symmetric polynomial. (This fact justifies the notation $v_n\mathrm{Sym}_n$ for the module of alternating polynomials.)

For a weak composition a of length n, define

$$\tilde{j}_a := \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) x^{\sigma(a)}.$$

Note that if $\mathbf{x}^a$ is a term of the alternating polynomial f, then so is every other term of $\tilde{j}_a$. Moreover if a has any repeated parts, then clearly $\tilde{j}_a = 0$. Hence $v_n\mathrm{Sym}_n$ has a natural basis of polynomials $\tilde{j}_\theta$, for θ ranging over **strict partitions**, that is partitions with distinct parts.

Every strict partition of length n may be written uniquely as $\delta + \lambda$, where $\delta = (n-1, n-2, \ldots, 0)$, λ is a partition, and the sum is componentwise. We write $j_\lambda := \tilde{j}_{\delta+\lambda}$, to obtain a basis of $v_n\mathrm{Sym}$ indexed by partitions. That is, the dimension of the space of alternating polynomials of degree $m + \binom{n}{2}$ equals the dimension of the space of symmetric polynomials of degree m. Indeed, we can even identify the isomorphism; it is just multiplication by $v_n = \tilde{j}_\delta = j_{(0)}$. If we shifted the grading of $v_n\mathrm{Sym}_n$ as suggested above (so that v_n is in degree 0), then multiplication by v_n is an isomorphism $\mathrm{Sym}_n \to v_n\mathrm{Sym}_n$ of *graded* Sym-modules.

The basis of Sym_n obtained by pulling back the j_λ basis of $v_n\mathrm{Sym}_n$ is not the basis of monomial symmetric polynomials, but rather something more interesting. These important objects

$$s_\lambda := \frac{j_\lambda}{v_n}$$

are called the **Schur polynomials**.

Although the Schur polynomials are clearly symmetric and hence can be expanded in the monomial basis, it is remarkable that the non-zero expansion coefficients are uniformly *positive*. A recurring theme in this survey will be such instances of positive basis changes between *a priori* unrelated bases.

A combinatorial formula manifesting the monomial-positivity of Schur polynomials was given by Littlewood. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we identify λ with its English-orientation Young diagram, consisting of λ_1 left-justified boxes in the top row, λ_2 left-justified boxes in the second row, etc. A **semistandard (Young) tableau** of shape λ is an assignment of a positive integer to each box of the Young diagram such that the labels weakly increase left to right across rows and strictly increase down columns. The **weight** of a tableau T is the weak composition $\mathrm{wt}(T) := (a_1, a_2, \ldots)$, where a_i records the number of boxes labeled i.

Theorem 2.1 (Littlewood, [47, 48]) *For any partition λ, we have*

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^{\mathrm{wt}(T)}.$$

□

Example 2.2 We have $s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$, owing to the two semistandard tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

◇

We now turn to the last two bases of Sym_n that we will consider. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we define the **elementary symmetric function** e_λ by

$$e_\lambda := \prod_i s_{\lambda_i}$$

and the **(complete) homogeneous symmetric function** by

$$h_\lambda := \prod_i s_{1^{\lambda_i}}.$$

It is not obvious that either of these families yields a basis of Sym_n; nevertheless, each of them does, as was originally established by Isaac Newton. It also is not obvious that, like the Schur basis, the elementary and homogeneous bases expand positively in the m_λ. However, in fact, something much stronger is true: each e_λ and each h_λ is a positive sum of Schur polynomials. This positivity is a consequence of an even more remarkable positivity property:

Theorem 2.3 *The Schur basis of* Sym_n *has positive structure coefficients. In other words, for any partitions* λ *and* μ*, the product* $s_\lambda \cdot s_\mu$ *expands as a positive sum of Schur polynomials.*

Corollary 2.4 *For any* λ*,* e_λ *and* h_λ *are both Schur-positive, and hence monomial-positive.*

Proof By Theorem 2.3, any product of Schur polynomials is Schur-positive. Since e_λ and h_λ are defined as products of special Schur polynomials, they are then Schur-positive. By Theorem 2.1, Schur polynomials are monomial-positive. Hence, any Schur-positive polynomial, in particular e_λ or h_λ, is also monomial-positive. □

By commutativity and the definitions, it is transparent that both the elementary and homogeneous bases of Sym_n also have positive structure coefficients.

There are a variety of distinct ways to establish Theorem 2.3. The most fundamental explanations involve interpreting the theorem representation-theoretically or geometrically.

For a representation-theoretic approach, one can establish that Sym is isomorphic to the ring of polynomial representations of the general linear group in such a way that the Schur functions are in one-to-one correspondence with the irreducible representations. Under this identification, decomposing the tensor product of two irreducible representations into irreducibles corresponds to expanding the product of two Schur functions in the Schur basis. Hence, Theorem 2.3 follows. Similarly, one can identify the Schur functions of homogeneous degree k with the irreducible representations of the symmetric group $\mathcal{S}_n$ in such a way that multiplying Schur functions corresponds to taking an 'induction product' of the corresponding representations. Again, since the induction product representation is necessarily a direct sum of irreducible representations, we recover Theorem 2.3. For more details on these representation-theoretic proofs, see, e.g., [23, 59].

A geometric approach is to identify Sym with the Chow ring of complex Grassmannians, the classifying spaces for complex vector bundles. A Grassmannian comes with a natural cell decomposition by certain subvarieties called *Schubert varieties*, yielding an effective basis of the Chow ring. Under the identification with Sym, this basis corresponds to the Schur polynomials. Multiplying Schur polynomials then corresponds to intersection product on Schubert varieties, and again Theorem 2.3 follows. For more details on these geometric constructions, see, e.g., [23, 25, 59].

The above interpretations of Theorem 2.3 deepen its significance and provide relatively easy proofs. Nonetheless, Theorem 2.3 is on its face a purely combinatorial statement and so one might hope it also had a purely combinatorial proof. Indeed, such a proof exists. Even better, it gives an explicit transparently-positive formula for the positive integers appearing in the Schur expansion. This formula can then be combined with the algebraic and geometric interpretations above to compute with and to better understand aspects of representation theory and enumerative geometry.

(Indeed, the combinatorics described here can be directly interpreted in terms of representations of *quantum groups* via M. Kashiwara's theory of *crystal bases* [35–37]; see [12] for an excellent combinatorially-flavored introduction to these connections.)

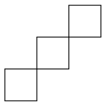
We write $\lambda \subseteq \nu$ to mean that the Young diagram of the partition λ is a subset of that for ν. The set-theoretic difference is called the **skew Young diagram** ν/λ. A **skew semistandard tableau** is a filling of a skew diagram by positive integers, such that rows weakly increase and columns strictly increase. Define the content of a skew tableau as for tableaux of partition shape. The **reading word** of a (skew) tableau T is the word given by reading the rows of T from top to bottom and from right to left (like the ordinary reading order in Arabic or Hebrew). We say that T is **Yamanouchi** if every initial segment of its reading word contains at least as many is as $(i+1)$s, for each positive integer i.

Theorem 2.5 (Littlewood–Richardson rule) *For partitions λ and μ, we have*

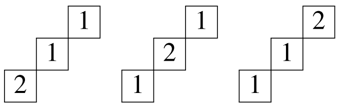
$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu,$$

where $c^\nu_{\lambda,\mu}$ counts the number of Yamanouchi semistandard tableaux of skew shape ν/λ and content μ.

Example 2.6 To compute the structure coefficient $c^{(3,2,1)}_{(2,1),(2,1)}$ via Theorem 2.5, we consider fillings of the skew shape $(3, 2, 1)/(2, 1)$:



using two 1s and one 2. There are three such fillings



all of which are semistandard. However, only the first two are Yamanouchi, as the reading word of the third is 211, which has a 2 before any 1. Hence, $c^{(3,2,1)}_{(2,1),(2,1)} = 2$. $\diamondsuit$

3 The Quasisymmetric World

In this section, we consider a third action of $\mathcal{S}_n$ on Poly_n. Here, the simple transposition $(i, i+1)$ acts on $f \in \mathrm{Poly}_n$ by swapping the variables x_i and x_{i+1} *in only those terms where at most one of the two variables appears.* The invariants of this action are the subalgebra QSym_n of **quasisymmetric polynomials**. Equivalently, quasisymmetric polynomials are those polynomials f such that, for all $a_1, \ldots, a_k$, the coefficient in f of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ equals the coefficient in f of $x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$, whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. Analogously to the symmetric case, one can also define the ring QSym of **quasisymmetric functions** in infinitely-many variables as the inverse limit of the QSym_n, but again our focus is on the essentially equivalent finite-variable case. For a much more detailed survey than we provide here of the state of the art in quasisymmetric function theory, see [61].

We will consider three important bases of QSym_n: the monomial, fundamental and quasiSchur bases. The relations among these three bases are illustrated in Fig. 2.

A **strong composition** α is a finite sequence of positive integers; we identify α with the weak composition obtained by appending infinitely many 0s to the end of α. For any weak composition a, its **positive part** is the strong composition a^+ given by deleting all 0s.

For any strong composition α, define the **monomial quasisymmetric polynomial** M_α by

$$M_\alpha(x_1, \ldots, x_n) := \sum_b \mathbf{x}^b \in \mathrm{QSym}_n,$$

where the sum is over all weak compositions b with $b^+ = \alpha$ and whose entries after position n are all zero. Clearly, the monomial quasisymmetric polynomials yield a basis of QSym_n.

Example 3.1 We have

$$M_{13}(x_1, x_2, x_3) = \mathbf{x}^{130} + \mathbf{x}^{103} + \mathbf{x}^{013} \in \mathrm{QSym}_3.$$

Note in particular that this polynomial is not an element of Sym_3. $\diamond$

It is clear that the monomial basis of QSym_n must have positive structure coefficients, as does the monomial basis of Sym_n discussed in Sect. 2. However, these structure coefficients are slightly more interesting than those for the monomial symmetric functions. They are given by the overlapping shuffles of M. Hazewinkel [29], which we now recall.

Let A and B be words in disjoint alphabets with A of length m and B of length n. An **overlapping shuffle** of A and B is a surjection

$$t : \{1, 2, \ldots, m+n\} \to \{1, 2, \ldots, k\}$$

(for some $\max\{m, n\} \leq k \leq m+n$) such that

$$t(i) < t(j) \text{ whenever } i < j \leq m \text{ or } m < i < j.$$

We write $A \sqcup\!\sqcup_o B$ for the **overlapping shuffle product** of A and B, the formal sum of all overlapping shuffles. The overlapping shuffle product $\alpha \sqcup\!\sqcup_o \beta$ of two strong compositions α and β is given by treating the strong compositions as words in disjoint alphabets. Here, we identify an overlapping shuffle $t : \{1, 2, \ldots, m+n\} \to \{1, 2, \ldots, k\}$ of α and β with the strong composition γ defined by

$$\gamma_i := \sum_{t(j)=i} (\alpha\beta)_j,$$

where $\alpha\beta$ denotes the concatenation of the two strong compositions.

Example 3.2 We compute the overlapping shuffle product of (2) and (1, 2):

$$(2) \sqcup\!\sqcup_o (1,2) = (2,1,2) + 2 \cdot (1,2,2) + (3,2) + (1,4).$$

$\diamond$

Although the relevant combinatorial construction was somewhat involved, the following positive multiplication formula is now essentially clear.

Theorem 3.3 *For strong compositions α and β, we have*

$$M_\alpha \cdot M_\beta = \sum_\gamma c^\gamma_{\alpha,\beta} M_\gamma,$$

where $c^\gamma_{\alpha,\beta}$ is the multiplicity of γ in the overlapping shuffle product $\alpha \sqcup\!\sqcup_o \beta$.

Example 3.4 To compute $M_{(2)} \cdot M_{(1,2)}$ via Theorem 3.3, we compute the overlapping shuffle product of (2) and (1, 2) as in Example 3.2. Then, the coefficients on the various strong compositions give the coefficients on the various monomial quasisymmetric polynomials in the product:

$$M_{(2)} \cdot M_{(1,2)} = M_{(2,1,2)} + 2M_{(1,2,2)} + M_{(3,2)} + M_{(1,4)}.$$

$\diamond$

Given two strong compositions α and β, we say β refines α and write $\beta \vDash \alpha$ if α can be obtained by summing *consecutive* entries of β, e.g. $(1, 2, 1) \vDash (1, 3)$ but $(2, 1, 1) \nvDash (1, 3)$.

Define the **fundamental quasisymmetric polynomial** F_α by

$$F_\alpha(x_1, \ldots, x_n) := \sum_b \mathbf{x}^b,$$

where the sum is over all distinct weak compositions b with $b^+ \vDash \alpha$ and whose entries after position n are all zero.

Example 3.5 We have

$$F_{13}(x_1, x_2, x_3) = \mathbf{x}^{130} + \mathbf{x}^{103} + \mathbf{x}^{013} + \mathbf{x}^{112} + \mathbf{x}^{121}.$$

$\diamond$

Theorem 3.6 *The fundamental quasisymmetric polynomials expand positively in the monomial quasisymmetric polynomials:*

$$F_\alpha(x_1, \ldots, x_n) = \sum_{\beta \vDash \alpha} M_\beta(x_1, \ldots, x_n).$$

A sign of the fundamental nature of the fundamental quasisymmetric polynomials is that they also have positive structure coefficients. Their multiplication is also governed by a shuffle product, that of S. Eilenberg and S. Mac Lane [19]. Let A and B be words in the disjoint alphabets $\mathcal{A}$ and $\mathcal{B}$, respectively. Recall a **shuffle** of A and B is a permutation of the concatenation AB such that the subword on the alphabet $\mathcal{A}$ is A and the subword on $\mathcal{B}$ is B. Alternatively, if A has length m and B has length n, we can think of a shuffle of A and B as a bijection

$$s : \{1, 2, \ldots, m+n\} \to \{1, 2, \ldots, m+n\}$$

such that

$$s(i) < s(j) \text{ whenever } i < j \leq m \text{ or } m < i < j.$$

The shuffle product of two strong compositions α and β is obtained as follows. Let $\mathcal{A}$ denote the alphabet of odd integers and let $\mathcal{B}$ denote the alphabet of even integers. Let A be the word in $\mathcal{A}$ consisting of α_1 copies of $2\ell(\alpha) - 1$, followed by α_2 copies of $2\ell(\alpha) - 3$, all the way to $\alpha_{\ell(\alpha)}$ copies of 1. Likewise, let B denote the word in $\mathcal{B}$ consisting of β_1 copies of $2\ell(\beta)$, followed by β_2 copies of $2\ell(\beta) - 2$, all the way to $\beta_{\ell(\beta)}$ copies of 2. Let $\mathrm{Sh}(A, B)$ denote the set of the $\binom{|\alpha|+|\beta|}{|\beta|}$ shuffles of A and B. For each $C \in \mathrm{Sh}(A, B)$, let $\mathrm{Des}(C)$ denote the **descent composition** of C, i.e. the strong composition obtained by decomposing C into maximal runs of increasing entries and letting $\mathrm{Des}(C)_i$ be the number of entries in the ith increasing run of C. Finally, define the **shuffle product** $\alpha \sqcup\!\sqcup\, \beta$ of the strong compositions α and β as the formal sum of strong compositions

$$\alpha \sqcup\!\sqcup\, \beta := \sum_{C \in \mathrm{Sh}(A,B)} \mathrm{Des}(C).$$

Example 3.7 Let $\alpha = (2)$ and $\beta = (1, 2)$. Then $A = 11$ and $B = 422$. We compute the set of shuffles of A and B:

$$\begin{aligned}\mathrm{Sh}(A, B) = \{&4|22|11, 4|2|12|1, 4|122|1, 14|22|1, 4|2|112,\\ &4|12|12, 14|2|12, 4|1122, 14|122, 114|22\},\end{aligned}$$

where we have placed bars to indicate the decomposition of each shuffle into maximally increasing runs. The corresponding descent compositions are thus, respectively,

$$\{(1,2,2),(1,1,2,1),(1,3,1),(2,2,1),(1,1,3),(1,2,2),(2,1,2),(1,4),(2,3),(3,2)\}.$$

Hence, we have

$$\begin{aligned}(2) \sqcup\!\sqcup (1,2) = 2(1,2,2) &+ (1,1,2,1) + (1,3,1) + (2,2,1)\\ &+ (1,1,3) + (2,1,2) + (1,4) + (2,3) + (3,2).\end{aligned}$$

Note that this sum is not multiplicity-free. $\diamond$

Theorem 3.8 *For strong compositions α and β, we have*

$$F_\alpha \cdot F_\beta = \sum_\gamma c^\gamma_{\alpha,\beta} F_\gamma,$$

where $c^\gamma_{\alpha,\beta}$ is the multiplicity of γ in the ordinary shuffle product $\alpha \sqcup\!\sqcup \beta$.

Proving this theorem is given as Exercise 7.93 in [78].

Example 3.9 To compute $F_{(2)} \cdot F_{(1,2)}$ via Theorem 3.8, we compute the shuffle product of (2) and $(1, 2)$ as in Example 3.7. Then, the coefficients on the various strong compositions give the coefficients on the various fundamental quasisymmetric polynomials in the product:

$$F_{(2)} \cdot F_{(1,2)} = 2F_{(1,2,2)} + F_{(1,1,2,1)} + F_{(1,3,1)} + F_{(2,2,1)} + F_{(1,1,3)} + F_{(2,1,2)} + F_{(1,4)} + F_{(2,3)} + F_{(3,2)}.$$ $\diamond$

The final basis for QSym_n that we will consider is the basis of quasiSchur polynomials introduced in [30]. For a detailed and readable survey of work related to this basis, see [50]. For those unfamiliar with quasiSchur polynomials, the definition may appear quite complicated; it originates as an important and tractable piece of the theory of *Macdonald polynomials*. It is not transparent from this definition that the quasiSchur polynomials are quasisymmetric, much less that they yield a basis of QSym_n.

First, we must extend the definition of the Young diagram of a partition to a general weak composition $a = (a_1, a_2, \ldots)$: Draw a_i left-justified boxes in row i. (Here, in accordance with our English orientation on Young diagrams for partitions, row 1 is the top row.) A **(composition) tableau** of shape a is an assignment of a positive integer to each box of the Young diagram for a. (Sometimes, we will augment such a tableau with an extra "zeroth" column of boxes (called the **basement**) immediately left of the first column, and write b_i for the positive integer labeling the basement box in row i. Basement entries do not contribute to the weight of a tableau.)

A **triple** of boxes in a composition tableau T is a set of three boxes in one of the two following configurations:

$$\begin{array}{cc} \boxed{Z}\boxed{X} \\ \quad\vdots \\ \quad\boxed{Y} \end{array} \qquad\qquad\qquad \begin{array}{cc} \boxed{Y} \\ \vdots \\ \boxed{Z}\boxed{X} \end{array}$$

upper row weakly longer — upper row strictly shorter

Note, in particular, that a triple has exactly two boxes sharing a row and exactly two boxes sharing a column. We say a triple is **inversion** if it is *not* the case that its labels satisfy $X \leq Y \leq Z$.

A composition tableau is **semistandard** if

(S.1) entries do not repeat in a column,
(S.2) rows weakly decrease from left to right,
(S.3) every triple is inversion,
(S.4) entries in the first column equal their row indices.

(Note that, in the case of partition shape, this definition unfortunately does not coincide with the definition of semistandard tableaux we have given previously.) For a weak composition a, let $\mathfrak{A}\mathsf{SST}(a)$ denote the set of semistandard tableaux of shape a. The **quasiSchur polynomial** for the strong composition α is then given by

$$S_\alpha(x_1, \ldots, x_n) = \sum_{a^+=\alpha} \sum_{T \in \mathfrak{A}\mathsf{SST}(a)} \mathbf{x}^{\mathrm{wt}(T)}, \tag{3.1}$$

where the first sum is over all weak compositions a of length n with positive part α.

Example 3.10 For $\alpha = (1, 3)$ and $n = 3$, we have

$$S_{(1,3)}(x_1, x_2, x_3) = \mathbf{x}^{130} + \mathbf{x}^{220} + \mathbf{x}^{103} + \mathbf{x}^{202} + 2\mathbf{x}^{112} + \mathbf{x}^{121} + \mathbf{x}^{211} + \mathbf{x}^{013} + \mathbf{x}^{022},$$

where the monomials are determined by the semistandard composition tableaux shown in Fig. 3. ◇

From the given definition of quasiSchur polynomials, it is not clear that they are natural objects that we should expect to exhibit any nice properties. Nonetheless, they participate in two beautiful positive combinatorial rules for basis expansion.

Since $\mathrm{Sym}_n \subset \mathrm{QSym}_n$, we can ask how bases of Sym_n expand in bases of QSym_n. First, observe the following straightforward formula for the M_α-expansion of the monomial symmetric polynomial m_λ. For a weak composition a, we write $\overleftarrow{a}$ for the partition formed by sorting the entries of a into weakly decreasing order.

Proposition 3.11 *The monomial symmetric polynomials expand positively in the monomial quasisymmetric polynomials:*

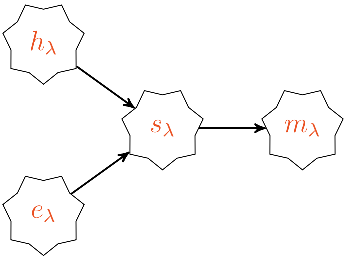


Fig. 1 The four bases of Sym_n considered here. The arrows denote that the basis at the head refines the basis at the tail. All four bases have positive structure coefficients

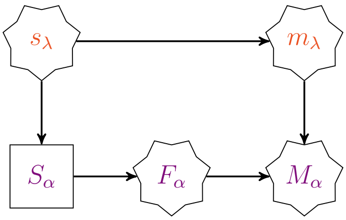


Fig. 2 The three bases of QSym_n considered here, together with some bases of Sym_n from Fig. 1. The arrows denote that the basis at the head refines the basis at the tail. The star-shaped nodes have positive structure coefficients

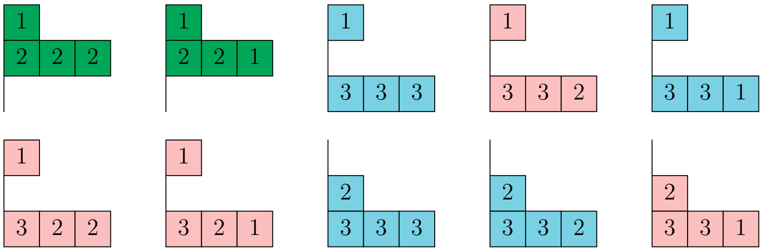


Fig. 3 The 10 semistandard composition tableaux associated to the quasiSchur polynomial $\mathcal{S}_{(1,3)}(x_1, x_2, x_3)$. The quasiYamanouchi tableaux are shaded in blue, the initial tableaux in pink, and those that are both quasiYamanouchi and initial in green

$$m_\lambda(x_1, \ldots, x_n) = \sum_{\overleftarrow{\alpha}=\lambda} M_\alpha(x_1, \ldots, x_n).$$

The quasiSchur expansion of a Schur polynomial is beautifully parallel to the formula of Proposition 3.11.

Theorem 3.12 ([30]) *The Schur polynomials expand positively in the quasiSchur polynomials:*

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\overleftarrow{\alpha}=\lambda} \mathcal{S}_\alpha(x_1, \ldots, x_n).$$

Remark 3.13 Considering Fig. 2 together with Proposition 3.11 and Theorem 3.12, one might be tempted to define polynomials

$$f_\lambda(x_1, \dots, x_n) = \sum_{\overleftarrow{\alpha}=\lambda} F_\alpha(x_1, \dots, x_n).$$

Extrapolating from Fig. 2, it might appear plausible that $\{f_\lambda\}$ should form a basis of Sym_n, perhaps even with positive structure coefficients. However, the polynomials f_λ are in general not even symmetric!

For example, in four or more variables, we have by Theorem 3.6 and Proposition 3.11 that

$$\begin{aligned} f_{31} &= F_{31} + F_{13} \\ &= M_{31} + M_{211} + 2M_{121} + M_{112} + M_{13} + 2M_{1111} \\ &= m_{31} + m_{211} + 2m_{1111} + M_{121}, \end{aligned}$$

a symmetric polynomial plus M_{121}.

To describe the expansion of quasiSchur polynomials into the fundamental basis, we isolate an important subclass of semistandard composition tableaux. Fix a strong composition α and consider $T \in \mathfrak{A}\mathsf{SST}(a)$ for some a with $a^+ = \alpha$. We say that T is **quasiYamanouchi** if for every integer i appearing in T, either

- an i appears in the first column, or
- there is an $i+1$ weakly right of an i.

We say T is **initial** if the set of integers i appearing in T is an initial segment of $\mathbb{Z}_{>0}$.

Theorem 3.14 *The quasiSchur polynomials expand positively in the fundamental quasisymmetric polynomials:*

$$S_\alpha(x_1, \dots, x_n) = \sum_T F_{\mathrm{wt}(T)}(x_1, \dots, x_n),$$

where the sum is over all initial quasiYamanouchi tableaux T such that $T \in \mathfrak{A}\mathsf{SST}(a)$ for some a with $a^+ = \alpha$.

A positive formula for the expansion of quasiSchur polynomials in fundamental quasisymmetric polynomials was first given in [30] in terms of *standard augmented fillings*. The formula in Theorem 3.14 above follows as a consequence of a result in [77]; we state the expansion in these terms for consistency with formulas in the upcoming sections.

Remark 3.15 Unlike the other two bases of QSym_n that we have considered, the quasiSchur basis does not have positive structure coefficients. For an example, see [30, Sect. 7.1]. However, [31] proves a slightly weaker form of positivity, giving a positive combinatorial formula for the quasiSchur expansion of the product of a quasiSchur polynomial by a *Schur* polynomial.

Just as the combinatorics of Sym_n is related to the representation theory of symmetric groups, the combinatorics of QSym_n turns out to be related to the representation theory of *0-Hecke algebras* (in type A). First, let us recall the standard Coxeter presentation of the symmetric group $\mathcal{S}_n$. It is easy to see that $\mathcal{S}_n$ is generated by the simple transpositions $s_i := (i, i+1)$ for $1 \leq i < n$. With a little more effort, one establishes that a generating set of relations is given by

- $s_i^2 = \mathrm{id}$,
- $s_i s_j = s_j s_i$ for $|i - j| > 1$, and
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

The 0-Hecke algebra $\mathcal{H}_n$ is the unital associative algebra over $\mathbb{C}$ defined by a very similar presentation: $\mathcal{H}_n$ is generated by symbols σ_i (for $1 \leq i < n$) subject to

- $\sigma_i^2 = \sigma_i$,
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$, and
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

That is, the $\{\sigma_i\}$ in $\mathcal{H}_n$ act exactly like the corresponding $\{s_i\}$ in $\mathcal{S}_n$, except that they are idempotent instead of being involutions.

The representation theory of $\mathcal{H}_n$ was first worked out in detail by P. Norton [65]. Despite the similarly between the descriptions of $\mathcal{S}_n$ and $\mathcal{H}_n$, their representation theory is rather different, as $\mathcal{H}_n$ is not semisimple. Indeed, the irreducible representations of $\mathcal{H}_n$ are all 1-dimensional, while $\mathcal{S}_n$ has irreducible representations of higher dimension. The irreducible representations of $\mathcal{H}_n$ are equinumerous with the set of compositions $\alpha \vDash (n)$.

There is a *quasisymmetric Frobenius character map* [18, 43] taking 0-Hecke-representations to quasisymmetric functions in such a way that the irreducible representations map to the fundamental quasisymmetric functions F_α. In this way, if the quasisymmetric function f corresponds to the representation M, then decomposing f as a sum of fundamental quasisymmetric functions corresponds to identifying the unique direct sum of irreducible 0-Hecke representations that is equivalent to M in the Grothendieck group of finite-dimensional representations. Certain explicit and combinatorial $\mathcal{H}_n$-representations are known whose quasisymmetric Frobenius characters are precisely the quasiSchur functions [79]; unfortunately, these representations are not generally indecomposable.

The geometry of QSym_n is much less well understood. In addition to its obvious product structure, QSym also possesses a compatible coproduct, turning it into a *Hopf algebra*. Although we won't describe it here, there is an important Hopf algebra NSym of *noncommutative symmetric functions* that is Hopf-dual to QSym. (For background on combinatorial Hopf algebras, see [27].) It is surprisingly easy to see that NSym is isomorphic to the homology of the loop space of the suspension of $\mathbb{CP}^\infty$, where the product structure on $H_\star(\Omega\Sigma\mathbb{CP}^\infty)$ is given by concatenation of loops [10]. Indeed, this isomorphism even holds on the level of Hopf algebras. Since $\Omega\Sigma\mathbb{CP}^\infty$ is an H-space, its homology and cohomology are dual Hopf algebras. (See, for example, [28, 85] for background on H-spaces and their associated Hopf algebras.) From this fact and the fact that QSym is Hopf-dual to NSym, it follows that $H^\star(\Omega\Sigma\mathbb{CP}^\infty)$

is isomorphic to QSym. This interpretation was used in [10] to give cohomological proofs of various properties of QSym; however, it seems that much more could be done from this perspective. A recent construction, which appears closely related, identifies QSym with the Chow ring of an algebraic stack of *expanded pairs* [66].

4 The Asymmetric World

In this section, we consider our fourth and final action of $\mathcal{S}_n$ on Poly_n, the trivial action. Although this action is trivial the associated combinatorics is not at all trivial, but full of rich internal structure and deep connections to geometry and representation theory. The invariant ring of this trivial action is, of course, the ring Poly_n itself. However, to emphasize analogies with the previous two sections, we will refer to the invariant ring Poly_n in this context as the **ring of asymmetric functions** ASym_n. That is, we write ASym_n for the ring $\mathbb{Z}[x_1, \ldots, x_n]$ when we think of it as an $\mathcal{S}_n$-module with the trivial $\mathcal{S}_n$ action, and we write Poly_n for the same ring considered as an object in the category of rings.

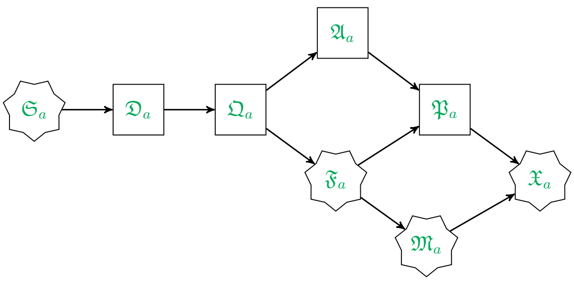
Bases of ASym_n are indexed by weak compositions a of length at most n, with the most obvious basis of ASym_n being given by individual monomials:

$$\mathfrak{X}_a := \mathbf{x}^a.$$

Just as $\{m_\lambda\}$ is not the most interesting basis of Sym_n, the $\{\mathfrak{X}_a\}$ basis of ASym_n is not very interesting either! Our goal in this section is to explore seven additional bases of rather less trivial nature.

Arguably, the most interesting basis of ASym_n is given by the *Schubert polynomials* of A. Lascoux and M.-P. Schützenberger [52]. We will use consideration of various formulas for Schubert polynomials to organize our discussion of the bases of Fig. 4. Instead of indexing Schubert polynomials by weak compositions, it is more convenient to index them by permutations. Hence, we first recall a standard way to translate between permutations and weak compositions. For a permutation $\pi \in \mathcal{S}_n$, let a_i denote the number of integers $j > i$ such that $w(i) > w(j)$. (Note that

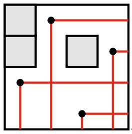
Fig. 4 The eight bases of ASym_n considered here. The arrows denote that the basis at the head refines the basis at the tail. The star-shaped nodes have positive structure coefficients



$a_i \leq n - i$.) The weak composition $a_\pi = (a_1, a_2, \ldots, a_n)$ is called the **Lehmer code** of π. Visually, one may determine the Lehmer code of a permutation π as follows.

Consider an $n \times n$ grid of boxes and place a laser gun (or dot) in each position $(i, \pi(i))$. Each laser gun fires to the right and down, destroying all boxes directly to its right and all boxes directly below itself (including its own box). The surviving boxes are the **Rothe diagram** $RD(\pi)$ of the permutation π. One checks that the Lehmer code of π records the number of boxes in each row of $RD(\pi)$.

Example 4.1 Let $\pi = 2413$. The Rothe diagram $RD(\pi)$ is shown below

,

where the surviving boxes are shaded in grey. Hence, the Lehmer code of $\pi = 2413$ is $(1, 2, 0, 0)$. $\diamond$

Consider the action of $\mathcal{S}_n$ on Poly_n from Sect. 2, where permutations act by permuting variables. Now, for each positive integer, define an operator ∂_i on Poly_n by

$$\partial_i(f) := \frac{f - s_i \cdot f}{x_i - x_{i+1}}.$$

Note that $\partial_i(f)$ is symmetric in the variables x_i and x_{i+1}. Now, for each permutation w of the form $n(n-1)\cdots 321$ (in one-line notation), the **Schubert polynomial** $\mathfrak{S}_w$ is defined to be

$$\mathfrak{S}_w := \prod_{i=1}^{n} x_i^{n-i} = \mathfrak{X}_{(n-1,n-2,\ldots,1,0)}. \tag{4.1}$$

(These permutations are exactly those that are longest in Coxeter length in $\mathcal{S}_n$ for some n, i.e., they have the largest possible number of inversions; their Lehmer code is $(n-1, n-2, \ldots, 1, 0)$.) For other permutations w, the corresponding Schubert polynomials are defined recursively by

$$\mathfrak{S}_w := \partial_i \mathfrak{S}_{w(i,i+1)},$$

for any i such that $w(i) < w(i+1)$. Amazingly, this recursive definition is self-consistent, so there is a uniquely defined Schubert polynomial $\mathfrak{S}_w$ for each permutation w.

Our first task in this section will be to obtain a more concrete understanding of Schubert polynomials by describing how to write them non-recursively in the monomial basis $\{\mathfrak{X}_a\}$. We'll describe three different combinatorial formulas for this expansion, exploring some other families of polynomials along the way.

The first such formula to be proven was given by S. Billey et al. [9]. For a permutation π, a **reduced factorization** of π is a way of writing π as a product $s_{i_1} s_{i_2} \cdots s_{i_k}$

of simple transpositions with k as small as possible. The sequence of subscripts $i_1 i_2 \cdots i_k$ is called a **reduced word** for π. We write $\mathrm{Red}(\pi)$ for the set of all reduced words of the permutation π. Note that every reduced word α is a strong composition. Given two strong compositions α and β, we say that β is α-**compatible** if

(R.1) α and β have the same length,
(R.2) β is weakly increasing (i.e., $\beta_i \leq \beta_j$ for $i < j$),
(R.3) β is bounded above by α (i.e., $\beta_i \leq \alpha_i$ for all i), and
(R.4) β strictly increases whenever α does (i.e., if $\alpha_i < \alpha_{i+1}$, then $\beta_i < \beta_{i+1}$).

In this case, we write $\beta \looparrowright \alpha$.

Example 4.2 If α is the strong composition 121 (a reduced word for the longest permutation in $\mathcal{S}_3$), then we claim that no strong composition is α-compatible. Suppose β were α-compatible. Since $\alpha_1 < \alpha_2$, we must have $\beta_1 < \beta_2$ by (R.4). Hence, $\beta_2 \geq 2$. Therefore, by (R.2), $\beta_3 \geq 2$. But this is incompatible with (R.3), since $\alpha_3 = 1$.

On the other hand, for $\gamma = 212$ (the other reduced word for this permutation), there is exactly one γ-compatible strong composition δ. By (R.3), we have $\delta_2 = 1$, and hence by (R.2) we also have $\delta_1 = 1$. By (R.4), $\delta_3 > \delta_2 = 1$, but by (R.3) $\delta_3 \leq 2$. Hence, $\delta = 112$ is the only γ-compatible strong composition. We write $112 \looparrowright 212$. ◇

Theorem 4.3 ([9, Theorem 1.1]) *The Schubert polynomials expand positively in monomials:*

$$\mathfrak{S}_\pi = \sum_{\alpha \in \mathrm{Red}(\pi)} \sum_{\beta \looparrowright \alpha} \prod_i x_{\beta_i}.$$

Example 4.4 Let $\pi = 321 = s_1 s_2 s_1 = s_2 s_1 s_2$ be the longest permutation in $\mathcal{S}_3$. Then, by Example 4.2, we have

$$\begin{aligned}
\mathfrak{S}_\pi &= \sum_{\alpha \in \mathrm{Red}(\pi)} \sum_{\beta \looparrowright \alpha} \prod_i x_{\beta_i} \\
&= \sum_{\beta \looparrowright 121} \prod_i x_{\beta_i} + \sum_{\delta \looparrowright 212} \prod_i x_{\delta_i} \\
&= 0 + x_1 x_1 x_2 = x_1^2 x_2 = \mathfrak{X}_{(2,1,0)}.
\end{aligned}$$

Note that this calculation is consistent with the definition given in Eq. (4.1). ◇

It might be reasonable to expect an important basis of ASym_n to restrict to an important basis of the subspace $\mathrm{Sym}_n \subset \mathrm{ASym}_n$. Indeed, one piece of evidence for the importance of Schubert polynomials is that those Schubert polynomials lying inside Sym_n are exactly its basis of Schur polynomials.

Definition 4.5 Suppose we have vector spaces $U \subseteq V$ with bases B_U and B_V, respectively. If $B_V \cap U = B_U$, we say the basis B_V **lifts** the basis B_U to V, or equivalently that B_V is a **lift** of B_U.

Since every Schur polynomial is a Schubert polynomial, in this language the Schubert basis of ASym_n is a lift of the Schur basis of Sym_n. To realize the Schur polynomial $s_\lambda \in \mathrm{Sym}_n$ as a Schubert polynomial, first realize the partition λ as a weak composition of length n by padding it by an appropriate number of final 0s. Now, reverse the letters of λ, so it becomes a weakly increasing sequence. The resulting weak composition is the Lehmer code of a unique permutation π_λ, and one has $s_\lambda = \mathfrak{S}_{\pi_\lambda}$. Equivalently, for $i \leq n$ one has $\pi_\lambda(i) = \lambda_{n-i+1} + i$ and for $i > n$ one has $\pi_\lambda(i) = \min\left(\mathbb{Z}_{>0} \setminus \{\pi_\lambda(j) : j < i\}\right)$.

Since the Schubert polynomials lift the Schur polynomials, one might wonder whether the Littlewood–Richardson rule (Theorem 2.5) also lifts to a positive combinatorial rule for the structure coefficients of the Schubert basis. Indeed, the Schubert basis of ASym_n, like the Schur basis of Sym_n, has positive structure coefficients! However, no combinatorial proof of this fact is known and we lack any sort of positive combinatorial rule (even conjectural) to describe these structure coefficients (except in a few very special cases, such as when the Schubert polynomials are actually Schur polynomials). Discovering and proving such a rule is one of the most important open problems in algebraic combinatorics. Part of our motivation for studying the combinatorial theory of ASym_n is the hope that such a theory will eventually lead to a Schubert structure coefficient rule, just as the Littlewood–Richardson rule for Schur polynomial structure coefficients eventually developed from the combinatorial theory of Sym_n.

Without such a combinatorial rule, how then do we know that the Schubert basis has positive structure coefficients? The answer comes, once again, from geometry and from representation theory. Geometrically, instead of looking at a complex Grassmannian, as we did for Schur polynomials, we should consider a complex flag variety Flags_n, the classifying space for complete flags $V_0 \subset V_1 \subset \cdots \subset V_n$ of nested complex vector bundles with V_k of rank k. This space has an analogous cell decomposition by Schubert varieties, yielding an effective basis of the Chow ring. By identifying Schubert varieties with corresponding Schubert polynomials, multiplying Schubert polynomials corresponds to the intersection product on Schubert varieties and positivity of structure coefficients follows. An alternative proof of positivity [82, 83] is given by interpreting Schubert polynomials as characters of certain *Kraśkiewicz–Pragacz modules* (introduced in [41, 42]) for Borel Lie algebras.

The formula of Theorem 4.3 naturally leads us to consider another family of polynomials. Suppose we fix a reduced word $\alpha \in \mathrm{Red}(\pi)$ for some $\pi \in \mathcal{S}_n$. Then, Theorem 4.3 suggests defining a polynomial

$$\mathfrak{F}(\alpha) := \sum_{\beta \hookleftarrow\!\!\rightarrow \alpha} \prod_i x_{\beta_i}, \tag{4.2}$$

so that Theorem 4.3 may be rewritten as

$$\mathfrak{S}_\pi = \sum_{\alpha \in \mathrm{Red}(\pi)} \mathfrak{F}(\alpha).$$

Indeed, the formula of Eq. (4.2) makes sense for any strong composition α, not necessarily a reduced word of a permutation.

Labeling these polynomials by strong compositions α is unnatural for at least two reasons. For some α, we have $\mathfrak{F}(\alpha) = 0$; for example, we have $\mathfrak{F}(1, 2, 1) = 0$ by Example 4.2. Those $\mathfrak{F}(\alpha)$ that are nonzero are called the **fundamental slide polynomials**; these were introduced in [2], although the alternate definition we give here follows [5]. Also, for $\alpha \neq \alpha'$, we can have $\mathfrak{F}(\alpha) = \mathfrak{F}(\alpha') \neq 0$; for example, by Example 4.2 we have $\mathfrak{F}(2, 1, 2) = \mathfrak{X}_{(2,1,0)}$, but it is also equally clear that $\mathfrak{F}(3, 1, 2) = \mathfrak{X}_{(2,1,0)}$.

For any strong composition α, note that, if α has any compatible sequences, then it has a unique such compatible sequence $\beta(\alpha)$ that is termwise maximal. Define the weak composition $a(\alpha)$ by letting $a(\alpha)_i$ denote the multiplicity of i in $\beta(\alpha)$. Then, we define

$$\mathfrak{F}_{a(\alpha)} := \mathfrak{F}(\alpha).$$

It is clear then that every fundamental slide polynomial is, in this fashion, uniquely indexed by a weak composition a. Moreover, every weak composition a appears as an index on some $\mathfrak{F}_a$, and we have $\mathfrak{F}_a \neq \mathfrak{F}_b$ if $a \neq b$. It is then not hard to see by triangularity in the $\mathfrak{X}_a$ basis that the set of fundamental slide polynomials forms another basis of ASym_n.

Clearly, the fundamental slide polynomials expand positively in the monomial basis $\{\mathfrak{X}_a\}$. It is useful to have a formula for this expansion of $\mathfrak{F}_a$, based only on the weak composition a. We first need a partial order on weak compositions: we write $a \geq b$ and say a **dominates** b if we have

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$$

for all k. (Note that the restriction of this partial order to the set of partitions recovers the usual notion of dominance order.)

Theorem 4.6 ([2, 5]) *The fundamental slide polynomials expand positively in monomials:*

$$\mathfrak{F}_a = \sum_{\substack{b \geq a \\ b^+ \vDash a^+}} \mathfrak{X}_b$$

Essentially by definition, the fundamental slide polynomials are pieces of Schubert polynomials. Although the basis of Schubert polynomials has positive structure constants, there is no reason to expect this property to descend to this basis of pieces. Remarkably, however, the fundamental slide polynomials also have positive structure constants! The first clue that this might be the case comes from considering the intersection of the fundamental slide basis with the subring $\mathrm{QSym}_n \subset \mathrm{ASym}_n$.

Theorem 4.7 ([2]) *We have* $\mathfrak{F}_a \in \mathrm{QSym}_n$ *if and only if* a *is of the form* $0^k\alpha$, *where* α *is a strong composition of length* $n-k$ *and* $0^k\alpha$ *denotes the weak composition obtained from* α *by prepending* k 0*s.*

Moreover, we have $\mathfrak{F}_{0^k\alpha} = F_\alpha$. *Thus, the fundamental slide basis of* ASym_n *is a lift of the fundamental quasisymmetric polynomial basis of* QSym_n.

In light of Theorem 4.7, one might hope to extend the combinatorial multiplication rule of Theorem 3.8 to fundamental slide polynomials. Indeed, this is possible. We need to extend the notion of the shuffle product of two strong compositions from Sect. 3 to the *slide product* or pairs of weak compositions a, b. Here, we borrow notation from [71]. As before, let $\mathcal{A}$ denote the alphabet of odd integers and let $\mathcal{B}$ denote the alphabet of even integers. Let A be the word in $\mathcal{A}$ consisting of a_1 copies of $2\ell(a)-1$, followed by a_2 copies of $2\ell(a)-3$, all the way to $a_{\ell(a)}$ copies of 1. Likewise, let B denote the word in $\mathcal{B}$ consisting of b_1 copies of $2\ell(b)$, followed by b_2 copies of $2\ell(b)-2$, all the way to $b_{\ell(b)}$ copies of 2.

For any word W in a totally ordered alphabet $\mathcal{Z}$, let $\mathrm{Runs}(W)$ denote the sequence of successive maximally weakly increasing runs of letters of W read from left to right. For a sequence S of words in $\mathcal{Z}$ and any subalphabet $\mathcal{Y} \subseteq \mathcal{Z}$, write $\mathrm{Comp}_{\mathcal{Y}}(S)$ for the weak composition whose ith coordinate is the number of letters of $\mathcal{Y}$ in the ith word of S.

Let $\mathrm{Sh}(a, b)$ denote the set of those shuffles C of A and B such that

$$\mathrm{Comp}_{\mathcal{A}}(\mathrm{Runs}(C)) \geq a \text{ and } \mathrm{Comp}_{\mathcal{B}}(\mathrm{Runs}(C)) \geq b.$$

For $C \in \mathrm{Sh}(a, b)$, let $\mathrm{BumpRuns}(C)$ denote the unique dominance-minimal way to insert words of length 0 into $\mathrm{Runs}(C)$ while preserving $\mathrm{Comp}_{\mathcal{A}}(\mathrm{BumpRuns}(C)) \geq a$ and $\mathrm{Comp}_{\mathcal{B}}(\mathrm{BumpRuns}(C)) \geq b$. Finally, define the **slide product** $a \sqcup b$ of the weak compositions a and b as the formal sum of weak compositions

$$a \sqcup b := \sum_{C \in \mathrm{Sh}(a,b)} \mathrm{Comp}_{\mathbb{Z}}(\mathrm{BumpRuns}(C)).$$

Example 4.8 Let $a = (0, 1, 0, 2)$ and $b = (1, 0, 0, 1)$. Then we consider the words $A = 511$ and $B = 82$. The set of all shuffles of A and B is

$$\{51182, 51812, 58112, 85112, 51821, 58121, 85121, 58211, 85211, 82511\}.$$

Many of these shuffles C fail $\mathrm{Comp}_{\mathcal{B}}(\mathrm{Runs}(C)) \geq b$; for example, with $C = 51821$, we have $\mathrm{Runs}(C) = (5, 18, 2, 1)$ and hence $\mathrm{Comp}_{\mathcal{B}}(\mathrm{Runs}(C)) = (0, 1, 1, 0) \not\geq b$. Thus we have

$$\mathrm{Sh}(a, b) = \{58112, 85112, 58121, 85121, 58211, 85211, 82511\}.$$

The corresponding $\mathrm{BumpRuns}(C)$ for $C \in \mathrm{Sh}(a, b)$ are

$$\{(58, \epsilon, \epsilon, 112), (8, 5, \epsilon, 112), (58, \epsilon, 12, 1), (8, 5, 12, 1), (58, \epsilon, 2, 11), (8, 5, 2, 11), (8, 25, \epsilon, 11)\},$$

where ϵ denotes the empty word. Thus, we have

$$(0,1,0,2) \sqcup\!\sqcup (1,0,0,1) = (2,0,0,3) + (1,1,0,3) + (2,0,2,1) + (1,1,2,1) + (2,0,1,2) + (1,1,1,2) + (1,2,0,2).$$

◇

Finally, we can state the multiplication rule for fundamental slide polynomials.

Theorem 4.9 ([2]) *For weak compositions a and b, we have*

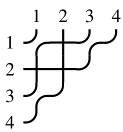
$$\mathfrak{F}_a \cdot \mathfrak{F}_b = \sum_c C^c_{a,b} \mathfrak{F}_c,$$

where $C^c_{a,b}$ is the multiplicity of c in the slide product $a \sqcup\!\sqcup b$.

It seems reasonable to expect that the positivity of Theorem 4.9 reflects some geometry or representation theory governed by fundamental slide polynomials. Sadly, no such interpretation of fundamental slide polynomials is known, except in the quasisymmetric case.

A second formula for Schubert polynomials uses the combinatorial model of pipe dreams. This model was successively developed in [7, 20, 21, 39]. A **pipe dream** P is a tiling of the grid of boxes (extending infinitely to the east and south) with **turning pipes** and finitely many **crossing pipes** $+$. Such a tiling gives rise to a collection of lines called **pipes**, which one imagines traveling from the left side of the grid (the negative y-axis) to the top side (the positive x-axis). A pipe dream P is called **reduced** if no two pipes cross each other more than once. The permutation corresponding to a reduced pipe dream is the permutation given (in one-line notation) by the columns in which the pipes end. The **weight** $\mathrm{wt}(P)$ of a pipe dream is the weak composition whose ith entry is the number of crossing pipe tiles in row i of P (where row 1 is the top row).

Example 4.10 The pipe dream

$P =$ 

corresponds to the permutation 1432. (Here, we omit the infinite collection of tiles extending uninterestingly to the southeast.) The pipe dream P has weight $(1,2,0)$.

◇

Given a permutation π, let $\mathrm{PD}(\pi)$ denote the set of reduced pipe dreams for π.

Theorem 4.11 ([7, 9]) The Schubert polynomial $\mathfrak{S}_\pi$ is the generating function of reduced pipe dreams for π, i.e.,

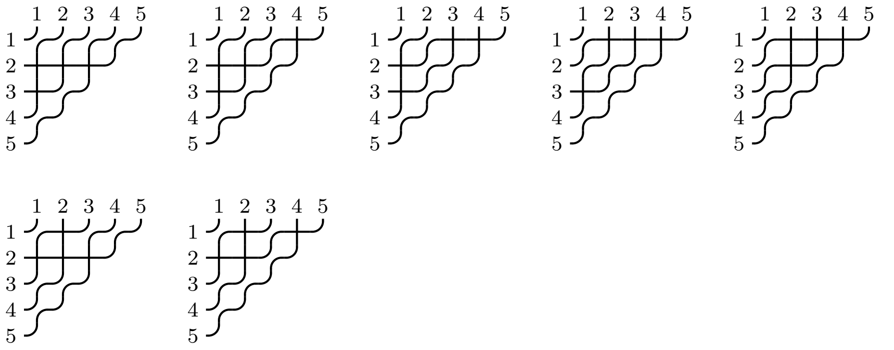


Fig. 5 The 7 reduced pipe dreams associated to the permutation 15324

$$\mathfrak{S}_\pi = \sum_{P \in \mathrm{PD}(\pi)} \mathbf{x}^{\mathrm{wt}(P)}.$$

Example 4.12 We have $\mathfrak{S}_{15324} = \mathbf{x}^{031} + \mathbf{x}^{121} + \mathbf{x}^{211} + \mathbf{x}^{310} + \mathbf{x}^{310} + \mathbf{x}^{130} + \mathbf{x}^{220}$, where the monomials are determined by the pipe dreams shown in Fig. 5. $\diamond$

A reduced pipe dream P is **quasiYamanouchi** if the following is true for the leftmost $+$ in every row: Either

(1) it is in the leftmost column, or
(2) it is weakly left of some $+$ in the row below it.

For a permutation π, write $\mathrm{QPD}(\pi)$ for the set of quasiYamanouchi reduced pipe dreams for π. We obtain then the following formula for the fundamental slide polynomial expansion of a Schubert polynomial.

Theorem 4.13 ([2]) *The Schubert polynomials expand positively in the fundamental slide polynomials:*

$$\mathfrak{S}_\pi = \sum_{P \in \mathrm{QPD}(\pi)} \mathfrak{F}_{\mathrm{wt}(P)}.$$

Before continuing to our third combinatorial formula for Schubert polynomials, let us consider another basis of ASym_n, closely related to the fundamental slide polynomials. Recall from Theorem 4.7 that the fundamental slide polynomials are a lift of the fundamental quasisymmetric polynomials. One might ask for an analogous lift to ASym_n of the monomial quasisymmetric polynomials. These are provided by the **monomial slide polynomials** of [2], which we now discuss.

Looking back at the combinatorial formulas for fundamental and monomial quasisymmetric polynomials, observe that they are identical, except that the formula for F_α looks at weak compositions b with $b^+ \vDash \alpha$ while the formula for M_α looks at weak compositions b with the more restrictive property $b^+ = \alpha$. It is easy then to guess the following modification of Theorem 4.6 that will yield the desired definition of monomial slide polynomials.

Definition 4.14 For any weak composition a, the **monomial slide polynomial** $\mathfrak{M}_a$ is defined by

$$\mathfrak{M}_a = \sum_{\substack{b \geq a \\ b^+ = a^+}} \mathfrak{X}_b.$$

By triangularity, it is straightforward that monomial slide polynomials form a basis of ASym_n. Moreover, we have the following analogue of Theorem 4.7:

Theorem 4.15 ([2]) *We have* $\mathfrak{M}_a \in \mathrm{QSym}_n$ *if and only if* a *is of the form* $0^k\alpha$*, where* α *is a strong composition of length* $n - k$.

Moreover, we have $\mathfrak{M}_{0^k\alpha} = M_\alpha$. *Thus, the monomial slide basis of* ASym_n *is a lift of the monomial quasisymmetric polynomial basis of* QSym_n.

Just as the combinatorial multiplication rule of Theorem 3.8 for fundamental quasisymmetric polynomials lifts to that of Theorem 4.9 for fundamental slide polynomials, the combinatorial multiplication rule of Theorem 3.3 for monomial quasisymmetric polynomials lifts to a rule for monomial slide polynomials.

First, we need to extend the overlapping shuffle product of Sect. 3 from strong compositions to general weak compositions. Given weak compositions a and b, treat them as words of some common finite length n by truncating at some position past all their nonzero entries. By $a + b$ we mean the weak composition that is the coordinatewise sum of a and b. Consider the set $S(a, b)$ of all pairs (a', b') of weak compositions of equal length $k \leq n$ such that

- $(a')^+ = a^+$ and $(b')^+ = b^+$;
- $a' \geq a$ and $b' \geq b$; and
- for all $1 \leq i \leq k$, we have $a'_i + b'_i > 0$.

Fix $(a', b') \in S(a, b)$. Let c be a weak composition of length r with zeros in positions $z_1, \ldots, z_m$ such that $c^+ = a' + b'$. Define c_a to be the weak composition of length r having zeros in the same positions $z_1, \ldots, z_m$ and the remaining positions of c_a are the entries of a', in order from left to right. Define c_b similarly, using the entries of b'. Then we have

- $c = c_a + c_b$, and
- $(c_a)^+ = (a')^+$ and $(c_b)^+ = (b')^+$.

For each such $(a', b') \in S(a, b)$, let $\mathrm{Bump}(a', b')$ denote the unique dominance-least weak composition satisfying

- $\mathrm{Bump}(a', b')^+ = a' + b'$, and
- $\mathrm{Bump}(a', b')_a \geq a$ and $\mathrm{Bump}(a', b')_b \geq b$.

The **overlapping slide product** of a and b is then the formal sum $a \sqcup\!\sqcup_o b$ of the $\mathrm{Bump}(a', b')$ for all $(a', b') \in S(a, b)$.

Example 4.16 Let $a = (0, 1, 0, 2)$ and $b = (1, 0, 0, 1)$, as in Example 4.8. Then $S(a, b)$ consists of the seven pairs

$$\big((0,1,0,2),(1,0,1,0)\big),\big((0,1,2,0),(1,0,0,1)\big),\big((0,1,2),(1,0,1)\big),\big((0,1,2),(1,1,0)\big)$$
$$\big((1,0,2),(1,1,0)\big),\big((1,2,0),(1,0,1)\big),\big((1,2),(1,1)\big)$$

The seven corresponding weak compositions $\mathrm{Bump}(a', b')$ are

$$(1,1,1,2),(1,1,2,1),(1,1,0,3),(1,2,0,2),(2,0,1,2),(2,0,2,1),(2,0,0,3).$$

$\diamondsuit$

Theorem 4.17 ([2]) *For weak compositions a and b, we have*

$$\mathfrak{M}_a \cdot \mathfrak{M}_b = \sum_c C^c_{a,b}\mathfrak{M}_c,$$

where $C^c_{a,b}$ is the multiplicity of c in the overlapping slide product $a \sqcup\!\sqcup_o b$.

The fundamental slide polynomials expand positively in the monomial slide basis. Say that $b \trianglerighteq a$ if $b \geq a$, and $c \geq b$ whenever $c \geq a$ and $c^+ = b^+$.

Theorem 4.18 *[2] The fundamental slide polynomials expand positively in the monomial slide polynomials:*

$$\mathfrak{F}_a = \sum_{\substack{b \trianglerighteq a \\ b^+ \vDash a^+}} \mathfrak{M}_b.$$

A third combinatorial formula for Schubert polynomials comes from a model introduced (conjecturally) by Axel Kohnert [40]. Let D be a box diagram, i.e., any subset of the boxes in an $n \times n$ grid. A **Kohnert move** on D selects the rightmost box in some row and moves it to the first available empty space above it in the same column (if such an empty space exists). Let $\mathrm{KD}(D)$ denote the set of all box diagrams that can be obtained from D by some sequence (possibly empty) of Kohnert moves. Define the weight $\mathrm{wt}(D)$ of a box diagram to be the weak composition where $\mathrm{wt}(D)_i$ records the number of boxes in the ith row of D from the top.

Example 4.19 Let D be the leftmost diagram in the top row of Fig. 6 (which happens to be the Rothe diagram $RD(15324)$). The set of diagrams in Fig. 6 is exactly $\mathrm{KD}(D)$.

$\diamondsuit$

Theorem 4.20 ([4, 86, 87]) *The Schubert polynomial $\mathfrak{S}_\pi$ is the generating function of the Kohnert diagrams for the Rothe diagram of π, i.e.,*

$$\mathfrak{S}_\pi = \sum_{D \in \mathrm{KD}(\mathrm{RD}(\pi))} \mathbf{x}^{\mathrm{wt}(D)}.$$

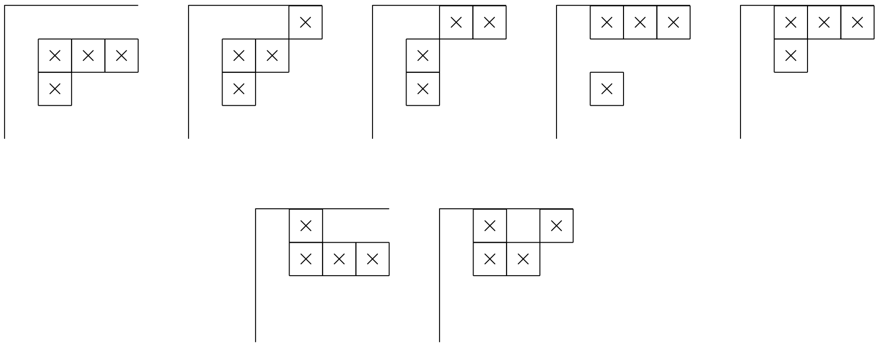


Fig. 6 The 7 Kohnert diagrams associated to $RD(15324)$

Example 4.21 We have $\mathfrak{S}_{15324} = \mathbf{x}^{031} + \mathbf{x}^{121} + \mathbf{x}^{211} + \mathbf{x}^{310} + \mathbf{x}^{310} + \mathbf{x}^{130} + \mathbf{x}^{220}$, where the monomials are determined by the Kohnert diagrams shown in Fig. 6. Note this is consistent with the computation in Example 4.12. $\diamondsuit$

Kohnert diagrams also yield another natural basis of ASym_n, which we now consider. Given a weak composition a, we can associate a box diagram D by first obtaining the permutation w corresponding to a and then taking $D = \mathrm{RD}(w)$. This construction, combined with the Kohnert moves, leads us to the characterization of Schubert polynomials from Theorem 4.20. However, there is also a much easier way to associate a box diagram $D(a)$ to a weak composition a–namely, just take $D(a)$ to be the Young diagram of a, as described in Sect. 3. This leads us to the following definition (really a theorem of Kohnert [40]): For a weak composition a, the **key polynomial** $\mathfrak{D}_a$ is the generating function

$$\mathfrak{D}_a := \sum_{D \in KD(D(a))} \mathbf{x}^{\mathrm{wt}(D)}.$$

Example 4.22 Let $a = (0, 2, 1)$. Then $\mathfrak{D}_a = \mathbf{x}^{021} + \mathbf{x}^{111} + \mathbf{x}^{201} + \mathbf{x}^{210} + \mathbf{x}^{120}$, as computed by the Kohnert diagrams in Fig. 7. $\diamondsuit$

The definition of key polynomials that we have given here is not the original one. These polynomials were first introduced in [17] where they were realized as characters of (type A) Demazure modules; for this reason, they are often referred to as

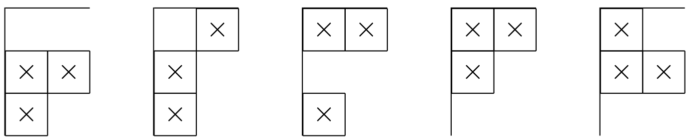


Fig. 7 The 5 Kohnert diagrams associated to the weak composition (0, 2, 1)

Demazure characters. Later they were studied from a more combinatorial perspective by Lascoux and Schützenberger [54], who coined the term 'key polynomial'. Key polynomials also arise as a specialization [34, 76] of the nonsymmetric Macdonald polynomials introduced in [15, 57, 67].

An alternative description of key polynomials is via a modification of the ∂_i operators that define Schubert polynomials. For each positive integer, define an operator π_i on Poly_n by

$$\pi_i(f) := \partial_i(x_i f).$$

Let w be a permutation and let $s_{i_1} \cdots s_{i_r}$ be any reduced word for w. Then define $\pi_w = \pi_{i_1} \cdots \pi_{i_r}$. This is independent of the choice of reduced word since these operators satisfy $\pi_i^2 = \pi_i$ and the usual commutation and braid relations for the symmetric group. (That is to say, the action of the π_i operators on Poly_n is a representation of the type A 0-Hecke algebra.) Let s_i act on a weak composition a by exchanging the ith and $(i+1)$st entries of a. Given a weak composition a, let $w(a)$ denote the permutation of minimal Coxeter length such that $w(a) \cdot a = \overleftarrow{a}$. Finally, the key polynomial $\mathfrak{D}_a$ is given by

$$\mathfrak{D}_a = \pi_{w(a)} \mathbf{x}^{\overleftarrow{a}}.$$

Example 4.23

$$\begin{aligned}
\mathfrak{D}_{(0,2,1)} &= \pi_1 \pi_2 (x_1^2 x_2) \\
&= \pi_1 (x_1^2 x_2 + x_1^2 x_3) \\
&= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3.
\end{aligned}$$

Compare this calculation with that of Example 4.22. ◇

As is the case for Schubert polynomials, there are several additional combinatorial formulas for key polynomials! An excellent overview can be found in [74]. Another such combinatorial formula for the key polynomials is given in [31] in terms of *semi-skyline fillings*. Let a be a weak composition of length n. We recall the definition of a triple of entries from the previous section; this extends verbatim from diagrams of compositions to diagrams of weak compositions.

A *semi-skyline filling* of $D(a)$ is a filling of the boxes of $D(a)$ with positive integers, one per box, such that

(S.1) entries do not repeat in a column,
(S.2) entries weakly decrease from left to right along rows, and
(S.3) every triple of entries is inversion.

The weight $\mathrm{wt}(T)$ of a semi-skyline filling is weak composition whose ith entry records the number of occurrences of the entry i in T.

Let $\overline{D}(a)$ denote the diagram of $D(a)$ augmented with an additional 0th column called a *basement*, and let $\mathrm{rev}(a)$ denote the weak composition obtained by reading the entries of a in reverse. Define a *key semi-skyline filling* for a to be a filling of

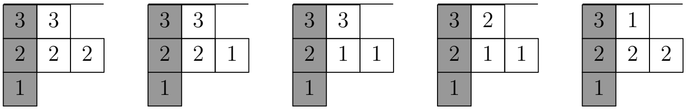


Fig. 8 The five key semi-skyline fillings associated to 021

$\overline{D}(\mathrm{rev}(a))$, where the ith basement entry is $n+1-i$, satisfying (S.1), (S.2) and (S.3) above (including on basement entries). Let $\mathfrak{D}\mathsf{SST}(a)$ denote the set of key semi-skyline fillings for a.

Example 4.24 Let $a = (0, 2, 1)$. The key semi-skyline fillings associated to a are shown in Fig. 8. The basement boxes are shaded in grey. $\diamondsuit$

Theorem 4.25 ([31]) *The key polynomial $\mathfrak{D}_a$ is given by*

$$\mathfrak{D}_a = \sum_{T \in \mathfrak{D}\mathsf{SST}(a)} \mathbf{x}^{\mathrm{wt}(T)}.$$

Example 4.26 From Example 4.24, we again compute $\mathfrak{D}_{(0,2,1)} = \mathbf{x}^{021} + \mathbf{x}^{111} + \mathbf{x}^{201} + \mathbf{x}^{210} + \mathbf{x}^{120}$. $\diamondsuit$

Yet another formula for key polynomials is given in terms of *Kohnert tableaux* [3]. Kohnert tableaux associate a canonical path in Kohnert's algorithm from the diagram of a weak composition to a given Kohnert diagram. In fact, the Kohnert tableaux are equivalent to the key semi-skyline fillings turned upside down, but are described by quite different local rules, which arise from Kohnert's algorithm as opposed to considerations in Macdonald polynomial theory. We omit the details of this construction, for which see [3].

One might naturally ask, analogously for Schubert polynomials, whether key polynomials restrict to an important basis of $\mathrm{Sym}_n \subset \mathrm{ASym}_n$. Exactly the same as for the Schubert polynomials, the key polynomials in Sym_n are exactly the basis of Schur polynomials. Thus the key basis of ASym_n is also a lift of the Schur basis of Sym_n. The key polynomial $\mathfrak{D}_a$ is a Schur polynomial if and only if a is weakly increasing; in this case, we have $\mathfrak{D}_a = s_{\overleftarrow{a}}$.

Moreover, the *stable limits* of the key polynomials are exactly the Schur functions. Given $m \in \mathbb{Z}_{\geq 0}$ and a weak composition a, recall that $0^m a$ denotes the weak composition obtained by prepending m zeros to a, e.g., $0^2(1, 0, 3) = (0, 0, 1, 0, 3)$. The stable limit of $\mathfrak{D}_a$ is the formal power series $\lim_{m\to\infty} \mathfrak{D}_{0^m a}$. The following result is implicit in work of Lascoux and Schützenberger [54]; an explicit proof is given in [3] with further details.

Theorem 4.27 ([3, 54]) *Let a be a weak composition. Then the stable limit of the key polynomial $\mathfrak{D}_a$ is the Schur function associated to the partition obtained by rearranging the entries of a into decreasing order. That is,*

$$\lim_{m\to\infty} \mathfrak{D}_{0^m a} = s_{\overleftarrow{a}}(X).$$

Unlike the Schubert basis, however, the key basis does not have positive structure constants.

Schubert polynomials expand in the key basis with positive coefficients, although we omit the details of this decomposition. Let T be a semistandard Young tableau. Define the *column reading word* $\mathrm{colword}(T)$ of T to be the word obtained by writing the entries of each column of T from bottom to top, starting with the leftmost column and proceeding rightwards. Then, as given in [74, Theorem 4], we have

$$\mathfrak{S}_\pi = \sum_{\mathrm{colword}(T)\in\mathrm{Red}(\pi^{-1})} \mathfrak{D}_{\mathrm{wt}(K^0_-(T))}$$

where the sum is over all semistandard Young tableaux T whose column reading word is a reduced word for π^{-1}, and $K^0_-(T)$ is a semistandard Young tableau of the same shape as T called the *left nil key* of T, as defined in [54, 74]. For another approach to this decomposition, see [6].

The Demazure atoms $\mathfrak{A}_a$ form another basis of ASym_n, introduced and studied in [54], where they are referred to as *standard bases*. Demazure atoms are characters of quotients of Demazure modules, and, like key polynomials, also arise as specializations of nonsymmetric Macdonald polynomials [60]. Just as we defined fundamental slide polynomials as the pieces of Schubert polynomials given by the summands of Theorem 4.3, we can define the Demazure atoms as the pieces of quasiSchur polynomials given by the summands of the definition in Eq. (3.1): Given a weak composition a, the **Demazure atom** $\mathfrak{A}_a$ is given by

$$\mathfrak{A}_a := \sum_{T\in\mathfrak{A}\mathsf{SST}(a)} \mathbf{x}^{\mathrm{wt}(T)}.$$

Example 4.28 We have

$$\mathfrak{A}_{(1,0,3)} = \mathbf{x}^{103} + \mathbf{x}^{112} + \mathbf{x}^{202} + \mathbf{x}^{121} + \mathbf{x}^{211},$$

where the monomials are determined by the five semistandard composition tableaux of shape $(1, 0, 3)$ from Fig. 3. ◇

The Demazure atom basis does not have positive structure coefficients. However, it does exhibit a variety of surprising positivity properties. First, notice that the definition that we have given immediately implies that the quasiSchur polynomial $S_\alpha(x_1,\ldots,x_n)$ expands positively in Demazure atoms.

Proposition 4.29 ([30]) *The quasiSchur polynomials expand positively in the Demazure atoms:*

$$S_\alpha(x_1,\ldots,x_n) = \sum_{a^+=\alpha} \mathfrak{A}_a,$$

where the sum is over weak compositions a of length n.

It is also the case that key polynomials expand positively in Demazure atoms. Let S_n act on weak compositions of length n via $v \cdot (a_1, \ldots, a_n) = (a_{v^{-1}(1)}, \ldots, a_{v^{-1}(n)})$. Given a weak composition a, let $v(a)$ denote the permutation of minimal Coxeter length such that $v(a) \cdot a = \overleftarrow{a}$.

Theorem 4.30 ([54]) *The key polynomials expand positively in the Demazure atoms:*

$$\mathfrak{D}_a = \sum_{\substack{v(b) \le v(a) \\ \overleftarrow{b} = \overleftarrow{a}}} \mathfrak{A}_b,$$

where $\le$ denotes the (strong) Bruhat order on permutations.

Since every Schur polynomial is also a key polynomial, Theorem 4.30 implies that Schur polynomials expand positively in the Demazure atoms.

Example 4.31 Let $a = (1, 0, 3)$. Then $v(a) = 231$ and

$$\mathfrak{D}_{(1,0,3)} = \mathfrak{A}_{(1,0,3)} + \mathfrak{A}_{(1,3,0)} + \mathfrak{A}_{(3,0,1)} + \mathfrak{A}_{(3,1,0)}.$$

$\diamond$

Finally, we mention the following remarkable conjecture of V. Reiner and M. Shimozono; for more details on this conjecture, see the work of A. Pun [72]. For a generalization, see [64]. Observe that the conjecture would follow trivially from Theorem 4.30 if either the key polynomial or the Demazure atom basis had positive structure coefficients; however, neither does, so the conjecture is quite mysterious.

Conjecture 4.32 *(Reiner–Shimozono)* The product $\mathfrak{D}_a \cdot \mathfrak{D}_b$ expands positively in Demazure atoms.

At this point, we have considered lifts to ASym_n of the Schur polynomials (two distinct lifts even), the fundamental quasisymmetric polynomials, and the monomial quasisymmetric polynomials. It is natural then to hope for an appropriate lift of the remaining basis of QSym_n that we considered in Sect. 3, namely the quasiSchur polynomials. The next basis we consider is exactly this desired lift, the quasikey polynomials of [3]. The quasikey polynomials are a lifting of the quasiSchur basis of QSym_n to ASym_n, and simultaneously a common coarsening of the fundamental slide polynomial and Demazure atom bases.

Let a be a weak composition of length n. The **quasikey polynomial** associated to a is given by

$$\mathfrak{Q}_a := \sum_{\substack{b^+ = a^+ \\ b \ge a}} \sum_{T \in \mathfrak{A}\mathsf{SST}(a)} \mathbf{x}^{\mathrm{wt}(T)}$$

where the first sum is over all weak compositions b of length n satisfying $b \ge a$ in dominance order and whose positive part is a^+. The form of this definition is due to [77]. The quasikey polynomials were originally defined in [3] as a weighted sum of *quasi-Kohnert tableaux*; we omit this alternate formulation.

Example 4.33 Let $a = (1, 0, 3)$. Then

$$\mathfrak{Q}_{(1,0,3)} = \mathbf{x}^{130} + \mathbf{x}^{220} + \mathbf{x}^{103} + \mathbf{x}^{112} + \mathbf{x}^{202} + \mathbf{x}^{121} + \mathbf{x}^{211},$$

where the monomials are determined by the first seven semistandard composition tableaux shown in Fig. 3. ◇

The definition we have given here immediately implies that each quasikey polynomial $\mathfrak{Q}_a$ expands positively in Demazure atoms. Specifically, we have the following.

Theorem 4.34 ([77, Theorem 3.4]) *The quasikey polynomials expand positively in the Demazure atoms: For a weak composition a of length n, we have*

$$\mathfrak{Q}_a = \sum_{\substack{b^+=a^+ \\ b \geq a}} \mathfrak{A}_b,$$

where the sum is over all weak compositions b of length n satisfying $b \geq a$ in dominance order and whose positive part is a^+.

Example 4.35 One easily calculates from Theorem 4.34 that

$$\mathfrak{Q}_{(1,0,3)} = \mathfrak{A}_{(1,0,3)} + \mathfrak{A}_{(1,3,0)}.$$

Compare this calculation to the tableaux of Fig. 3. ◇

Less clear from our definition is the following additional positivity property of quasikey polynomials.

Theorem 4.36 ([2]) *The quasikey polynomials expand positively in the fundamental slide polynomials: For a weak composition a of length n, we have*

$$\mathfrak{Q}_a = \sum_T \mathfrak{F}_{\mathrm{wt}(T)},$$

where the sum is over all quasiYamanouchi tableaux T whose support contains the support of a, and such that $T \in \mathfrak{A}\mathsf{SST}(b)$ for some weak composition b of length n with $b^+ = a^+$ and $b \geq a$.

Example 4.37

$$\mathfrak{Q}_{(1,0,3)} = \mathfrak{F}_{(1,0,3)} + \mathfrak{F}_{(2,0,2)}$$

where the two fundamental slides correspond to the 3rd and 5th composition tableaux in Fig. 3. ◇

Although we claimed that the quasikey polynomials were to be a lift from QSym_n to ASym_n of the quasiSchur polynomials, we have not yet explained this fact. The sense of this lift is given in the following proposition.

Proposition 4.38 ([3, Theorem 4.16]) *We have* $\mathfrak{Q}_a \in \mathrm{QSym}_n$ *if and only if* a *is of the form* $0^k\alpha$*, where* α *is a strong composition of length* $n-k$*. Moreover, we have*

$$\mathfrak{Q}_{0^k\alpha} = S_\alpha(x_1, \ldots, x_n).$$

Thus, the quasikey polynomial basis of ASym_n *is a lift of the quasiSchur polynomial basis of* QSym_n.

Moreover, the quasikey polynomials stabilize to the quasiSchur functions:

Theorem 4.39 ([3, Theorem 4.17]) *Let* a *be a weak composition. Then*

$$\lim_{m\to\infty} \mathfrak{Q}_{0^m a} = S_{a^+}.$$

As Schur polynomials expand positively in the quasiSchur polynomial basis, one might hope for the same positivity to hold for their respective lifts, the key polynomials (or Schubert polynomials) and quasikey polynomials. Indeed, these expansions are positive, with positive combinatorial formulas mirroring the expansion of Theorem 3.12. To provide a formula for this expansion, we need the concept of a left swap on weak compositions. A **left swap** on a weak composition a exchanges two entries a_i and a_j such that $a_i < a_j$ and $i < j$. In essence, left swaps move larger entries leftwards. Given a weak composition a of length n, define the set $\mathrm{lswap}(a)$ to be all weak compositions b of length n that can be obtained from a by a sequence of left swaps.

Example 4.40 We have

$$\mathrm{lswap}(1, 0, 3) = \{(1, 0, 3), (1, 3, 0), (3, 0, 1), (3, 1, 0)\}.$$

$\diamond$

In fact, the elements of $\mathrm{lswap}(a)$ are exactly those weak compositions b such that $\overleftarrow{b} = \overleftarrow{a}$ and $w(b) \le w(a)$ in Bruhat order. Hence, in this new language, the formula in Theorem 4.30 for expanding key polynomials in Demazure atoms may be re-expressed as follows.

Proposition 4.41 ([77, Lemma 3.1])

$$\mathfrak{D}_a = \sum_{b \in \mathrm{lswap}(a)} \mathfrak{A}_b.$$

Define $\mathrm{Qlswap}(a)$ to be those $b \in \mathrm{lswap}(a)$ such that for all $c \in \mathrm{lswap}(a)$ with $c^+ = b^+$, one has $c \ge b$ in dominance order.

Example 4.42 The set lswap$(1, 0, 3)$ is computed in Example 4.40. Partitioning lswap$(1, 0, 3)$ into equivalence classes under the relation $c \sim b$ when $c^+ = b^+$ yields classes $\{(1, 0, 3), (1, 3, 0)\}$ and $\{(3, 0, 1), (3, 1, 0)\}$. Taking the weak composition smallest in dominance order from each class then yields

$$\mathrm{Qlswap}(1, 0, 3) = \{(1, 0, 3), (3, 0, 1)\}.$$

◇

Theorem 4.43 ([3, Theorem 3.7]) *Let a be a weak composition of length n. Then*

$$\mathfrak{D}_a = \sum_{b \in \mathrm{Qlswap}(a)} \mathfrak{Q}_b.$$

A key polynomial $\mathfrak{D}_a$ is a Schur polynomial if and only if the entries of a are weakly increasing. In this case, by Proposition 4.38 and the definition of Qlswap, the formula of Theorem 4.43 reduces to the formula of Theorem 3.12 for the quasiSchur polynomial expansion of a Schur polynomial.

Quasikey polynomials do not have positive structure constants. This is immediate from the fact that the quasikey polynomial basis contains all the quasiSchur polynomials, which themselves do not have positive structure constants. However, [77] gives a positive combinatorial formula for the quasikey expansion of the product of a quasikey polynomial and a Schur polynomial, extending the analogous formula of [31] for quasiSchur polynomials.

The final basis of ASym_n that we consider here is the basis of *fundamental particles* introduced in [77]. Note that the formula for the Demazure atom expansion of a quasikey polynomial given in Theorem 4.34 is identical to the formula for the monomial expansion of a monomial slide polynomial given in Definition 4.14. The main motivating property of fundamental particles is that this same formula will give the fundamental particle expansion of a fundamental slide polynomial.

Given a weak composition a, let $\mathfrak{P}\mathsf{SST}(a)$ denote the set of those semistandard composition tableaux of shape a satisfying the property that whenever $i < j$, every label in row i is smaller than every label in row j. For example, $\mathfrak{P}\mathsf{SST}(1, 0, 3)$ consists of the 3rd, 4th and 6th tableaux in Fig. 3.

Let a be a weak composition of length n. The **fundamental particle** associated to a is given by

$$\mathfrak{P}_a := \sum_{T \in \mathfrak{P}\mathsf{SST}(a)} \mathbf{x}^{\mathrm{wt}(T)}.$$

Example 4.44 Let $a = (1, 0, 3)$. Then

$$\mathfrak{P}_{(1,0,3)} = \mathbf{x}^{103} + \mathbf{x}^{112} + \mathbf{x}^{121}.$$

◇

The following is straightforward from the definitions.

Theorem 4.45 ([77]) *The fundamental slide polynomials expand positively in the fundamental particles: For a weak composition a of length n, we have*

$$\mathfrak{F}_a = \sum_{\substack{b^+ = a^+ \\ b \geq a}} \mathfrak{P}_b,$$

where the sum is over all weak compositions b of length n satisfying $b \geq a$ in dominance order and whose positive part is a^+.

The fundamental particles were constructed to be a refinement of the fundamental slide polynomials (with a particular positive expansion formula), as in Theorem 4.45; remarkably, the fundamental particles are also a refinement of the Demazure atoms. Given a weak composition a, define the set $\mathsf{HSST}(a)$ of **particle-highest** semistandard composition tableaux of shape a to be the set of those $T \in \mathfrak{A}\mathsf{SST}(a)$ such that for each integer i appearing in T, either

- an i appears in the first column, or
- there is an i^+ weakly right of an i, where i^+ is the smallest integer greater than i appearing in T.

Notice the particle-highest condition is a weakening of the quasiYamanouchi condition: every quasiYamanouchi tableau is necessarily particle-highest.

Establishing the last of the arrows shown in Fig. 4, we have the following additional positivity.

Theorem 4.46 ([77]) *The Demazure atoms expand positively in the fundamental particles:*

$$\mathfrak{A}_a = \sum_{T \in \mathsf{HSST}(a)} \mathfrak{P}_{\mathrm{wt}(T)}.$$

Fundamental particles do not have positive structure constants, however, in analogy with quasikey polynomials and Demazure atoms, there is a positive combinatorial formula for the fundamental particle expansion of the product of a fundamental particle and a Schur polynomial given in [77].

In this section, we have given three formulas for Schubert polynomials. Each formula lead us to consider certain other families of polynomials described by similar combinatorics. There is additionally a fourth, fundamentally different, formula for Schubert polynomials given by A. Lascoux [45] in terms of the *square-ice/6-vertex model* of statistical physics. This formula was recently rediscovered in [49] in the guise of *bumpless pipe dreams*. (The connection between these two combinatorial descriptions is detailed in [84].) We won't describe this rule here, as its context is currently unclear, although we think it is likely to be important. Connections to Gröbner geometry appear in [32].

5 The Mirror Worlds: K-Theoretic Polynomials

A trend in modern Schubert calculus is to look at Flags_n, Grassmannians, and other generalized flag varieties, not through the lens of ordinary cohomology as in Sects. 2–4, but through the sharper yet more mysterious lenses of other complex oriented cohomology theories [16, 26, 55, 56].

Particularly well studied over the past 20 years are combinatorial aspects of the K-theory rings of these spaces. Early work here includes [20, 22, 52, 53]; however, the area only became very active after the influential work of [14, 46]. Following [8, 20, 33], it turns out that one can slightly generalize this setting to *connective K-theory* with almost no extra combinatorial complexity (indeed, in some ways the more general combinatorics seems easier). In general, each complex oriented cohomology theory is determined by its formal group law, which describes how to write the Chern class of a tensor product of two line bundles in terms of the two original Chern classes. In the case of connective K-theory, the formal group law is

$$c_1(L \otimes M) = c_1(L) + c_1(M) + \beta c_1(L)c_1(M), \tag{5.1}$$

where β is a formal parameter and L, M are complex line bundles on the space in question. In this notation, the ordinary cohomology ring is recovered by setting $\beta = 0$ and the ordinary K-theory ring is recovered (up to convention choices) by setting $\beta = -1$. For more background on connective K-theory, see the appendix to [1].

Just as the Schubert classes in the ordinary cohomology of Flags_n are represented by the Schubert polynomials (as described in Sect. 4), we would like to have such polynomial representatives for the corresponding connective K-theory classes. These are provided by the *β-Grothendieck polynomials* $\{\overline{\mathfrak{S}}_a\}$ of S. Fomin and A. Kirillov [20], as identified in [33]. These polynomials form a basis of $\mathrm{ASym}_n[\beta]$, where β is the formal parameter from Eq. (5.1). This basis is homogeneous if the parameter β is understood to live in degree -1. Specializing at $\beta = 0$, one recovers the Schubert basis $\{\mathfrak{S}_a\}$ of ASym_n. The usual Grothendieck polynomials of A. Lascoux and M.-P. Schützenberger [52] are realized at $\beta = -1$. (To help the reader track relations among bases, we deviate from established practice by denoting connective K-analogues by applying an 'overbar' to their cohomological specializations.) Like the Schubert basis of ASym_n, the β-Grothendieck polynomial basis has positive structure coefficients; this is, of course, currently only known by geometric arguments [11], as there is no combinatorial proof of this fact even in the case $\beta = 0$.

Intersecting $\{\overline{\mathfrak{S}}_a\}$ with $\mathrm{Sym}_n[\beta]$ yields the basis $\{\overline{s}_\lambda\}$ of *symmetric Grothendieck polynomials*. These represent connective K-theory Schubert classes on Grassmannians. In this setting, like the Schur polynomial setting, a number of Littlewood–Richardson rules for $\{\overline{s}_\lambda\}$ are now known (e.g., [73, 80, 81]), following the first found by A. Buch [14].

There is a beautiful combinatorial formula for $\overline{s}_\lambda$ given by Buch [14], as a direct extension of the Littlewood formula of Theorem 2.1 for Schur polynomials. A **set-**

valued tableau T of shape λ is a filling of each cell of the Young diagram λ by a nonempty set of positive integers. Deleting all but one number from each set resolves T to an ordinary tableau, and T is called **semistandard** if all all such resolutions are semistandard in the sense of Sect. 2. In other words, T is semistandard if the greatest entry in each box is not larger than the least entry in the box to its right and is strictly smaller than the least entry in the box directly below it. Let $\mathrm{SV}(\lambda)$ denote the set of semistandard set-valued tableaux of shape λ. The **weight** of a set-valued tableau T is the weak composition $\mathrm{wt}(T) := (a_1, a_2, \ldots)$, where a_i records the number of instances of the number i among all boxes of T.

Theorem 5.1 ([14, Theorem 3.1]) *For any partition λ, we have*

$$\overline{s}_\lambda = \sum_{T \in \mathrm{SV}(\lambda)} \mathbf{x}^{\mathrm{wt}(T)}.$$

The remaining families of polynomials discussed in Sects. 2–4 are not currently understood well in term of cohomology. Remarkably, however, from a combinatorial perspective they all appear to have natural 'connective K-analogues'. That is, for each basis, there is a combinatorially-natural β-deformation that is homogeneous (with the understanding that β has degree -1), forms a basis of $\mathrm{ASym}_n[\beta]$, and (at least conjecturally) shares the positivity properties of the original basis. These deformed bases are presented in Table 1.

It is a mystery as to whether these various apparently K-theoretic families of polynomials in fact have a geometric interpretation in terms of K-theory. If they did, then presumably the $\beta = 0$ specialization considered in the previous sections would similarly have a cohomological interpretation. Such an interpretation would be rather surprising, as currently these specializations are only understood through combinatorics and (in some cases) representation theory. Alternatively, perhaps there is a general representation-theoretic construction that yields all of these various β-deformations. No such construction is currently known, but for some ideas along these lines see [24, 63, 69].

For details of the definitions of the bases from Table 1 and their relations, see the references given there, especially [64] which contains a partial survey. Here, we only briefly sketch hints of this theory. (However, the theory is in many ways exactly parallel to that given in Sects. 2–4, so the astute reader can likely guess approximations to many of the structure theorems.)

The notion of semistandard *set-valued* skyline fillings was introduced in [62], and employed there to provide an explicit combinatorial definition of the *Lascoux polynomials* $\overline{\mathfrak{D}}_a$ and the *Lascoux atoms* $\overline{\mathfrak{A}}_a$: K-theoretic analogues of key polynomials and Demazure atoms, respectively. Such K-theoretic analogues have also been studied in [13, 38, 44, 63, 64, 70, 75].

The basis $\overline{\mathfrak{F}}_a$ of *glide polynomials* was introduced in [71]. Glide polynomials are simultaneously a K-theoretic analogue of the fundamental slide basis and a polynomial lift of the *multi-fundamental quasisymmetric* basis [51] of quasisymmetric polynomials. Remarkably, the glide basis also has positive structure constants, which

Table 1 Bases from Sects. 2–4, together with their corresponding connective K-analogues. For each K-theoretic family of polynomials, we have a given a few major references; however, these references are generally not exhaustive

Sym_n	Schur polynomial s_λ	Symmetric Grothendieck polynomial $\overline{s}_\lambda$ [14, 63]
QSym_n	Monomial quasisymmetric polynomial M_α	Multimonomial polynomial $\overline{M}_\alpha$ [51]
	Fundamental quasisymmetric polynomial F_α	Multifundamental polynomial $\overline{F}_\alpha$ [51, 68, 71]
	QuasiSchur polynomial S_α	QuasiGrothendieck polynomial $\overline{S}_\alpha$ [62, 64]
ASym_n	Schubert polynomial $\mathfrak{S}_a$	Grothendieck polynomial $\overline{\mathfrak{S}}_a$ [20, 39, 52]
	Demazure character/key polynomial $\mathfrak{D}_a$	Lascoux polynomial $\overline{\mathfrak{D}}_a$ [38, 62, 64, 75]
	Quasikey polynomial $\mathfrak{Q}_a$	QuasiLascoux polynomial $\overline{\mathfrak{Q}}_a$ [64]
	Demazure atom/standard basis $\mathfrak{A}_a$	Lascoux atom $\overline{\mathfrak{A}}_a$ [13, 62, 64, 70]
	Fundamental particle/pion $\mathfrak{P}_a$	kaon $\overline{\mathfrak{P}}_a$ [64]
	Fundamental slide polynomial $\mathfrak{F}_a$	glide polynomial $\overline{\mathfrak{F}}_a$ [64, 71]

can be described in terms the *glide product* [71] of weak compositions. The glide product is a simultaneous generalization of the slide product of [2] and the *multi-shuffle* product of [51] on strong compositions.

The *quasiLascoux* basis $\overline{\mathfrak{Q}}_a$ and *kaon* basis $\overline{\mathfrak{P}}_a$ were introduced in [64]. These are K-analogues of the quasi-key and fundamental particle bases. Remarkably, the positivity relations between the bases in Fig. 4 have been proven to hold for their K-analogues mentioned above [64, 71], with the exception of the expansion of Grothendieck polynomials in Lascoux polynomials, whose positivity remains conjectural.

As outlined in the previous section, the product of an element of any basis in Fig. 4 with a Schur polynomial expands positively in that basis. It would be interesting to know if the analogous result is true in the K-theory world: that product of an element of a K-theoretic analogue and a symmetric Grothendieck polynomial expands positively in that K-theoretic basis. This is obviously true for the Grothendieck basis by geometry. It is also true for the glide basis, since the glide basis has positive structure constants and refines Grothendieck polynomials. We believe this question remains, however, open for the Lascoux, quasiLascoux, Lascoux atom and kaon bases.

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References

1. Anderson, David: K-theoretic Chern class formulas for vexillary degeneracy loci. Adv. Math. **350**, 440–485 (2019)
2. Assaf, S., Searles, D.: Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams. Adv. Math. **306**, 89–122 (2017)
3. Assaf, S., Searles, D.: Kohnert tableaux and a lifting of quasi-Schur functions. J. Combin. Theory Ser. A **156**, 85–118 (2018)
4. Sami Assaf, *Combinatorial models for Schubert polynomials*, preprint (2017), 29 pages, arXiv:1703.00088
5. Sami Assaf, *Multiplication of a Schubert polynomial by a Stanley symmetric polynomial*, preprint (2017), 9 pages, arXiv:1702.00132
6. Sami Assaf, *A generalization of Edelman–Greene insertion for Schubert polynomials*, preprint (2019), 28 pages, arXiv:1903.05802
7. Bergeron, Nantel, Billey, Sara: RC-graphs and Schubert polynomials. Experiment. Math. **2**(4), 257–269 (1993)
8. Bressler, Paul, Evens, Sam: The Schubert calculus, braid relations, and generalized cohomology. Trans. Amer. Math. Soc. **317**(2), 799–811 (1990)
9. Billey, Sara C., Jockusch, William, Stanley, Richard P.: Some combinatorial properties of Schubert polynomials. J. Algebraic Combin. **2**(4), 345–374 (1993)
10. Baker, Andrew, Richter, Birgit: Quasisymmetric functions from a topological point of view. Math. Scand. **103**(2), 208–242 (2008)
11. Michel Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), no. 1, 137–159, Special issue in celebration of Claudio Procesi's 60th birthday
12. Bump, Daniel, Schilling, Anne: Crystal bases: Representations and combinatorics, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2017)
13. Valentin Buciumas, Travis Scrimshaw, and Katherine Weber, *Colored five-vertex models and Lascoux polynomials and atoms*, J. Lond. Math. Soc. 19 pages, preprint (2019) arXiv:1908.07364
14. Anders Skovsted Buch: A Littlewood-Richardson rule for the K-theory of Grassmannians. Acta Math. **189**(1), 37–78 (2002)
15. Ivan Cherednik, *Nonsymmetric Macdonald polynomials*, Int. Math. Res. Not. (1995), no. 10, 483–515
16. B. Calmès, K. Zainoulline, and C. Zhong, *Equivariant oriented cohomology of flag varieties*, Doc. Math. (2015), no. Extra vol.: Alexander S. Merkurjev's sixtieth birthday, 113–144
17. M. Demazure, *Une nouvelle formule des caractères*, Bull. Sci. Math. (2) **98** (1974), no. 3, 163–172
18. Gérard Duchamp, Daniel Krob, Bernard Leclerc, and Jean-Yves Thibon, *Fonctions quasi-symétriques, fonctions symétriques non commutatives et algèbres de Hecke à $q = 0$*, C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), no. 2, 107–112
19. Samuel Eilenberg and Saunders Mac Lane, *On the groups of $H(\Pi, n)$. I*, Ann. of Math. (2) **58** (1953), 55–106
20. Fomin, Sergey, Kirillov, Anatol N.: Grothendieck polynomials and the Yang-Baxter equation, pp. 183–189. DIMACS, Piscataway, NJ, Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique (1994)
21. Sergey Fomin and Anatol N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), vol. 153, 1996, pp. 123–143
22. Fulton, William, Lascoux, Alain: A Pieri formula in the Grothendieck ring of a flag bundle. Duke Math. J. **76**(3), 711–729 (1994)
23. William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry

24. Galashin, Pavel: A Littlewood-Richardson rule for dual stable Grothendieck polynomials. J. Combin. Theory Ser. A **151**, 23–35 (2017)
25. Maria Gillespie, *Variations on a theme of Schubert calculus*, Recent trends in algebraic combinatorics, Assoc. Women Math. Ser., vol. 16, Springer, Cham, 2019, pp. 115–158
26. Ganter, N., Ram, A.: Generalized Schubert calculus. J. Ramanujan Math. Soc. **28A**, 149–190 (2013)
27. Darij Grinberg and Victor Reiner, *Hopf algebras in combinatorics*, preprint (2018), 235 pages, arXiv:1409.8356v5
28. Hatcher, Allen: Algebraic topology. Cambridge University Press, Cambridge (2002)
29. Hazewinkel, Michiel: The algebra of quasi-symmetric functions is free over the integers. Adv. Math. **164**(2), 283–300 (2001)
30. Haglund, J., Luoto, K., Mason, S., van Willigenburg, S.: Quasisymmetric Schur functions. J. Combin. Theory Ser. A **118**(2), 463–490 (2011)
31. Haglund, J., Luoto, K., Mason, S., van Willigenburg, S.: Refinements of the Littlewood-Richardson rule. Trans. Amer. Math. Soc. **363**(3), 1665–1686 (2011)
32. Zachary Hamaker, Oliver Pechenik, and Anna Weigandt, *Gröbner geometry of Schubert polynomials through ice*, in preparation (2019)
33. Hudson, Thomas: A Thom-Porteous formula for connective K-theory using algebraic cobordism. J. K-Theory **14**(2), 343–369 (2014)
34. Ion, Bogdan: Nonsymmetric Macdonald polynomials and Demazure characters. Duke Math. J. **116**(2), 299–318 (2003)
35. Kashiwara, Masaki: Crystalizing the q-analogue of universal enveloping algebras. Comm. Math. Phys. **133**(2), 249–260 (1990)
36. Kashiwara, Masaki: On crystal bases of the Q-analogue of universal enveloping algebras. Duke Math. J. **63**(2), 465–516 (1991)
37. Kashiwara, Masaki: The crystal base and Littelmann's refined Demazure character formula. Duke Math. J. **71**(3), 839–858 (1993)
38. A.N. Kirillov, *Notes on Schubert, Grothendieck and key polynomials*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), Paper No. 034, 1–56
39. Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318
40. A. Kohnert, *Weintrauben, Polynome, Tableaux*, Bayreuth. Math. Schr. (1991), no. 38, 1–97, Dissertation, Universität Bayreuth, Bayreuth, 1990
41. Witold Kraśkiewicz and Piotr Pragacz, *Foncteurs de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 9, 209–211
42. Kraśkiewicz, Witold, Pragacz, Piotr: Schubert functors and Schubert polynomials. European J. Combin. **25**(8), 1327–1344 (2004)
43. Daniel Krob and Jean-Yves Thibon, *Noncommutative symmetric functions. IV. Quantum linear groups and Hecke algebras at q=0*, J. Algebraic Combin. **6** (1997), no. 4, 339–376
44. Lascoux, A., Transition on Grothendieck polynomials, Physics and combinatorics, : (Nagoya), World Sci. Publ. River Edge, NJ **2001**, 164–179 (2000)
45. Alain Lascoux, *Chern and Yang through ice*, preprint (2002), www-igm.univ-mlv.fr/ al/ARTICLES/ChernYang.ps.gz
46. Cristian Lenart, *Combinatorial aspects of the K-theory of Grassmannians.*, Ann. Comb. **4** (2000), no. 1, 67–82
47. D. E. Littlewood, *The construction of invariant matrices*, Proc. London Math. Soc. (2) **43** (1937), no. 3, 226–240
48. Dudley, E.: Littlewood. Oxford University Press, New York, The Theory of Group Characters and Matrix Representations of Groups (1940)
49. Thomas Lam, Seung Jin Lee, and Mark Shimozono, *Back stable Schubert calculus*, preprint (2018), 63 pages, arXiv:1806.11233
50. Kurt Luoto, Stefan Mykytiuk, and Stephanie van Willigenburg, *An introduction to quasisymmetric Schur functions*, SpringerBriefs in Mathematics, Springer, New York, 2013, Hopf algebras, quasisymmetric functions, and Young composition tableaux

51. Thomas Lam and Pavlo Pylyavskyy, *Combinatorial Hopf algebras and K-homology of Grassmannians*, Int. Math. Res. Not. IMRN (2007), no. 24, Art. ID rnm125, 48
52. Alain Lascoux and Marcel-Paul Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633
53. Alain Lascoux and Marcel-Paul Schützenberger, *Symmetry and flag manifolds.*, Invariant theory, Proc. 1st 1982 Sess. C.I.M.E., Montecatini/Italy, Lect. Notes Math. 996, 118-144 (1983)., 1983
54. Lascoux, A., Schützenberger, M.-P., Keys & standard bases, Invariant theory and tableaux (Minneapolis, MN, : IMA Vol. Math. Appl., vol. 19. Springer, New York **1990**, 125–144 (1988)
55. Lenart, C., Zainoulline, K.: A Schubert basis in equivariant elliptic cohomology. New York J. Math. **23**, 711–737 (2017)
56. Cristian Lenart and Kirill Zainoulline, *Towards generalized cohmology Schubert calculus via formal root polynomials.*, Math. Res. Lett. **24** (2017), no. 3, 839–877
57. I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Astérisque **237** (1996), 189–207, Séminaire Bourbaki, Exp. No. 797 Vol. 1994/95
58. I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015, With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition
59. Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001, Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3
60. Mason, S.: A decomposition of Schur functions and an analogue of the Robinson-Schensted-Knuth algorithm. Sém. Lothar. Combin. **57**, B57e (2008)
61. Sarah K. Mason, *Recent trends in quasisymmetric functions*, Recent trends in algebraic combinatorics, Assoc. Women Math. Ser., vol. 16, Springer, Cham, 2019, pp. 239–279
62. Cara Monical, *Set-valued skyline fillings*, preprint (2016), 17 pages, arXiv:1611.08777
63. Cara Monical, Oliver Pechenik, and Travis Scrimshaw, *Crystal structures for symmetric Grothendieck polynomials*, Transformation Groups, preprint (2018), 47 pages arXiv:1807.03294
64. Cara Monical, Oliver Pechenik, and Dominic Searles, *Polynomials from combinatorial K-theory*, Canad. J. Math., to appear (2019), 35 pages, arXiv:1806.03802
65. Norton, P.N.: 0-Hecke algebras. J. Austral. Math. Soc. Ser. A **27**(3), 337–357 (1979)
66. Jakob Oesinghaus, *Quasisymmetric functions and the Chow ring of the stack of expanded pairs*, Res. Math. Sci. **6** (2019), no. 1, Paper No. 5, 18
67. Opdam, Eric: Harmonic analysis for certain representations of graded Hecke algebras. Acta Math. **1**, 75–121 (1995)
68. Rebecca Patrias, *Antipode formulas for some combinatorial Hopf algebras*, Electron. J. Combin. **23** (2016), no. 4, Paper 4.30, 32
69. Oliver Pechenik, *The genomic Schur function is fundamental-positive*, Ann. Comb. **24**, 95–108 (2020), arXiv:1810.04727, https://link.springer.com/article/10.1007/s00026-019-00483-2
70. Oliver Pechenik and Travis Scrimshaw, *K-theoretic crystals for set-valued tableaux of rectangular shapes*, preprint (2019), 20 pages, arXiv:1904.09674
71. Oliver Pechenik and Dominic Searles, *Decompositions of Grothendieck polynomials*, Int. Math. Res. Not. IMRN (2019), no. 10, 3214–3241
72. A. Pun, *On deposition of the product of Demazure atoms and Demazure characters*, preprint (2016), 86 pages, arXiv:1606.02291
73. Pechenik, Oliver, Yong, Alexander: Genomic tableaux. J. Algebraic Combin. **45**(3), 649–685 (2017)
74. Reiner, V., Shimozono, M.: Key polynomials and a flagged Littlewood-Richardson rule. J. Combin. Theory Ser. A **70**(1), 107–143 (1995)

75. C. Ross and A. Yong, *Combinatorial rules for three bases of polynomials*, Sém. Lothar. Combin. **74** (2015), Art. B74a, 11 pages
76. Yasmine, B.: Sanderson, On the connection between Macdonald polynomials and Demazure characters. J. Algebraic Combin. **11**(3), 269–275 (2000)
77. Dominic Searles, *Polynomial bases: positivity and Schur multiplication*, Trans. Amer. Math. Soc., in press (2019), 29 pages, https://doi.org/10.1090/tran/7670
78. Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin
79. Tewari, Vasu V., van Willigenburg, Stephanie J.: Modules of the 0-Hecke algebra and quasisymmetric Schur functions. Adv. Math. **285**, 1025–1065 (2015)
80. Thomas, Hugh, Yong, Alexander: A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus. Algebra Number Theory **3**(2), 121–148 (2009)
81. R. Vakil, *A geometric Littlewood-Richardson rule*, Ann. of Math. (2) **164** (2006), no. 2, 371–421, Appendix A written with A. Knutson
82. Watanabe, Masaki: Tensor product of Kraśkiewicz-Pragacz modules. J. Algebra **443**, 422–429 (2015)
83. Watanabe, Masaki: An approach towards Schubert positivities of polynomials using Kraśkiewicz-Pragacz modules. European J. Combin. **58**, 17–33 (2016)
84. Anna Weigandt, *Bumpless pipe dreams and alternating sign matrices*, in preparation (2019)
85. George, W.: Whitehead, Elements of homotopy theory, Graduate Texts in Mathematics, vol. 61. Springer-Verlag, New York-Berlin (1978)
86. Winkel, Rudolf: Diagram rules for the generation of Schubert polynomials. J. Combin. Theory Ser. A **86**(1), 14–48 (1999)
87. Rudolf Winkel, *A derivation of Kohnert's algorithm from Monk's rule*, Sém. Lothar. Combin. **48** (2002), Art. B48f, 14 pages

Minuscule Schubert Calculus and the Geometric Satake Correspondence

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Abstract We describe a relationship between work of Gatto, Laksov, and their collaborators on realizations of (generalized) Schubert calculus of Grassmannians, and the geometric Satake correspondence of Lusztig, Ginzburg, and Mirković and Vilonen. Along the way we obtain new proofs of equivariant Giambelli formulas for the ordinary and orthogonal Grassmannians, as well as a simple derivation of the "rim-hook" rule for computing in the equivariant quantum cohomology of the Grassmannian.

Keywords Geometric Satake correspondence · Schubert calculus · Affine Grassmannian · Pfaffian · Equivariant quantum cohomology

1 Introduction

The goal of this article is to illustrate connections between several circles of ideas in Schubert calculus, representation theory, and symmetric functions. The driving force behind the connections we explore is the *geometric Satake correspondence*—which, in the special cases we examine, matches Schubert classes in the Grassmannian (and related spaces) with weight vectors in exterior products (and related representations). We will focus especially on an equivariant version of this correspondence, where additional bases appear on both sides. An analysis of the transition matrices for these bases leads us directly to well-known symmetric functions: the Schur S- and P-functions, along with their factorial generalizations.

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The approach we take is primarily expository, at least in the sense that all the main theorems have appeared previously. However, in describing a relationship among notions which do not often appear side by side, we obtain some novel consequences: new and simple proofs of the equivariant Giambelli formulas for ordinary and orthogonal Grassmannians, as well as of a rule for computing in the equivariant quantum cohomology of the Grassmannian. This perspective should find further applications within Schubert calculus. One theme we wish to emphasize is this: when some aspect of $H_T^*Gr(k,n)$ appears related to exterior algebra, one can expect generalizations via the Satake correspondence, either to other minuscule spaces, or to other subvarieties of the affine Grassmannian.

Our starting point is a very simple observation, which must be quite old. The Grassmannian $Gr(k,n)$ has a decomposition into Schubert cells, indexed by k-element subsets of $[n] := \{1, \dots, n\}$. As $\mathbb{C}$-vector spaces, therefore, one has

$$H_*(Gr(k,n), \mathbb{C}) = H^*(Gr(k,n), \mathbb{C}) = \bigwedge\nolimits^k \mathbb{C}^n, \tag{1}$$

and the cohomology ring $H^*(Gr(k,n), \mathbb{C})$ acts as a certain ring of operators on the exterior power. The rest of this story is an attempt to add as much structure as possible to this identification.

As a first step, for $k = 1$, let us identify the basis of linear subspaces of $\mathbb{P}^{n-1}$ with the standard basis of $\mathbb{C}^n$:

$$[\mathbb{P}^{n-i}] = \varepsilon_i \quad \text{in} \quad H_{n-i}\mathbb{P}^{n-1} = H^{i-1}\mathbb{P}^{n-1}.$$

(Since the spaces we consider have no odd-degree singular (co)homology, we will economize notation by always writing H_i and H^i for singular homology and cohomology in degree $2i$.) This induces a grading on $\mathbb{C}^n$, and makes (1) an isomorphism of graded vector spaces. More generally, writing Ω_I for the Schubert variety corresponding to $I = \{i_1 < \cdots < i_k\} \subseteq [n]$, we identify

$$[\Omega_I] = \varepsilon_I := \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}$$

in $H_*Gr(k,n) = H^*Gr(k,n)$.

To be a little more specific, since it will matter later, these are the *opposite* Schubert varieties: Ω_I is the closure of the cell Ω_I° whose representing $k \times n$ matrices have pivots in columns I, with zeroes to the left of the pivots. For example, in $Gr(3,8)$ we have

$$\Omega^\circ_{\{2,4,7\}} = \begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}.$$

There is a standard bijection between $I \subseteq [n]$ and partitions λ fitting inside the $k \times (n-k)$ rectangle; one puts

$$\lambda_{k+1-j} = i_j - j.$$

Then, writing $\Omega_\lambda = \Omega_I$, the grading is realized by codim $\Omega_\lambda = |\lambda| = \lambda_1 + \cdots + \lambda_k$. We will write σ_λ for the (co)homology class $[\Omega_\lambda]$.

The essential idea is to exploit the isomorphism (1) and use linear algebra to develop the basic ingredients of Schubert calculus—including Giambelli and Pieri formulas. This can be done using elementary (though still nontrivial) methods.

Again, the first step is to examine the easiest case, where $k = 1$. The action of the divisor class $\sigma_\square$ translates into an operator ξ on $\mathbb{C}^n$, given by

$$\xi \cdot \varepsilon_i = \begin{cases} \varepsilon_{i+1} & \text{if } i < n; \\ 0 & \text{if } i = n. \end{cases}$$

So this has matrix

$$\xi = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ 0 & \ddots & \ddots & \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2}$$

and one can view it as an element of the Lie algebra $\mathfrak{gl}_n$ (or $\mathfrak{sl}_n$). This Lie algebra acts naturally on exterior powers $\bigwedge^k \mathbb{C}^n$, and the basic observation is that for any k, $\sigma_\square$ acts on $H^*Gr(k,n)$ (via cup product) just as the matrix ξ acts on $\bigwedge^k \mathbb{C}^n$ (via the Lie algebra action).

For example, consider σ_1 acting on $H^*Gr(2,n)$. We have

$$\begin{aligned} \xi \cdot \varepsilon_1 \wedge \varepsilon_2 &= \varepsilon_2 \wedge \varepsilon_2 + \varepsilon_1 \wedge \varepsilon_3 \\ &= \varepsilon_1 \wedge \varepsilon_3, \end{aligned}$$

corresponding to $\sigma_\square \cdot \sigma_\emptyset = \sigma_{(1)}$. Similarly,

$$\xi \cdot \varepsilon_1 \wedge \varepsilon_3 = \varepsilon_2 \wedge \varepsilon_3 + \varepsilon_1 \wedge \varepsilon_4,$$

corresponding to $\sigma_\square \cdot \sigma_{(1)} = \sigma_{(1,1)} + \sigma_{(2)}$. Note, however, that the Lie algebra action of the matrix ξ^2 does not correspond to multiplication by $\sigma_\square^2$. For instance, the Lie algebra action gives

$$\begin{aligned} \xi^2 \cdot \varepsilon_2 \wedge \varepsilon_3 &= \varepsilon_4 \wedge \varepsilon_3 + \varepsilon_2 \wedge \varepsilon_5 \\ &= \varepsilon_2 \wedge \varepsilon_5 - \varepsilon_3 \wedge \varepsilon_4, \end{aligned}$$

which corresponds to the computation $p_2 \cdot \sigma_{(1,1)} = \sigma_{(3,1)} - \sigma_{(2,2)}$, where p_2 is the *power sum* symmetric polynomial.

The question thus arises: are the above calculations merely coincidental? In the next two sections, we will see that they are not, by viewing them as a shadow of the geometric Satake correspondence, a major construction in modern representation theory.

In Sect. 4, we show how the equivariant Giambelli formula—which computes a Schubert class in $H_T^*Gr(k,n)$ as a factorial Schur polynomial $s_\lambda(x|t)$—follows directly from the defining properties of the exterior power, putting computations of Gatto, Laksov, Thorup, and others into the general context of the Satake correspondence. Having done this, it is natural to proceed to the minuscule spaces of type D: in Sect. 6, we apply similar methods to compute equivariant Schubert classes via the factorial Schur P-functions $P_\lambda(x|t)$, using computations on even-dimensional quadrics (carried out in Sect. 5) in place of projective spaces. The functions $s_\lambda(x|t)$ are frequently defined as a certain "Jacobi-Trudi" determinant, but they may also be written as a ratio of two determinants (as was originally done by Cauchy); similarly, $P_\lambda(x|t)$ may be written either as a Pfaffian or as a ratio of two Pfaffians. A curious aspect of our arguments is that the ratio description of these functions appears naturally, in contrast to most geometric arguments (dating to Giambelli), where the Jacobi-Trudi formulation is used.

We turn to quantum cohomology in Sect. 7, where we give a short proof of the equivariant rim-hook rule for computing in $QH_T^*Gr(k,n)$. Here the Satake isomorphism serves only a psychological function, and is not logically necessary: the main point is that the combinatorial operation of removing a rim hook from a partition (and picking up a corresponding sign) is precisely that of reducing the indices of a pure wedge modulo n.

Most of these ideas have appeared in the work of other authors, at least in some form; as mentioned above, our primary aim is to indicate connections and extract a few new consequences. We first learned of the possibility of a "formal" Schubert calculus on exterior powers from a series of papers by Gatto and Laksov–Thorup in the 2000s, and this point of view has been developed further by these authors and their collaborators [14–17, 30–33]. More detailed references are given throughout the article, and we point to further connections in a closing section (Sect. 8).

2 The Geometric Satake Correspondence

A second simple observation is the following: On one hand, the vector space $\bigwedge^k \mathbb{C}^n$ is a fundamental (and in fact, minuscule) representation V_{ϖ_k} of $\mathfrak{sl}_n$. On the other hand, $Gr(k,n) = PGL_n/P_{\varpi_k}$, where P_{ϖ_k} is the parabolic corresponding[1] to the cocharacter ϖ_k of PGL_n. In fact, if $V = \mathbb{C}^n$ is the standard representation of $\mathfrak{sl}_n$, it is most natural to regard $Gr(k,n) = Gr(k, V^*)$ as parametrizing k-planes in the dual vector space.

[1] In general, a cocharacter $\varpi \colon \mathbb{C}^* \to T \subseteq G$ determines a reductive subgroup $G_\varpi \subseteq G$, the centralizer of its image; the corresponding parabolic P_ϖ is generated by G_ϖ together with the Borel.

Work from the 1990's by Ginzburg [18] and Mirković and Vilonen [41]—which in turn builds on work of Lusztig [35] from the early 1980's—puts this into a more general context. To describe it we need some terminology and basic facts.

Any reductive group G with maximal torus T comes with a root datum R. Root data have a built-in duality, and exchanging R with $R^\vee$ yields a *Langlands dual group* $G^\vee$. The details will not be too important for now, beyond this: the roots of $G^\vee$ are the coroots of G, and the characters of the maximal torus $T^\vee \subseteq G^\vee$ are the co-characters of $T \subseteq G$. For example, $(PGL_n)^\vee \cong SL_n$, $(Sp_{2n})^\vee \cong SO_{2n+1}$, $(PSO_{2n})^\vee = Spin_{2n}$, and $(GL_n)^\vee \cong GL_n$. (A significant part of Ginzburg and Mirković–Vilonen's program was to give a more intrinsic construction of $G^\vee$.)

Let $\mathcal{K} = \mathbb{C}((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$. The *affine Grassmannian* of a complex reductive group G is the infinite-dimensional orbit space

$$\mathcal{G}r_G = G(\mathcal{K})/G(\mathcal{O}),$$

topologized as an ind-variety. The essence of geometric Satake is to relate the geometry of $\mathcal{G}r_G$ with the representation theory of $G^\vee$. We will describe a very small part of this correspondence which suffices for our purposes.

The group $G(\mathcal{O})$ acts on $\mathcal{G}r_G$ via left multiplication, and its orbits are naturally parametrized by *dominant co-characters* $\varpi\colon \mathbb{C}^* \to T$. These, by duality, are the same as dominant characters of $T^\vee$. Since there is a well-known indexing of irreducible representations of $G^\vee$ by dominant characters, we have a bijection of sets

$$\begin{array}{ccc} \{G(\mathcal{O})\text{-orbit closures in } \mathcal{G}r_G\} & \leftrightarrow & \left\{\text{irreducible representations of } G^\vee\right\} \\ \overline{\mathcal{G}r^\varpi} & \leftrightarrow & V_\varpi \end{array}$$

and as before the goal is to endow this with more structure.

Let $\mathfrak{g}^\vee = \mathrm{Lie}(G^\vee)$, and take a regular nilpotent element $\xi \in \mathfrak{g}^\vee$. (Up to conjugation, in $\mathfrak{sl}_n$ such an element is the matrix from (2). More generally, one can write $\xi = \sum a_i E_{\alpha_i}$ as a sum of simple root vectors.) Let $\mathfrak{a} \subseteq \mathfrak{g}^\vee$ be the centralizer of ξ, an abelian Lie subalgebra of dimension equal to the rank of $\mathfrak{g}$. (In the case of $\xi \in \mathfrak{sl}_n$, this subalgebra is spanned by the matrix powers $\xi, \xi^2, \ldots, \xi^{n-1}$.) Its universal enveloping algebra, denoted $\mathfrak{U}(\mathfrak{a})$, acts naturally on any representation of $\mathfrak{a}$. Since $\mathfrak{a}$ is abelian, $\mathfrak{U}(\mathfrak{a})$ is just the polynomial algebra $\mathrm{Sym}^*_{\mathbb{C}}\, \mathfrak{a}$.

Finally, IH_*X denotes the (middle-perversity) intersection homology of a space X, with coefficients in $\mathbb{C}$. This is a graded vector space which exhibits Poincaré duality, and which comes with an action of H^*Y, for any $X \to Y$, via cap product.

Theorem 1 (Geometric Satake [18, 41]) *There are graded isomorphisms of algebras*

$$H^*(\mathcal{G}r_G^\circ) \cong \mathfrak{U}(\mathfrak{a}) = \mathrm{Sym}^*_{\mathbb{C}}\, \mathfrak{a}$$

and vector spaces

$$IH_*(\overline{\mathcal{G}r^\varpi}) \cong V_\varpi,$$

for all dominant ϖ, and these isomorphisms are compatible with the natural actions of $H^*(\mathcal{G}r_G)$ on $IH_*(\overline{\mathcal{G}r^\varpi})$ (via cap product) and of $\mathfrak{U}(\mathfrak{a})$ on V_ϖ (via the representation of $\mathfrak{g}^\vee$). Furthermore, there is a natural basis of MV-cycles in $IH_*(\overline{\mathcal{G}r^\varpi})$ which corresponds to a weight basis of V_ϖ.

The statements proved by Ginzburg and Mirković–Vilonen are vastly stronger: they establish an equivalence of tensor categories between the category of $G(\mathcal{O})$-equivariant perverse sheaves and the representation category of $G^\vee$. We will not need this level of generality, however.

The connected components of $\mathcal{G}r_G$ are indexed by elements of $\pi_1(G)$, and in fact there is a natural group structure on the set of components. In the statement of the theorem, $\mathcal{G}r_G^\circ$ means the identity component, although in fact all components are isomorphic—as spaces, but not compatibly with the left $G(\mathcal{O})$-action.

In general, the orbit closure $\overline{\mathcal{G}r^\varpi}$ is singular, hence the appearance of intersection homology. However, the minimal orbit in each connected component of $\mathcal{G}r_G$ is closed, so such $\overline{\mathcal{G}r^\varpi} = \mathcal{G}r^\varpi$ are smooth, and one has $IH_* = H_* = H^*$. When G is adjoint (so $G^\vee$ is simply connected), these minimal orbits correspond to the *minuscule weights* of $G^\vee$. For a minuscule weight ϖ, one has $\mathcal{G}r^\varpi = G/P_\varpi$. Furthermore, in this case the MV-cycles are precisely the Schubert varieties in G/P_ϖ (as noted in [26, Sect. 1]).

Example 2 The minuscule weights of $\mathfrak{sl}_n$ are $0, \varpi_1, \ldots, \varpi_{n-1}$, corresponding to the n elements of $\pi_1(PGL_n) = \mathbb{Z}/n\mathbb{Z}$. The representations are the exterior powers $\bigwedge^k \mathbb{C}^n$, and the orbits are the Grassmannians $PGL_n/P_{\varpi_k} = Gr(k, n)$, for $0 \le k \le n-1$.

The minuscule weights of $\mathfrak{so}_{2n}$ are $0, \varpi_1, \varpi_{n-1}, \varpi_n$, where the nonzero ones are the fundamental weights corresponding to the three end-nodes of the D_n Dynkin diagram. The representations are the standard one, $V_{\varpi_1} = \mathbb{C}^{2n}$, and the half-spin representations, $V_{\varpi_{n-1}} = \mathbb{S}_n^+$ and $V_{\varpi_n} = \mathbb{S}_n^-$. The orbits are the quadric $Q^{2n-2} = PSO_{2n}/P_{\varpi_1}$, and the two maximal orthogonal Grassmannians $OG^+(n, 2n) = PSO_{2n}/P_{\varpi_{n-1}}$ and $OG^-(n, 2n) = PSO_{2n}/P_{\varpi_n}$.

In type C_n, the weight ϖ_1 corresponding to the standard representation of $\mathfrak{sp}_{2n}$ is minuscule, and in type B_n, the weight ϖ_n corresponding to the spin representation of $\mathfrak{so}_{2n+1}$ is minuscule. The minimal orbits are isomorphic to $\mathbb{P}^{2n-1}$ and $OG^+(n+1, 2n+2)$, respectively, so they already occur in types A and D.

Among simple groups G, there are only a few other instances of nonzero minuscule weights. In type E_6, the weights ϖ_1 and ϖ_6 are minuscule, corresponding to the 27-dimensional Jordan algebra representation and its dual; the corresponding varieties are the octonionic projective plane and its dual. In type E_7, there is one nonzero minuscule weight, whose corresponding representation is the 56-dimensional Brown algebra, and whose corresponding homogeneous space is known as the Freudenthal variety.

As a special case of Theorem 1, we have the isomorphism

$$H_*Gr(k, n) = \bigwedge^k \mathbb{C}^n,$$

together with the compatible actions by divisor class and regular nilpotent, described in the introduction—in particular, it is no coincidence that one can do this. We will push this further to obtain a new perspective on Laksov's computation of the equivariant cohomology of $Gr(k, n)$ in Sect. 4.

Remark 3 Let $\Lambda_{\mathbb{C}}$ be the ring of symmetric functions with coefficients in $\mathbb{C}$. It can be identified with the infinite polynomial ring $\mathbb{C}[p_1, p_2, \ldots]$, where $p_r = x_1^r + x_2^r + \cdots$ is the power sum symmetric function. For $G = PGL_n$, Bott [5] showed that there is a natural map

$$\Lambda_{\mathbb{C}} \to H^*\mathcal{G}r_G^\circ,$$

identifying the RHS as $\Lambda_{\mathbb{C}}/(p_n, p_{n+1}, \ldots) \cong \mathbb{C}[p_1, \ldots, p_{n-1}]$. Furthermore, this identifies the subspace $\mathfrak{P} \subseteq H^*\mathcal{G}r_G^\circ$ of *primitive classes* with the space spanned by $\{p_1, \ldots, p_{n-1}\}$. Ginzburg's proof of the first part of Theorem 1 establishes an isomorphism $\mathfrak{a} \cong \mathfrak{P}$. So the power sum symmetric functions play a central role in this story; we will see an echo of this in the rim-hook rule for quantum cohomology (Sect. 7). (A word of caution: this isomorphism does not hold when one takes cohomology with coefficients in $\mathbb{Z}$. See [5, Proposition 8.1] for a more precise statement.)

3 The Equivariant Correspondence

There is an equivariant version of Theorem 1, whose proof is sketched in [18]. We will write t for the generic element of the Cartan subalgebra $\mathfrak{t}^\vee \subseteq \mathfrak{g}^\vee$ and use the notation $\mathfrak{g}^\vee[t] = \mathfrak{g}^\vee \otimes \mathbb{C}[\mathfrak{t}^\vee]$ for the Lie algebra over the polynomial ring. Given a $\mathfrak{g}^\vee$-module V, there is an induced $\mathfrak{g}^\vee[t]$-module $V[t] := V \otimes \mathbb{C}[\mathfrak{t}^\vee]$, where the action is given by

$$(x \otimes f) \cdot (v \otimes g) = (x \cdot v) \otimes (fg),$$

for $x \in \mathfrak{g}^\vee$, $v \in V$, and $f, g \in \mathbb{C}[\mathfrak{t}^\vee]$.

Next suppose $\mathfrak{b}^\vee \subseteq \mathfrak{g}^\vee$ is a Borel subalgebra containing $\mathfrak{t}^\vee$. Any character χ of $\mathfrak{t}^\vee$ extends to one of $\mathfrak{b}^\vee$, and also to $\mathfrak{b}^\vee[t]$. If V is a $\mathfrak{b}^\vee$-module, we can twist it by the character χ to obtain modules $V(\chi)$ and $V(\chi)[t]$ for $\mathfrak{b}^\vee$ and $\mathfrak{b}^\vee[t]$, respectively. Concretely, if one writes an element of $\mathfrak{b}^\vee = \mathfrak{n}^\vee \oplus \mathfrak{t}^\vee$ as $x = n + t$, then for $f, g \in \mathbb{C}[\mathfrak{t}^\vee]$ and $v \in V(\chi)$ a weight vector for $\mathfrak{t}^\vee$, we have

$$\begin{aligned}(x \otimes f) \cdot (v \otimes g) &= (n \otimes f + t \otimes f) \cdot (v \otimes g)\\ &= (n \cdot v) \otimes (fg) + (t \cdot v) \otimes (fg) + \chi(t \otimes f)(v \otimes g).\end{aligned}$$

Now let $\xi_t = \xi - t$ in $\mathfrak{g}^\vee[t] = \mathfrak{g}^\vee \otimes \mathbb{C}[\mathfrak{t}^\vee]$, where ξ is a principal nilpotent as before, and t is the generic element of the Cartan subalgebra $\mathfrak{t}^\vee \subseteq \mathfrak{g}^\vee$. Concretely, for $\mathfrak{gl}_n$ this is

$$\xi_t = \begin{bmatrix} -t_1 & & & \\ 1 & \ddots & & \\ 0 & \ddots & \ddots & \\ 0 & 0 & 1 & -t_n \end{bmatrix}. \tag{3}$$

Let $\mathfrak{a}_t \subseteq \mathfrak{g}^\vee[t]$ be the centralizer of ξ_t. For $\mathfrak{gl}_n$, this subalgebra is spanned over $\mathbb{C}[\mathfrak{t}^\vee] = \mathrm{Sym}^* \mathfrak{t}^\vee$ by the matrix powers $1, \xi_t, \xi_t^2, \ldots, \xi_t^{n-1}$.

Theorem 4 (Equivariant Geometric Satake) *There are isomorphisms*

$$H_T^* \mathcal{G}r_G^\circ \cong \mathfrak{U}_{\mathbb{C}[\mathfrak{t}^\vee]}(\mathfrak{a}_t) = \mathrm{Sym}^*_{\mathbb{C}[\mathfrak{t}^\vee]} \mathfrak{a}_t$$

inducing compatible actions on

$$IH_*^T(\overline{\mathcal{G}r^\varpi}) \cong V_\varpi(-\varpi)[t].$$

The effect of twisting by the character $-\varpi$ is to move the highest weight vector of V_ϖ to weight zero. This corresponds to endowing $IH_*^T(\overline{\mathcal{G}r^\varpi})$ with a $\mathfrak{t}^\vee$-module structure so that the fundamental class $[\overline{\mathcal{G}r^\varpi}]$ has weight 0. This choice has the advantage of identifying the action of the element ξ_t with equivariant multiplication by the divisor class $\sigma_\square$. (Of course, a similar isomorphism holds without the twist; noting that $\sigma_\square = c_1^T(\mathcal{O}(1) \otimes (-\varpi))$, where $\mathcal{O}(1)$ is the ample line bundle corresponding to the weight ϖ, the untwisted version identifies the action of ξ_t with multiplication by $c_1^T(\mathcal{O}(1))$.)

Example 5 For $\mathfrak{g} = \mathfrak{gl}_n$, the action of ξ_t on $V_{\varpi_2}(-\varpi_2)[t] = (\bigwedge^2 \mathbb{C}^n \otimes (-t_1 - t_2)) \otimes \mathbb{C}[\mathfrak{t}^\vee]$ is as follows.

$$\begin{aligned} \xi_t \cdot (\varepsilon_1 \wedge \varepsilon_2) &= \varepsilon_2 \wedge \varepsilon_2 + \varepsilon_1 \wedge \varepsilon_3 - (t_1 + t_2 - t_1 - t_2)\varepsilon_1 \wedge \varepsilon_2 \\ &= \varepsilon_1 \wedge \varepsilon_3, \end{aligned}$$

corresponding to $\sigma_\square \cdot \sigma_\emptyset = \sigma_{(1)}$ in $H_T^* Gr(2, n)$. (The cancellation of the last term shows why the twist by $-\varpi$ is necessary.) Similarly,

$$\begin{aligned} \xi_t \cdot (\varepsilon_1 \wedge \varepsilon_3) &= \varepsilon_2 \wedge \varepsilon_3 + \varepsilon_1 \wedge \varepsilon_4 - (t_1 + t_3 - t_1 - t_2)\varepsilon_1 \wedge \varepsilon_3 \\ &= \varepsilon_2 \wedge \varepsilon_3 + \varepsilon_1 \wedge \varepsilon_4 + (t_2 - t_3)\varepsilon_1 \wedge \varepsilon_3, \end{aligned}$$

corresponding to $\sigma_\square \cdot \sigma_{(1)} = \sigma_{(1,1)} + \sigma_{(2)} + (t_2 - t_3)\sigma_{(1)}$.

As in the non-equivariant case, one needs to beware of the notation: matrix powers ξ_t^j do not correspond to iterates of the Lie algebra action, e.g., $\xi_t^2 \cdot \varepsilon_I$ is generally not equal to $\xi_t \cdot (\xi_t \cdot \varepsilon_I)$.

Example 6 Still in the case $\mathfrak{g} = \mathfrak{gl}_n$, let us consider higher powers of ξ_t. One computes the entries of the matrix powers as

$$\begin{aligned}\xi_t^j \varepsilon_i &= \varepsilon_{i+j} - h_1(t_i, \ldots, t_{i+j-1})\, \varepsilon_{i+j-1} + \cdots + (-1)^j\, h_j(t_i)\, \varepsilon_i \\ &= \sum_{a=0}^{j} (-1)^a\, h_a(t_i, \ldots, t_{i+j-a})\, \varepsilon_{i+j-a},\end{aligned}$$

where the h_a are complete homogeneous symmetric polynomials in the indicated variables. (That is, the $(i + j - a, i)$ matrix entry of ξ_t^j is $(-1)^a h_a(t_i, \ldots, t_{i+j-a})$.) Incorporating the twist by $-\varpi_k$, the Lie algebra action on $V_{\varpi_k}(-\varpi_k)[t]$ is

$$\begin{aligned}\xi_t^j \cdot \varepsilon_I = &\left(\sum_{a=0}^{j} (-1)^a\, h_a(t_{i_1}, \ldots, t_{i_1+j-a})\, \varepsilon_{i_1+j-a}\right) \wedge \varepsilon_{i_2} \wedge \cdots \wedge \varepsilon_{i_k} \\ &+ \varepsilon_{i_1} \wedge \left(\sum_{a=0}^{j} (-1)^a\, h_a(t_{i_2}, \ldots, t_{i_2+j-a})\, \varepsilon_{i_2+j-a}\right) \wedge \cdots \wedge \varepsilon_{i_k} \\ &+ \cdots + \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots \wedge \left(\sum_{a=0}^{j} (-1)^a\, h_a(t_{i_k}, \ldots, t_{i_k+j-a})\, \varepsilon_{i_k+j-a}\right) \\ &- ((-t_1)^j + \cdots + (-t_k)^j)\, \varepsilon_I.\end{aligned}$$

For instance,

$$\begin{aligned}\xi_t^2 \cdot (\varepsilon_2 \wedge \varepsilon_3) &= (\varepsilon_4 - (t_2 + t_3)\, \varepsilon_3 + t_2^2\, \varepsilon_2) \wedge \varepsilon_3 + \varepsilon_2 \wedge (\varepsilon_5 - (t_3 + t_4)\, \varepsilon_4 + t_3^2\, \varepsilon_3) \\ &\quad - (t_1^2 + t_2^2)\, \varepsilon_2 \wedge \varepsilon_3 \\ &= \varepsilon_2 \wedge \varepsilon_5 - \varepsilon_3 \wedge \varepsilon_4 - (t_3 + t_4)\, \varepsilon_2 \wedge \varepsilon_4 + (t_3^2 - t_2^2)\, \varepsilon_2 \wedge \varepsilon_3.\end{aligned}$$

The leading term agrees with the computation $p_2 \cdot \sigma_{(1,1)} = \sigma_{(3,1)} - \sigma_{(2,2)}$ in $H^*Gr(2, n)$ done in the introduction.

A new feature appears in the equivariant correspondence. Let us pass to the fraction field $\mathbb{C}(\mathfrak{t}^\vee)$, and consider $\mathfrak{g}^\vee(t)$, etc., as Lie algebras over this field. Since the element ξ_t is regular semisimple, its centralizer $\mathfrak{h} \subseteq \mathfrak{g}^\vee(t)$ is a Cartan subalgebra. (In fact, $\mathfrak{h}$ is just the extension of $\mathfrak{a}_t$ to $\mathbb{C}(\mathfrak{t}^\vee)$.) So our setup leads naturally to another basis for $V_\varpi(-\varpi)(t)$, a basis of weight vectors for $\mathfrak{h}(t)$, diagonalizing ξ_t.

What is this basis on the geometric side of the correspondence? By the localization theorem (see [22, (6.3)]), there is a fixed-point basis for $IH_*^T(\mathcal{G}r^\varpi) \otimes \mathbb{C}(\mathfrak{t}^\vee)$, and in fact this basis corresponds to a (suitably chosen) weight basis for $\mathfrak{h}$.

Theorem 7 (Equivariant Satake, continued) *Under the Satake isomorphism $IH_*^T(\overline{\mathcal{G}r^\varpi}) \cong V_\varpi(-\varpi)[t]$, equivariant MV-cycles correspond to a weight basis of $V_\varpi(-\varpi)[t]$ with respect to $\mathfrak{t}^\vee$, and the fixed point basis corresponds to a weight basis with respect to $\mathfrak{h}$.*

In general, there is ambiguity in choosing a weight basis. However, for minuscule ϖ, all weight spaces of V_ϖ are one-dimensional, so a weight basis is determined (up to scaling) by the Cartan. As noted before, in this case $\mathcal{G}r^{\varpi} = G/P_{\varpi}$ is homogeneous, and the MV basis consists of (opposite) Schubert classes. This is the situation we will consider for the remainder of the paper. Let us write $X = G/P_{\varpi}$.

Let us write $\{\sigma_\lambda\}$ for the basis of Schubert classes in H_T^*X. The fixed point set is $X^T = \{p_\lambda\}$ (with λ running over the same set indexing Schubert classes), and we will write $\{\mathbf{1}_\lambda\}$ for the corresponding idempotent basis of $H_T^*X^T = \bigoplus H_T^*(p_\lambda)$. The localization theorem says that the restriction homomorphism (of $\mathbb{C}[\mathfrak{t}^\vee]$-algebras)

$$\iota^*\colon H_T^*X \to H_T^*X^T$$

becomes an isomorphism after tensoring with $\mathbb{C}(\mathfrak{t}^\vee)$. An important part of equivariant Schubert calculus is to compute the restriction of a Schubert class σ_λ to a fixed point p_μ. Formulas for these restrictions have been given by Billey for complete flag varieties [4], and by Ikeda-Naruse, who consider special cases that are related to the focus of this article [24].

The fact that the fixed-point classes form a basis of eigenvectors for ξ_t is part of a general phenomenon, with a simple proof. Consider any nonsingular variety X with finite fixed locus X^T, and any class $\alpha \in (H_T^*X) \otimes \mathbb{C}(\mathfrak{t}^\vee)$.

Lemma 8 *The idempotent classes $\mathbf{1}_p \in (H_T^*X) \otimes \mathbb{C}(\mathfrak{t}^\vee) = (H_T^*X^T) \otimes \mathbb{C}(\mathfrak{t}^\vee)$ form a basis of eigenvectors for the endomorphism $x \mapsto \alpha \cdot x$.*

This is almost a tautology. Simply observe that for distinct fixed points $p \neq q$, we have $\mathbf{1}_p \cdot \mathbf{1}_q = 0$. Writing $\alpha = \sum \alpha_p \cdot \mathbf{1}_p$, the statement follows.

Returning to minuscule Schubert calculus, the restrictions $\sigma_\lambda|_\mu$ may be regarded as matrix entries for the homomorphism ι^*, with respect to the Schubert and fixed-point bases. The Satake correspondence translates the problem of computing this matrix into the following:

> Find the change-of-basis matrix relating weight bases of the minuscule representation $V_{\varpi}(-\varpi)(t)$, with respect to two (specific) Cartan subalgebras, $\mathfrak{t}^\vee$ and $\mathfrak{h}$, of $\mathfrak{g}^\vee(t)$.

This perspective also suggests a framework for setting up and solving the problem of computing restrictions of Schubert classes $\sigma_\lambda|_\mu$. We will work this out in types A and D below.

4 A Giambelli Formula for Grassmannians

We will describe a proof of the "equivariant Giambelli formula"

$$\sigma_\lambda = s_\lambda(x|t) \tag{4}$$

identifying the Schubert class $\sigma_\lambda \in H_T^*Gr(k,n)$ with a *factorial Schur polynomial*, in the spirit of Laksov's approach to equivariant Schubert calculus [30]. The following definition of the factorial Schur polynomial can be found in Macdonald's book [36, Sect. I.3, Ex. 20]. The *(generalized) factorial power* is defined as

$$(x|t)^a = (x + t_1)(x + t_2) \cdots (x + t_a).$$

Let $I = \{i_1, \ldots, i_k\} \subseteq [n]$ be the subset corresponding to the partition λ; recall that this means $\lambda_{k+1-a} = i_a - a$. One defines

$$s_\lambda(x|t) = \frac{\det\left((x_j|t)^{i-1}\right)_{i\in I, 1\leq j\leq k}}{\det\left((x_j|t)^{i-1}\right)_{1\leq i,j\leq k}}.$$

An easy computation shows the denominator is

$$\det\left((x_j|t)^{i-1}\right)_{1\leq i,j\leq k} = \det(x_j^{i-1}) = \prod_{1\leq a<b\leq k}(x_a - x_b) =: \Delta,$$

the Vandermonde determinant, so the factorial Schur polynomial can also be written as

$$s_\lambda(x|t) = \frac{\det\left((x_j|t)^{i-1}\right)_{i\in I, 1\leq j\leq k}}{\Delta}.$$

The meaning of the Giambelli formula is this. By the localization theorem, the equivariant cohomology of $Gr(k, n)$ embeds in that of its fixed locus:

$$H_T^* Gr(k, n) \hookrightarrow H_T^*(Gr(k, n)^T) = \bigoplus_J \mathbb{C}[t],$$

the sum being over all k-element subsets $J \subset [n]$. On the other hand, there is a presentation of $H_T^* Gr(k, n)$ as a quotient of $\mathbb{C}[t][x_1, \ldots, x_k]^{S_k}$ (symmetric polynomials in x, with coefficients in $\mathbb{C}[t]$). Composing with the localization homomorphism gives

$$\mathbb{C}[t][x_1, \ldots, x_k]^{S_k} \to \bigoplus_J \mathbb{C}[t],$$

defined on the Jth summand by sending $x_a \mapsto -t_{j_a}$. The precise statement is this:

Theorem 9 *Under the homomorphism $\mathbb{C}[t][x_1, \ldots, x_k]^{S_k} \to H_T^* Gr(k, n)$, we have $s_\lambda(x|t) \mapsto \sigma_\lambda$. Equivalently, for each $J = \{j_1 < \cdots < j_k\}$, we have*

$$\sigma_\lambda|_{p_J} = s_\lambda(-t_{j_1}, \ldots, -t_{j_k}|t).$$

Proof There are three simple steps. We describe them informally first, since we will follow the same pattern in proving a type D formula later.

(1) Work out the case $k = 1$, corresponding to projective space. Here, by construction, the element ξ_t corresponds to multiplication by the hyperplane class $\sigma_\square$

on $H_T^*\mathbb{P}^{n-1}$, written in the Schubert basis $\varepsilon_i = [\mathbb{P}^{n-i}]$. We choose a basis $\overline{f}_i$ diagonalizing the semisimple element ξ_t; by Lemma 8, this basis coincides with the basis of idempotents $\mathbf{1}_i$, up to scalar. We normalize the $\overline{f}_i$ so that $\overline{f}_i = \mathbf{1}_i$, by requiring $\varepsilon_1 = \overline{f}_1 + \cdots + \overline{f}_n$ (since ε_1 corresponds to $\mathbf{1} \in H_T^*\mathbb{P}^{n-1}$). The expansion of ε_i in the $\overline{f}_j$ basis is then a localization calculation, which is easy for projective space.

(2) For each $k > 1$, take the weight basis $\{\varepsilon_I\}$ of $V_{\varpi_k}(-\varpi_k)[t] = \bigwedge_{\mathbb{C}[t]}^k \mathbb{C}[t]^n$ to be $\varepsilon_I = \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}$. Verify that the action of ξ_t agrees with the known formula for multiplication by $\sigma_{\square}$ on the Schubert basis of $H_T^*Gr(k,n)$, so that we can identify $\sigma_I = \varepsilon_I$. (The latter formula is often called the *equivariant Chevalley formula*.)

(3) By Lemma 8 again, the vectors $\overline{f}_{j_1} \wedge \cdots \wedge \overline{f}_{i_k}$ agree with the basis of idempotents $\mathbf{1}_J$, up to scalar; normalize it so that $\overline{f}_J = \mathbf{1}_J$ by requiring $\varepsilon_{\{1,\ldots,k\}} = \sum \overline{f}_J$ (since $\varepsilon_{\{1,\ldots,k\}}$ corresponds to $\mathbf{1} \in H_T^*Gr(k,n)$). On the other hand, formulas from Step (1) expressing ε_i in terms of $\overline{f}_j$ yield (determinantal) formulas for ε_I in terms of $\overline{f}_{j_1} \wedge \cdots \wedge \overline{f}_{j_k}$; comparing with the normalized vectors $\overline{f}_J$ proves the theorem.

Now we proceed to work this out in detail. It is not hard to see that

$$f_i = \varepsilon_i + \frac{1}{(t_{i+1} - t_i)}\varepsilon_{i+1} + \cdots + \frac{1}{(t_n - t_i)\cdots(t_{i+1} - t_i)}\varepsilon_n$$

is a basis of eigenvectors for ξ_t acting on $\mathbb{C}^n \otimes \mathbb{C}(t)$ (inverting nonzero characters). This is related to ε_i by a unitriangular change of basis. However, note that $\sigma_i|_{p_i} = (t_1 - t_i)\cdots(t_{i-1} - t_i)$ (since $\Omega_i = \mathbb{P}^{n-i}$ is defined by the vanishing of the first $i-1$ coordinates); this means that we must rescale to obtain the idempotent basis. In fact,

$$\overline{f}_i = \frac{1}{(t_1 - t_i)\cdots(t_{i-1} - t_i)} f_i$$

identifies with the idempotent basis $\mathbf{1}_i \in H_T^*(\mathbb{P}^{n-1})^T$. Since we know the restrictions of $\varepsilon_i = [\mathbb{P}^{n-i}]$ to fixed points, we see

$$\begin{aligned} \varepsilon_i &= \sum_j (t_1 - t_j)\cdots(t_{i-1} - t_j)\,\overline{f}_j \\ &= \sum_j (x_j|t)^{i-1}|_{x_j = -t_j}\,\overline{f}_j. \end{aligned}$$

using the generalized factorial power notation. This completes the first step.

For the second step, we take $\varepsilon_I = \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}$ as our basis for $V_{\varpi_k}(-\varpi_k)[t] = H_T^*Gr(k,n)$. The verification that ξ_t acts on this basis as $\sigma_{\square}$ does on the Schubert basis is left to the reader. (Illustrative examples were done above.) We note that the Chevalley formula says

$$\sigma_{\Box} \cdot \sigma_I = \sum_{I^+} \sigma_{I^+} + (t_1 + \cdots + t_k - t_{i_1} - \cdots - t_{i_k})\sigma_I,$$

where the sum is over I^+ obtained from I by replacing some $i_a \in I$ such that $i_a + 1 \notin I$ by $i_a + 1$. (In terms of the corresponding partitions, $\lambda(I^+)$ is obtained from $\lambda(I)$ by adding a single box.)

Finally, from the definition of exterior product, we get

$$\varepsilon_I := \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k} = \sum_J \det\left((x_j|t)^{i-1}|_{x_j=-t_j}\right)_{i\in I,\, j\in J} \overline{f}_{j_1} \wedge \cdots \wedge \overline{f}_{j_k}.$$

In particular,

$$\begin{aligned}\varepsilon_{\{1,\ldots,k\}} &= \sum_J \det\left((x_j|t)^{i-1}|_{x_j=-t_j}\right)_{1\le i\le k,\, j\in J} \overline{f}_{j_1} \wedge \cdots \wedge \overline{f}_{j_k} \\ &= \sum_J \Delta_J \overline{f}_{j_1} \wedge \cdots \wedge \overline{f}_{j_k},\end{aligned}$$

where

$$\Delta_J = \prod_{1\le a<b\le k} (t_{j_a} - t_{j_b})$$

is the specialization of the Vandermonde determinant Δ. Since $\varepsilon_{\{1,\ldots,k\}}$ should be identified with $\mathbf{1} = [Gr(k,n)]$ in $H_T^*Gr(k,n)$, this tells us that the idempotent classes are

$$\mathbf{1}_J = \overline{f}_J := \Delta_J \overline{f}_{j_1} \wedge \cdots \wedge \overline{f}_{j_k},$$

and we can rewrite the above formula as

$$\begin{aligned}\varepsilon_I := \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k} &= \sum_J \left(\frac{\det\left((x_j|t)^{i-1}|_{x_j=-t_j}\right)_{i\in I,\, j\in J}}{\Delta_J}\right) \overline{f}_J \\ &= \sum_J s_\lambda(x_1,\ldots,x_k|t)|_{x_a=-t_{j_a}} \overline{f}_J,\end{aligned}$$

as required.

Remark 10 It is hard to give clear attribution to the equivariant Giambelli formula; certainly it was known by around 2000. Reference to it appears in [28], and a proof is in [40]. In retrospect, the Kempf–Laksov formula [27] is equivalent to (4). See also [1] for more discussion and an alternative proof.

Likewise, it is difficult to identify the earliest appearance of the connection between the $\mathfrak{gl}_n$-module $\bigwedge^k \mathbb{C}^n$ and the cohomology of $H^*Gr(k,n)$. While surely known long before, it appears in several sources by the 2000s [14, 26, 30, 48].

Remark 11 Recall that the Grassmannian $Gr(k,n) = Gr(k, V^*)$ embeds naturally in $\mathbb{P}(\bigwedge^k V^*)$ as the locus of "pure wedges" $v_1 \wedge \cdots \wedge v_k$. Since our basis $\varepsilon_1, \ldots, \varepsilon_n$ for V is a weight basis for $\mathfrak{t}^\vee$, the dual basis $\varepsilon_1^*, \ldots, \varepsilon_n^*$ is a weight basis for the action of T on V^*; the points of $Gr(k, V^*)$ corresponding to $\varepsilon_I^* = \varepsilon_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$ are therefore precisely the T-fixed points. Thus the Satake correspondence exchanges T-fixed points in $Gr(k, V^*) \subseteq \mathbb{P}(\bigwedge^k V^*)$ with Schubert classes in $\mathbb{P}(H^*Gr(k, V)) = \mathbb{P}(\bigwedge^k V)$. When considered as cohomology classes on $Gr(k, V^*)$, do pure wedges in $\mathbb{P}(\bigwedge^k V)$ have a natural geometric meaning? What does this correspondence look like when upgraded to the equivariant setting?

5 Quadrics

As another example, we describe the correspondence for minuscule varieties of type D. To set things up, fix a basis

$$\varepsilon_{\overline{n-1}}, \ldots, \varepsilon_{\overline{1}}, \varepsilon_{\overline{0}}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$$

for $V \cong \mathbb{C}^{2n}$, and equip this vector space with the symmetric bilinear form defined by $\langle \varepsilon_{\overline{\imath}}, \varepsilon_j \rangle = \delta_{i,j}$. (The barred indices should be regarded as notation for negative integers.) The form identifies the dual basis for $V^* = V$ as $\varepsilon_i^* = \varepsilon_{\overline{\imath}}$. The subspaces $E = \operatorname{span}\{\varepsilon_0, \ldots, \varepsilon_{n-1}\}$ and $\overline{E} = \operatorname{span}\{\varepsilon_{\overline{n-1}}, \ldots, \varepsilon_{\overline{0}}\}$ are maximal isotropic subspaces of V, and we have $V = \overline{E} \oplus E$.

We will take $\mathfrak{so}_{2n} \subseteq \mathfrak{sl}_{2n}$ to be the algebra preserving the given bilinear form; there is also a canonical identification $\mathfrak{so}_{2n} = \bigwedge^2 V$ (see, e.g., [12, Sect. 20]). We take our principal nilpotent element ξ and generic t so that

$$\xi_t = \xi - t = \begin{bmatrix} t_{n-1} & & & & & & & \\ -1 & \ddots & & & & & & \\ & \ddots & t_1 & & & & & \\ & & -1 & t_0 & 0 & & & \\ & & -1 & 0 & -t_0 & & & \\ & & & 1 & 1 & \ddots & & \\ & & & & & \ddots & -t_{n-2} & \\ & & & & & & 1 & -t_{n-1} \end{bmatrix}. \tag{5}$$

Using $\varepsilon_{\overline{\imath}}^* = \varepsilon_i$, ξ can also be written as

$$\xi = -\sum_{i=1}^{n-1} \varepsilon_{\bar{i}}^* \otimes \varepsilon_{\overline{i-1}} - \varepsilon_{\bar{1}}^* \otimes \varepsilon_0 + \varepsilon_{\bar{0}}^* \otimes \varepsilon_1 + \sum_{i=1}^{n-1} \varepsilon_{i-1}^* \otimes \varepsilon_i$$
$$= \varepsilon_0 \wedge \varepsilon_1 + \sum_{i=1}^{n-1} \varepsilon_{\overline{i-1}} \wedge \varepsilon_i,$$

which exhibits it as an element of $\bigwedge^2 V$. Similarly, we have

$$\xi_t = \varepsilon_0 \wedge \varepsilon_1 + \sum_{i=1}^{n-1} \varepsilon_{\overline{i-1}} \wedge \varepsilon_i - \sum_{i=0}^{n-1} t_i\, \varepsilon_{\bar{i}} \wedge \varepsilon_i. \tag{6}$$

Note that our indexing conventions and choice of form $\langle\ ,\ \rangle$ make it natural to identify elements of $\bigwedge^2 V$ with matrices which are skew-symmetric about the *anti-diagonal*.

The odd matrix powers $\xi_t, \xi_t^3, \ldots, \xi_t^{2n-3}$ all lie in $\mathfrak{so}_{2n}$ as well, and they are easily seen to be linearly independent elements of the centralizer $\mathfrak{a}_t$ of ξ_t. Since ξ_t is regular, one knows dim $\mathfrak{a}_t = n$; the missing element is

$$\eta_t = -\sum_{j=1}^{n-1}(t_{j+1}\cdots t_{n-1})\varepsilon_0 \wedge \varepsilon_j + \sum_{0\le i\le j\le n-1} (t_0 \cdots t_{i-1} t_{j+1} \cdots t_{n-1})\varepsilon_{\bar{i}} \wedge \varepsilon_j. \tag{7}$$

For example, when $n = 4$, this is

$$\eta_t = \begin{bmatrix} -t_0t_1t_2 &&&&&&& \\ -t_0t_1 & -t_0t_1t_3 &&&&&& \\ -t_0 & -t_0t_3 & -t_0t_2t_3 &&&&& \\ -1 & -t_3 & -t_2t_3 & -t_1t_2t_3 &&&& \\ 1 & t_3 & t_2t_3 & 0 & t_1t_2t_3 &&& \\ 0 & 0 & 0 & -t_2t_3 & t_2t_3 & t_0t_2t_3 && \\ 0 & 0 & 0 & -t_3 & t_3 & t_0t_3 & t_0t_1t_3 & \\ 0 & 0 & 0 & -1 & 1 & t_0 & t_0t_1 & t_0t_1t_2 \end{bmatrix}.$$

When discussing homogeneous spaces, we will assume $n \ge 3$ to avoid setting conventions for special cases.[2] Consider the $(2n-2)$-dimensional quadric $\mathcal{Q} = \mathcal{Q}^{2n-2} \subseteq \mathbb{P}(V^*) \cong \mathbb{P}^{2n-1}$ of isotropic vectors for the given bilinear form. (In coordinates, $\mathcal{Q}$ is defined by the vanishing of the quadratic form $\sum_{i=0}^{n-1} X_{\bar{i}} X_i$, where $X_i = \varepsilon_i^*$.) The torus $T \cong (\mathbb{C}^*)^n$ acts on V with weights $-t_{n-1}, \ldots, -t_0, t_0, \ldots, t_{n-1}$, inducing an action on $\mathcal{Q}$.

The quadric $\mathcal{Q}$ is homogeneous for PSO_{2n}, and the Satake correspondence identifies $H_T^*\mathcal{Q}$ with $V_{\varpi_1}(-\varpi_1)[t]$, where $\varpi_1 = -t_{n-1}$ and $V_{\varpi_1} = V$ is the standard

[2]When $n = 1$, the spaces are 0-dimensional; when $n = 2$, they coincide with type A spaces. For $n = 3$, there are coincidences $\mathcal{Q}^4 = Gr(2,4)$ and $OG^+(3,6) = OG^-(3,6) = \mathbb{P}^3$. For $n = 4$, there are also coincidences $\mathcal{Q}^6 = OG^+(4,8) = OG^-(4,8)$. The reader may use these to verify our claims, but beware that the torus actions are usually written differently.

representation of $\mathfrak{so}_{2n}$. To see this explicitly, define Schubert varieties in $\mathcal{Q}$ by

$$\Omega_{\bar{\imath}} = \{\varepsilon^*_{\overline{n-1}} = \cdots = \varepsilon^*_{\overline{i+1}} = 0\},$$
$$\Omega_i = \{\varepsilon^*_{\overline{n-1}} = \cdots = \varepsilon^*_{\bar{0}} = \varepsilon^*_0 = \cdots = \varepsilon^*_{i-1} = 0\}$$

for $i > 0$; and

$$\Omega_{\bar{0}} = \{\varepsilon^*_{\overline{n-1}} = \cdots = \varepsilon^*_{\bar{1}} = \varepsilon^*_{0} = 0\}$$
$$\Omega_0 = \{\varepsilon^*_{\overline{n-1}} = \cdots = \varepsilon^*_{\bar{1}} = \varepsilon^*_{\bar{0}} = 0\}.$$

Identify the Schubert classes $\sigma_i = [\Omega_i]$ in $H^T_{n-1-i}\mathcal{Q} = H_T^{n-1+i}\mathcal{Q}$ with basis elements ε_i by

$$\sigma_{\bar{\imath}} = (-1)^i\, \varepsilon_{\bar{\imath}} \quad \text{and} \quad \sigma_i = \varepsilon_i, \tag{8}$$

for $i \geq 0$. Using the twist by t_{n-1}, we have

$$\xi_t \cdot \varepsilon_{\bar{\imath}} = -\varepsilon_{\overline{i+1}} - (-t_i + t_{n-1})\varepsilon_{\bar{\imath}} \qquad \text{for } i > 1,$$
$$\xi_t \cdot \varepsilon_{\bar{1}} = -\varepsilon_{\bar{0}} - \varepsilon_0 - (-t_1 + t_{n-1})\varepsilon_{\bar{1}},$$
$$\xi_t \cdot \varepsilon_{\bar{0}} = \varepsilon_1 - (-t_0 + t_{n-1})\varepsilon_{\bar{0}}, \quad \text{and}$$
$$\xi_t \cdot \varepsilon_i = \varepsilon_{i+1} - (t_i + t_{n-1})\varepsilon_i \qquad \text{for } i \geq 0.$$

On the other hand, taking $\sigma_\square = \sigma_{\overline{n-2}} \in H^1_T\mathcal{Q}$ to be the hyperplane class,

$$\sigma_\square \cdot \sigma_{\bar{\imath}} = \sigma_{\overline{i+1}} + (t_i - t_{n-1})\sigma_{\bar{\imath}} \qquad \text{for } i > 1,$$
$$\sigma_\square \cdot \sigma_{\bar{1}} = \sigma_{\bar{0}} + \sigma_0 + (t_1 - t_{n-1})\sigma_{\bar{1}},$$
$$\sigma_\square \cdot \sigma_{\bar{0}} = \sigma_1 + (t_0 - t_{n-1})\sigma_{\bar{0}}, \quad \text{and}$$
$$\sigma_\square \cdot \sigma_i = \sigma_{i+1} + (-t_i - t_{n-1})\sigma_i \qquad \text{for } i \geq 0,$$

so (8) compatibly identifies the action of ξ_t with the product by $\sigma_\square$.

Iterating the above computation of $\xi_t \cdot \varepsilon_i$ leads to a formula for the matrix entries of ξ_t^{2j-1}, for $j = 1, \ldots, n-1$:

$$\begin{aligned}\xi_t^{2j-1} &= \sum_{k=0}^{n-1}\sum_{i=0}^{n-1-k} (-1)^{2j-1-k} h_{2j-1-k}(t_i,\dots,t_{i+k})\,\varepsilon_{\bar{\imath}}\wedge\varepsilon_{i+k} \\ &\quad+\sum_{k=1}^{n-1}(-1)^{2j-1-k}h_{2j-1-k}(-t_0,t_1,\dots,t_k)\,\varepsilon_0\wedge\varepsilon_k \\ &\quad+2\sum_{i=1}^{n-2}\sum_{k=i+1}^{n-1}(-1)^{2j-1-k}h_{2j-1-i-k}(-t_i,-t_{i-1},\dots,-t_0,t_0,\dots,t_k)\,\varepsilon_i\wedge\varepsilon_k,\end{aligned} \tag{9}$$

where in the last sum, the complete homogeneous symmetric functions are in $i+k+2$ variables, specialized as indicated to consecutive t's.

The fixed points in the quadric $\mathcal{Q}$ are the $2n$ coordinate points:

$$\mathcal{Q}^T = \{p_{\overline{n-1}},\dots,p_{\overline{0}},p_0,\dots,p_{n-1}.\}$$

That is, $p_j \in \mathcal{Q} \subseteq \mathbb{P}^{2n-1}$ is the point with 1 in the jth coordinate and 0 elsewhere. For each i, one computes the weights of T acting on the tangent space $T_{p_i}\mathcal{Q}$ to be $\{t_j - t_i \mid j \neq i, \bar{\imath}\}$, using the notation $t_{\bar{\jmath}} = -t_j$.

Using the defining equations, it is easy to write down the restrictions of Schubert classes to fixed points. For $i > 0$, and any j, we have

$$\begin{aligned}\sigma_{\bar{\imath}}|_{p_j} &= (-t_{n-1}-t_j)\cdots(-t_{i+1}-t_j)\\ &= \prod_{k=i+1}^{n-1}(-t_k-t_j),\end{aligned}$$

Thus $\sigma_{\bar{\imath}}|_{p_j} = 0$ for $j < \bar{\imath}$, since the factor $(-t_{\bar{\jmath}} - t_j)$ is zero—and indeed, in this case $p_j \notin \Omega_{\bar{\imath}}$.

For $i = \overline{0}$, we have

$$\begin{aligned}\sigma_{\overline{0}}|_{p_{\overline{0}}} &= \prod_{k=1}^{n-1}(-t_k+t_0);\\ \sigma_{\overline{0}}|_{p_j} &= (t_0-t_j)\prod_{\substack{k=0\\k\neq j}}^{n-1}(-t_k-t_j) \quad \text{for } j>0;\end{aligned}$$

and all other restrictions are zero. Finally, for $i \geq 0$,

$$\sigma_i|_{p_j} = \left(\prod_{\substack{k=0\\k\neq j}}^{n-1}(-t_k-t_j)\right)\prod_{k=0}^{i-1}(t_k-t_j) \quad \text{for } j \geq i,$$

and all other restrictions are zero.

Finally, we translate these calculations into a change of bases for the representation V. Using $\varepsilon_{\bar{\imath}} = (-1)^i \sigma_{\bar{\imath}}$ and $\varepsilon_i = \sigma_i$ for $i \geq 0$, we have $[\mathcal{Q}] = \sigma_{\overline{n-1}} = (-1)^{n-1}\varepsilon_{\overline{n-1}}$, so by setting $\overline{f}_i = (-1)^{n-1}\mathbf{1}_i$, where $\mathbf{1}_i \in H_T^*\mathcal{Q}^T$ is the idempotent class at p_i, we have

$$\varepsilon_{\overline{n-1}} = \overline{f}_{\overline{n-1}} + \cdots + \overline{f}_{\overline{0}} + \overline{f}_0 + \cdots + \overline{f}_{n-1}.$$

More generally, for $i > 0$ we have

$$\varepsilon_{\bar{\imath}} = \sum_{j=\bar{\imath}}^{n-1} \left(\prod_{k=i+1}^{n-1} (t_k + t_j) \right) \overline{f}_j;$$

for $i = \overline{0}$ we have

$$\varepsilon_{\overline{0}} = \left(\prod_{k=1}^{n-1} (t_k - t_0) \right) \overline{f}_{\overline{0}} + \sum_{j=1}^{n-1} \left((t_j - t_0) \prod_{k>0, k\neq j} (t_k + t_j) \right) \overline{f}_j;$$

and for $i \geq 0$ we have

$$\varepsilon_i = \sum_{j=i}^{n-1} \left(\prod_{k=0}^{i-1} (t_k^2 - t_j^2) \prod_{\substack{k=i \\ k\neq j}}^{n-1} (t_k + t_j) \right) \overline{f}_j.$$

It will be convenient to rescale the basis $\{\overline{f}_i\}$ so that $\{\varepsilon_i\}$ is related by a unitriangular change of basis. To this end, for each i let

$$f_i = \alpha_i \overline{f}_i,$$

where, for $i \geq 0$, the scaling coefficients are $\alpha_{\bar{\imath}} = (-1)^{n-1-i}\sigma_{\bar{\imath}}|_{p_{\bar{\imath}}}$ and $\alpha_i = (-1)^{n-1}\sigma_i|_{p_i}$. Now we may write, for $i \geq 0$,

$$\varepsilon_{\bar{\imath}} = \sum_{j=0}^{i} \overline{c}_{ji} f_{\bar{\jmath}} + \sum_{j=0}^{n-1} c_{ji} f_j,$$

and

$$\varepsilon_i = \sum_{j=0}^{i} \overline{b}_{ji} f_j.$$

Explicitly, the matrices $\overline{C} = (\overline{c}_{ji})$, $C = (c_{ji})$, and $\overline{B} = (\overline{b}_{ji})$ are computed as follows. For $i > 0$,

$$\bar{c}_{ji} = \frac{1}{\prod_{k=j+1}^{i}(t_k - t_j)} \quad \text{for } 0 \le j \le i;$$
$$c_{ji} = \frac{1}{\prod_{k=0}^{j-1}(t_k^2 - t_j^2)\prod_{k=j+1}^{i-1}(t_k + t_j)} \quad \text{for } 0 \le j \le i;$$

and

$$c_{ji} = \frac{2t_j}{\prod_{k=0}^{i}(t_k^2 - t_j^2)\prod_{k=i+1}^{j-1}(t_k - t_j)} \quad \text{for } j > i.$$

For $i = 0$,

$$\bar{c}_{00} = 1;$$
$$c_{00} = 0;$$

and

$$c_{j0} = \frac{1}{(-t_0 - t_j)\prod_{k=1}^{j-1}(t_k - t_j)}.$$

Finally, for $i \ge 0$,

$$\bar{b}_{ji} = \frac{1}{\prod_{k=i}^{j-1}(t_k - t_j)}.$$

Note the matrices $\overline{C}$ and $\overline{B}$ are indeed unitriangular.

6 Orthogonal Grassmannians

Now we turn to the maximal orthogonal Grassmannians $OG^{\pm}(n, 2n)$, also known as *spinor varieties*. We will maintain the notation from the previous section, so $V = \mathbb{C}^{2n}$ has a bilinear form and basis ε_i so that $\langle \varepsilon_{\bar{i}}, \varepsilon_j \rangle = \delta_{ij}$. As noted above, the subspace $E \subseteq V$ spanned by $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$ is isotropic with respect to the bilinear form, as is the complementary subspace $\overline{E} = \text{span}\{\varepsilon_{\bar{0}}, \dots, \varepsilon_{\overline{n-1}}\}$. The orthogonal Grassmannian $OG^+(n, 2n)$ (respectively, $OG^-(n, 2n)$) parametrizes all n-dimensional isotropic subspaces $L \subseteq V$ such that $\dim(E \cap L)$ is even (resp., odd). We will focus on the "+" case, and write $OG(n) = OG^+(n, 2n)$ from now on.

The torus $T = (\mathbb{C}^*)^n$ acts on $OG(n)$ via its action on $V = \mathbb{C}^{2n}$; recall that this is given by weights $-t_{n-1}, \dots, -t_0, t_0, \dots, t_{n-1}$. The T-fixed points in $OG(n)$ are indexed by subsets $I \subseteq \{0, \dots, n-1\}$ such that the cardinality of I is even. For such a subset, the fixed point p_I corresponds to the subspace

$$E_I = \mathrm{span}\left(\{\varepsilon_i \mid i \in I\} \cup \{\varepsilon_{\overline{j}} \mid j \notin I\}\right).$$

For example, $p_\emptyset = \overline{E}$.

Schubert varieties in $OG(n)$ are also indexed by subsets $I \subseteq \{0, \dots, n-1\}$ of even cardinality. As for the ordinary Grassmannian, the elements of I index pivots for Schubert cells Ω_I°, and Ω_I is the closure. The isotropicity conditions mean that exactly one of i or $\overline{\imath}$ occurs as a pivot, for $0 \leq i \leq n-1$, and we record the positive ones. For example, in $OG(4)$ we have

$$\Omega^\circ_{\{1,3\}} = \begin{bmatrix} 0 & 1 & * & 0 & * & 0 & \bullet & 0 \\ 0 & 0 & 0 & 1 & \bullet & 0 & \bullet & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \bullet & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(From left to right, the columns are numbered $\overline{3}, \overline{2}, \overline{1}, \overline{0}, 0, 1, 2, 3$. Stars are free entries, and bullets indicate entries that are dependent on the others, by the isotropicity condition.) Similarly, $\Omega_\emptyset = OG(n)$, and $\Omega_{\{(0),1,\dots,n-1\}} = \{p_{\{(0),1,\dots,n-1\}}\}$, where 0 is included or not, depending on the parity of n. Schubert varieties are T-invariant.

Frequently one interprets the subsets I as *strict partitions* λ, simply by reversing order from increasing to decreasing. In this context, we will usually prefer the subset notation to partition notation, although the latter is useful for indicating containment relations: if I and J are subsets corresponding to partitions λ and μ, respectively, then $\Omega_I \subseteq \Omega_J$ if and only if $\lambda \supseteq \mu$ as Young diagrams. We will write $I \geq J$ in this case. (If $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_s\}$, then $I \geq J$ is equivalent to $r \geq s$ and $i_a \geq j_a$ for $1 \leq a \leq s$.) We write $\sigma_I = [\Omega_I]$ for the equivariant class of a Schubert variety; it has degree $|I| := \sum i_a$.

Our main goal in this section is to compute formulas for the restrictions $\sigma_I|_{p_J}$. Since p_I is the unique fixed point in the Schubert cell Ω_I°, we have $p_I \in \Omega_J$ if and only if $I \geq J$. From matrix representatives, it is easy to see that the normal space to $\Omega_I^\circ \subset OG(n)$ at the point p_I has weights $\{-t_i + t_j \mid i > j;\ i \in I, j \notin I\} \cup \{-t_i - t_j \mid i > j;\ i, j \in I\}$. It follows that

$$\sigma_I|_{p_I} = \prod_{i \in I} \left(\prod_{\substack{j \notin I \\ j < i}} (-t_i + t_j) \prod_{\substack{j \in I \\ j < i}} (-t_i - t_j) \right). \tag{10}$$

The corresponding minuscule representation is the *half-spin representation* $\mathbb{S}^+$ of $\mathfrak{so}_{2n}$. A brief description, suitable for our purposes, is in the appendix; to see this worked out in detail, we recommend [12, Sect. 20], [10], or [37].

Recall our standard representation V of $\mathfrak{so}_{2n}$ splits into maximal isotropic subspaces $V = \overline{E} \oplus E$, and we have fixed a basis ε_i so that $\varepsilon_{\overline{n-1}}, \dots, \varepsilon_{\overline{0}}$ span $\overline{E}$, and $e_0, \dots, e_{n-1}$ span E. As noted in the appendix, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ is an ideal of the Clifford algebra $Cl(V)$, and $\mathbb{S}^+$ has a basis of elements

$$\varepsilon_I := \varepsilon_{\overline{i'_1}} \cdots \varepsilon_{\overline{i'_{n-r}}} \cdot \varepsilon, \tag{11}$$

for $I = \{i_1 < \cdots < i_r\} \subseteq \{0, \ldots, n-1\}$ of even cardinality, with complement $I' = \{i'_1 < \cdots < i'_{n-r}\}$. Here $\varepsilon = \varepsilon_0 \cdots \varepsilon_{n-1}$.

The Cartan subalgebra is spanned by $\varepsilon_{\overline{i}} \wedge \varepsilon_i$, and its eigenvalue on the weight vector ε_I is computed to be

$$\frac{1}{2}\left(\sum_{j \notin I} t_j - \sum_{i \in I} t_i\right).$$

In particular, the highest weight vector $\varepsilon_\emptyset$ has weight $\varpi_n = \frac{1}{2}\sum_j t_j$. Twisting by $-\varpi_n$, the action of $\mathfrak{t}^\vee$ on $V_{\varpi_n}(-\varpi_n)[t]$ has $t \cdot \varepsilon_I = (-\sum_{i\in I} t_i)\,\varepsilon_I$. Straightforward computations also show

$$\begin{aligned} \varepsilon_{\overline{i}} \wedge \varepsilon_{i+1} \cdot \varepsilon_I &= \varepsilon_{\overline{i}} \cdot \varepsilon_{i+1} \cdot \varepsilon_I \\ &= \begin{cases} \varepsilon_{I^+} & \text{if } i \in I \text{ and } i+1 \notin I; \\ 0 & \text{otherwise;} \end{cases} \end{aligned}$$

here $I^+ = (I \smallsetminus \{i\}) \cup \{i+1\}$. Similarly, $\varepsilon_0 \wedge \varepsilon_1 \cdot \varepsilon_I = \varepsilon_{I \cup \{0,1\}}$ if $0, 1 \notin I$, and is zero otherwise.

This is enough to compute the action of ξ_t on $\mathbb{S}^+$. For example,

$$\begin{aligned} \xi_t \cdot \varepsilon_\emptyset &= \varepsilon_{\{0,1\}}; \\ \xi_t \cdot \varepsilon_{\{0,1\}} &= (\varepsilon_{\overline{1}} \wedge \varepsilon_2) \cdot \varepsilon_{\{0,1\}} - t \cdot \varepsilon_{\{0,1\}} \\ &= \varepsilon_{\{0,2\}} + (t_0 + t_1)\,\varepsilon_{\{0,1\}}; \\ \xi_t \cdot \varepsilon_{\{0,2\}} &= (\varepsilon_{\overline{0}} \wedge \varepsilon_1 + \varepsilon_{\overline{2}} \wedge \varepsilon_3) \cdot \varepsilon_{\{0,2\}} - t \cdot \varepsilon_{\{0,2\}} \\ &= \varepsilon_{\{1,2\}} + \varepsilon_{\{0,3\}} + (t_0 + t_2)\,\varepsilon_{\{0,2\}}. \end{aligned}$$

The Schubert classes are identified by

$$\sigma_I = \varepsilon_I,$$

so writing $\sigma_{\square} = \sigma_{\{0,1\}}$ for the divisor class, the above calculation agrees with

$$\begin{aligned} \sigma_{\square} \cdot 1 &= \sigma_{\{0,1\}}; \\ \sigma_{\square} \cdot \sigma_{\{0,1\}} &= \sigma_{\{0,2\}} + (t_0 + t_1)\,\sigma_{\{0,1\}}; \\ \sigma_{\square} \cdot \sigma_{\{0,2\}} &= \sigma_{\{1,2\}} + \sigma_{\{0,3\}} + (t_0 + t_2)\,\sigma_{\{0,2\}}, \end{aligned}$$

and once again the action of ξ_t corresponds to multiplication by $\sigma_{\square}$.

Now we are ready for the equivariant Giambelli formula for orthogonal Grassmannians. This says

$$\sigma_I = P_\lambda(x|t), \tag{12}$$

where λ is the strict partition corresponding to I (i.e., write I in decreasing order), and $P_\lambda(x|t)$ is the *factorial Schur P-function*. These polynomials were studied by Ivanov [25] and were shown to represent Schubert classes in $H_T^*OG(n)$ by Ikeda [23]; setting $t = 0$, they specialize to Schur's P-functions (see [36, III.8]), which were shown to represent Schubert classes by Pragacz [45].

Ivanov gives a formula for $P_\lambda(x|t)$ as a ratio of Pfaffians, similar to the one defining $s_\lambda(x|t)$, and inspired by Nimmo's formula for $P_\lambda(x) = P_\lambda(x|0)$ [42]. Let $A(x) = (a_{ij}(x))$ be the $n \times n$ skew-symmetric matrix with

$$a_{ij}(x) = \frac{x_i - x_j}{x_i + x_j},$$

for $0 \le i, j \le n-1$. Given $I = \{i_1 < \cdots < i_r\}$, let $B_I(x|t) = (b_{kl}(x|t))$ be the $n \times r$ matrix with

$$b_{kl}(x|t) = (x_k|t)^{i_l}$$

Form the skew-symmetric matrix[3]

$$A_I(x|t) = \left[\begin{array}{c|c} A(x) & B_I(x|t) \\ \hline -B_I(x|t)^t & 0 \end{array}\right].$$

Then the Ivanov–Nimmo formula is

$$P_\lambda(x|t) = \frac{\mathrm{Pf}(A_I(x|t))}{\mathrm{Pf}(A(x))}. \tag{13}$$

By Schur's identity, the denominator is

$$\mathrm{Pf}(A(x)) = \prod_{i<j} \frac{x_i - x_j}{x_i + x_j}.$$

Writing $\mathrm{Pf}_K(A)$ for the Pfaffian of the submatrix on any subset of rows and columns $K \subseteq [n]$, note that

$$\mathrm{Pf}_K(A(x)) = \prod_{k<k' \text{ in } K} \frac{x_k - x_{k'}}{x_k + x_{k'}}.$$

[3]The indexing most natural to our setup is slightly nonstandard. The rows and columns of $A(x)$ are labelled $n-1, \ldots, 0$ (left to right, top to bottom); similarly, the rows of $B_I(x|t)$ are labelled $n-1, \ldots, 0$ (top to bottom) and its columns are labelled $i_1, \ldots, i_k$ (left to right).

Now we can state the theorem. The setting is analogous to that for ordinary Grassmannians. For each even subset $K \subseteq [n]$, there is a fixed point $p_K \in OG(n)$ and corresponding restriction homomorphism $H_T^* OG(n) \to \mathbb{C}[t]$. On the other hand, there is a presentation of the cohomology in terms of symmetric functions, i.e., a surjective homomorphism $\mathbb{C}[t][x_0, \ldots, x_{n-1}]^{S_n} \to H_T^* OG(n)$, and composition with the fixed-point restriction to p_K is the evaluation

$$\mathbb{C}[t][x_0, \ldots, x_{n-1}]^{S_n} \to \mathbb{C}[t]$$

given by $x_i \mapsto -t_i$ if $i \in K$, and $x_i \mapsto 0$ if $i \notin K$.

Theorem 12 *Under the homomorphism* $\mathbb{C}[t][x_0, \ldots, x_{n-1}]^{S_n} \to H_T^* OG(n)$*, we have* $P_\lambda(x|t) \mapsto \sigma_I$*, where* I *is the subset corresponding to the strict partition* λ*. Equivalently, for each* $K = \{k_1 < \cdots < k_r\}$*, we have*

$$\sigma_I|_{p_K} = P_\lambda(x|t)|_{x=-t_K},$$

where the specialization $x = -t_K$ *means* $x_i \mapsto -t_i$ *if* $i \in K$*, and* $x_i \mapsto 0$ *if* $i \notin K$*.*

Proof We follow the same outline as in type A. The first step has been done in the previous section, where we worked out the cohomology of quadrics and the corresponding change of basis between ε_i and f_i. The second step has been done above: we identify the spinor $\varepsilon_I \in \mathbb{S}^+[t] = V_{\varpi_n}(-\varpi_n)[t]$ with the Schubert class $\sigma_I \in H_T^* OG(n)$.

It remains to compute the idempotent basis, and the expansion of ε_I in this basis. With f_i as before, we define spinors f_I by the same formula (11) defining ε_I:

$$f_I = f_{\overline{i'_1}} \cdots f_{\overline{i'_{n-r}}} \cdot f, \tag{14}$$

where $f = f_0 \cdots f_{n-1}$. In fact, $f = \varepsilon$, since the change of basis is unitriangular.

Since $\varepsilon_\emptyset = \sigma_\emptyset = \mathbf{1} \in H_T^* OG(n)$, we compute this case first (to normalize the $\overline{f}_I$ basis). Using notation of the previous section,

$$\varepsilon_{\overline{\imath}} = \overline{C} \cdot f_{\overline{\imath}} + C \cdot f_i.$$

Since $\overline{C}$ is unitriangular, if we multiply by its inverse and instead consider

$$\widetilde{\varepsilon}_{\overline{\imath}} = f_{\overline{\imath}} + \overline{C}^{-1} C \cdot f_i,$$

we have

$$\varepsilon_\emptyset = \varepsilon_{\overline{0}} \cdots \varepsilon_{\overline{n-1}} \cdot \varepsilon = \widetilde{\varepsilon}_{\overline{0}} \cdots \widetilde{\varepsilon}_{\overline{n-1}} \cdot \varepsilon$$

in $\mathbb{S}^+$. On the other hand, now we are in the situation of the Theorem of the Appendix, which says

$$\varepsilon_\emptyset = \sum_K \mathrm{Pf}_K(\overline{C}^{-1} C) f_K.$$

To compute these Pfaffians, it helps to introduce the $n \times n$ diagonal matrix S, with diagonal entries

$$(t_0 - t_{n-1}) \cdots (t_{n-2} - t_{n-1}),\ (t_0 - t_{n-2}) \cdots (t_{n-3} - t_{n-2}),\ \ldots,\ t_0 - t_1, 1.$$

Then $S\overline{C}^{-1} C S = A = (a_{ji})$, where $a_{ji} = \frac{t_j - t_i}{t_j + t_i}$. So for any $K \subseteq [n]$,

$$\mathrm{Pf}_K(A) = \mathrm{Pf}_K(S\overline{C}^{-1} C S) = \det_K(S) \cdot \mathrm{Pf}_K(\overline{C}^{-1} C),$$

where $\det_K(S)$ means the determinant of the submatrix on rows and columns K. Combining the formulas

$$\det_K(S) = \prod_{\substack{i<k\\ k\in K}} (t_i - t_k) \quad \text{and } \mathrm{Pf}_K(A) = \prod_{\substack{i<j\\ i,j\in K}} \frac{t_j - t_i}{t_j + t_i},$$

we obtain

$$\mathrm{Pf}_K(\overline{C}^{-1} C) = \frac{1}{\displaystyle\prod_{\substack{i<k\\ i,k\in K}} (-t_i - t_k) \prod_{\substack{i<k\\ i\notin K, k\in K}} (t_i - t_k)}.$$

Therefore, the idempotent classes $\overline{f}_K$ are determined by writing

$$\varepsilon_\emptyset = \sum_K \overline{f}_K,$$

that is, by setting

$$\overline{f}_K = \mathrm{Pf}_K(\overline{C}^{-1} C) f_K.$$

Equivalently,

$$f_K = \left(\prod_{\substack{i<k\\ i,k\in K}} (-t_i - t_k) \prod_{\substack{i<k\\ i\notin K, k\in K}} (t_i - t_k) \right) \overline{f}_K.$$

Now we compute ε_I. Using notation from our computations at the end of Sect. 5, the transition between ε_i and f_i has the form

$$\left[\begin{array}{c|c} C & \overline{B} \\ \hline \overline{C} & 0 \end{array} \right].$$

Since the Theorem of the Appendix computes ε_I as $(-1)^{|I|}\varepsilon_{i_1}\cdots\varepsilon_{i_r}\cdot\varepsilon_\emptyset$, we may replace $\varepsilon_{\overline{\imath}}$ by $\widetilde{\varepsilon_{\overline{\imath}}}$, which amounts to using the matrix

$$\left[\begin{array}{c|c} \overline{C}^{-1}C & \overline{B} \\ \hline w_\circ & 0 \end{array}\right],$$

where $w_\circ$ is the $n\times n$ matrix with 1's on the antidiagonal and 0's elsewhere. Now formula (A.4) says

$$\varepsilon_I = \sum_K \mathrm{Pf}_K(\overline{A}_I) f_K, \tag{15}$$

where

$$\overline{A}_I = \left[\begin{array}{c|c} \overline{C}^{-1}C & \overline{B}_I \\ \hline -\overline{B}_I^t & 0 \end{array}\right]$$

and $\overline{B}_I$ is the submatrix of $\overline{B}$ on columns I.

(Note that unitriangularity of the matrix relating ε_i and f_i implies that

$$\begin{aligned}\varepsilon_I &= f_I + \sum_{K>I} \mathrm{Pf}_K(\overline{A}_I) f_K \\ &= \left(\prod_{\substack{j<i \\ i,j\in I}} (-t_j - t_i) \prod_{\substack{j<i \\ j\notin I, i\in I}} (t_j - t_i)\right) \overline{f}_I + \cdots,\end{aligned}$$

which agrees with the formula (10) for $\sigma_I|_{p_I}$.)

To conclude the proof, we must relate the coefficients $\mathrm{Pf}_K(\overline{A}_I)$ to the evaluations $P_\lambda(x|t)|_{x=-t_K}$. Observe first that

$$\begin{aligned}\mathrm{Pf}(A(x))|_{x=-t_K} &= \prod_{k'<k \text{ in } K} \frac{-t_{k'}+t_k}{-t_{k'}-t_k} \\ &= \mathrm{Pf}_K(A) \\ &= \det{}_K(S)\,\mathrm{Pf}_K(\overline{C}^{-1}C).\end{aligned}$$

Furthermore, scaling the first n rows and columns of $\overline{A}_I$ by S, one computes

$$\left[\begin{array}{c|c} S\overline{C}^{-1}CS & S\overline{B}_I \\ \hline -\overline{B}_I^t S & 0 \end{array}\right] = \left[\begin{array}{c|c} A(x) & B_I(x|t) \\ \hline -B_I(x|t)^t & 0 \end{array}\right]\Bigg|_{x_0=-t_0,\ldots,x_{n-1}=-t_{n-1}},$$

and also that

$$\mathrm{Pf}_K\left(\left[\begin{array}{c|c} A(x) & B_I(x|t) \\ \hline -B_I(x|t)^t & 0 \end{array}\right]\Bigg|_{x_j=-t_j,\ \text{all } j}\right)$$
$$= \left(\mathrm{Pf}\left[\begin{array}{c|c} A(x) & B_I(x|t) \\ \hline -B_I(x|t)^t & 0 \end{array}\right]\right)\Bigg|_{x=-t_K}.$$

Taking Pfaffians on rows and columns K, it follows that

$$\mathrm{Pf}_K(\overline{A}_I)\cdot \det_K(S) = \left(\mathrm{Pf}\left[\begin{array}{c|c} A(x) & B_I(x|t) \\ \hline -B_I(x|t)^t & 0 \end{array}\right]\right)\Bigg|_{x=-t_K}.$$

So the Ivanov–Nimmo formula gives

$$P_\lambda(x|t)|_{x=-t_K} = \frac{\mathrm{Pf}_K(\overline{A}_I)}{\mathrm{Pf}_K(\overline{C}^{-1}C)},$$

which is precisely the coefficient of $\overline{f}_K = \mathrm{Pf}_K(\overline{C}^{-1}C)f_K$ in the expansion of ε_I in (15).

7 Rim Hook Rules for Quantum Cohomology

Consider a homogeneous space $X = G/P$ such that $H^2(X) \cong \mathbb{C}$. (This includes all minuscule G/P.) The (small) quantum cohomology ring of X is a $\mathbb{C}[q]$-algebra $QH^*(X) = \mathbb{C}[q] \otimes H^*(X)$, where q is a formal parameter whose degree depends on X, equipped with a product which deforms the usual cup product on $H^*(X)$; the structure constants in $QH^*(X)$ are 3*-point Gromov-Witten invariants*, counting certain rational curves in X. Using the action of a maximal torus $T \subseteq G$ on X, there are also equivariant versions $QH^*_T(X)$. We will write $T = (\mathbb{C}^*)^n$, and

$$S = S_n = H^*_T(\mathrm{pt}) \cong \mathbb{C}[t_1, \ldots, t_n],$$

so $QH^*_T(X)$ is an algebra over $S[q]$. (We continue to use $\mathbb{C}$ coefficients for cohomology, but all results of this section are equally valid with $\mathbb{Z}$ coefficients.)

For any N, consider the embedding of Grassmannians

$$\iota\colon Gr(k, N) \hookrightarrow Gr(k, N+1)$$

corresponding to the standard embedding of $\mathbb{C}^N$ in $\mathbb{C}^{N+1}$ (as the span of the first N standard basis vectors). Let us write $T_N = (\mathbb{C}^*)^N$, and $S_N = H^*_{T_N}(\mathrm{pt})$. There is a similar embedding of T_N in T_{N+1} (corresponding to the map on character groups $\mathbb{Z}^{N+1} \to \mathbb{Z}^N$ sending t_{N+1} to 0 and all other t_i to t_i). The embedding ι is equivariant with respect to the inclusion of tori and their natural actions

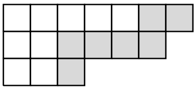

Fig. 1 A 7-rim hook of height 2

on the Grassmannians, so we have an inverse system of graded homomorphisms $\iota^*\colon H^*_{T_{N+1}} Gr(k, N+1) \to H^*_{T_N} Gr(k, N)$. Let $H^*_{T_\infty} Gr(k, \infty)$ be the graded inverse limit. This ring can be identified with the ring of symmetric polynomials in variables $x_1, \ldots, x_k$, with coefficients in S_∞.

Writing $\Omega^{(N)}_\lambda$ for the Schubert variety in $Gr(k, N)$ and $\sigma^{(N)}_\lambda$ for its equivariant cohomology class, one checks that $\iota^{-1}\Omega^{(N+1)}_\lambda = \Omega^{(N)}_\lambda$, and it follows that $\iota^*\sigma^{(N+1)}_\lambda = \sigma^{(N)}_\lambda$. Therefore we have well-defined classes $\sigma_\lambda \in H^*_{T_\infty} Gr(k, \infty)$.

Fixing n and $T = T_n$, for $N > n$, consider a (different) inclusion of tori $T \hookrightarrow T_N$ given by

$$(z_1, z_2, \ldots, z_n) \mapsto (z_1, z_2, \ldots, z_n, z_1, z_2, \ldots);$$

equivalently, take the map on character groups $\mathbb{Z}^N \to \mathbb{Z}^n$ given by $t_i \mapsto t_{i \pmod n}$, where the representatives mod n are taken to be $1, \ldots, n$. This also defines a ring homomorphism $S_\infty \to S_n$, inducing an algebra homomorphism $H^*_{T_\infty} Gr(k, \infty) \to H^*_T Gr(k, \infty)$. We will see how to extend this to a homomorphism $H^*_T Gr(k, \infty) \to QH^*_T Gr(k, n)$.

The *rim hook rule*[4] for the equivariant quantum cohomology of $Gr(k, n)$ is a recipe for computing quantum products in terms of ordinary products on $Gr(k, N)$, for $N > n$. Given any partition λ, an n-**rim hook** (also called an n-**border strip**) is a connected collection of n boxes on the southeast border of its diagram, with exactly one box in each diagonal, such that the complement μ is also the diagram of a partition. If the boxes appear in rows i through j of the diagram, the **height** of the rim hook is $j - i$. See Fig. 1 for an example, and [36, Sect. I.1] for more details.

Any partition λ has a well-defined n-**core** μ, which is the partition obtained by removing all possible n-rim hooks from λ, in any order. The parity of the sum of the heights of these rim hooks is also independent of choices, so we can define $\epsilon(\lambda/\mu) = \sum_\delta \mathrm{height}(\delta) \pmod 2$, taking the sum over a pieces of a decomposition of the skew shape λ/μ into n-rim hooks δ.

Under the bijection between partitions λ with k parts and k-element subsets I of $\{1, 2, \ldots\}$, the operation of removing an n-rim hook corresponds to replacing some element $i \in I$ with $i - n$ (and an n-rim hook exists only if there is an $i > n$ such that $i - n$ is not in I); the height of such a rim hook is the length of the permutation needed to sort the resulting set into increasing order. Taking the n-core of λ can be described as follows. Write the elements of I as $s_i n + r_i$, for $1 \le i \le k$ and $1 \le r_i \le n$. Next consider the multiset of residues $\{r_i\}$. For each r that appears in

[4] The non-equivariant rim hook rule was discovered by Bertram, Ciocan-Fontanine, and Fulton during the 1996–7 program on quantum cohomology at Institut Mittag-Leffler [3]. Bertiger, Milićević, and Taipale gave an equivariant generalization [2].

this multiset, replace the jth occurrence of r by $r + (j-1)n$, obtaining a set $\overline{I}$ of k distinct integers with the same multiset of residues modulo n. The n-core of λ is the partition μ corresponding to $\overline{I}$, and the sign $(-1)^{\epsilon(\lambda/\mu)}$ is the sign of the permutation needed to sort $\overline{I}$ (see [36, Sect. I.1, Ex. 8]).

For example, consider $\lambda = (7, 6, 3)$. For $k = 3$, this corresponds to the set $I = \{4, 8, 10\}$. There are two ways to remove a 7-rim hook: replace 10 with 3 (obtaining $I' = \{4, 8, 3\}$) or replace 8 with 1 (obtaining $I' = \{4, 1, 10\}$). In the first case, the permutation which sorts I' has length 2, and in the second it has length 1. To find the 7-core, write residues modulo 7 to obtain $\{4, 1, 3\}$; this is sorted to $\overline{I} = \{1, 3, 4\}$ by a permutation of length 2, so $\mu = (1, 1)$ and $\epsilon(\lambda/\mu)$ is even.

Given a class $\sigma_\lambda \in H^*_{T_\infty} Gr(k, \infty)$, for $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, let μ be the n-core of λ and define a linear map

$$\varphi\colon H^*_{T_\infty} Gr(k, \infty) \to QH^*_T Gr(k, n)$$
$$\sigma_\lambda \mapsto \begin{cases} (-1)^{(k-1)s+\epsilon(\lambda/\mu)}\, q^s\, \sigma_\mu & \text{if } \mu \subseteq \rho_{k,n-k}, \\ 0 & \text{otherwise,} \end{cases}$$

where $s = (|\lambda| - |\mu|)/n$ is the number of rim hooks removed from λ to obtain μ. This map factors into the specialization homomorphism $H^*_{T_\infty} Gr(k, \infty) \to H^*_T Gr(k, \infty)$ described above, followed by a linear map $\psi\colon H^*_T Gr(k, \infty) \to QH^*_T Gr(k, n)$ which is given by the same formula as φ. The first is clearly an algebra homomorphism (with respect to the ring map $S_\infty \to S$ which cyclically specializes the t variables), while the second is *a priori* a homomorphism of S-modules.

Theorem 13 (Equivariant rim hook rule [2, 3]) *The map φ respects multiplication: it is a surjective homomorphism of algebras, compatible with the cyclic specialization $S_\infty \to S$.*

It suffices to show that the second factor ψ is a surjective ring homomorphism. We will give a simple proof, inspired by the Satake correspondence. (A similar construction was described by Gatto [14]. The phrasing in terms of reduction modulo n also appears in work by Buch [7, Corollary 1] and Sottile [47].) The basic idea is to show that the kernel of ψ is an ideal, so that ψ induces a ring structure on its image; then apply Mihalcea's characterization of the quantum product [39] to conclude that the product induced by ψ is the quantum product.

Before proving the theorem, we first consider the easier case of projective space, so $k = 1$. Writing a class in $H^*_T \mathbb{P}^\infty$ as σ_{sn+i}, for $0 \leq i \leq n-1$, the rim hook rule says $\psi(\sigma_{sn+i}) = q^s \sigma_i$. (In this case, ψ is an isomorphism of S-modules.) Both $H^*_{T_n} \mathbb{P}^\infty$ and $QH^*_T \mathbb{P}^{n-1}$ are free S-algebras, generated by the divisor class $\sigma_\square = \sigma_1$, so to see that ψ is an isomorphism of algebras it suffices to check that it respects multiplication by $\sigma_\square$. This is a simple application of a special case of the equivariant (quantum) Pieri rule [39]: in $H^*_T \mathbb{P}^\infty$ one has

$$\sigma_\square \cdot \sigma_i = \sigma_{i+1} + (t_1 - t_{(i+1) \pmod n})\, \sigma_i,$$

and in $QH^*_{T_n}\mathbb{P}^{n-1}$ one has

$$\sigma_{\square} \cdot \sigma_i = \begin{cases} \sigma_{i+1} + (t_1 - t_{i+1})\,\sigma_i & \text{for } i < n-1, \\ q\,\sigma_0 + (t_1 - t_n)\,\sigma_{n-1} & \text{for } i = n-1. \end{cases}$$

This computation can be rephrased to make it analogous to the one we did in the introduction. There are homomorphisms with compatible identifications

$$\begin{array}{ccccc} H^*_{T_\infty}\mathbb{P}^\infty & \longrightarrow & H^*_T\mathbb{P}^\infty & \xrightarrow{\psi} & QH^*_T\mathbb{P}^{n-1} \\ \| & & \| & & \| \\ (S_\infty)^\infty & \longrightarrow & (S)^\infty & \xrightarrow{\psi} & (S[q])^n, \end{array} \tag{16}$$

where the first horizontal map is given by the cyclic specialization $t_i \mapsto t_{i \pmod n}$ as above. Identifying the standard basis ε_i with σ_{i-1} as before, the rim hook rule says ψ is given by $\varepsilon_{sn+i} \mapsto q^s\varepsilon_i$, for $1 \le i \le n$. (Recalling that ε_i is a weight vector with weight t_i, one sees that the cyclic specialization is necessary to make the second map compatible with the S_n-module structure.)

Multiplication by $\sigma_{\square}$ on $H^*_{T_\infty}\mathbb{P}^\infty$ is given by the infinite matrix ξ_t with $-t_1, -t_2, \ldots$ on the diagonal, 1's on the subdiagonal, and 0's elsewhere. (One should twist $(S_\infty)^\infty$ by e^{-t_1} as in Sect. 3 to get the correct action.) Specializing the diagonal variables modulo n and applying the homomorphism ψ, the action of ξ_t on $(S_\infty)^\infty$ is transformed into the action of the $n \times n$ matrix

$$\xi_{q,t} = \begin{bmatrix} -t_1 & & & q \\ 1 & \ddots & & \\ 0 & \ddots & \ddots & \\ 0 & 0 & 1 & -t_n \end{bmatrix} \tag{17}$$

on $(S[q])^n$, which agrees with multiplication by $\sigma_{\square}$ in $QH^*_T\mathbb{P}^{n-1}$.

Lemma 14 *The S-module homomorphism* $\psi\colon H^*_TGr(k,\infty) \to QH^*_TGr(k,n)$ *is surjective, and its kernel is an ideal.*

Proof It is straightforward to see that ψ is surjective, and we leave this to the reader. It is also easy to see that the kernel is generated (as an S-module) by two types of elements:

(1) classes σ_λ such that the n-core of λ does not fit in $\rho_{k,n-k}$; and
(2) differences $(-1)^{\epsilon(\lambda/\mu)}\sigma_\lambda - (-1)^{\epsilon(\lambda'/\mu)}\sigma_{\lambda'}$, for two partitions λ, λ' of the same size and with the same n-core μ.

We will show that the products of such elements with S-algebra generators for $H^*_TGr(k,\mathbb{C}^\infty)$ are also in the kernel of ψ.

Making identifications

$$\begin{array}{ccc} H_T^* Gr(k,\infty) & \xrightarrow{\psi} & QH_T^* Gr(k,n) \\ \| & & \| \\ \bigwedge_S^k S^\infty & \xrightarrow{\psi} & \bigwedge_{S[q]}^k (S[q])^n, \end{array} \tag{18}$$

up to sign the homomorphism ψ is induced by the corresponding map described above for projective space: if one sends ε_{sn+i} to $((-1)^{k-1}q)^s\,\varepsilon_i$, then ψ is the induced map on the kth exterior power. Furthermore, from this point of view, the operators $\xi_t, \xi_t^2, \ldots, \xi_t^k$—or rather, the cohomology classes they correspond to—form a set of S-algebra generators for $H_T^* Gr(k,\infty)$. (To see this, recall that $H_T^* Gr(k,\infty)$ is isomorphic to the ring of symmetric polynomials in k variables with coefficients in S, and that the leading term of ξ_t^j corresponds to multiplication by the power sum function p_j; these generate the ring of symmetric polynomials, for $1 \le j \le k$.)

Now let us consider the generators of the kernel of ψ, using the correspondence between partitions and k-element sets to write classes in $H_T^* Gr(k,\infty)$ as $\sigma_\lambda = \varepsilon_I$. For $I = \{i_1 < \cdots < i_k\}$, extract residues by writing $i_a = s_a n + r_a$. The n-core of λ fits inside $\rho_{k,n-k}$ if and only if the residues r_a are all distinct, and two partitions λ, λ' share the same n-core if and only if the corresponding sets I, I' share the same multiset of residues.

Suppose ε_I is the first type of generator for the kernel, so I has at least two elements i, i' with the same residue modulo n. Reordering as necessary, we can write

$$\varepsilon_I = \pm(\varepsilon_i \wedge \varepsilon_{i'} \wedge \cdots).$$

Using the formula from Example 6,

$$\begin{aligned} \xi_t^j \cdot (\varepsilon_i \wedge \varepsilon_{i'} \wedge \cdots) = & \left(\sum_{a=0}^{j} (-1)^a\, h_a(t_i,\ldots,t_{i+j-a})\,\varepsilon_{i+j-a}\right) \wedge \varepsilon_{i'} \wedge \cdots \\ & + \varepsilon_i \wedge \left(\sum_{a=0}^{j} (-1)^a\, h_a(t_{i'},\ldots,t_{i'+j-a})\,\varepsilon_{i'+j-a}\right) \wedge \cdots \\ & + (\text{ terms involving } \varepsilon_i \wedge \varepsilon_{i'} \wedge \cdots); \end{aligned}$$

Due to the cyclic specialization, the subscripts on the t variables should be read modulo n, and since $i \equiv i' \pmod n$, the coefficients $h_a(t_i,\ldots,t_{i+j-a})$ and $h_a(t_{i'},\ldots,t_{i'+j-a})$ are equal. After applying ψ, the first two terms cancel, and all the others go to zero. It follows that $\xi_t^j \cdot \varepsilon_I$ also lies in the kernel.

The second case is similar. Recall that the sign $(-1)^{\epsilon}(\lambda/\mu)$ is equal to the sign of the permutation needed to sort the residues of I into increasing order. Suppose I and I' have the same (distinct) residues modulo n, and are such that the corresponding partitions λ and λ' have the same size. Then

$$(-1)^{\epsilon(\lambda/\mu)}\,\varepsilon_I - (-1)^{\epsilon(\lambda'/\mu)}\,\varepsilon_{I'} = (\varepsilon_{s_1 n + r_1} \wedge \cdots \wedge \varepsilon_{s_k n + r_k}) - (\varepsilon_{s'_1 n + r_1} \wedge \cdots \wedge \varepsilon_{s'_k n + r_k}),$$

for $r_1 < \cdots < r_k$. A calculation analogous to the previous one shows that the product of this difference with ξ_t^j remains in the kernel of ψ.

Proof of Theorem 13 As remarked above, it suffices to show that ψ is a ring homomorphism. By the lemma, $\ker(\psi)$ is an ideal of $H_T^* Gr(k, \infty)$. Let $A = H_T^* Gr(k, \infty)/\ker(\psi)$, and write $q \in A$ for the image of $(-1)^{k-1}\sigma_n$. This makes A an $S[q]$-algebra, with an $S[q]$-module basis indexed by partitions fitting inside $\rho_{k,n-k}$. The homomorphism ψ maps this basis onto the Schubert basis for $QH_T^* Gr(k, n)$. We must check that this is an isomorphism of algebras.

By Mihalcea's characterization of $QH_T^* Gr(k, n)$ [39, Corollary 7.1], it suffices to check that the isomorphism respects multiplication by the divisor class σ_1. On one hand, multiplication by σ_1 on A is given by the action of a matrix $\xi_{q,t}$ on $\bigwedge_{S[q]}^k (S[q])^n \otimes \mathrm{e}^{-\varpi_k}$. (The matrix $\xi_{q,t}$ is the one of (17), with q replaced by $(-1)^{k-1}q$.) That is,

$$\begin{aligned}\xi_{q,t} \cdot \varepsilon_I = {} & \varepsilon_{i_1+1} \wedge \varepsilon_{i_2} \wedge \cdots \wedge \varepsilon_{i_k} + \cdots + \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots \wedge \varepsilon_{i_k+1} \\ & + (t_1 + \cdots + t_k - t_{i_1} - \cdots - t_{i_k})\, \varepsilon_I \\ & (+ q\varepsilon_1 \wedge \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{k-1}}),\end{aligned}$$

with the last term occurring only if $i_k = n$ and $i_1 > 1$. Writing this in terms of partitions, one recovers exactly the equivariant quantum Pieri formula [39, Theorem 1] for multiplication by σ_1 in $QH_T^* Gr(k, n)$, as desired.[5]

8 Closing Remarks and Other Directions

8.1 Relation to Quantum Integrability

In [20, 21], Gorbounov, Korff, and Stroppel indicate an alternative approach to the (equivariant) rim-hook rule, based on quantum integrability of the six-vertex model. This model can be seen as a degeneration of the Yangian Hopf algebra (a certain quantum group) which governs the cohomology of the cotangent space of the Grassmannian. This fits in a major program initiated by Maulik and Okounkov to relate representation theory of quantum groups to enumerative geometry [38, 44, 46].

The authors of [20, 21] consider a vector space $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$. The standard basis $\{v_{\omega_1} \otimes v_{\omega_1} \otimes \ldots \otimes v_{\omega_n}\}$ of $V^{\otimes n}$, where $\omega_i \in \{0, 1\}$, can be identified with set of 01-words $\{\omega = \omega_1\omega_2 \ldots \omega_n\}$. The set of words $\{\omega : \sum_{i=1}^n \omega_i = k\}$ is in bijection with the set of multi-indexes $I = \{i_1 < \cdots < i_k\} \subseteq [n]$. The bijection is given by sending I to the 01-word with 1's in positions $i_1, \ldots, i_k$ and 0's elsewhere. Through

[5] Our indexing of t variables is reversed when compared with that of [39], since we use opposite Schubert classes.

this identification, let $V_k \subseteq V^{\otimes n}$ be the vector subspace generated by vectors indexed by this set. In [21], two bases for $V_k \otimes \mathbb{C}[t_1, \dots, t_n]$ are constructed, which are called respectively standard or spin basis[6] and the Bethe vector basis. Then a connection is made with equivariant cohomology of the Grassmannian, by identifying those bases with the Schubert basis and the torus fixed point basis, respectively. The authors construct certain (generating series of) operators denoted $A + qD$ ([21, Sect. 5]) acting on the spin basis which match the action of quantum multiplication by the divisor class on the Schubert basis.

This is analogous to the setup from the previous section, where the principal nilpotent ξ_t is corrected by the nilpotent matrix

$$q \begin{bmatrix} 0 & & & 1 \\ 0 & \ddots & & \\ 0 & \ddots & \ddots & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in order to match the quantum equivariant Chevalley rule after a twist.

We should also mention another closely related result by Lam and Templier, [32, Proposition 4.13] where essentially the same presentation of the quantum Chevalley operator is given.

8.2 Mirror Symmetry and the Gamma Conjectures

Mirror symmetry for Fano varieties roughly claims a correspondence which associate to each Fano variety X a smooth quasi-projective variety Y endowed with a flat map $w : Y \longrightarrow \mathbb{A}^1$ with quasi-projective fibres, called superpotential. Under this correspondence critical points of w give rise to vanishing cycles which through homological mirror symmetry correspond to objects of the derived category $D^b(X)$ of coherent sheaves on X.

The Gamma conjectures were formulated by Galkin, Golyshev and Iritani in [13], refining previous conjectures by Dubrovin [11]. They claim that simple eigenvalues of a certain variation of the quantum Chevalley operator denoted $\star_0$ are related to characteristic classes $\widehat{\Gamma}_X Ch(E)$, where E is an exceptional object of $D^b(X)$, which in turn are constructed through flat sections of Dubrovin's quantum connection

$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}(c_1(X)\star_0) + \mu, \tag{19}$$

[6] This "spin basis" is not directly connected to spin representations and the orthogonal Grassmannian, as far as we are aware.

a meromorphic flat connection on the trivial bundle $H^*(X) \times \mathbb{P}^1 \to \mathbb{P}^1$. In (19) μ is the grading operator on $H^*(X)$ and z is a local coordinate on $\mathbb{P}^1$.

The Gamma conjectures were proved for Grassmannians in [13], (see also [9]), by making fundamental use of the quantum correction to the Satake correspondence.

8.3 Other Representations

We have focused on minuscule representations in types A and D, with their corresponding (minuscule) homogeneous spaces G/P: Grassmannians, quadrics, and orthogonal Grassmannians. It is natural to ask what can be said for other spaces.

In other types (E_6 and E_7), there is only one minuscule space, up to isomorphism, so the approach we used in Sects. 4 and 6 does not seem productive. Isomorphisms between quantum cohomology of these spaces and corresponding representations were worked out by Golyshev and Manivel [19]. On the other hand, it would be interesting to see an analogue of the rim-hook rule for orthogonal Grassmannians, presumably related to an infinite-dimensional spin representation.

In a different direction, one can ask for the transition matrices between the "ε" and "$\overline{f}$" bases of non-minuscule representations. This should correspond to localization formulas for MV-cycles. We have seen Schur S- and P-functions appear naturally in the exterior and spin representations; what other symmetric functions arise this way?

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Appendix: Pfaffians and Spinors

In this appendix, we prove a change-of-basis formula in the spin representation, where Pfaffians play the role analogous to determinants in the exterior algebra. This refines similar formulas of Chevalley and Manivel [10, 37].

First, for any complex vector space V with symmetric bilinear form $\langle\ ,\ \rangle$, the associated **Clifford algebra** is the quotient of the tensor algebra by two-sided ideal forcing the relation $v \cdot w + v \cdot w = \langle v, w\rangle 1$ for all vectors $v, w \in V$:

$$Cl(V) := T^\bullet(V)/(v \otimes w + w \otimes v - \langle v, w\rangle 1).$$

(Note that this definition differs slightly from the standard one, where $\langle v, w\rangle 1$ would be replaced by $2\langle v, w\rangle 1$—but it is equivalent, up to rescaling the form $\langle\ ,\ \rangle$, since we are not in characteristic 2.) If the bilinear form is zero, this is just the exterior algebra: $Cl(V) \cong \bigwedge^\bullet V$. In general, $\dim Cl(V) = 2^{\dim V}$, with a basis consisting of products $v_I = v_{i_1} \cdots v_{i_k}$ of distinct basis elements of V. (One can prove this by degenerating to the exterior algebra.)

We note the following general formulas in $Cl(V)$, which are immediate from the defining relations. First, if x and y are orthogonal vectors in V (i.e., $\langle x, y\rangle = 0$), then

$$x \cdot y = -y \cdot x \tag{A.1}$$

in $Cl(V)$. Next, suppose $x = x_1 \cdots x_r \in Cl(V)$ is a product of vectors in V such that $X = \mathrm{span}\{x_1, \ldots, x_r\} \subseteq V$ is isotropic. Then for any vector $y \in X$,

$$y \cdot x = 0 \tag{A.2}$$

in $Cl(V)$.

From now on, we assume $\dim V = 2n$ and its bilinear form is nondegenerate. Let $y_{\overline{n-1}}, \ldots, y_{\overline{0}}, y_0, \ldots, y_{n-1}$ be an "orthogonal basis", meaning $\langle y_{\bar{\imath}}, y_j\rangle = \delta_{ij}$. (We interpret the bar as a negative sign, so $\bar{\bar{\imath}} = i$.) Let $y = y_0 \cdots y_{n-1}$. The **spin representation** is the left ideal

$$\mathbb{S} = Cl(V) \cdot y.$$

A **(pure) spinor** is an element of the form $z \cdot y \in \mathbb{S}$, where $z = z_1 \cdots z_n$ is a product of vectors $z_1, \ldots, z_n \in V$ which span a maximal isotropic subspace of V.

Let $[m] = \{0, \ldots, m-1\}$ for any integer $m \geq 1$. For a subset $I = \{i_1 < \cdots < i_r\} \subseteq [n]$, let $I' = \{i'_1 < \cdots < i'_{n-r}\} = [n] \smallsetminus I$. Given such an I, the corresponding standard spinor is

$$y_I := y_{\bar{\imath}'_1} \cdots y_{\bar{\imath}'_{n-r}} \cdot y.$$

These form a basis for $\mathbb{S}$, as I ranges over all subsets of $[n]$. Note that $y_\emptyset = y_{\overline{0}} \cdots y_{\overline{n-1}} \cdot y$, and

$$y_{i_1} \cdots y_{i_r} \cdot y_\emptyset = (-1)^{i_1 + \cdots + i_r} y_I, \tag{A.3}$$

which is sometimes useful.

The spin representation becomes an $\mathfrak{so}_{2n}$-module via the embedding $\mathfrak{so}_{2n} \cong \bigwedge^2 V \hookrightarrow Cl(V)$ by

$$v \wedge w \mapsto \frac{1}{2}(v \cdot w - w \cdot v) = v \cdot w - \frac{1}{2}\langle v, w\rangle 1.$$

So $\mathfrak{so}_{2n}$ preserves the parity of basis vectors of $Cl(V)$, and the spin representation breaks into two irreducible **half-spin representations**

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-.$$

Our convention will be that $\mathbb{S}^+$ is spanned by standard spinors y_I, with I an even subset of $[n]$, i.e., it has even cardinality.

Now let $x_{\overline{n-1}}, \ldots, x_{\overline{0}}, x_0, \ldots, x_{n-1}$ be another orthogonal basis, and assume it is related to the y_i's by a unitriangular matrix:

$$x_{\overline{\imath}} = y_{\overline{\imath}} + \sum_{j<i} \overline{c}_{ji} y_{\overline{\jmath}} + \sum_{j=0}^{n-1} c_{ji} y_j$$

and

$$x_i = y_i + \sum_{j>i} b_{ji} y_j.$$

Thus $x = x_0 \cdots x_{n-1}$ is equal to $y = y_0 \cdots y_{n-1}$.

We define pure spinors x_I analogously, by

$$x_I := x_{\overline{\imath}'_1} \cdots x_{\overline{\imath}'_{n-r}} \cdot y.$$

Our goal is to compute the expansion of x_I in the basis y_K.

In fact, we will be free to multiply by the inverse of the matrix $\overline{C} = (\overline{c}_{ji})$ and assume that $x_{\overline{\imath}}$ is written

$$x_{\overline{\imath}} = y_{\overline{\imath}} + \sum_{j=0}^{n-1} a_{ji} y_j.$$

In this case, the fact that the $x_{\overline{\imath}}$ span an isotropic space is equivalent to the fact that the matrix $A = (a_{ji})$ is skew-symmetric. Up to appropriately indexing rows and columns, the x basis is related to the y basis by a matrix of the form

$$\left(\begin{array}{c|c} A & B \\ \hline w_\circ & 0 \end{array}\right),$$

where $w_\circ$ is the matrix with 1's on the antidiagonal and zeroes elsewhere. The top n rows determine a skew-symmetric matrix (by replacing $w_\circ$ with $-B^t$). For a subset $I \subseteq [n]$, let $B(I)$ be the submatrix formed by taking columns I of B, and let $A(I)$ be the skew-symmetric matrix

$$A(I) = \left(\begin{array}{c|c} A & B(I) \\ \hline -B(I)^t & 0 \end{array}\right).$$

Only the top n rows are needed to perform operations with such a matrix. For instance, if $n = 3$ and $I = \{1\}$, the top rows are

$$\left(\begin{array}{ccc|ccc} 0 & a_{21} & a_{20} & b_{20} & b_{21} & 1 \\ & 0 & a_{10} & b_{10} & 1 & 0 \\ & & 0 & 1 & 0 & 0 \end{array}\right)$$

and

$$A(I) = \left(\begin{array}{ccc|c} 0 & a_{21} & a_{20} & b_{21} \\ -a_{21} & 0 & a_{10} & 1 \\ -a_{20} & -a_{10} & 0 & 0 \\ \hline -b_{21} & -1 & 0 & 0 \end{array}\right)$$

For even r, the **Pfaffian** of any $r \times r$ skew-symmetric matrix A may be computed recursively using the Laplace-type expansion formula

$$\mathrm{Pf}(A) = \sum_{j=1}^{r-1} (-1)^{j-1} a_{jr} \, \mathrm{Pf}_{\widehat{j,r}}(A),$$

where $\mathrm{Pf}_{\widehat{j,r}}(A)$ is the Pfaffian of the submatrix of A obtained by removing the jth and rth rows and columns. Let $\mathrm{Pf}_K(A)$ denote the Pfaffian of the submatrix on rows and columns K, and we always use the convention $\mathrm{Pf}_\emptyset(A) = 1$.

Theorem *For a subset $I \subseteq [n]$, we have*

$$x_I = \sum_{\substack{K \subseteq [n] \\ K \text{ even}}} \mathrm{Pf}_K(A(I)) \, y_K, \tag{A.4}$$

where the sum is over subsets K of even cardinality.

See also [43, Corollary 2.4] for a related Pfaffian identity.

Proof First we establish the formula for $x_\emptyset$. In fact, for any $m \leq n$, we have

$$x_{\overline{0}} \cdots x_{\overline{m-1}} \cdot y = \sum_{\substack{K \subseteq [m] \\ K \text{ even}}} \mathrm{Pf}_K(A) \, y_{\overline{k'_1}} \cdots y_{\overline{k'_s}} \cdot y,$$

where $K' = \{k'_1 < \cdots < k'_s\} = [m] \smallsetminus K$. This is proved by induction on m, the base case $m = 0$ being the tautology $y = y$. For the induction step, we compute using the Laplace expansion formula:

$$\begin{aligned} x_{\overline{0}} \cdots x_{\overline{m}} \cdot y &= (-1)^m x_{\overline{m}} \cdot x_{\overline{0}} \cdots x_{\overline{m-1}} \cdot y \\ &= (-1)^m \left(y_{\overline{m}} + \sum_{j=0}^{m-1} a_{jm} y_j \right) \sum_{K \subseteq [m]} \mathrm{Pf}_K(A) y_{\overline{k'_1}} \cdots y_{\overline{k'_r}} \cdot y \\ &= \sum_{K \subseteq [m]} \mathrm{Pf}_K(A) y_{\overline{k'_1}} \cdots y_{\overline{k'_r}} \cdot y_{\overline{m}} \cdot y \end{aligned}$$

$$+\sum_{j=0}^{m-1}\sum_{K\subseteq[m]}(-1)^m a_{jm}\,\mathrm{Pf}_K(A)y_j\cdot y_{\overline{k}'_1}\cdots y_{\overline{k}'_r}\cdot y.$$

Note

$$y_j\cdot y_{\overline{k}'_1}\cdots y_{\overline{k}'_r}\cdot y=\begin{cases}0 & \text{if } j\in K;\\ (-1)^{a-1}y_{\overline{k}'_1}\cdots\widehat{y_{\overline{k}'_a}}\cdots y_{\overline{k}'_r}\cdot y & \text{if } j=k'_a\in K'.\end{cases}$$

So the second sum above becomes

$$\sum_K\sum_{j\notin K}(-1)^{m+a-1}a_{jm}\,\mathrm{Pf}_K(A)y_{\overline{k}''_1}\cdots y_{\overline{k}''_{r-1}}\cdot y,$$

where $K''=K'\smallsetminus j$. Combining the two and using Laplace expansion yields the claimed formula for $x_\emptyset$.

The general case is similar. Using (A.3), it is equivalent to show

$$x_{i_1}\cdots x_{i_r}\cdot x_\emptyset=(-1)^{i_1+\cdots+i_r}\sum_{\substack{K\subseteq[n]\\ K\text{ even}}}\mathrm{Pf}_K(A(I))\,y_K,$$

and we do this by induction on r. Let $I=\{i_1<\cdots<i_r\}$ and consider $i>i_r$. We compute:

$$\begin{aligned}x_{i_1}\cdots x_{i_r}\cdot x_i\cdot x_\emptyset&=(-1)^r x_i\cdot x_{i_1}\cdots x_{i_r}\cdot x_\emptyset\\ &=(-1)^r\left(\sum_j b_{ji}y_j\right)(-1)^{i_1+\cdots+i_r}\sum_K\mathrm{Pf}_K(A(I))y_K\\ &=(-1)^{r+i_1+\cdots+i_r}\sum_j\sum_{K\not\ni j}b_{ji}\,\mathrm{Pf}_K(A(I))y_j\cdot y_K\\ &=\cdots\\ &=(-1)^{i+i_1+\cdots+i_r}\sum_K\mathrm{Pf}_K(A(I\cup\{i\}))y_K,\end{aligned}$$

as desired.

References

1. Anderson, D.: Introduction to equivariant cohomology in algebraic geometry," lectures from a mini-course given at the IMPANGA Summer School on Algebraic Geometry, July 2010. In: Pragacz, P. (ed.) Contributions to Algebraic Geometry (2012)
2. Bertiger, A., Milićević, E., Taipale, K.: Algebr. Comb. **1**(3), 327–352 (2018)

3. Bertram, A., Ciocan-Fontanine, I., Fulton, W.: Quantum multiplication of Schur polynomials. J. Algebr. **219**(2), 728–746 (1999)
4. Billey, S.: Kostant polynomials and the cohomology ring for G/B. Duke Math. J. **96**(1), 205–224 (1999)
5. Bott, R.: The space of loops on a Lie group. Michigan Math. J. **5**, 35–61 (1958)
6. Brion, M., Carrell, J.B.: The equivariant cohomology ring of regular varieties. Michigan Math. J. **52**(1), 189–203 (2004)
7. Buch, A.: Quantum cohomology of Grassmannians. Compositio Math. **137**(2), 227–235 (2003)
8. Buch, A., Kresch, A., Tamvakis, H.: Quantum Pieri rules for isotropic Grassmannians. Invent. Math. **178**(2), 345–405 (2009)
9. Cotti, G., Dubrovin, B., Guzzetti, D.: Helix structures in quantum cohomology of fano varieties (2018).arXiv: 1811.09235
10. Chevalley, C.: In: Cartier, P., Chevalley, C (eds.) The Algebraic Theory of Spinors and Clifford Algebras, Collected Works Vol. 2. Springer, Berlin (1997)
11. Dubrovin, B.: Geometry and analytic theory of Frobenius manifolds. In: Proceedings of the ICM, Vol. II (Berlin), pp. 315–326 (1998)
12. Fulton, W., Harris, J.: Representation Theory: A First Course. Springer, Berlin (1991)
13. Galkin, S., Golyshev, V., Iritani, H.: Gamma classes and quantum cohomology of Fano manifolds: Gamma conjectures. Duke Math. J. **165**, 2005–2077 (2016)
14. Gatto, L.: Schubert calculus via Hasse-Schmidt derivations. Asian J. Math. **9**(3), 315–321 (2005)
15. Gatto, L., Salehyan, P.: Schubert derivations on the infinite wedge power (2019). arXiv:1901.06853
16. Gatto, L., Salehyan, P.: The Cohomology of the Grassmannian is a gl_n-module. Comm. Alg. (2019). arXiv:1902.03824
17. Gatto, L., Santiago, T.: Schubert calculus on a Grassmann algebra. Canad. Math. Bull. **52**(2), 200–212 (2009)
18. Ginzburg, V.: Perverse sheaves on a loop group and Langlands duality (2000). arXiv:alg-geom/9511007v4
19. Golyshev, V., Manivel, L.: Quantum cohomology and the Satake isomorphism (2011). arXiv:1106.3120v1
20. Gorbounov, V., Korff, C.: Quantum integrability and generalised quantum Schubert calculus. Adv. Math. **313**, 282–356 (2017)
21. Gorbounov, V., Korff, C., Stroppel, C.: Yang-Baxter algebras as convolution algebras: the Grassmannian case (2018). arXiv:1802.09497
22. Goresky, M., Kottwitz, R., MacPherson, R.: Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math. **131**(1), 25–83 (1998)
23. Ikeda, T.: Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian. Adv. Math. **215**(1), 1–23 (2007)
24. Ikeda, T., Naruse, H.: Excited Young diagrams and equivariant Schubert calculus. Trans. Amer. Math. Soc. **361**, 5193–5221 (2009)
25. Ivanov, V.N.: Interpolation analogues of Schur Q-functions. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov **307**, 99–119, 281–282 (2004); translation in J. Math. Sci. (N. Y.) **131**, 5495–5507 (2005)
26. Kamnitzer, J.: Mirković-Vilonen cycles and polytopes. Ann. of Math. **171**(1), 245–294 (2010)
27. Kempf, G., Laksov, D.: The determinantal formula of Schubert calculus. Acta Math. **132**, 153–162 (1974)
28. Knutson, A., Tao, T.: Puzzles and (equivariant) cohomology of Grassmannians. Duke Math. J. **119**(2), 221–260 (2003)
29. Laksov, D.: Schubert calculus and equivariant cohomology of Grassmannians. Adv. Math. **217**(4), 1869–1888 (2008)
30. Laksov, D.: A formalism for equivariant Schubert calculus. Algebr. Number Theory **3**(6), 711–727 (2009)

31. Laksov, D., Thorup, A.: A determinantal formula for the exterior powers of the polynomial ring. Indiana Univ. Math. J. **56**(2), 825–845 (2007)
32. Laksov, D., Thorup, A.: Schubert calculus on Grassmannians and exterior powers. Indiana Univ. Math. J. **58**(1), 283–300 (2009)
33. Laksov, D., Thorup, A.: Splitting algebras and Schubert calculus. Indiana Univ. Math. J. **61**(3), 1253–1312 (2012)
34. Lam, T., Templier, N.: The mirror conjecture for minuscule flag varieties (2017). arXiv:1705.00758
35. Lusztig, G.: Singularities, character formulas, and a q-analog of weight multiplicities, Analysis and topology on singular spaces, II, III (Luminy, 1981), 208–229, Astérisque, 101–102. Soc. Math. France, Paris (1983)
36. Macdonald, I.G.: Symmetric Functions and Hall polynomials, 2nd edn. Oxford University Press, Oxford (1995)
37. Manivel, L.: On spinor varieties and their secants. SIGMA Symmetry Integr. Geom. Methods Appl. **5**, Paper 078, 22 pp. (2009)
38. Maulik, D., Okounkov, A.: Quantum groups and quantum cohomology (2012). arXiv:1211.1287
39. Mihalcea, L.: Equivariant quantum Schubert calculus. Adv. Math. **203**(1), 1–33 (2006)
40. Mihalcea, L.: Giambelli formulae for the equivariant quantum cohomology of the Grassmannian. Trans. Amer. Math. Soc. **360**(5), 2285–2301 (2008)
41. Mirković, I., Vilonen, K.: Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. Math. **166**(1), 95–143 (2007)
42. Nimmo, J.J.C.: Hall-Littlewood symmetric functions and the BKP equation. J. Phys. A **23**(5), 751–760 (1990)
43. Okada, S.: Pfaffian formulas and Schur Q-Function identities (2017). arXiv:1706.01029
44. Okounkov, A.: Enumerative geometry and geometric representation theory. Proc. Symp. Pure Math. **97**(1), 419–457 (2018)
45. Pragacz, P.: Algebro-geometric applications of Schur S- and Q-polynomials. Topics in Invariant Theory, Paris, 1989/1990. Lecture Notes in Mathematics, vol. 1478, pp. 130–191 (1991)
46. Rimányi, R., Smirnov, A., Varchenko, A., Zhou, Z.: 3d mirror symmetry and elliptic stable envelopes (2019). arXiv:1902.03677
47. Sottile, F.: Rational curves on Grassmannians: systems theory, reality, and transversality. Advances in Algebraic Geometry Motivated by Physics, pp. 9–42. Lowell, MA (2000); Contemp. Math. **276**, Amer. Math. Soc., Providence, RI (2001)
48. Varagnolo, M.: Quiver varieties and Yangians. Lett. Math. Phys. **53**(4), 273–283 (2000)

Positive Level, Negative Level and Level Zero

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Abstract This is a survey on the combinatorics and geometry of integrable representations of quantum affine algebras with a particular focus on level 0. Pictures and examples are included to illustrate the affine Weyl group orbits, crystal graphs and Macdonald polynomials that provide detailed understanding of the structure of the extremal weight modules and their characters. The final section surveys the alcove walk method of working with the positive level, negative level and level zero affine flag varieties and describes the corresponding actions of the affine Hecke algebra.

Keywords Affine flag varieties · Integrable representations · Quantum affine algebras

AMS Subject Classifications Primary 17B37 · Secondary 17B67

1 Introduction

This paper is about positive level, negative level and level 0. It was motivated by the striking result of [23], which establishes a Pieri-Chevalley formula for the K-theory of the semi-infinite (level 0) affine flag variety. This made us want to learn more about the level 0 integrable modules of quantum affine algebras. Our trek brought us face to face with a huge literature, including important contributions from Drinfeld, Kashiwara, Beck, Chari, Nakajima, Lenart–Schilling–Shimozono, Cherednik–Orr,

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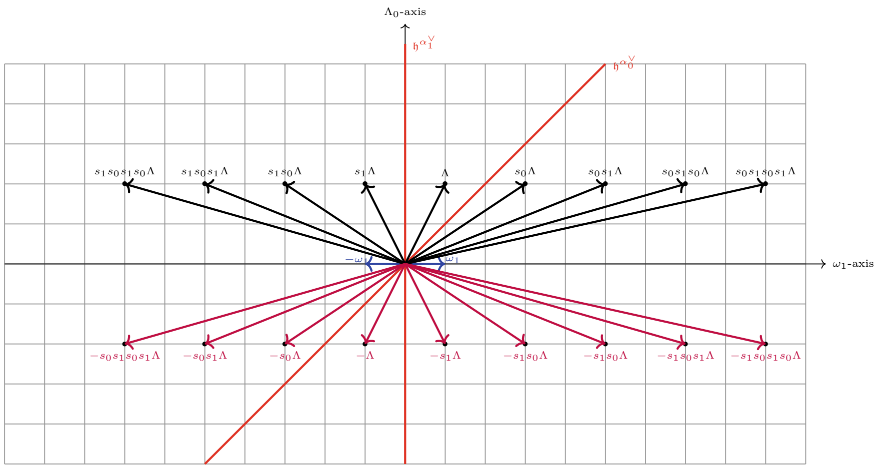
Naito–Sagaki, Feigin–Makedonskyi, Kato, their coauthors and many others. It is a beautiful theory and we count ourselves lucky to have been drawn into it.

The main point is that the integrable modules for quantum affine algebras $\mathbf{U}$ naturally partition themselves into families: positive level, negative level and level zero. Their structure is shadowed by the orbits of the affine Weyl group on the lattice of weights for the affine Lie algebra, which take the shape of a concave up paraboloid at positive level, a concave down paraboloid at negative level and a tube at level 0. These integrable modules have crystal bases which provide detailed control of their characters. At level 0 the characters are (up to a factor similar to a Weyl denominator) Macdonald polynomials specialised at $t = 0$. The next amazing feature is that there are Borel–Weil–Bott theorems for each case: positive level, negative level and level 0, where, respectively, the appropriate geometry is a positive level (thin) affine flag variety, a negative level (thick) affine flag variety, and a level zero (semi-infinite) affine flag variety.

This paper is a survey of the general picture of positive level, negative level, and level zero, in the context of the combinatorics of affine Weyl groups and crystals, of the representation theory of integrable modules for quantum affine algebras, and of the geometry of affine flag varieties. In recent years, the picture has become more and more rich and taken clearer focus. We hope that this paper will help to bring this story to a wider audience by providing pictures and some explicit small examples for $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sl}}_3$.

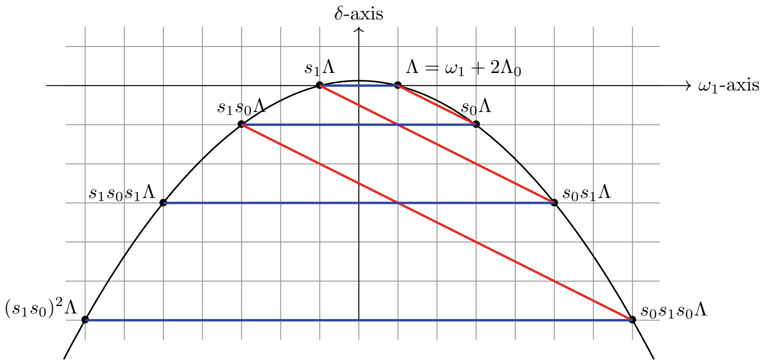
1.1 Orbits of the Affine Weyl Group W Action on $\mathfrak{h}^*$

For $\widehat{\mathfrak{sl}}_2$ the vector space $\mathfrak{h}^*$ is three dimensional with basis $\{\delta, \omega_1, \Lambda_0\}$, and the orbits of the action of the affine Weyl group W^{ad} on $\mathfrak{h}^*$ on different levels look like



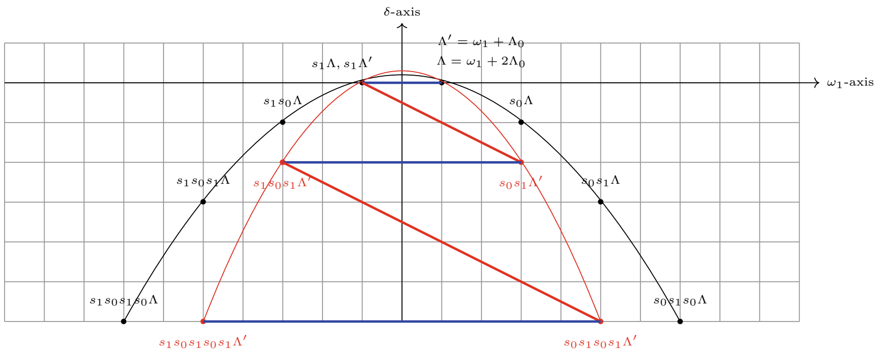
$W^{\mathrm{ad}}(\omega_1 + 2\Lambda_0)$ (positive level), $W^{\mathrm{ad}}\omega_1$ (level 0), and $W^{\mathrm{ad}}(-\omega_1 - 2\Lambda_0)$ (negative level) mod δ

Although informative, the picture above is misleading as it is a two dimensional projection (mod δ) of what is actually going on. The W^{ad}-action fixes the level (the coefficient of Λ_0) but it actually changes the δ coordinate significantly. Let us look at the orbits in δ and ω_1 coordinates (i.e. mod Λ_0).

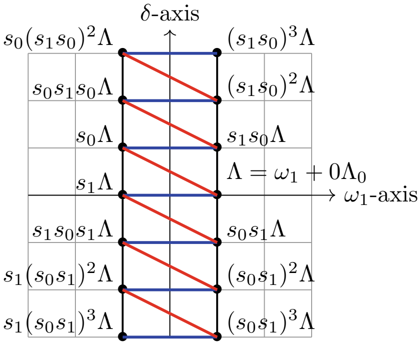


The positive level orbit $W^{\mathrm{ad}}(\omega_1 + 2\Lambda_0)$ mod Λ_0

When the level (coefficient of Λ_0) is large the parabola is wide, and it gets tighter as the level decreases.

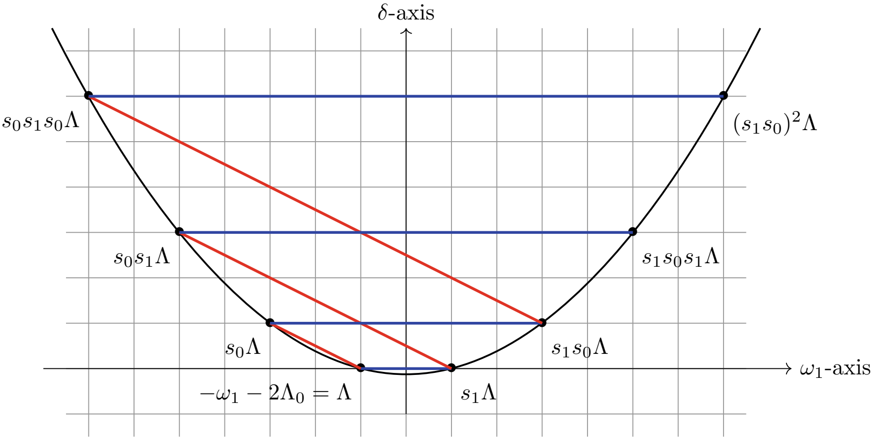


At level 0, the parabola pops and becomes two straight lines.



The level 0 orbit $W^{\mathrm{ad}}\omega_1 \bmod \Lambda_0$

At negative level the parabola forms again, but this time facing the opposite way, and getting wider as the level gets more and more negative.



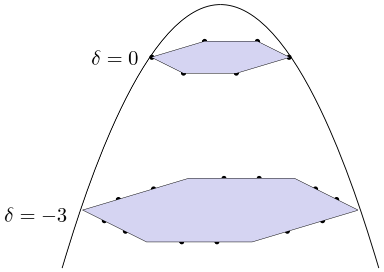
The orbit $W^{\mathrm{ad}}(-\omega_1 - 2\Lambda_0)$ (negative level) mod Λ_0

The three different Bruhat orders on the affine Weyl group are visible on the W^{ad}-orbits:

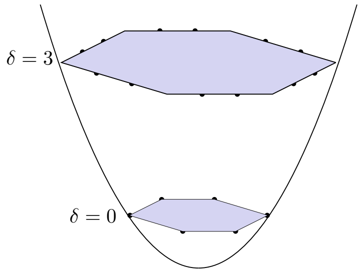
$$
\begin{aligned}
v &\overset{+}{\leq} w && \text{if } v(\omega_1 + \Lambda_0) \text{ is higher than } (\omega_1 + \Lambda_0) \text{ in } W^{\mathrm{ad}}(\omega_1 + \Lambda_0),\\
v &\overset{0}{\leq} w && \text{if } v\omega_1 \text{ is higher than } w(\omega_1 + 0\Lambda_0) \text{ in } W^{\mathrm{ad}}(\omega_1 + 0\Lambda_0).\\
v &\leq w && \text{if } v(-\omega_1 - \Lambda_0) \text{ is higher than } w(-\omega_1 - \Lambda_0) \text{ in } W^{\mathrm{ad}}(-\omega_1 - \Lambda_0).
\end{aligned}
$$

The definitions of the Bruhat orders on W^{ad} and their relation to the closure order for Schubert cells in affine flag varieties is made precise in Sect. 2.3. Indicative relations illustrating the from of the Hasse diagrams of the positive level, negative level, and level zero Bruhat orders for the Weyl group of $\widehat{\mathfrak{sl}}_3$ are pictured in Plate A (there are additional relations which are not displayed in the pictures—in an effort to make the periodicity pattern easily visible).

For the case of $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$ the affine Weyl group orbits take a similar form, with the points sitting on a downward paraboloid at positive level, on an upward paraboloid at negative level and with the paraboloid popping and becoming a tube at level 0 (for an example tube see the picture of $B(\omega_1 + \omega_2)$ for $\widehat{\mathfrak{sl}}_3$ in Plate D).



Positive level orbit $W^{\mathrm{ad}}(\omega_1 + \omega_2 + 2\Lambda_0)$ for $\widehat{\mathfrak{sl}}_3$



Negative level orbit $W^{\mathrm{ad}}(-\omega_1 - \omega_2 - 2\Lambda_0)$ for $\widehat{\mathfrak{sl}}_3$

1.2 Extremal Weight Modules $L(\Lambda)$ and Their Crystals $B(\Lambda)$

For the affine Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ the weights of integrable $\mathfrak{g}$-modules always lie in the set

$$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{C}\delta + \mathbb{Z}\omega_1 + \mathbb{Z}\Lambda_0.$$

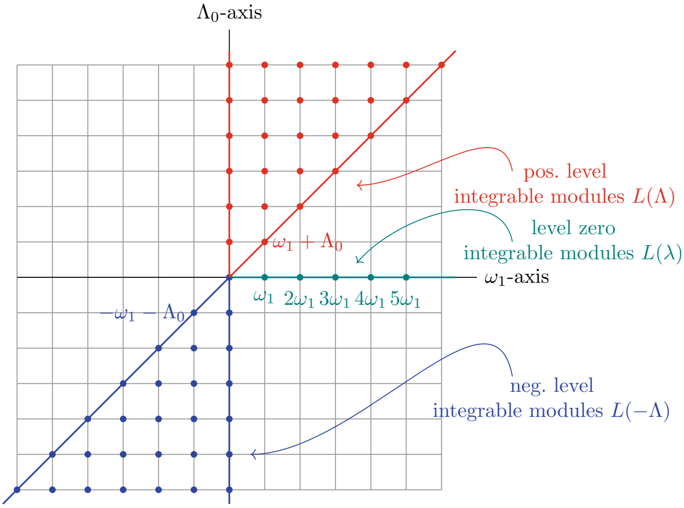
A set of representatives for the orbits of the action of W^{ad} on $\mathfrak{h}^*_{\mathbb{Z}}$ is

$$(\mathfrak{h}^*)_{\mathrm{int}} = (\mathfrak{h}^*)^+_{\mathrm{int}} \cup (\mathfrak{h}^*)^0_{\mathrm{int}} \cup (\mathfrak{h}^*)^-_{\mathrm{int}}, \quad \text{where} \quad \begin{aligned} (\mathfrak{h}^*)^0_{\mathrm{int}} &= \{a\delta + n\omega_1 \in \mathfrak{h}^*_{\mathbb{Z}} \mid 0 \le n\}, \\ (\mathfrak{h}^*)^+_{\mathrm{int}} &= \{a\delta + m\omega_1 + n\Lambda_0 \in \mathfrak{h}^*_{\mathbb{Z}} \mid 0 \le m \le n\}, \\ (\mathfrak{h}^*)^-_{\mathrm{int}} &= \{a\delta - m\omega_1 - n\Lambda_0 \in \mathfrak{h}^*_{\mathbb{Z}} \mid 0 \le m \le n\}. \end{aligned}$$

These sets are illustrated (mod δ) below.

For each of the $\Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}$, there is a (universal) integrable *extremal weight* module $L(\Lambda)$, which is highest weight if $\Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}}$, is lowest weight (and not highest weight) if $\Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}}$, and which is neither highest or lowest weight when $\Lambda \in (\mathfrak{h}^*)^0_{\mathrm{int}}$. The module $L(\Lambda)$ has a crystal $B(\Lambda)$.

At positive level and negative levels the crystals $B(\Lambda)$ are connected, but the crystal $B(\Lambda)$ is usually not connected in level 0. The connected components and their structure are known explicitly from a combination of results of Kashiwara, Beck–Chari–Pressley, Nakajima, Beck–Nakajima, Fourier–Littelmann, Ion and others. These results are collected in Theorem 5 and Eq. (28) expresses the characters of the $L(\Lambda)$ in terms of Macdonald polynomials specialised at $t = 0$.



1.3 Affine Flag Varieties G/I^+, G/I^0 and G/I^-

There are three kinds of affine flag varieties for the loop group $G = G(\mathbb{C}((\epsilon)))$: the positive level (thin) affine flag variety G/I^+, the negative level (thick) affine flag variety G/I^- and the level 0 (semi-infinite) affine flag variety G/I^0. A combination of results of Kumar, Mathieu, Kashiwara, Kashiwara–Tanisaki, Kashiwara–Shimozono, Varagnolo–Vasserot, Lusztig and Braverman–Finkelberg have made it clear that there is a Borel–Weil–Bott theorem for each of these:

$$\begin{aligned} H^0(G/I^+, \mathcal{L}_\Lambda) &\cong L(\Lambda), &&\text{for positive level } \Lambda \in (\mathfrak{h}^*)^+_{\text{int}}, \\ H^0(G/I^0, \mathcal{L}_\lambda) &\cong L(\lambda), &&\text{for level zero } \lambda \in (\mathfrak{h}^*)^0_{\text{int}}, \\ H^0(G/I^-, \mathcal{L}_{-\Lambda}) &\cong L(-\Lambda), &&\text{for negative } -\Lambda \in (\mathfrak{h}^*)^-_{\text{int}}. \end{aligned}$$

These Borel–Weil–Bott theorems tightly connect the representation theory with the geometry. In all essential aspects the combinatorics of the positive level affine flag variety and the loop Grassmannian generalizes to the negative level and the level 0 affine flag varieties.

Section 6 extends the results of [38] and displays the alcove walk combinatorics for each of the three cases (positive level, negative level and level 0) in parallel. In addition it describes the method for deriving the natural affine Hecke algebra actions on the function spaces $C(G/I^+)$, $C(I^+\backslash G/I^+)$, $C(I^+\backslash G/I^+)$ and $C(I^-\backslash G/I^+)$.

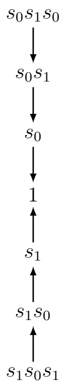
1.4 References and Technicalities

Section 2.1 introduces the affine Lie algebra and the homogeneous Heisenberg subalgebra following [17] and Sect. 2.2 gives explicit matrices describing the actions of the affine Weyl group W^{ad} on the affine Cartan $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. Section 2.3 defines the Bruhat orders on the affine Weyl group and explains their relation to the corresponding affine flag varieties following [24] and [27, Sects. 7 and 11]. Sections 2.4 and 2.5 introduce the affine braid groups and Macdonald polynomials following [40]. Section 2.6 treats the specializations of (nonsymmetric) Macdonald polynomials at $t = 0$, $t = \infty$, $q = 0$ and $q = \infty$ and reviews the result of Ion [15] that relates Macdonald polynomials at $t = 0$ to Demazure operators.

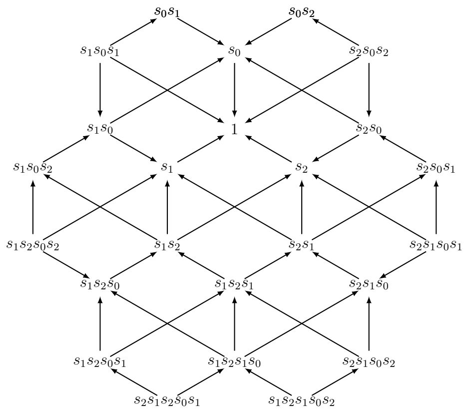
Section 2 follows [1, 3, 4], introducing the quantum affine algebra $\mathbf{U}$, the conversion to its loop presentation, the PBW-type elements and the quantum homogeneous Heisenberg subalgebra. Section 3.2 defines integrable $\mathbf{U}$-modules and Sect. 3.3 introduces the extremal weight modules $L(\Lambda)$ following [19, (8.2.2)] and [20, Sect. 3.1]. Section 3.4 reviews the Demazure character formulas for extremal weight modules. Section 3.5 discusses the loop presentation of the level 0 extremal weight modules and the fact that these coincide with the universal standard modules of [34] and the global Weyl modules of [9]. Letting $\mathbf{U}'$ be $\mathbf{U}$ without the element D, Sect. 3.6 explains how to shrink the extremal weight module to a local Weyl module and how this provides a classification of finite dimensional simple $\mathbf{U}'$-modules by Drinfeld polynomials.

We have made a concerted effort to make a useful survey. In order to simplify the exposition we have brushed under the rug a number of technicalities which are wisely ignored when one learns the subject (particularly (a) the difference between simply laced cases and the general case requires proper attention to the diagonal matrix which symmetrizes the affine Cartan matrix [17, (2.1.1)] causing the constants d_i which pepper the quantum group literature and (b) the machinations necessary for allowing multiple parameters $t_i^{\frac{1}{2}}$ in Macdonald polynomials). The reader who needs to sort out these features is advised to drink a strong double espresso to optimise clear thinking, consult the references (particularly [4, 40]) and *not trust our exposition*. Perhaps in the future a more complete (probably book length) version of this paper will be completed which allows us to attend more carefully to these nuances and include more detailed proofs. Having made this point, we can say that a careful effort has been made to provide specific references to the literature at every step and we hope that this will be useful for the reader that wishes to go further.

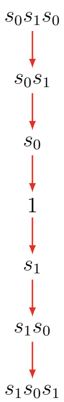
PLATE A: Bruhat orders on the affine Weyl group (partial, indicative, relations)



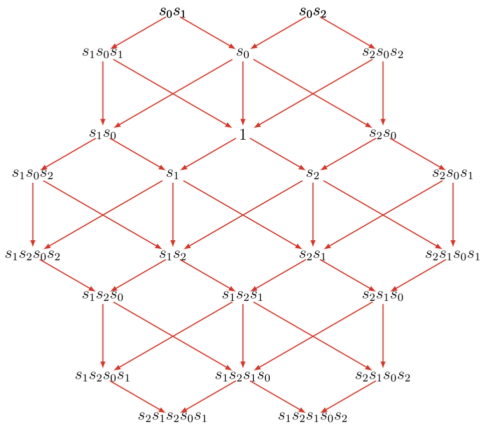
postive level Bruhat order for $\widehat{\mathfrak{sl}}_2$
1 is minimal



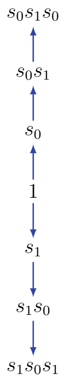
postive level Bruhat order for $\widehat{\mathfrak{sl}}_3$
1 is minimal



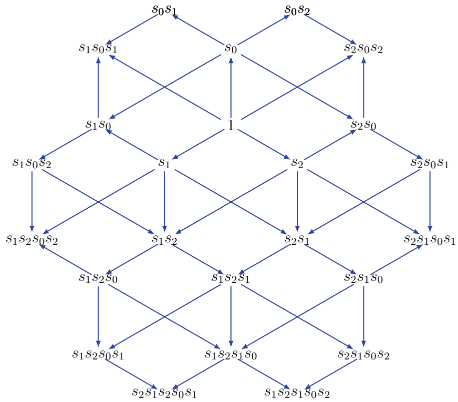
level zero Bruhat order for $\widehat{\mathfrak{sl}}_2$
translation invariant



level zero Bruhat order for $\widehat{\mathfrak{sl}}_3$
translation invariant

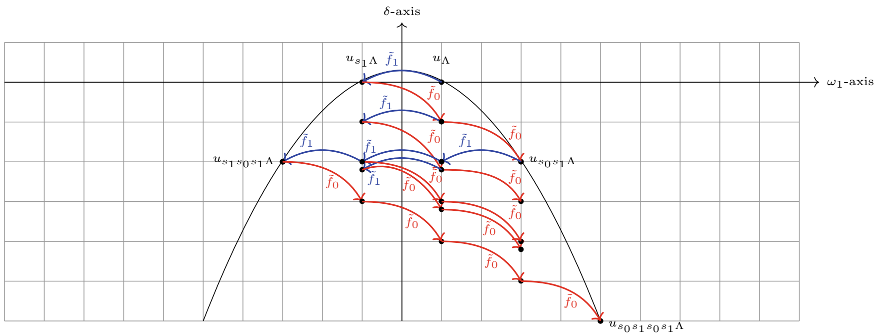


negative level Bruhat order for $\widehat{\mathfrak{sl}}_2$
1 is maximal

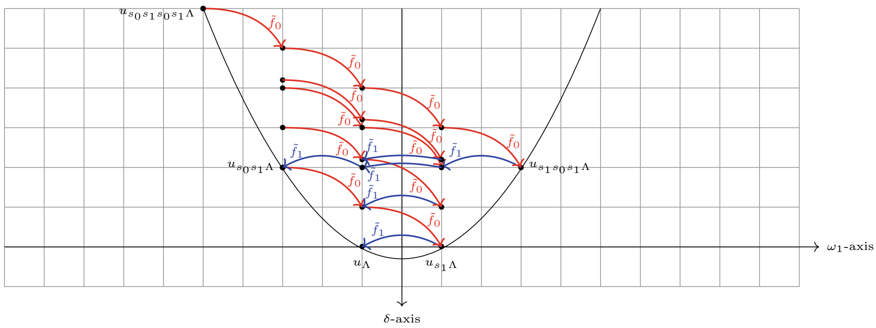


negative level Bruhat order for $\widehat{\mathfrak{sl}}_3$
1 is maximal

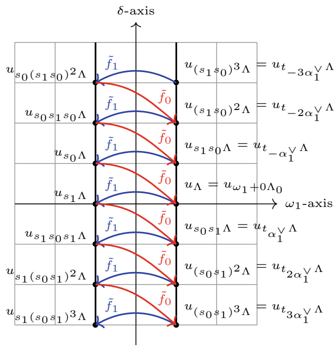
PLATE B: Pictures of $B(\omega_1 + \Lambda_0)$, $B(\omega_1 + 0\Lambda_0)$ and $B(-\omega_1 - \Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$



Initial portion of the crystal graph of $B(\omega_1 + \Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$

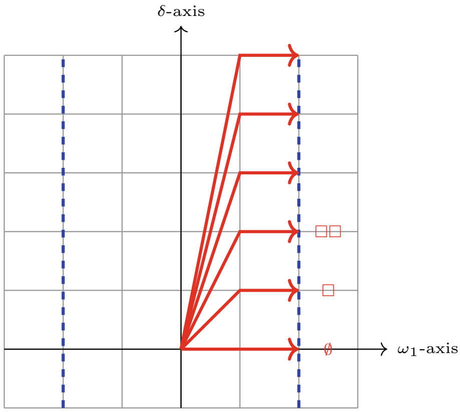


Final portion of the crystal graph of $B(-\omega_1 - \Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$

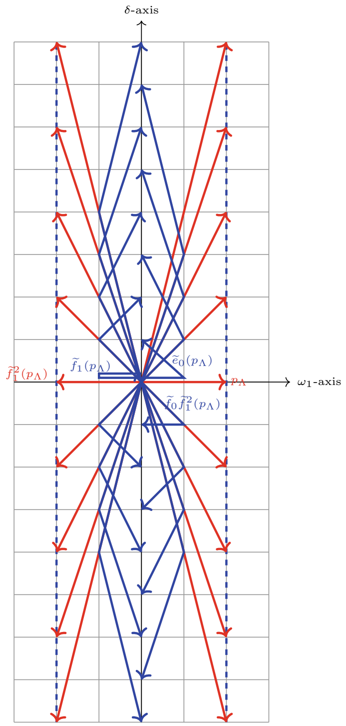


Middle portion of the crystal graph of $B(\omega_1 + 0\Lambda_0)$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$

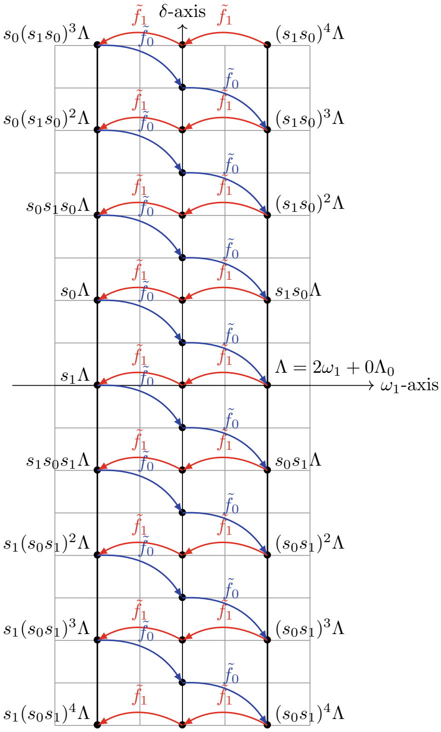
PLATE C: Pictures for $B(2\omega_1)$. Representative paths from the (first five) connected components of $B(2\omega_1)$ are



and the paths in $B(2\omega_1)_0 \subseteq B(\omega_1) \otimes B(\omega_1)$ are

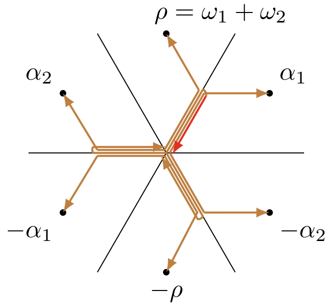


Paths in $B(2\omega_1)_0$



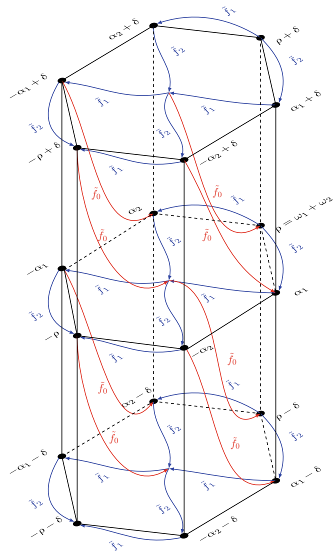
The crystal graph of $B(2\omega_1)_0$

PLATE D: Pictures and characters of $B(\omega_1+\omega_2)$ **for** $\widehat{\mathfrak{sl}}_3$**.** The colour red indicates change in the δ-coordinate.



$B^{\mathrm{fin}}(\omega_1+\omega_2) \subseteq B(\omega_1) \otimes B(\omega_2)$

$$\mathrm{gchar}(B^{\mathrm{fin}}(\omega_1+\omega_2)) = X^{\rho} + X^{\alpha_1} + X^{\alpha_2} + X^{-\alpha_1} + X^{-\alpha_2} + X^{-\rho} + 2 + q^{-1}$$



$B(\omega_1+\omega_2)$ crystal graph

At $t=0$ and $t=\infty$ the normalized nonsymmetric Macdonald polynomials $\tilde{E}_{s_1s_2s_1\rho}(q,t)$ are

$$\tilde{E}_{s_1s_2s_1\rho}(q,0) = X^{s_1s_2s_1\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} + X^{s_2\rho} + X^{s_1\rho} + X^{\rho} + 2 + q,$$
$$\tilde{E}_{s_1s_2s_1\rho}(q,\infty) = X^{s_1s_2s_1\rho} + q^{-1}(X^{s_1s_2\rho} + X^{s_2s_1\rho}) + q^{-2}(X^{s_1\rho} + X^{s_2\rho} + X^{\rho}) + 2q^{-2} + q^{-1}.$$

Letting $q = e^{-\delta}$ (as in [17, (12.1.9)]), the Demazure module $L(\omega_1+\omega_2)_{\le s_1s_2s_1}$ has character

$$\mathrm{char}(L(\omega_1+\omega_2)_{\le s_1s_2s_1}) = \frac{1}{(1-q^{-1})^2}\tilde{E}_{s_1s_2s_1\rho}(q^{-1},0) \quad \text{and}$$
$$\mathrm{char}(L(\omega_1+\omega_2)) = 0_q 0_q \tilde{E}_{s_1s_2s_1\rho}(q^{-1},0) = 0_q 0_q \tilde{E}_{s_1s_2s_1\rho}(q^{-1},\infty),$$

where $0_q = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots$ as in Remark 2.

Remark 1 The expansion

$$\frac{1}{(1-q^{-1})^2} = (1+q^{-1}+q^{-2}+\cdots)(1+q^{-1}+q^{-2}+\cdots) = 1 + 2q^{-1} + 3q^{-2} + 4q^{-3} + 5q^{-4} + \cdots$$

show that the sizes of the weight spaces of $L(\omega_1+\omega_2)_{\le s_1s_2s_1}$ are growing as δ increases. Similarly, in the character formula of $L(\omega_1+\omega_2)$, the factor $0_q 0_q$ has coefficient of q^n equal to $\mathrm{Card}\big(\{(k_1,k_2)\in\mathbb{Z}^2 \mid k_1+k_2=n\}\big) = \infty$. This shows that *every weight space of the extremal weight module* $L(\omega_1+\omega_2)$ *is infinite dimensional*.

2 Affine Weyl Groups, Braid Groups and Macdonald Polynomials

2.1 The Affine Lie Algebra $\mathfrak{g}$

Let $\mathring{\mathfrak{g}}$ be a finite dimensional complex semisimple Lie algebra and fix a Cartan subalgebra $\mathfrak{a} \subseteq \mathring{\mathfrak{g}}$ and a symmetric, ad-invariant, nondegenerate, bilinear form $\langle , \rangle : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C}$. The *affine Kac–Moody algebra* is

$$\mathfrak{g} = \left(\bigoplus_{k \in \mathbb{Z}} \mathring{\mathfrak{g}} \epsilon^k \right) \oplus \mathbb{C}K \oplus \mathbb{C}d, \qquad \text{with bracket given by} \quad [K, x\epsilon^k] = 0, \quad [K, d] = 0,$$

$$[d, x\epsilon^k] = kx\epsilon^k, \quad \text{and} \quad [x\epsilon^k, y\epsilon^\ell] = [x, y]\epsilon^{k+\ell} + k\delta_{k,-\ell}\langle x, y\rangle K, \tag{1}$$

for $x, y \in \mathring{\mathfrak{g}}$ and $k, \ell \in \mathbb{Z}$ (see [17, (7.2.2)]). Let θ be the highest root of $\mathring{\mathfrak{g}}$ (the highest weight of the adjoint representation) and define

$$e_0 = f_\theta \epsilon, \qquad f_0 = e_\theta \epsilon^{-1}, \quad \text{and} \quad h_0 = [e_0, f_0] = -h_\theta + K.$$

The miracle is that $\mathfrak{g}$ is a Kac–Moody Lie algebra with Chevalley generators

$$e_0, \dots, e_n, \ h_0, \dots h_n, \ d, \ f_0, \dots f_n, \qquad \text{which satisfy Serre relations.} \tag{2}$$

Because of (2), $\mathfrak{g}$ has a corresponding quantum enveloping algebra $\mathbf{U} = U_q\mathfrak{g}$.

The Cartan subalgebra of $\mathfrak{g}$ is

$$\mathfrak{h} = \mathfrak{a} \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \text{where } \mathfrak{a} \subseteq \mathring{\mathfrak{g}} \text{ is the Cartan subalgebra of } \mathring{\mathfrak{g}}.$$

Let $\mathring{R}^+$ be the set of positive roots for $\mathring{\mathfrak{g}}$. For $\alpha \in \mathring{R}^+$, $k \in \mathbb{Z}$, $\ell \in \mathbb{Z}_{\neq 0}$ and $i \in \{1, \dots, n\}$, let

$$x_{\alpha+k\delta} = e_\alpha \epsilon^k, \qquad x_{-\alpha+k\delta} = f_\alpha \epsilon^k, \qquad h_{i,\ell} = h_i \epsilon^\ell.$$

The *homogeneous Heisenberg subalgebra* (see [17, Sects. 8.4 and 14.8]) is

$$\mathbb{C}K \oplus \mathfrak{a}[\epsilon, \epsilon^{-1}] \quad \text{with} \quad [h_i\epsilon^k, h_j\epsilon^\ell] = k\delta_{k,-\ell}\frac{2}{\langle \alpha_i, \alpha_i\rangle}\alpha_i(h_j)K, \tag{3}$$

and $\mathfrak{a}[\epsilon]$ is a commutative Lie algebra with basis $\{h_i\epsilon^k \mid i \in \{1, \dots, n\}, \ k \in \mathbb{Z}_{\geq 0}\}$.

2.2 The Affine Weyl Group W^{ad} and Its Action on $\mathfrak{h}^*$ and $\mathfrak{h}$

Let $\delta, \omega_1, \ldots, \omega_n, \Lambda_0$ be the basis in $\mathfrak{h}^*$ which is the dual basis to the basis $d, h_1, \ldots, h_n, K$ of $\mathfrak{h}$. The affine Weyl group W^{ad} is the subgroup of $GL(\mathfrak{h}^*)$ generated by the linear transformations $s_0, s_1, \ldots, s_n$ which, in the basis $\delta, \omega_1, \ldots, \omega_n, \Lambda_0$, are

$$s_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & -\alpha_i(h_1) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_2) & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_i(h_{i-1}) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_{i+1}) & 1 & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_i(h_n) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad \text{for } i \in \{1, \ldots, n\} \tag{4}$$

and, writing $\theta = a_1\alpha_1 + \cdots + a_n\alpha_n$ and $h_\theta = [e_\theta, f_\theta] = a_1^\vee h_1 + \cdots + a_n^\vee h_n$,

$$s_0 = \begin{pmatrix} 1 & a_1^\vee & a_2^\vee & \cdots & a_n^\vee & -1 \\ 0 & 1 - a_1 a_1^\vee & -a_1 a_2^\vee & \cdots & -a_1 a_n^\vee & a_1 \\ 0 & -a_2 a_1^\vee & 1 - a_2 a_2^\vee & \cdots & -a_2 a_n^\vee & a_2 \\ \vdots & & & \vdots & & \vdots \\ 0 & -a_n a_1^\vee & -a_n a_2^\vee & \cdots & 1 - a_n a_n^\vee & a_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{5}$$

Let $\mathfrak{a}_{\mathbb{R}}^* = \mathbb{R}\text{-span}\{\alpha_1, \ldots, \alpha_n\}$. An *alcove* is a fundamental region for the action of W^{ad} on $(\mathbb{R}\delta + \mathfrak{a}_{\mathbb{R}}^* + \Lambda_0)/\mathbb{R}\delta$. As explained (for example) in [39, 40], there is a bijection

$$\begin{matrix} W^{\mathrm{ad}} & \longleftrightarrow & \{\text{alcoves}\} \\ 1 & \longmapsto & \{x + \Lambda_0 \in \mathfrak{a}_{\mathbb{R}}^* + \Lambda_0 \mid x(h_i) > 0 \text{ for } i \in \{0, \ldots, n\}\} \end{matrix} \tag{6}$$

Let $\mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}} = \mathbb{Z}\text{-span}\{h_1, \ldots, h_n\}$. The *finite Weyl group* W_{fin} is generated by $s_1, \ldots, s_n$. The translation presentation of the affine Weyl group is

$$W^{\mathrm{ad}} = \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}} \rtimes W_{\mathrm{fin}} = \{t_{\mu^\vee} u \mid \mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}, u \in W_{\mathrm{fin}}\} \quad \text{with} \quad \begin{matrix} t_{\mu^\vee} t_{\nu^\vee} = t_{\mu^\vee + \nu^\vee} \text{ and} \\ u t_{\mu^\vee} = t_{u\mu^\vee} u, \end{matrix} \tag{7}$$

for $\mu^\vee, \nu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$ and $u \in W_{\mathrm{fin}}$.

Let $\alpha_i^\vee$ be the image of h_i under the isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$ coming from the nondegenerate bilinear form on $\mathfrak{a}$ which is the restriction of the nondegerate bilinear form on $\mathring{\mathfrak{g}}$. In matrix form with respect to the basis $\delta, \omega_1, \ldots, \omega_n, \Lambda_0$ of $\mathfrak{h}^*$ the action of W^{ad} on $\mathfrak{h}^*$ is given by

$$t_{\mu^\vee} = \left(\begin{array}{c|ccc|c} 1 & k_1 & \cdots & k_n & -\frac{1}{2}\langle\mu^\vee, \mu^\vee\rangle \\ \hline \vdots & & & & \mu_1^\vee \\ 0 & & 1 & & \vdots \\ \vdots & & & & \mu_n^\vee \\ \hline 0 & \cdots & 0 & \cdots & 1 \end{array}\right), \quad \text{for} \quad \begin{aligned} \mu^\vee &= k_1 h_1 + \cdots + k_n h_n \\ &= k_1 \alpha_1^\vee + \cdots + k_n \alpha_n^\vee \\ &= \tfrac{k_1}{d_1}\alpha_1 + \cdots + \tfrac{k_n}{d_n}\alpha_n \\ &= \mu_1^\vee \omega_1 + \cdots + \mu_n^\vee \omega_n, \end{aligned} \tag{8}$$

so that $-\frac{1}{2}\langle\mu^\vee, \mu^\vee\rangle = -\frac{1}{2}(\mu_1^\vee k_1 + \cdots \mu_n^\vee k_n)$. (In (8) $d_1, \ldots, d_n$ are the minimal positive integers such that the product of the diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$ with the Cartan matrix is symmetric, see [17, (2.1.1)].)

The basis $\{d, h_1, \ldots, h_n, K\}$ of $\mathfrak{h}$ is the dual basis to the basis $\{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}$ of $\mathfrak{h}^*$. Using the W^{ad}-action on $\mathfrak{h}$ given by

$$s_i\mu^\vee = \mu^\vee - \alpha_i(\mu^\vee)h_i, \qquad \text{for } i \in \{0, \ldots, n\} \text{ and } \mu^\vee \in \mathfrak{h},$$

the matrices for the action of $s_0, s_1, \ldots, s_n$ on $\mathfrak{h}$, in the basis $\{d, h_1, \ldots, h_n, K\}$, are the *transposes of the matrices in* (4) *and* (5).

2.3 The Positive Level, Negative Level and Level 0 Bruhat Orders on W^{ad}

In the framework of Sect. 6, where $G = \mathring{G}(\mathbb{C}((\epsilon)))$ is the loop group, the closure orders for the Schubert cells in the positive level (thin) affine flag variety G/I^+, the negative level (thick) affine flag variety G/I^-, and the level 0 (semi-infinite) affine flag variety G/I^0 give partial orders on the affine Weyl group W^{ad}:

$$\overline{I^+wI^+} = \bigsqcup_{x \overset{+}{\leq} w} I^+xI^+, \qquad \overline{I^+wI^0} = \bigsqcup_{x \overset{0}{\leq} w} I^+xI^0, \qquad \overline{I^+wI^-} = \bigsqcup_{x \overset{-}{\leq} w} I^+xI^-.$$

These orders can be described combinatorially as follows.

An element $w \in W^{\mathrm{ad}}$ is *dominant* if

$$w(\rho + \Lambda_0) \in \mathbb{R}_{\geq 0}\text{-span}\{\omega_1, \ldots, \omega_n\} + \Lambda_0, \qquad \text{where} \quad \rho = \omega_1 + \cdots + \omega_n.$$

In the identification (6) of elements of W^{ad} with alcoves, the dominant elements of W^{ad} are the alcoves in the dominant Weyl chamber.

Let $x, w \in W^{\mathrm{ad}}$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w in the generators $s_0, \ldots, s_n$. The *positive level Bruhat order on W^{ad}* is defined by

$$x \overset{+}{\leq} w \quad \text{if } x \text{ has a reduced word which is a subword of } w = s_{i_1} \cdots s_{i_\ell}$$

The *negative level Bruhat order* on W^{ad} is defined by $x \overset{-}{\leq} w$ if $x \overset{+}{\geq} w$.
The *level 0* Bruhat order on W^{ad} is determined by

(a) $\overset{0}{\leq}$ for dominant elements: If x, w are dominant then $x \overset{0}{\leq} w$ if and only if $x \overset{+}{\leq} w$,
(b) $\overset{0}{\leq}$ translation invariance: If $\mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$ and $x, w \in W$ then $x \overset{0}{\leq} w$ if and only if $xt_{\mu^\vee} \overset{0}{\leq} wt_{\mu^\vee}$.

The *positive level length* is $\ell^+ \colon W^{\mathrm{ad}} \to \mathbb{Z}_{\geq 0}$ given by $\ell^+(w) = (length\ of\ a\ reduced\ word\ for\ w)$.
The *negative level length* is $\ell^- \colon W^{\mathrm{ad}} \to \mathbb{Z}_{\leq 0}$ given by $\ell^-(w) = -\ell^+(w)$.
The *level 0 length* is $\ell^0 \colon W^{\mathrm{ad}} \to \mathbb{Z}$ given by

$$\ell^0(w) = \ell^+(w) \text{ if } w \text{ is dominant} \qquad \text{and} \qquad \ell^0(xt_{\mu^\vee}) - \ell^0(yt_{\mu^\vee}) = \ell^0(x) - \ell^0(y),$$

for $x, y \in W^{\mathrm{ad}}$ and $\mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$. Using the formula for ℓ^+ given in [30, (2.8)], gives a formula for ℓ^0,

$$\ell^0(ut_{\mu^\vee}) = \ell^+(u) + 2\langle \rho, \mu^\vee \rangle, \qquad \text{for } u \in W_{\mathrm{fin}}, \mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}. \tag{9}$$

The length functions ℓ^+, ℓ^- and ℓ^0 return, respectively, the dimension, the codimension and the relative dimension of Schubert cells in the positive level, negative level and level 0 affine flag varieties.

2.4 The Affine Braid Groups $\mathcal{B}^{\mathrm{sc}}$ and $\mathcal{B}^{\mathrm{ad}}$

Let $\omega_1^\vee, \ldots, \omega_n^\vee$ be the basis of $\mathfrak{a}$ which is dual to the basis $\alpha_1, \ldots, \alpha_n$ of $\mathfrak{a}^*$. Let

$$\mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}} = \mathbb{Z}\text{-span}\{h_1, \ldots, h_n\} \quad \subseteq \quad \mathfrak{a}_{\mathbb{Z}}^{\mathrm{sc}} = \mathbb{Z}\text{-span}\{\omega_1^\vee, \ldots, \omega_n^\vee\}.$$

The affine braid group $\mathcal{B}^{\mathrm{ad}}$ (resp. $\mathcal{B}^{\mathrm{sc}}$) is generated by $T_1, \ldots, T_n$ and $Y^{\lambda^\vee}$, $\lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$ (resp. $\lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{sc}}$), with relations

$$Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee + \sigma^\vee}, \qquad \underbrace{T_i T_j \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i \cdots}_{m_{ij} \text{ factors}}, \qquad \begin{array}{l} T_i^{-1} Y^{\lambda^\vee} = Y^{s_i \lambda^\vee} T_i^{-1}, \text{ if } \langle \lambda^\vee, \alpha_i \rangle = 0, \\ T_i^{-1} Y^{\lambda^\vee} T_i^{-1} = Y^{s_i \lambda^\vee}, \text{ if } \langle \lambda^\vee, \alpha_i \rangle = 1, \end{array}$$

for $i \in \{1, \ldots, n\}$ and $\lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$ (resp. $\mathfrak{a}_{\mathbb{Z}}^{\mathrm{sc}}$) and $m_{ij} = \alpha_i(h_j)\alpha_j(h_i)$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

2.5 Macdonald Polynomials

Let

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\} \qquad \text{and} \qquad \mathfrak{a}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\omega_1, \ldots, \omega_n\}.$$

The *double affine Hecke algebra* $\tilde{H}$ is presented by generators $T_0, \ldots, T_n$ and X^μ, $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, with relations

$$X^\lambda X^\mu = X^{\lambda+\mu}, \qquad \underbrace{T_i T_j \cdots}_{m_{ij}\ \text{factors}} = \underbrace{T_j T_i \cdots}_{m_{ij}\ \text{factors}}, \qquad T_i^2 = (t^{\frac12} - t^{-\frac12})T_i + 1, \tag{10}$$

$$T_i X^\mu = X^{s_i\mu} T_i + (t^{\frac12} - t^{-\frac12}) \frac{X^\mu - X^{s_i\mu}}{1 - X^{\alpha_i}}, \qquad T_i^{-1} X^\mu = X^{s_i\mu} T_i^{-1} - (t^{\frac12} - t^{-\frac12}) \frac{X^\mu - X^{s_i\mu}}{1 - X^{-\alpha_i}}.$$

for $i \in \{0, \ldots, n\}$ and $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$. For $w \in W^{\text{ad}}$ put

$$Y^w = \begin{cases} Y^{ws_i} T_i^{-1}, & \text{if } w \lessdot ws_i, \\ Y^{ws_i} T_i, & \text{if } w \gtrdot ws_i, \end{cases} \qquad \text{and let} \qquad Y^{\lambda^\vee} = Y^{t_{\lambda^\vee}} \text{ for } \lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}. \tag{11}$$

Putting $q = X^\delta = Y^{-K}$ then, as an algebra over $\mathbb{C}[q^{\pm 1}, t^{\pm\frac12}]$,

$$\tilde{H} \text{ has basis} \quad \{X^\mu T_u Y^{\lambda^\vee} \mid \mu \in \mathfrak{a}_{\mathbb{Z}}^* + \mathbb{Z}\Lambda_0, u \in W_{\text{fin}}, \lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}\}, \quad \text{where} \quad T_u = T_{i_1} \cdots T_{i_\ell},$$

for a reduced word $u = s_{i_1} \cdots s_{i_\ell}$. The *affine Hecke algebra* is the subalgebra H of $\tilde{H}$ with basis $\{T_u Y^{\lambda^\vee} \mid u \in W_{\text{fin}}, \lambda^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\text{ad}}\}$. The *polynomial representation of* $\tilde{H}$ is

$$\mathbb{C}[X] = \text{Ind}_H^{\tilde{H}}(\mathbf{1}) \qquad \text{with basis} \qquad \{X^\mu \mathbf{1} \mid \mu \in \mathfrak{a}_{\mathbb{Z}}^*\}, \tag{12}$$

and $Y^K \mathbf{1} = q^{-1}\mathbf{1}$, $Y^{-\alpha_i^\vee}\mathbf{1} = t\mathbf{1}$, and $T_i \mathbf{1} = t^{\frac12}\mathbf{1}$ for $i \in \{1, \ldots, n\}$.

Let

$$T_0^\vee = Y^{\alpha_0^\vee} X^{-\Lambda_0} T_0^{-1} X^{\Lambda_0} \quad \text{and} \quad T_i^\vee = T_i \text{ for } i \in \{1, \ldots, n\}. \tag{13}$$

The automorphism of $\tilde{H}$ given by conjugation by $X^{-\Lambda_0}$ is the automorphism $\tau\colon \tilde{H} \to \tilde{H}$ of [10, (2.8)]. Extend $\tilde{H}$ to allow rational functions in the $Y^{\lambda^\vee}$. For each $i \in \{0, 1, \ldots, n\}$, *the intertwiner* $\tau_i^\vee \in \tilde{H}$ is

$$\tau_i^\vee = T_i^\vee + \frac{t^{-\frac12}(1-t)}{1 - Y^{-\alpha_i^\vee}} = (T_i^\vee)^{-1} + \frac{t^{-\frac12}(1-t)Y^{-\alpha_i^\vee}}{1 - Y^{-\alpha_i^\vee}} \quad \text{so that} \quad Y^{\lambda^\vee}\tau_i^\vee = \tau_i^\vee Y^{s_i\lambda^\vee}. \tag{14}$$

Let, for simplicity, $\mu \in \mathbb{Z}\text{-span}\{\alpha_1, \ldots, \alpha_n\}$ (the general case $\mu \in \mathfrak{a}_{\mathbb{Z}}^*$ requires consideration of the group $\Omega^\vee$, the quotient of the weight lattice by the root lattice, and is treated in detail in [40]). The *nonsymmetric Macdonald polynomial* $E_\mu = E_\mu(q,t)$ is

$$E_\mu = E_\mu(q,t) = \tau_{i_1}^\vee \dots \tau_{i_\ell}^\vee \mathbf{1}, \qquad \text{where } m_\mu = s_{i_1} \dots s_{i_\ell} \text{ is a reduced word} \tag{15}$$

for the minimal length element in the coset $t_\mu W_{\mathrm{fin}}$. The E_μ form a basis of $\mathbb{C}[X]$ consisting of eigenvectors for the $Y^{\lambda^\vee}$ (the Cherednik–Dunkl operators).

Fix a reduced word $m_\mu = s_{i_1} \dots s_{i_\ell}$ as in (15). Identifying the elements of W^{ad} with alcoves as in (6), an alcove walk of type $\vec{m}_\mu = (i_1, \dots, i_\ell)$ beginning at 1 (the fundamental alcove) is a sequence of steps, of types $i_1, \dots, i_\ell$, where a step of type j is (the signs – and + indicate that $zs_j \gtrdot z$)

positive j-crossing negative j-crossing positive j-fold negative j-fold with $zs_j \gtrdot z$.

Let $\mathcal{B}(1, \vec{m}_\mu)$ be the set of alcove walks of type $\vec{m}_\mu = (i_1, \dots, i_\ell)$ beginning at 1. For a walk $p \in \mathcal{B}(1, \vec{m}_\mu)$ let

$$\begin{aligned}
f^+(p) &= \{k \in \{1, \dots, \ell\} \mid \text{the } k\text{th step of } p \text{ is a positive fold}\},\\
f^-(p) &= \{k \in \{1, \dots, \ell\} \mid \text{the } k\text{th step of } p \text{ is a negative fold}\},\\
f(p) &= f^+(p) \cup f^-(p) = \{k \in \{1, \dots, \ell\} \mid \text{the } k\text{th step of } p \text{ is a fold}\}.
\end{aligned}$$

For $p \in \mathcal{B}(1, \vec{m}_\mu)$ let $\mathrm{end}(p)$ be the endpoint of p (an element of W^{ad}) and define the *weight* $\mathrm{wt}(p)$ and the *final direction* $\varphi(p)$ of p by

$$X^{\mathrm{end}(p)} = X^{\mathrm{wt}(p)} T^\vee_{\varphi(p)}, \qquad \text{with } \mathrm{wt}(p) \in \mathfrak{a}^*_{\mathbb{Z}} \text{ and } \varphi(p) \in W_{\mathrm{fin}}.$$

Using (14) and doing a left to right expansion of the terms of $\tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \mathbf{1}$ produces the monomial expansion of E_μ as sum over alcove walks as given in the following theorem. For simplicity we state the following theorem for $\mu \in \mathbb{Z}\text{-span}\{\alpha_1, \dots, \alpha_n\}$. It holds, after a small technical adjustment to the statement, for all $\mu \in \mathfrak{a}^*_{\mathbb{Z}}$, see [40] for details.

Theorem 1 ([40, Theorem 3.1 and Remark 3.3]) *Let $\mu \in \mathbb{Z}$-span$\{\alpha_1, \dots, \alpha_n\}$ and let m_μ be the minimal length element in the coset $t_\mu W_{\mathrm{fin}}$. Fix a reduced word $\vec{m}_\mu = s_{i_1} \cdots s_{i_\ell}$, let*

$$\beta_1^\vee = s_{i_\ell} \cdots s_{i_2} \alpha_{i_1}^\vee, \qquad \beta_2^\vee = s_{i_\ell} \cdots s_{i_3} \alpha_{i_2}^\vee, \quad \dots, \quad \beta_\ell^\vee = \alpha_{i_\ell}^\vee,$$

and let $\mathrm{sh}(\beta_k^\vee)$ *and* $\mathrm{ht}(\beta_k^\vee)$ *be defined by* $Y^{\beta_k^\vee}\mathbf{1} = q^{\mathrm{sh}(\beta_k^\vee)} t^{\mathrm{ht}(\beta_k^\vee)} \mathbf{1}$ *for* $k \in \{1, \dots, \ell\}$. *Then*

$$E_\mu(q,t) = \sum_{p\in\mathcal{B}(1,\vec{m}_\mu)} X^{wt(p)} t^{\frac{1}{2}(\ell(\varphi(p))} \prod_{k\in f^+(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}} \prod_{k\in f^-(p)} \frac{t^{-\frac{1}{2}}(1-t)q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}{1-q^{\mathrm{sh}(\beta_k^\vee)}t^{\mathrm{ht}(\beta_k^\vee)}}.$$

2.6 Specializations of the Normalized Macdonald Polynomials $\tilde{E}_\mu(q,t)$

If $m_\mu = t_\mu m$ with $m \in W_{\mathrm{fin}}$ then $E_\mu(q,t)$ has top term $t^{\frac{1}{2}\ell(m)}X^\mu$ (this term is the term corresponding to the unique alcove walk in $\mathcal{B}(\vec{m}_\mu)$ with no folds). The *normalized nonsymmetric Macdonald polynomial* is

$$\tilde{E}_\mu(q,t) = t^{-\frac{1}{2}\ell(m)}E_\mu(q,t) \quad \text{so that } \tilde{E}_\mu(q,t) \text{ has top term } X^\mu.$$

A path $p \in \mathcal{B}(1,\vec{m}_\mu)$ is *positively folded* if there are no negative folds, i.e. $\#f^-(p) = 0$.
A path $p \in \mathcal{B}(1,\vec{m}_\mu)$ is *negatively folded* if there are no positive folds, i.e. $\#f^+(p) = 0$.
A path $p \in \mathcal{B}(1,\vec{m}_\mu)$ is *positive semi-infinite* if $\ell(\varphi(p)) - \ell(m) - \#f(p) + 2\sum_{k\in f^-(p)} ht(\beta_k^\vee) = 0$. A path $p \in \mathcal{B}(1,\vec{m}_\mu)$ is *negative semi-infinite* if $\ell(m) - \ell(\varphi(p)) + \#f(p) + 2\sum_{k\in f^+(p)} ht(\beta_k^\vee) = 0$.

Proposition 1 *Let $\mu \in \mathfrak{a}_\mathbb{Z}^*$. The specializations $q = 0$, $t = 0$, $q^{-1} = 0$ and $t^{-1} = 0$ are well defined and given, respectively, by*

$$\tilde{E}_\mu(0,t) = \sum_{\substack{p\in\mathcal{B}(1,\vec{m}_\mu)\\ p \text{ pos folded}}} t^{\frac{1}{2}(\ell(\varphi(p))-\ell(m)-\#f(p))}(1-t)^{\#f(p)}X^{\mathrm{wt}(p)},$$

$$\tilde{E}_\mu(q,0) = \sum_{\substack{p\in\mathcal{B}(1,\vec{m}_\mu)\\ p \text{ neg semi-inf}}} q^{\sum_{k\in f^-(p)}\mathrm{sh}(\beta_k^\vee)}X^{\mathrm{wt}(p)},$$

$$\tilde{E}_\mu(\infty,t) = \sum_{\substack{p\in\mathcal{B}(1,\vec{m}_\mu)\\ p \text{ neg folded}}} t^{-\frac{1}{2}(\ell(\varphi(p))-\#f(p))}(1-t^{-1})^{\#f(p)}X^{\mathrm{wt}(p)},$$

$$\tilde{E}_\mu(q,\infty) = \sum_{\substack{p\in\mathcal{B}(1,\vec{m}_\mu)\\ p \text{ pos semi-inf}}} q^{-\sum_{k\in f^+(p)}\mathrm{sh}(\beta_k^\vee)}X^{\mathrm{wt}(p)}.$$

For $i \in \{0,1,\ldots,n\}$ and $f \in \mathbb{C}[\mathfrak{h}_\mathbb{Z}^*]$ define

$$\Delta_i f = \frac{f - s_i f}{1-X^{-\alpha_i}} \qquad \text{and} \qquad D_i f = (1+s_i)\frac{1}{1-X^{-\alpha_i}}f = \frac{f - X^{-\alpha_i}s_i f}{1-X^{-\alpha_i}}. \quad (16)$$

Equations (14), (13) and the last relation in (10) give that, as operators on the polynomial representation,

$$t^{\frac{1}{2}}\tau_i^\vee = X^{-\Lambda_0}(D_i - t\Delta_i)X^{\Lambda_0} + \frac{(1-t)Y^{-\alpha_i^\vee}}{1-Y^{-\alpha_i^\vee}} \quad \text{for } i \in \{1,\dots,n\}, \quad \text{and}$$

$$t^{\frac{1}{2}}\tau_0^\vee Y^{\alpha_0^\vee} = Y^{-\alpha_0^\vee}t^{\frac{1}{2}}\tau_0^\vee = X^{-\Lambda_0}(D_0 - t\Delta_0)X^{\Lambda_0} + \frac{(1-t)Y^{-\alpha_0^\vee}}{1-Y^{-\alpha_0^\vee}}.$$

When applied in the formula of (15), these formulas for (normalized) intertwiners are specializable at $t = 0$, giving the following result (see the examples computed in Sect. 4.2).

Theorem 2 ([15, Sect. 4.1]) *Let $\mu \in \mathfrak{a}_\mathbb{Z}^*$. There are unique $\nu \in \mathfrak{a}_\mathbb{Z}^*$ and $j \in \mathbb{Z}$ such that*

$$\Lambda_0 + \nu \in (\mathfrak{h}^*)^+_{\mathrm{int}} \quad \textit{and} \quad -j\delta + \mu + \Lambda_0 \in W^{\mathrm{ad}}(\Lambda_0 + \nu).$$

Let $w \in W^{\mathrm{ad}}$ be minimal length such that $-j + \mu + \Lambda_0 = w(\Lambda_0 + \nu)$ and let $w = s_{i_1}\cdots s_{i_\ell}$ be a reduced word. Letting $D_0, \dots, D_n$ be the Demazure operators given in (16)*,*

$$\tilde{E}_\mu(q,0) = q^{-j}X^{-\Lambda_0}D_{i_1}\cdots D_{i_\ell}X^\nu X^{\Lambda_0}\mathbf{1}.$$

3 Quantum Affine Algebras U and Integrable Modules

3.1 The Quantum Affine Algebra U

The *quantum affine algebra* $\mathbf{U}$ is the $\mathbb{C}(q)$-algebra generated by

$$E_0,\dots,E_n,\ F_0,\dots,F_n,,\ K_0^{\pm 1},\dots,K_n^{\pm 1},\ C^{\pm\frac{1}{2}},\ D^{\pm 1}$$

with Chevalley-Serre type relations corresponding to the affine Dynkin diagram. Following [12, (11)], [28, Sect. 39], [1, end of Sect. 1], there is an action of the affine braid group $\mathcal{B}^{\mathrm{sc}}$ on $\mathbf{U}$ by automorphisms.

Let $\mathbf{U}^+$ be the subalgebra of $\mathbf{U}$ generated by $E_0, E_1, \dots, E_n$. As explained in [3, Lemma 1.1 (iv)] and [4, (3.1)], there is a doubly infinite "longest element" for the affine Weyl group with a favourite reduced expression $w_\infty = \cdots s_{i_{-1}}s_{i_0}s_{i_1}\cdots$. This reduced word is used, with the braid group action, to define *root vectors* in $\mathbf{U}^+$ by

$$E_{\beta_0} = E_{i_0}, \quad \text{and} \quad E_{\beta_{-k}} = T_{i_0}^{-1}T_{i_{-1}}^{-1}\cdots T_{i_{-(k-1)}}^{-1}E_{i_{-k}} \quad \text{and} \quad E_{\beta_k} = T_{i_1}T_{i_2}\dots T_{i_{k-1}}E_{i_k}, \tag{17}$$

for $k \in \mathbb{Z}_{>0}$. For $i \in \{1,\dots,n\}$ and $r, s \in \mathbb{Z}$ define the *loop generators in* $\mathbf{U}^+$

$$\mathbf{x}_{i,r}^+ = E_{\alpha_i + r\delta} = Y^{-r\omega_i^\vee}E_i \quad \text{and} \quad \mathbf{x}_{i,s}^- = E_{-\alpha_i+s\delta} = Y^{s\omega_i^\vee}F_i, \tag{18}$$

where $Y^{r\omega_i^\vee}$ and $Y^{s\omega_i^\vee}$ are elements of the braid group $\mathcal{B}^{\mathrm{sc}}$ as defined in Sect. 2.4. For $r \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{>0}$ these are special cases of the root vectors in (17). For $i \in \{1, \ldots, n\}$ and $r \in \mathbb{Z}_{>0}$ define $\mathbf{q}_r^{(i)}$ by

$$\mathbf{q}_r^{(i)} = \mathbf{x}_{i,r}^- \mathbf{x}_{i,0}^+ - q^{-2} \mathbf{x}_{i,0}^+ \mathbf{x}_{i,r}^-, \quad \text{let} \quad \mathbf{q}_+^{(i)}(z) = 1 + (q - q^{-1}) \sum_{s \in \mathbb{Z}_{>0}} \mathbf{q}_s^{(i)} z^s, \tag{19}$$

and define $\mathbf{p}_r^{(i)}$ and $\mathbf{e}_r^{(i)}$ by

$$\mathbf{q}_+^{(i)}(z) = \exp\Big(\sum_{r \in \mathbb{Z}_{>0}} (q - q^{-1}) \mathbf{p}_r^{(i)} z^r \Big) \quad \text{and} \quad \exp\Big(\sum_{r \in \mathbb{Z}_{>0}} \frac{\mathbf{p}_r^{(i)}}{[r]} z^r \Big) = 1 + \sum_{k \in \mathbb{Z}_{>0}} \mathbf{e}_k^{(i)} z^k. \tag{20}$$

For a sequence of partitions $\vec{\kappa} = (\kappa^{(1)}, \ldots, \kappa^{(n)})$ define

$$\mathbf{s}_{\vec{\kappa}} = \mathbf{s}_{\kappa^{(1)}} \cdots \mathbf{s}_{\kappa^{(n)}}, \qquad \text{where} \quad \mathbf{s}_{\kappa^{(i)}} = \det(\mathbf{e}^{(i)}_{(\kappa^{(i)})'_r - r + s})_{1 \leq r,s \leq m_i}, \tag{21}$$

where $(\kappa^{(i)})'_r$ is the length of the rth column of $\kappa^{(i)}$ and $m_i = \ell(\kappa^{(i)})$ (see [29, Chap. I (3.5)]). For a sequence $\mathbf{c} = (\cdots, c_{-3}, c_{-2}, c_{-1}, \vec{\kappa}, c_1, c_2, c_3, \ldots)$ with $c_i \in \mathbb{Z}_{\geq 0}$ and all but a finite number of c_i equal to 0. The corresponding *PBW-type element of* $\mathbf{U}^+$ is

$$E_{\mathbf{c}} = (E_{\beta_0}^{(c_0)} E_{\beta_{-1}}^{(c_{-1})} E_{\beta_{-2}}^{(c_{-2})} \cdots) \, (\mathbf{s}_{\kappa^{(1)}} \cdots \mathbf{s}_{\kappa^{(n)}}) \, (\cdots E_{\beta_3}^{(c_3)} E_{\beta_2}^{(c_2)} E_{\beta_1}^{(c_1)}). \tag{22}$$

The *Cartan involution* is the $\mathbb{C}$-linear anti-automorphism $\Omega \colon \mathbf{U} \to \mathbf{U}$ given by

$$\Omega(E_i) = F_i, \quad \Omega(F_i) = E_i, \quad \Omega(K_i) = K_i^{-1}, \quad \Omega(D) = D^{-1}, \quad \Omega(q) = q^{-1},$$

and $\mathbf{U}^- = \Omega(\mathbf{U}^+)$. Putting $\mathbf{p}_{-r}^{(i)} = \Omega(\mathbf{p}_r^{(i)})$, then (see [12, Th. 2 (6)] and [1, Th. 4.7 (2)])

$$[\mathbf{p}_k^{(i)}, \mathbf{p}_l^{(i)}] = \delta_{k,-l} \frac{1}{k} \Big(\frac{q^{k\alpha_i(h_j)} - q^{-k\alpha_i(h_j)}}{q - q^{-1}} \Big) \frac{C^k - C^{-k}}{q - q^{-1}}. \tag{23}$$

Define $q_{-s}^{(i)} = \Omega(q_s^{(i)})$ and

$$\mathbf{q}_-^{(i)}(z^{-1}) = 1 + (q - q^{-1}) \sum_{s \in \mathbb{Z}_{>0}} \mathbf{q}_{-s}^{(i)} z^{-s} \tag{24}$$

The *Heisenberg subalgebra* $\mathbf{H}$ is the subalgebra of $\mathbf{U}$ generated by $\{\mathbf{p}_k^{(i)} \mid i \in \{1, \ldots, n\},\ k \in \mathbb{Z}_{\neq 0}\}$. In $\mathbf{H} \cap \mathbf{U}^+$ the $\mathbf{p}_r^{(i)}$ $(r \in \mathbb{Z}_{>0})$ are the *power sums*, the $\mathbf{q}_r^{(i)}$ $(r \in \mathbb{Z}_{>0})$ are the *Hall-Littlewoods* and the $\mathbf{e}_r^{(i)}$ $(r \in \mathbb{Z}_{>0})$ are the *elementary symmetric functions* and the $\mathbf{s}_{\vec{\kappa}}$ are the *Schur functions*.

3.2 Integrable U-Modules

As in (5), let $h_\theta = a_1^\vee h_1 + \cdots + a_n^\vee h_n$ be the highest root of $\mathring{\mathfrak{g}}$ and let

$$\Lambda_i = \omega_i + a_i^\vee \Lambda_0, \quad \text{for } i \in \{1, \ldots, n\},$$

so that $\{\delta, \Lambda_1, \ldots, \Lambda_n, \Lambda_0\}$ is the dual basis in $\mathfrak{h}^*$ to the basis $\{d, h_1, \ldots, h_n, h_0\}$ of $\mathfrak{h}$. Let

$$\mathfrak{h}^*_{\mathbb{Z}} = \{\Lambda \in \mathfrak{h}^* \mid \langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for } i \in \{0, 1, \ldots, n\}\} = \mathbb{C}\delta + \mathbb{Z}\text{-span}\{\Lambda_0, \ldots, \Lambda_n\}.$$

A set of representatives for the W^{ad}-orbits on $\mathfrak{h}^*_{\mathbb{Z}}$ is

$$(\mathfrak{h}^*)_{\mathrm{int}} = (\mathfrak{h}^*)^+_{\mathrm{int}} \cup (\mathfrak{h}^*)^0_{\mathrm{int}} \cup (\mathfrak{h}^*)^-_{\mathrm{int}}, \quad \text{where} \quad \begin{aligned} (\mathfrak{h}^*)^+_{\mathrm{int}} &= \mathbb{C}\delta + \mathbb{Z}_{\geq 0}\text{-span}\{\Lambda_0, \ldots, \Lambda_n\}, \\ (\mathfrak{h}^*)^0_{\mathrm{int}} &= \mathbb{C}\delta + 0\Lambda_0 + \mathbb{Z}_{\geq 0}\text{-span}\{\omega_1, \ldots, \omega_n\}, \\ (\mathfrak{h}^*)^-_{\mathrm{int}} &= \mathbb{C}\delta + \mathbb{Z}_{\leq 0}\text{-span}\{\Lambda_0, \ldots, \Lambda_n\}. \end{aligned} \tag{25}$$

For $\widehat{\mathfrak{sl}}_2$ these sets are pictured (mod δ) in (0.2).

For $i \in \{0, 1, \ldots, n\}$ let $\mathbf{U}_{(i)}$ be the subalgebra of $\mathbf{U}$ generated by $\{E_i, F_i, K_i^{\pm 1}\}$. An *integrable* $\mathbf{U}$*-module* is a $\mathbf{U}$-module M such that if $i \in \{0, \ldots, n\}$ then

$$\mathrm{Res}^{\mathbf{U}}_{\mathbf{U}_{(i)}}(M) \qquad \text{is a direct sum of finite dimensional } \mathbf{U}_{(i)}\text{-modules},$$

where $\mathrm{Res}^{\mathbf{U}}_{\mathbf{U}_{(i)}}(M)$ denotes the restriction of the $\mathbf{U}$-module M to a $\mathbf{U}_{(i)}$-module.

Let M be an integrable $\mathbf{U}$-module. Following [28, Sect. 5], for each $w \in W^{\mathrm{ad}}$ there is a linear map

$$T_w \colon M \to M \quad \text{such that} \quad T_w(um) = T_w(u)T_w(m),$$

for $u \in \mathbf{U}$ and $m \in M$ (here $T_w(u)$ refers to the braid group action on $\mathbf{U}$). Thus, every integrable module M is a module for the semidirect product $\mathcal{B}^{\mathrm{ad}} \ltimes \mathbf{U}$ where $\mathcal{B}^{\mathrm{ad}}$ is the braid group of W^{ad}.

3.3 Extremal Weight Modules $L(\Lambda)$

Let $\Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}$. Following [19, (8.2.2)] and [20, Sect. 3.1], the *extremal weight module* $L(\Lambda)$ is the $\mathbf{U}$-module

$$\begin{aligned} &\text{generated by } \{u_{w\Lambda} \mid w \in W\} \quad \text{with relations} \quad K_i(u_{w\Lambda}) = q^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda}, \\ &E_i u_{w\Lambda} = 0, \quad \text{and} \quad F_i^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad \text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}, \\ &F_i u_{w\Lambda} = 0, \quad \text{and} \quad E_i^{-\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad \text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0}, \end{aligned} \tag{26}$$

for $i \in \{0, \cdots, n\}$. The module $L(\Lambda)$ has a crystal $B(\Lambda)$ ([19, Prop. 8.2.2(ii)], [20, Sect. 3.1]).

- If $\Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}}$ then $L(\Lambda)$ is the simple $\mathbf{U}$-module of highest weight Λ (see [17, (10.4.6)]).
- If $\Lambda \notin (\mathfrak{h}^*)^+_{\mathrm{int}}$ then $L(\Lambda)$ is not a highest weight module.
- If $\Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}}$ then $L(\Lambda)$ is the simple $\mathbf{U}$-module of lowest weight Λ.

The finite dimensional simple modules, denoted $L^{\mathrm{fin}}(a(u)))$, are integrable weight modules which are not extremal weight modules. The connection between the $L^{\mathrm{fin}}(a(u))$ and the $L(\lambda)$ for $\lambda \in (\mathfrak{h}^*)^0_{\mathrm{int}}$ is given by Theorem 4 below.

The module $L(\Lambda)$ is universal (see [21, Sect. 2.6], [2, Sect. 2.1], [36, Sect. 2.5]). One way to formulate this universality is to let $\mathbf{U}_0$ be the subalgebra generated by $K_1^{\pm 1}, \ldots, K_n^{\pm 1}, C^{\pm\frac{1}{2}}, D^{\pm 1}$, let intInd be an induction functor in the category of integrable $\mathbf{U}$-modules and write

$$L(\Lambda) = \mathrm{intInd}^{\mathbf{U}}_{\mathbf{U}_0 \rtimes \mathcal{B}^{\mathrm{ad}}}(S(\Lambda)), \qquad \text{where} \quad S(\Lambda) = \mathrm{span}\{u_{w\Lambda} \mid w \in W^{\mathrm{ad}}\}$$

is the $\mathbf{U}_0 \rtimes \mathcal{B}^{\mathrm{ad}}$-module with action given by $T_i u_{w\Lambda} = (-q)^{\langle w\Lambda_i, \alpha_i^\vee\rangle} u_{s_i w\Lambda}$ and $K_i u_{w\Lambda} = q^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda}$, for $i \in \{0, 1, \ldots, n\}$ and $w \in W^{\mathrm{ad}}$.

3.4 Demazure Submodules $L(\Lambda)_{\leq w}$

Let $w \in W^{\mathrm{ad}}$. The *Demazure module* $L(\Lambda)_{\leq w}$ is the $\mathbf{U}^+$-submodule of $L(\Lambda)$ given by

$$L(\Lambda)_{\leq w} = \mathbf{U}^+ u_{w\Lambda} \qquad \text{and} \qquad \mathrm{char}(L(\Lambda)_{\leq w}) = \sum_{p \in B(\Lambda)_{\leq w}} e^{\mathrm{wt}(p)},$$

since $L(\Lambda)_{\leq w}$ has a crystal $B(\Lambda)_{\leq w}$. The *BGG-Demazure operator* on $\mathbb{C}[\mathfrak{h}^*_{\mathbb{Z}}] = \mathbb{C}\text{-span}\{X^\lambda \mid \lambda \in \mathfrak{h}^*_{\mathbb{Z}}\}$ is given by

$$D_i = (1 + s_i)\frac{1}{1 - X^{-\alpha_i}}, \qquad \text{for } i \in \{0, 1, \ldots, n\}.$$

Let $\Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}$, $w \in W$ and $i \in \{0, 1, \ldots, n\}$.

$$\text{If } \Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}} \quad \text{then} \qquad D_i \mathrm{char}(L(\Lambda)_{\leq w})) = \begin{cases} \mathrm{char}(L(\Lambda)_{\leq s_i w}), & \text{if } s_i w \overset{+}{\geq} w, \\ \mathrm{char}(L(\Lambda)_{\leq w}), & \text{if } s_i w \overset{+}{\leq} w; \end{cases}$$

$$\text{if } \lambda \in (\mathfrak{h}^*)^0_{\mathrm{int}} \quad \text{then} \qquad D_i \mathrm{char}(L(\lambda)_{\le w}) = \begin{cases} \mathrm{char}(L(\lambda)_{\le s_i w}), & \text{if } s_i w \trianglerighteq w, \\ \mathrm{char}(L(\lambda)_{\le w}), & \text{if } s_i w \trianglelefteq w; \end{cases}$$

$$\text{if } \Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}} \quad \text{then} \qquad D_i \mathrm{char}(L(\Lambda)_{\le w}) = \begin{cases} \mathrm{char}(L(\Lambda)_{\le s_i w}), & \text{if } s_i w \gtrsim w, \\ \mathrm{char}(L(\Lambda)_{\le w}), & \text{if } s_i w \lesssim w; \end{cases}$$

(see [24, Theorem 8.2.9], [18], [21, Sect. 2.8] and [22, Theorems 4.7 and 4.11]).

3.5 An Alternate Presentation for Level 0 Extremal Weight Modules

For $\lambda = m_1\omega_1 + \cdots + m_n\omega_n \in (\mathfrak{h}^*)_{\mathrm{int}}$ and let $x_{1,1}, \ldots, x_{m_1,1}, x_{1,2}, \ldots, x_{m_2,2}, \ \ldots$ $x_{1,n}, \ldots, x_{m_n,n}$ be n sets of formal variables. Letting $e_i^{(j)} = e_i(x_{1,j}, \ldots, x_{m_j,j})$ denote the elementary symmetric function in the variables $x_{1,j}, \ldots, x_{m_j,j}$ define

$$\begin{aligned} RG_\lambda &= \mathbb{C}[x_{1,1}^{\pm 1}, \ldots, x_{m_1,1}^{\pm 1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_{1,n}^{\pm 1}, \ldots, x_{m_n,n}^{\pm 1}]^{S_{m_n}} \\ &= \mathbb{C}[e_1^{(1)}, \ldots, e_{m_1-1}^{(1)}, (e_{m_1}^{(1)})^{\pm 1}] \otimes \cdots \otimes \mathbb{C}[e_1^{(n)}, \ldots, e_{m_n-1}^{(n)}, (e_{m_n}^{(n)})^{\pm 1}], \\ RG_\lambda^+ &= \mathbb{C}[x_{1,1}, \ldots, x_{m_1,1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_{1,n}, \ldots, x_{m_n,n}]^{S_{m_n}}, \quad \text{and} \\ RG_\lambda^- &= \mathbb{C}[x_{1,1}^{-1}, \ldots, x_{m_1,1}^{-1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_{1,n}^{-1}, \ldots, x_{m_n,n}^{-1}]^{S_{m_n}}. \end{aligned}$$

Let

$$\begin{aligned} e_+^{(i)}(u) &= (1 - x_{1,i}u)(1 - x_{2,i}u)\cdots(1 - x_{m_i,i}u) \quad \text{and} \\ e_-^{(i)}(u^{-1}) &= (1 - x_{1,i}^{-1}u^{-1})(1 - x_{2,i}^{-1}u^{-1})\cdots(1 - x_{m_i,i}^{-1}u^{-1}). \end{aligned}$$

Let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ without the generator D.

Theorem 3 (see [36, Sect. 3.4]) *The extremal weight module $L(\lambda)$ is the $(\mathbf{U}' \otimes_{\mathbb{Z}} RG_\lambda)$-module generated by a single vector m_λ with relations*

$$\mathbf{x}_{i,r}^+ m_\lambda = 0, \qquad K_i m_\lambda = q^{m_i} m_\lambda, \quad C m_\lambda = m_\lambda,$$

$$\mathbf{q}_+^{(i)}(u) m_\lambda = K_i \frac{e_+^{(i)}(q^{-1}u)}{e_+^{(i)}(qu)} m_\lambda \quad \textit{and} \quad \mathbf{q}_-^{(i)}(u^{-1}) m_\lambda = K_i \frac{e_-^{(i)}(qu^{-1})}{e_-^{(i)}(q^{-1}u^{-1})} m_\lambda,$$

where $\mathbf{q}_+^{(i)}(u)$ and $\mathbf{q}_-^{(i)}(u^{-1})$ are as defined in (19) *and* (24).

In this form $L(\lambda)$ has been termed the *universal standard module* [36, Sect. 3.4]) or the *global Weyl module* [9, Sect. 2]. See [36, Theorem 2] and [36, Remark 2.15] for discussion of how to see that the extremal weight module, the universal standard module and the global Weyl module coincide.

Remark 2 Let

$$0_q = \frac{1}{1-q} + \frac{q^{-1}}{1-q^{-1}} = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots,$$

(although $\frac{q^{-1}}{1-q^{-1}} = \frac{1}{q-1} = \frac{-1}{1-q}$, it is important to note that 0_q is *not* equal to 0, it is a doubly infinite formal series in q and q^{-1}). Since $\deg(e_j^{(i)}) = j$,

$$\mathrm{gchar}(RG_\lambda^+) = \left(\prod_{i=1}^{n}\prod_{k=1}^{m_i} \frac{1}{1-q^k}\right) \qquad \mathrm{gchar}(RG_\lambda^-) = \left(\prod_{i=1}^{n}\prod_{k=1}^{m_i} \frac{1}{1-q^{-k}}\right) \quad \text{and}$$

$$\mathrm{gchar}(RG_\lambda) = \left(0_{q^{m_1}} \prod_{k=1}^{m_1-1} \frac{1}{1-q^k}\right)\left(0_{q^{m_2}} \prod_{k=1}^{m_2-1} \frac{1}{1-q^k}\right)\cdots\left(0_{q^{m_n}} \prod_{k=1}^{m_n-1} \frac{1}{1-q^k}\right).$$

3.6 Level 0 $L(\lambda)$ and Finite Dimensional Simple $\mathbf{U}$-Modules $L^{\mathrm{fin}}(a(u))$

The loop presentation provides a triangular decomposition of $\mathbf{U}$ (different form the usual triangular decomposition coming from the Kac–Moody presentation). The extremal weight module $L(\lambda)$ is the standard (Verma type) module for the loop triangular decomposition (see [8, Theorem 2.3(b)], [36, Lemma 2.14] and [7, outline of proof of Theorem 12.2.6]).

A *Drinfeld polynomial* is an n-tuple of polynomials $a(u) = (a^{(1)}(u), \ldots, a^{(n)}(u))$ with $a^{(i)}(u) \in \mathbb{C}[u]$, represented as

$$a(u) = a^{(1)}(u)\omega_1 + \cdots + a^{(n)}(u)\omega_n, \qquad \text{with} \quad a^{(i)}(u) = (u - a_{1,i})\cdots(u - a_{m_i,i})$$

so that the coefficient of u^j in $a^{(i)}(u)$ is $e_{m_i-j}^{(i)}(a_{1,i}, \ldots, a_{m_i,i})$, the $(m_i - j)$th elementary symmetric function evaluated at the values $a_{1,i}, \ldots, a_{m_i,i}$. The *local Weyl module* (a finite dimensional standard module) is defined by

$$M^{\mathrm{fin}}(a(u)) = L(\lambda) \otimes_{RG_\lambda} m_{a(u)}, \quad \text{where} \quad e_k^{(i)}(x_{1,i}, x_{2,i}, \ldots) m_{a(u)} = e_k^{(i)}(a_{1,i}, \ldots, a_{m_i,i}) m_{a(u)}$$

specifies the RG_λ-action on $m_{a(u)}$. In other words, the module $M^{\mathrm{fin}}(a(u))$ is $L(\lambda)$ except that variables $x_{j,i}$ from Sect. 3.5 have been specialised to the values $a_{j,i}$. As in Theorem 3, let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ without the generator D.

Theorem 4 (see [12, Theorem 2] and [8, Theorem 3.3]) *The standard module $M^{\mathrm{fin}}(a(u))$ has a unique simple quotient $L^{\mathrm{fin}}(a(u))$ and*

$$\begin{array}{ccc} \{\textit{Drinfeld polynomials}\} & \longrightarrow & \{\textit{finite dimensional simple } \mathbf{U}'\textit{-modules}\} \\ a(u) = a^{(1)}(u)\omega_1 + \cdots + a^{(n)}(u)\omega_n & \longmapsto & L^{\mathrm{fin}}(a(u)) \end{array}$$

is a bijection.

3.7 Path Models for the Crystals $B(\Lambda)$

The work of Littelmann [25, 26] provided a particularly convenient model for the crystals $B(\Lambda)$ when Λ is positive or negative level. This model realizes the crystal as a set of paths $p\colon \mathbb{R}_{[0,1]} \to \mathfrak{h}^*$ with combinatorially defined Kashiwara operators $\tilde{e}_0, \ldots, \tilde{e}_n, \tilde{f}_0, \ldots, \tilde{f}_n$. In the LS (Lakshmibai-Seshadri) model the generator of the crystal $B(\Lambda)$ is the straight line path to Λ.

When $\lambda \in (\mathfrak{h}^*)^0_{\mathrm{int}}$ using

$$\begin{array}{rcl} p_\lambda\colon \mathbb{R}_{[0,1]} & \to & \mathfrak{h}^* \\ t & \to & t\lambda\,, \end{array} \qquad \text{the straight line path from 0 to } \lambda,$$

as a generator for $B(\lambda)$ may not be the optimal choice. Remarkably, Naito and Sagaki (see [16, Definition 3.1.4 and Theorem 3.2.1] and [32, Theorem 4.6.1(b)]), have shown that $B(\lambda)$ can be constructed with sequences of Weyl group elements and rational numbers as in [25, Sect. 1.2, 1.3 and 2.2] but with the positive level length ℓ^+ and Bruhat order $\trianglelefteq^{+}$ replaced by the level zero length ℓ^0 and Bruhat order $\trianglelefteq^{0}$. However, when working with the Naito–Sagaki construction one must be very careful not to identify the Naito–Sagaki sequences with actual paths (piecewise linear maps from $\mathbb{R}_{[0,1]}$ to $\mathfrak{h}^*$) because the natural map from Naito–Sagaki sequences to paths is *not always injective* (an example is provided by [20, Remark 5.10]).

If $\Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}}$ then $-\Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}}$ and

$$B(\Lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} p_\Lambda \mid k \in \mathbb{Z}_{\geq 0} \text{ and } i_1, \ldots, i_k \in \{0, 1, \ldots, n\}\}$$

$$B(-\Lambda) = \{\tilde{e}_{i_1} \cdots \tilde{e}_{i_k} p_{-\Lambda} \mid k \in \mathbb{Z}_{\geq 0} \text{ and } i_1, \ldots, i_k \in \{0, 1, \ldots, n\}\}.$$

are each a single connected component and their characters are determined by the Weyl-Kac character formula [17, Theorem 11.13.3].

3.8 Crystals for Level 0 Extremal Weight Modules $L(\lambda)$

For general $\lambda \in (\mathfrak{h}^*)^0_{\mathrm{int}}$ the crystal $B(\lambda)$ is not connected (as a graph with edges determined by the Kashiwara operators $\tilde{e}_0, \ldots, \tilde{e}_n, \tilde{f}_0, \ldots \tilde{f}_n$). Let $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$, with $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$. By [4, Corollary 4.15], the map

$$\begin{array}{rcl} \Phi_\lambda\colon L(\lambda) & \longrightarrow & L(\omega_1)^{\otimes m_1} \otimes \cdots \otimes L(\omega_n)^{\otimes m_n} \\ u_\lambda & \longmapsto & u_{\omega_1}^{\otimes m_1} \otimes \cdots \otimes u_{\omega_n}^{\otimes m_n} \end{array} \quad \text{is injective} \tag{27}$$

and gives rise to an injection of crystals

$$B(\lambda) \hookrightarrow B(\omega_1)^{\otimes m_1} \otimes \cdots \otimes B(\omega_n)^{\otimes m_n}$$

which takes the connected component of $B(\lambda)$ containing b_λ to the connected component of $B(\omega_1)^{\otimes m_1} \otimes \cdots \otimes B(\omega_n)^{\otimes m_n}$ containing $b_{\omega_1}^{\otimes m_1} \otimes \cdots \otimes b_{\omega_n}^{\otimes m_n}$. Kashiwara [20, Theorem 5.15] fully described the structure of $L(\omega_i)$ (see [4, Theorem 2.16]). Beck-Nakajima analyzed the PBW basis of (22) and use (27) to show that the connected components of $B(\lambda)$ are labeled by n-tuples of partitions $\kappa = (\kappa^{(1)}, \ldots, \kappa^{(n)})$ such that $\ell(\kappa^{(i)}) < m_i$. Together with a result of Fourier–Littelmann which shows that the crystal of the level 0 module $M^{\text{fin}}(a(u))$ is isomorphic to a level one Demazure crystal $B(\nu + \Lambda_0)_{\leq w}$, the full result is as detailed in Theorem 5 below.

A labeling set for a basis of $RG_\lambda / \langle e_{m_i}^{(i)} = 1 \rangle$ is

$$S^\lambda = \{\vec{\kappa} = (\kappa^{(1)}, \ldots, \kappa^{(n)}) \mid \kappa^{(i)} \text{ is a partition with } \ell(\kappa^{(i)}) < m_i \text{ for } i \in \{1, \ldots, n\}\}.$$

The connected component of b_λ in $B(\lambda)$ is

$$B(\lambda)_0 = \left\{ \tilde{r}_{i_1} \cdots \tilde{r}_{i_k} b_\lambda \mid k \in \mathbb{Z}_{\geq 0} \text{ and } \tilde{r}_{i_1}, \ldots, \tilde{r}_{i_k} \in \{\tilde{e}_0, \ldots, \tilde{e}_n, \tilde{f}_0, \ldots, \tilde{f}_n\} \right\}.$$

Define $B^{\text{fin}}(\lambda)$ to be the "crystal" which has a crystal graph which is the "quotient" of the crystal graph of $L(\lambda)$ obtained by identifying the vertices b and b' if there is an element $\mathbf{s} \in RG_\lambda$ such that $\mathbf{s}G(b) = G(b')$, where $G(b)$ denotes the canonical basis element of $L(\lambda)$ corresponding to b.

$B^{\text{fin}}(\lambda)$ is the crystal of the finite dimensional standard module $M^{\text{fin}}(a(u))$.

Theorem 5 (see [4, Theorem 4.16], [36, Sect. 3.4]) and [14, Proposition 3]) *Let* $\lambda = m_1\omega_1 + \cdots + m_n\omega_n \in (\mathfrak{h}^*)^0_{\text{int}}$. *As in Theorem 2, let* $\nu \in \mathfrak{a}^*_{\mathbb{Z}}$, $j \in \mathbb{Z}_{\geq 0}$ *and* $w \in W^{\text{ad}}$ *such that* $w(\nu + \Lambda_0) = -j\delta + \lambda + \Lambda_0$ *and* w *is minimal length. Then*

$$B(\lambda) \simeq B(\lambda)_0 \times S^\lambda \quad \textit{and} \quad B(\lambda)_0 \simeq \mathbb{Z}^k \times B^{\text{fin}}(\lambda) \quad \textit{and} \quad B^{\text{fin}}(\lambda) \simeq B(\nu + \Lambda_0)_{\leq w},$$

where k *is the number of elements of* $m_1, \ldots, m_n$ *which are nonzero.*

Additional useful references for Theorem 5 are [36, Theorem 1] and [2, Theorem 1]. The first two statements in Theorem 5 are reflections of the very important fact that $L(\lambda)$ is free as an RG_λ-module. This fact that $L(\lambda)$ is free as an RG_λ-module was deduced geometrically, via the quiver variety, in [34, Theorem 7.3.5] (note that property $(T_{G_{\mathbf{w}} \times \mathbb{C}^\times})$ there includes the freeness, see the definition of property (T_G) after [34, (7.1.1)]). This freeness was understood more algebraically in the work of Fourier–Littelmann [14] and Chari–Ion [6, Cor. 2.10]. Further understanding of the RG_λ-action in terms of the geometry of the semi-infinite flag variety is in [5, Sect. 5.1].

The last statement of Theorem 5 is proved by considering the map

$$\begin{array}{ccc} B(\lambda) & \longrightarrow & B(\Lambda_0)\otimes B(\lambda) \\ \cup| & & \cup| \\ B^{\mathrm{fin}}(\lambda) & \xrightarrow{\sim} & B(\nu+\Lambda_0)_{\le w} \end{array} \qquad \text{given by} \qquad b\mapsto b_{\Lambda_0}\otimes b.$$

where b_{Λ_0} is the highest weight of the crystal $B(\Lambda_0)$. Combining the isomorphism $B^{\mathrm{fin}}(\lambda)\simeq B(\nu+\Lambda_0)_{\le w}$ with Theorem 2 and the positive level formula in Sect. 3.4 gives

$$\mathrm{char}(M(a(u))) = \mathrm{char}(L(\nu+\Lambda_0)_{\le w}) = q^{-j}X^{-\Lambda_0}\tilde{E}_\lambda(q,0) \qquad \text{and}$$

$$\mathrm{char}(L(\lambda)_{\le w_0}) = \mathrm{gchar}(RG^+_\lambda)\tilde{E}_{w_0\lambda}(q,0), \qquad \mathrm{char}(L(\lambda)) = \mathrm{gchar}(RG_\lambda)\tilde{E}_{w_0\lambda}(q,0). \tag{28}$$

4 Examples for $\mathfrak{g}=\widehat{\mathfrak{sl}}_2$

Let $\mathfrak{g}=\widehat{\mathfrak{sl}}_2$ with $\mathfrak{h}^*=\mathbb{C}\omega_1\oplus\mathbb{C}\Lambda_0\oplus\mathbb{C}\delta$ with affine Cartan matrix

$$\begin{pmatrix}\alpha_0(h_0) & \alpha_0(h_1)\\ \alpha_1(h_0) & \alpha_1(h_1)\end{pmatrix} = \begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix} \qquad \text{and} \qquad \begin{array}{l}\theta=\alpha_1=2\omega_1,\ \ \theta^\vee=\alpha_1^\vee=h_1,\\ \Lambda_1=\omega_1+\Lambda_0,\ \ \alpha_0=-\alpha_1+\delta,\end{array}$$

Using that $\langle\alpha_1,\alpha_1\rangle=2$, if $k\in\mathbb{Z}$ and $\mu^\vee=k\alpha_1^\vee=k\alpha_1=k2\omega_1=2k\omega_1$ so that $\mu_1^\vee=2k$ and $-\frac{1}{2}\langle\mu^\vee,\mu^\vee\rangle=-\frac{1}{2}(k2k)=-k^2$. Thus, following (8), (4) and (5), in the basis $\{\delta,\omega_1,\Lambda_0\}$ of $\mathfrak{h}^*$,

$$t_{k\alpha^\vee}=\begin{pmatrix}1 & -k & -k^2\\ 0 & 1 & 2k\\ 0 & 0 & 1\end{pmatrix}, \qquad s_1=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad s_0=\begin{pmatrix}1 & 1 & -1\\ 0 & -1 & 2\\ 0 & 0 & 1\end{pmatrix}.$$

These matrices are used to compute the W-orbits pictured in Sect. 1.1.

4.1 Macdonald Polynomials

The Demazure operators are given by

$$D_1 f = \frac{f-X^{-2\omega_1}(s_1 f)}{1-X^{-2\omega_1}} \qquad \text{and} \qquad D_0 f = \frac{f-X^{-\alpha_0}(s_0 f)}{1-X^{-\alpha_0}} = \frac{f-q^{-1}X^{2\omega_1}(s_0 f)}{1-q^{-1}X^{2\omega_1}}.$$

The normalized Macdonald polynomials $\tilde{E}_{\omega_1}(q, t)$ and $\tilde{E}_{-\omega_1}(q, t)$ are

$\tilde{E}_{\omega_1}(q,t)$ and $\tilde{E}_{-\omega_1}(q,t)$ are

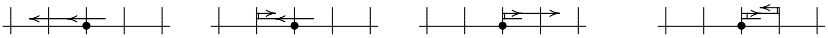
$$\tilde{E}_{\omega_1}(q,t) = X^{\omega_1} \quad \text{and} \quad \begin{aligned} \tilde{E}_{-\omega_1}(q,t) &= X^{-\omega_1} + \frac{(1-t)}{1-qt} X^{\omega_1} \\ &= X^{-\omega_1} + \frac{(1-t^{-1})q^{-1}}{1-q^{-1}t^{-1}} X^{\omega_1} \end{aligned}$$

giving $\tilde{E}_{\omega_1}(0, t) = \tilde{E}_{\omega_1}(\infty, t) = \tilde{E}_{\omega_1}(q, 0) = \tilde{E}_{\omega_1}(q, \infty) = X^{\omega_1}$ and

$$\begin{array}{ll} \tilde{E}_{-\omega_1}(0, t) = X^{-\omega_1} + (1-t)X^{\omega_1}, & \tilde{E}_{-\omega_1}(\infty, t) = X^{-\omega_1} \\ \tilde{E}_{-\omega_1}(q, 0) = X^{-\omega_1} + X^{\omega_1}, & \tilde{E}_{-\omega_1}(q, \infty) = X^{-\omega_1} + q^{-1}X^{\omega_1}. \end{array}$$

The normalized Macdonald polynomials $\tilde{E}_{2\omega_1}(q, t)$ and $\tilde{E}_{-2\omega_1}(q, t)$ are

$$\begin{aligned} \tilde{E}_{2\omega_1}(q,t) &= X^{2\omega_1} + \frac{(1-t)q}{1-qt} \\ &= X^{2\omega_1} + \frac{1-t^{-1}}{1 \quad q^{-1}t^{-1}} \end{aligned} \qquad \text{and}$$



$$\begin{aligned} \tilde{E}_{-2\omega_1}(q,t) &= X^{-2\omega_1} + \frac{1-t}{1-qt} + \frac{1-t}{1-q^2t} X^{2\omega_1} + \frac{(1-t)}{(1-q^2t)} \frac{(1-t)q}{(1-qt)} \\ &= X^{-2\omega_1} + \frac{(1-t^{-1})q^{-1}}{1-q^{-1}t^{-1}} + \frac{(1-t^{-1})q^{-2}}{1-q^{-2}t^{-1}} X^{2\omega_1} + \frac{(1-t^{-1})q^{-2}}{(1-q^{-2}t^{-1})} \frac{(1-t^{-1})}{(1-q^{-1}t^{-1})} \end{aligned}$$

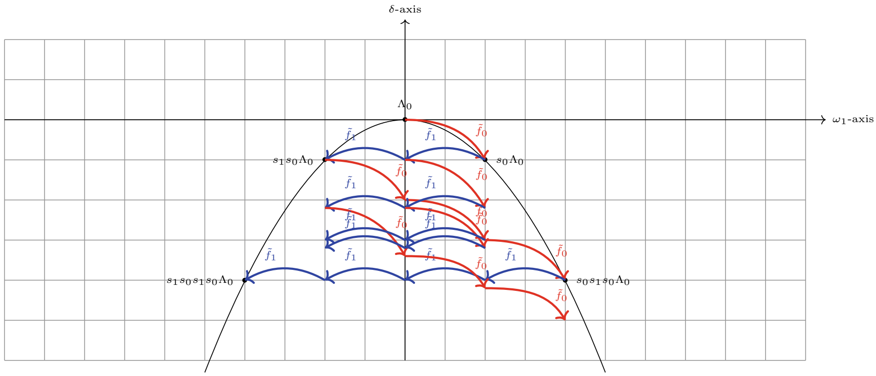
giving

$$\begin{array}{ll} \tilde{E}_{2\omega_1}(0, t) = X^{2\omega_1}, & \tilde{E}_{2\omega_1}(\infty, t) = X^{2\omega_1} + (1-t^{-1}) \\ \tilde{E}_{2\omega_1}(q, 0) = X^{2\omega_1} + q, & \tilde{E}_{2\omega_1}(q, \infty) = X^{2\omega_1} + 1 \end{array}$$

and

$$\begin{array}{ll} \tilde{E}_{-2\omega_1}(0, t) = X^{-2\omega_1} + (1-t)X^{2\omega_1} + (1-t), & \tilde{E}_{-2\omega_1}(\infty, t) = X^{-2\omega_1}, \\ \tilde{E}_{-2\omega_1}(q, 0) = X^{-2\omega_1} + X^{2\omega_1} + 1 + q, & \tilde{E}_{-2\omega_1}(q, \infty) = X^{-2\omega_1} + q^{-2}X^{2\omega_1} + q^{-1} + q^{-2}. \end{array}$$

4.2 The Crystal $B(\Lambda_0)$ and $B(\omega_1 + \Lambda_0)$



Initial portion of the crystal graph of $B(\Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$

The characters of the first few Demazure modules in $L(\Lambda_0)$ are

$$\begin{aligned}
\mathrm{char}(L(\Lambda_0)_{\le 1}) &= X^{\Lambda_0} = X^{\Lambda_0}\tilde{E}_0(q^{-1}, 0),\\
\mathrm{char}(L(\Lambda_0)_{\le s_0}) &= D_0 X^{\Lambda_0} = X^{\Lambda_0}(1 + qX^{2\omega_1}) = qX^{\Lambda_0}(X^{2\omega_1} + q^{-1}) = qX^{\Lambda_0}\tilde{E}_{2\omega_1}(q^{-1}, 0),\\
\mathrm{char}(L(\Lambda_0)_{\le s_1 s_0}) &= D_1 D_0 X^{\Lambda_0} = X^{\Lambda_0}(1 + qX^{2\omega_1} + q + qX^{-2\omega_1})\\
&= qX^{\Lambda_0}(X^{-2\omega_1} + X^{2\omega_1} + 1 + q^{-1}) = qX^{\Lambda_0}\tilde{E}_{-2\omega_1}(q^{-1}, 0).
\end{aligned}$$

The crystal graph of $B(\omega_1 + \Lambda_0)$ is pictured in Plate B and

$$\begin{aligned}
\mathrm{char}(L(\omega_1 + \Lambda_0)_{\le 1}) &= X^{\Lambda_0}X^{\omega_1} = X^{\Lambda_0}\tilde{E}_{\omega_1}(q^{-1}, 0),\\
\mathrm{char}(L(\omega_1 + \Lambda_0)_{\le s_1}) &= D_1 X^{\omega_1+\Lambda_0} = X^{\Lambda_0}(X^{\omega_1} + X^{-\omega_1}) = X^{\Lambda_0}\tilde{E}_{-\omega_1}(q^{-1}, 0),\\
\mathrm{char}(L(\omega_1 + \Lambda_0)_{\le s_0 s_1}) &= D_0 D_1 X^{\omega_1+\Lambda_0} = X^{\Lambda_0}(X^{\omega_1} + X^{-\omega_1} + qX^{\omega_1} + q^2 X^{3\omega_1})\\
&= q^2 X^{\Lambda_0}\tilde{E}_{3\omega_1}(q^{-1}, 0),
\end{aligned}$$

4.3 The Crystal $B(\omega_1)$

The crystal $B(\omega_1) = \{p_{-\omega_1+k\delta}, p_{\omega_1+k\delta} \mid k \in \mathbb{Z}\}$ is a single connected component. The crystal graph of $B(\omega_1)$ is pictured in Plate B. Following [17, (12.1.9)] and putting $q = X^{-\delta}$ noting that $\tilde{E}_{-\omega_1}(q^{-1}, 0) = X^{\omega_1} + X^{-\omega_1}$ and $\tilde{E}_{-\omega_1}(q^{-1}, \infty) = X^{-\omega_1} + qX^{\omega_1}$, then

$$\mathrm{char}(L(\omega_1)_{\le s_1}) = \frac{1}{1-q^{-1}}\tilde{E}_{-\omega_1}(q^{-1},0) \qquad \text{and}$$
$$\mathrm{char}(L(\omega_1)) = 0_q\tilde{E}_{-\omega_1}(q^{-1},0) = 0_q\tilde{E}_{-\omega_1}(q^{-1},\infty),$$

where $0_q = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots$ as in Remark 2.

4.4 The Crystal $B(2\omega_1)$

On crystals, the injective **U**-module homomorphism $L(2\omega_1) \hookrightarrow L(\omega_1) \otimes L(\omega_1)$ given in (27) is the inclusion

$$\begin{array}{ll} B(\omega_1) \otimes B(\omega_1) & = \{p_{w_1\omega_1} \otimes p_{w_2\omega_1} \mid w_1, w_2 \in W^{\mathrm{ad}}\} \\ \quad\cup\!| & \\ \quad B(2\omega_1) & = \{p_{w_1\omega_1} \otimes p_{w_2\omega_1} \mid w_1, w_2 \in W^{\mathrm{ad}} \text{ with } w_1 \trianglerighteq w_2\} \end{array}$$

The connected components of $B(2\omega_1)$ are determined by

$$B(2\omega_1) = B(2\omega_1)_0 \times S^{2\omega_1}, \qquad \text{where} \quad S^{2\omega_1} = \{\text{partitions } \kappa \text{ with } \ell(\kappa) < 2\} = \mathbb{Z}_{\ge 0}.$$

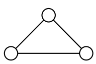
For $\kappa \in \mathbb{Z}_{\ge 0}$, the connected component corresponding to κ, as a subset of $B(\omega_1) \otimes B(\omega_1)$, is

$$B(2\omega_1)_\kappa = \{p_{w_1\omega_1} \otimes p_{w_2\omega_1} \mid w_1 \trianglerighteq w_2 \text{ and } \ell^0(w_1) - \ell^0(w_2) \in \{\kappa, \kappa+1\}\ \}$$

Representatives of the components and the crystal graph of the connected component $B(2\omega_1)_0$ are pictured in Plate C. Inspection of the crystal graphs gives

$$\mathrm{char}(L(2\omega_1)_{\le s_1}) = \frac{1}{1-q^{-2}}\frac{1}{1-q^{-1}}\tilde{E}_{-2\omega_1}(q^{-1},0), \quad \text{and}$$
$$\mathrm{char}(L(2\omega_1)) = 0_q\frac{1}{1-q}\tilde{E}_{-2\omega_1}(q^{-1},0) = 0_q\frac{1}{1-q}\tilde{E}_{-2\omega_1}(q^{-1},\infty).$$

5 Examples for $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$

Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$. The affine Dynkin diagram is  and the affine Cartan matrix is

$$\begin{pmatrix} \alpha_0(h_0) & \alpha_0(h_1) & \alpha_0(h_2) \\ \alpha_1(h_0) & \alpha_1(h_1) & \alpha_1(h_2) \\ \alpha_2(h_0) & \alpha_2(h_1) & \alpha_2(h_2) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

and $h_0 = -(h_1 + h_2) + K$, $\alpha_0 = -(\alpha_1 + \alpha_2) + \delta$, $\Lambda_1 = \omega_1 + \Lambda_0$, $\Lambda_2 = \omega_2 + \Lambda_0$.

In the basis $\{\delta, \omega_1, \omega_2, \Lambda_0\}$ of $\mathfrak{h}^*$ the action of the affine Weyl group W^{ad} is given by

$$s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad s_0 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and}$$

$$t_{k_1h_1+k_2h_2} = \begin{pmatrix} 1 & -k_1 & -k_2 & -k_1^2 - k_2^2 + k_1k_2 \\ 0 & 1 & 0 & 2k_1 - k_2 \\ 0 & 0 & 1 & 2k_2 - k_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{for } k_1, k_2 \in \mathbb{Z}.$$

These matrices are used to compute the orbits pictured at the end of Sect. 1.1.

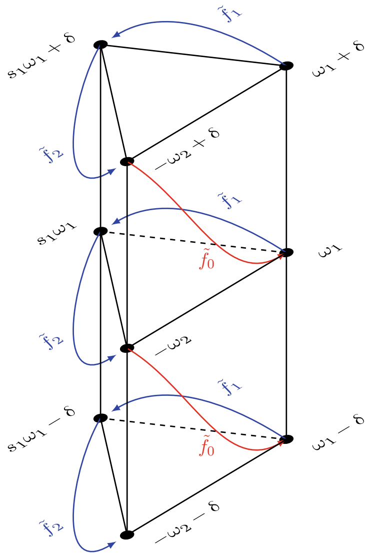
5.1 The Extremal Weight Modules $L(\omega_1)$ and $L(\omega_2)$

Letting $\mathbb{C}^3 = \mathbb{C}\text{-span}\{v_1, v_2, v_3\}$ be the standard representation of $\mathring{\mathfrak{g}} = \mathfrak{sl}_3$, the extremal weight modules $L(\omega_1)$ and $L(\omega_2)$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_3$ are

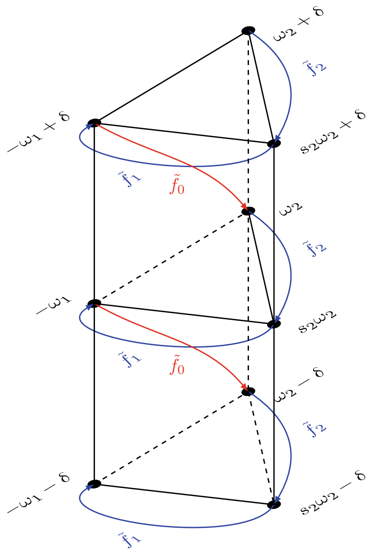
$$L(\omega_1) = \mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{C}[\epsilon, \epsilon^{-1}] \qquad \text{and} \qquad L(\omega_2) = (\Lambda^2\mathbb{C}^3) \otimes_{\mathbb{C}} \mathbb{C}[\epsilon, \epsilon^{-1}],$$

with $u_{\omega_1} = v_1$ and $u_{\omega_2} = v_1 \wedge v_2$, respectively. The crystals $B(\omega_1)$ and $B(\omega_2)$ have realizations as sets of straight line paths:

$$B(\omega_1) = \{p_{\omega_1+k\delta}, p_{s_1\omega_1+k\delta}, p_{-\omega_2+k\delta} \mid k \in \mathbb{Z}\}, \qquad B(\omega_2) = \{p_{\omega_2+k\delta}, p_{s_2\omega_2+k\delta}, p_{-\omega_1+k\delta} \mid k \in \mathbb{Z}\}.$$



$B(\omega_1)$ crystal graph



$B(\omega_2)$ crystal graph

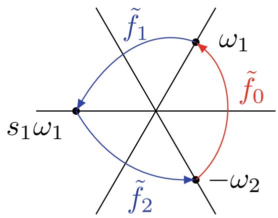
Each of the crystals $B(\omega_1)$ and $B(\omega_2)$ has a single connected component, all weight spaces are one dimensional and

$$\operatorname{char}(L(\omega_1)_{\le s_2 s_1}) = \frac{1}{1-q^{-1}}(X^{\omega_1} + X^{s_1\omega_1} + X^{-\omega_2}) = \frac{1}{1-q^{-1}}\tilde{E}_{-\omega_2}(q^{-1},0), \quad \text{and}$$

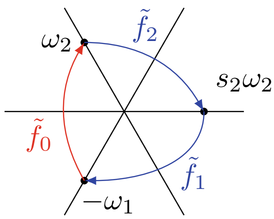
$$\begin{aligned}\operatorname{char}(L(\omega_1)) &= 0_q(X^{\omega_1} + X^{-\omega_2} + X^{s_1\omega_1}) = 0_q\tilde{E}_{-\omega_2}(q^{-1},0)\\ &= 0_q(q^{-1}X^{\omega_1} + q^{-1}X^{\omega_2-\omega_1} + X^{-\omega_2}) = 0_q\tilde{E}_{-\omega_2}(q^{-1},\infty).\end{aligned}$$

where $0_q = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots$ as in Remark 2.

For $a \in \mathbb{C}$, the crystals of $M^{\mathrm{fin}}((u-a)\omega_1) \cong \mathbb{C}^3$ and $M^{\mathrm{fin}}((u-a)\omega_2) \cong \Lambda^2(\mathbb{C}^3)$ have crystal graphs $B^{\mathrm{fin}}(\omega_1)$ and $B^{\mathrm{fin}}(\omega_2)$.



$B^{\mathrm{fin}}(\omega_1)$



$B^{\mathrm{fin}}(\omega_2)$

5.2 The Extremal Weight Module $L(\omega_1 + \omega_2)$

To construct the crystal $B(\omega_1 + \omega_2)$ use

$$B(\omega_1) \otimes B(\omega_2) = \{p_{v\omega_1+k\delta} \otimes p_{w\omega_2+\ell\delta} \mid v \in \{1, s_1, s_2s_1\}, w \in \{1, s_2, s_1s_2\}, k \in \mathbb{Z}, \ell \in \mathbb{Z}\}$$

with the tensor product action for crystals given by (see, for example, [39, Prop. 5.7])

$$\tilde{f}_i(p_1 \otimes p_2) = \begin{cases} \tilde{f}_i p_1 \otimes p_2, & \text{if } d_i^+(p_1) > d_i^-(p_2), \\ p_1 \otimes \tilde{f}_i p_2, & \text{if } d_i^+(p_1) \le d_i^-(p_2), \end{cases} \quad \tilde{e}_i(p_1 \otimes p_2) = \begin{cases} \tilde{e}_i p_1 \otimes p_2, & \text{if } d_i^+(p_1) \ge d_i^-(p_2), \\ p_1 \otimes \tilde{e}_i p_2, & \text{if } d_i^+(p_1) < d_i^-(p_2). \end{cases}$$

where $d_i^\pm(p)$ are determined by

$$\tilde{f}_i^{d_i^+(p)} p \neq 0 \text{ and } \tilde{f}_i^{d_i^+(p)+1} p = 0, \qquad \text{and} \qquad \tilde{e}_i^{d_i^-(p)} p \neq 0 \text{ and } \tilde{e}_i^{d_i^-(p)+1} p = 0.$$

The crystal $B(\omega_1 + \omega_2)$ is realized as a subset of $B(\omega_1) \otimes B(\omega_2)$ via the crystal embedding

$$\begin{array}{ccc} B(\omega_1 + \omega_2) & \hookrightarrow & B(\omega_1) \otimes B(\omega_2) \\ b_{\omega_1+\omega_2} & \longmapsto & p_{\omega_1} \otimes p_{\omega_2} \end{array}$$

By Theorem 5, $B(\omega_1 + \omega_2)$ is connected and is generated by $b_{\omega_1+\omega_2}$. The crystal $B(\omega_1 + \omega_2)$ is pictured and its character is computed in Plate D.

6 Alcove Walks for Affine Flag Varieties

6.1 The Affine Kac–Moody Group G

The most visible form of the loop group is $G = \mathring{G}(\mathbb{C}((\epsilon)))$, where $\mathbb{C}((\epsilon))$ is the field of formal power series in a variable ϵ and $\mathring{G}$ is a reductive algebraic group. The favourite example is when

$$\text{when} \quad \mathring{G} = GL_n \qquad \text{and the loop group is} \quad G = GL_n(\mathbb{C}((\epsilon))),$$

the group of $n \times n$ invertible matrices with entries in $\mathbb{C}((\epsilon))$. A slightly more extended, and extremely powerful, point of view is to let G be the Kac–Moody group whose Lie algebra is the affine Lie algebra $\mathfrak{g}$ of Sect. 2.1, so that G is a central extension of a semidirect product (where the semidirect product comes from the action of $\mathbb{C}^\times$ on $\mathring{G}(\mathbb{C}((\epsilon))$ by "loop rotations")

$$\{1\} \to \mathbb{C}^\times \to G \to \mathring{G}(\mathbb{C}((\epsilon))) \rtimes \mathbb{C}^\times \to \{1\} \qquad \text{so that} \qquad G = \exp(\mathfrak{g}). \tag{29}$$

In this section we wish to work with G via generators and relations. Fortunately, presentations of G are well established in the work of Steinberg [41] and Tits [42] and others (see [38, Sect. 3] for a survey). More specifically, up to an extra (commutative) torus $T \cong (\mathbb{C}^\times)^k$ for some k, the group G is generated by subgroups isomorphic to $SL_2(\mathbb{C})$, the images of homomorphisms

$$\begin{array}{rccc} \varphi_i\colon & SL_2(\mathbb{C}) & \longrightarrow & G \\ & \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} & \longmapsto & x_{\alpha_i}(c) \\ & \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} & \longmapsto & x_{-\alpha_i}(c) \\ & \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} & \longmapsto & h_{\alpha_i^\vee}(d) \\ & \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix} & \longmapsto & y_i(c) \end{array} \qquad \begin{array}{l} \text{for each vertex} \quad \overset{i}{\circ} \\ \text{of the affine Dynkin diagram.} \end{array} \tag{30}$$

For each $\alpha \in \mathring{R}^+$ there is a homomorphism $\varphi_\alpha\colon SL_2(\mathbb{C}((\epsilon))) \to G$ and we let

$$x_\alpha(f) = \varphi_\alpha \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad x_{-\alpha}(f) = \varphi_\alpha \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \quad \text{for } f \in \mathbb{C}((\epsilon)). \tag{31}$$

For each $z \in W^{\mathrm{ad}}$ fix a reduced word $z = s_{j_1} \cdots s_{j_r}$, define

$$n_z = y_{j_1}(0) \cdots y_{j_r}(0) \quad \text{and define} \quad x_{\pm\alpha+r\delta}(c) = x_{\pm\alpha}(c\epsilon^r) \quad \text{for } \alpha \in \mathring{R}^+ \text{ and } r \in \mathbb{Z}. \tag{32}$$

In the following sections, for simplicity, we will work only with the loop group $G = \mathring{G}(\mathbb{C}((\epsilon)))$ rather than the full affine Kac–Moody group as in (29). We shall also use a slightly more general setting where $\mathbb{C}$ is replaced by an arbitrary field $\Bbbk$ so that, in Proposition 4, we may let $\Bbbk = \mathbb{F}_q$ be the finite field with q elements.

6.2 *The Affine Flag Varieties G/I^+, G/I^0 and G/I^-*

Let $\Bbbk$ be a field. Define subgroups of $\mathring{G}(\Bbbk)$ by

$$\begin{array}{rl} \text{“unipotent upper triangular matrices”} & U^+(\Bbbk) = \langle x_\alpha(c) \mid \alpha \in \mathring{R}^+, c \in \Bbbk \rangle, \\ \text{“diagonal matrices”} & H(\Bbbk^\times) = \langle h_{\alpha_i^\vee}(c) \mid i \in \{1, \ldots, n\},\ c \in \Bbbk^\times \rangle, \\ \text{“unipotent lower triangular matrices”} & U^-(\Bbbk) = \langle x_{-\alpha}(c) \mid \alpha \in \mathring{R}^+, c \in \Bbbk \} \rangle. \end{array}$$

Let

$$\mathbb{F} = \Bbbk((\epsilon)), \quad \text{which has} \quad \mathfrak{o} = \Bbbk[[\epsilon]] \ \text{ and } \mathfrak{o}^\times = \{p \in \Bbbk[[\epsilon]] \mid p(0) \neq 0\},$$
$$\text{or let } \mathbb{F} = \Bbbk[\epsilon, \epsilon^{-1}], \quad \text{which has } \mathfrak{o} = \Bbbk[\epsilon] \quad \text{and} \quad \mathfrak{o}^\times = \Bbbk^\times.$$

Define subgroups of $G = \mathring{G}(\mathbb{F})$ by

$$U^-(\mathbb{F}) = \langle x_{-\alpha+k\delta}(c) \mid c \in \Bbbk, \alpha \in \mathring{R}^+, k \in \mathbb{Z} \rangle \quad \text{and} \quad H(\mathfrak{o}^\times) = \langle h_{\alpha_i^\vee}(c) \mid c \in \mathfrak{o}^\times \rangle.$$

Let $g(0)$ denote g evaluated at $\epsilon = 0$ and let $g(\infty)$ denote g evaluated at $\epsilon^{-1} = 0$. Following, for example, [37, Lect. 11 Theorem (B)] and [27, Sect. 11], define subgroups of $G(\mathbb{F})$ by

$$\begin{array}{ll} \text{(positive Iwahori)} & I^+ = \{g \in G \mid g(0) \text{ exists and } g(0) \in U^+(\Bbbk)H(\Bbbk^\times)\}, \\ \text{(level 0 Iwahori)} & I^0 = U^-(\mathbb{F})H(\mathfrak{o}^\times), \\ \text{(negative Iwahori)} & I^- = \{g \in G \mid g(\infty) \text{ exists and } g(\infty) \in U^-(\Bbbk)H(\Bbbk^\times)\}, \end{array}$$

Then

$$\begin{aligned} &G/I^+ \text{ is the positive level (thin) affine flag variety,} \\ &G/I^0 \text{ is the level 0 (semi-infinite) affine flag variety,} \\ &G/I^- \text{ is the negative level (thick) affine flag variety.} \end{aligned}$$

These are studied with the aid of the decompositions (relax notation and write zI^+ for $n_z I^+$)

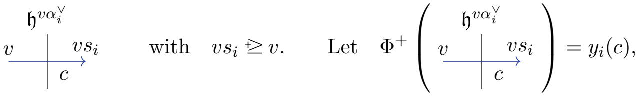
$$G = \bigsqcup_{x \in W^{\mathrm{ad}}} I^+ x I^+, \qquad G = \bigsqcup_{y \in W^{\mathrm{ad}}} I^0 y I^+, \qquad G = \bigsqcup_{z \in W^{\mathrm{ad}}} I^- z I^+,$$

6.3 Labeling Points of I^+wI^+, $I^+wI^+ \cap I^0vI^+$ and $I^+wI^+ \cap I^-zI^+$

Recall the bijection between elements of W^{ad} and alcoves given in (6) and note that if $v \in W^{\mathrm{ad}}$ and $i \in \{0, \ldots, n\}$ then the hyperplane separating the alcoves corresponding to v and $s_i v$ is

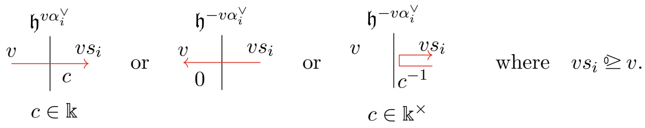
$$\mathfrak{h}^{v\alpha_i^\vee} = \{x + \Lambda_0 \mid x \in \mathfrak{a}_{\mathbb{R}}^* \text{ and } \langle x + \Lambda_0, v\alpha_i^\vee \rangle = 0\}.$$

A *blue labeled step of type i* is

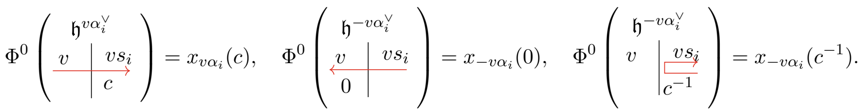


where $y_i(c)$ is as in (30). A *blue labeled walk of type* $\vec{w} = (i_1, \ldots, i_\ell)$ is a sequence $(p_1, \ldots, p_\ell)$ where p_k is a blue labeled step of type i_k and which begins at the alcove 1.

A *red labeled step of type* i is

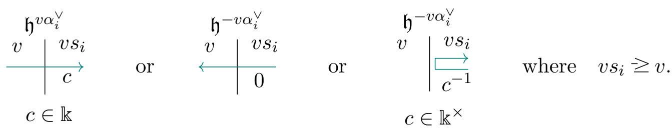


With notations as in (32), let

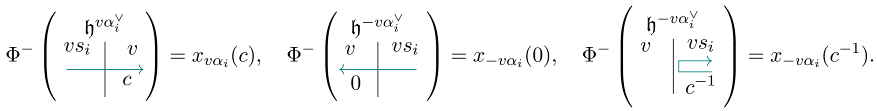


A *red labeled walk of type* $\vec{w} = (i_1, \ldots, i_\ell)$ is a sequence $(p_1, \ldots, p_\ell)$ where p_k is a red labeled step of type i_k and which begins at the alcove 1.

A *green labeled step of type* i is



Let



A *green labeled walk of type* $\vec{w} = (i_1, \ldots, i_\ell)$ is a sequence $(p_1, \ldots, p_\ell)$ where p_k is a green labeled step of type i_k and which begins at the alcove 1.

Theorem 6 *Let $v, w \in W^{\mathrm{ad}}$ and fix a reduced expression $w = s_{i_1} \dots s_{i_\ell}$ for w. The maps $\Psi^+_{\vec{w}}$, $\Psi^0_{\vec{w},v}$, $\Psi^-_{\vec{w},v}$ are bijections.*

$$\begin{array}{rcll}\Phi^+_{\vec{w}}: & \left\{\begin{array}{l}\text{blue labeled paths of type}\\ \vec{w} = (i_1, \dots, i_\ell)\end{array}\right\} & \xrightarrow{\sim} & (I^+wI^+)/I^+\\ & (p_1, \dots, p_\ell) & \longmapsto & \Phi^+(p_1)\dots\Phi^+(p_\ell)I^+\end{array}$$

$$\begin{array}{rcll}\Phi^0_{\vec{w},v}: & \left\{\begin{array}{l}\text{red labeled paths of type}\\ \vec{w} = (i_1, \dots, i_\ell) \text{ ending in } v\end{array}\right\} & \xrightarrow{\sim} & (I^0vI^+ \cap I^+wI^+)/I^+\\ & (p_1, \dots, p_\ell) & \longmapsto & \Phi^0(p_1)\dots\Phi^0(p_\ell)n_vI^+\end{array}$$

$$\begin{array}{rcll}\Phi^-_{\vec{w},v}: & \left\{\begin{array}{l}\text{green labeled paths of type}\\ \vec{w} = (i_1, \dots, i_\ell) \text{ ending in } v\end{array}\right\} & \xrightarrow{\sim} & (I^-vI^+ \cap I^+wI^+)/I^+\\ & (p_1, \dots, p_\ell) & \longmapsto & \Phi^-(p_1)\dots\Phi^-(p_\ell)n_vI^+\end{array}$$

The proof of Theorem 6 is by induction on the length of w following [38, Theorem 4.1 and §7] where (a) and (b) are proved. The induction step can be formulated as the following proposition.

Proposition 2 *Let $v, w \in W^{\mathrm{ad}}$ and fix a reduced expression $\vec{w} = s_{i_1} \dots s_{i_\ell}$ for w. Let $j \in \{0, \dots, n\}$ and $c \in \mathbb{C}$. If $ws_j \lessdot_+ w$ then assume that the reduced word for w is chosen with $i_\ell = j$. Let*

$$\tilde{c} \in \mathbb{C} \text{ and } \tilde{b}_1 \in I^+ \quad \text{be the unique elements such that} \quad b_1 y_j(c) = y_j(\tilde{c})\tilde{b}_1.$$

(a) Let $p = (p_1, \dots, p_\ell)$ be a blue labeled path of type $(i_1, \dots, i_\ell)$ and let $\Phi^+_{\vec{w}}(p) = y_{i_1}(c_1)\dots y_{i_\ell}(c_\ell)$. Then

$$\begin{aligned}&(y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)b_1)(y_j(c)b_2)\\ &= \begin{cases} y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)y_j(\tilde{c})\tilde{b}_1b_2, & \text{if } w \lessdot_+ ws_j,\\ y_{i_1}(c_1)\cdots y_{i_{\ell-1}}(c_{\ell-1})y_{i_\ell}(c_\ell - \tilde{c}^{-1})h_{\alpha_1^\vee}(\tilde{c})x_{\alpha_{i_\ell}}(-\tilde{c}^{-1})\tilde{b}_1b_2, & \text{if } ws_j \lessdot_+ w \text{ and } \tilde{c} \neq 0,\\ y_{i_1}(c_1)\cdots y_{i_{\ell-1}}(c_{\ell-1})h_{\alpha_{i_\ell}^\vee}(-1)x_{\alpha_{i_\ell}}(c)\tilde{b}_1b_2, & \text{if } ws_j \lessdot_+ w \text{ and } \tilde{c} = 0,\end{cases}\end{aligned}$$

(b) Let $p = (p_1, \dots, p_\ell)$ be a red labeled path of type $(i_1, \dots, i_\ell)$ ending in v and let $\Phi^0_{\vec{w},v}(p) = x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)n_vI^+$ where the notation is as in (32). Then

$$\begin{aligned}&(x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)n_vb_1)(y_j(c)b_2)\\ &= \begin{cases} x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)x_{v\alpha_j}(\pm\tilde{c})n_{vs_j}\tilde{b}_1b_2. & \text{if } w \lessdot_0 ws_j,\\ x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)x_{-v\alpha_j}(\tilde{c}^{-1})n_vx_{\alpha_j}(-\tilde{c})h_{\alpha_j^\vee}(\tilde{c})\tilde{b}_1b_2, & \text{if } ws_j \lessdot_0 w \text{ and } \tilde{c} \neq 0,\\ x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)x_{-v\alpha_j}(0)n_{vs_j}\tilde{b}_1b_2, & \text{if } ws_j \lessdot_0 w \text{ and } \tilde{c} = 0,\end{cases}\end{aligned}$$

(c) Let $p = (p_1, \dots, p_\ell)$ be a green labeled path of type $(i_1, \dots, i_\ell)$ ending in v and let $\Phi^-_{\vec{w},v}(p) = x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)n_vI^+$ where the notation is as in (32). Then

$$(x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)n_v b_1)(y_j(c)b_2)$$
$$=\begin{cases} x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)x_{v\alpha_j}(\pm\tilde{c})n_{vs_j}\tilde{b}_1 b_2. & \text{if } w \lessdot ws_j,\\ x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)x_{-v\alpha_j}(\tilde{c}^{-1})n_v x_{\alpha_j}(-\tilde{c})h_{\alpha_j^\vee}(\tilde{c})\tilde{b}_1 b_2, & \text{if } ws_j \lessdot w \text{ and } \tilde{c}\neq 0,\\ x_{\gamma_1}(c_1)\cdots x_{\gamma_\ell}(c_\ell)x_{-v\alpha_j}(0)n_{vs_j}\tilde{b}_1 b_2, & \text{if } ws_j \lessdot w \text{ and } \tilde{c}=0,\end{cases}$$

6.4 Actions of the Affine Hecke Algebra

The *affine flag representation* is

$$C(G/I^+) = \mathbb{C}\text{-span}\left\{y_{\vec{w}}(\vec{c})I^+ \mid w \in W^{\mathrm{ad}}, \vec{c} = (c_1,\ldots,c_\ell) \in \mathbb{C}^{\ell(w)}\right\},$$

where, for a fixed reduced word $w = s_{i_1}\cdots s_{i_\ell}$

$$y_{\vec{w}}(\vec{c}) = y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)I^+ = \sum_{g\in y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)I^+} g \qquad \text{(a formal sum).}$$

Let

$$\begin{aligned} C(I^+\backslash G/I^+) &= \mathbb{C}\text{-span}\{T_w \mid w \in W^{\mathrm{ad}}\}, \text{ where } T_w = I^+ w I^+,\\ C(I^0\backslash G/I^+) &= \mathbb{C}\text{-span}\{X^w \mid w \in W^{\mathrm{ad}}\}, \text{ where } X^w = I^0 w I^+,\\ C(I^-\backslash G/I^+) &= \mathbb{C}\text{-span}\{L^w \mid w \in W^{\mathrm{ad}}\}, \text{ where } L^w = I^- w I^+,\end{aligned}$$

The *affine Hecke algebra* is $\quad h_{I^+}(G/I^+) = \mathbb{C}\text{-span}\{T_w \mid w \in W\}$,

$$\text{where} \quad T_w = I^+ n_w I^+ = \sum_{x\in I^+ n_w I^+} x \qquad \text{(a formal sum).}$$

The formal sums allow us to view all of these elements as elements of the group algebra of G, where infinite formal sums are allowed (to do this precisely one should use Haar measure and a convolution product). Proposition 3 computes the (right) action of $h_{I^+}(G/I^+)$ on $C(G/I^+)$, and Proposition 4 computes the (right) action of $h_{I^+}(G/I^+)$ on $C(I^+\backslash G/I^+)$ on $C(I^0\backslash G/I^+)$, and on $C(I^-\backslash G/I^+)$. Proposition 3 follows from Proposition 2(a) by summing over c and Proposition 4 follows from Proposition 2 by summing over the appropriate double cosets. We use the convention that the normalization (Haar measure) is such that $I^+ \cdot I^+ = I^+$.

Proposition 3 *Let $w \in W^{\mathrm{ad}}$, let $\vec{w} = s_{i_1}\cdots s_{i_\ell}$ be a reduced word for w and let $j \in \{0,\ldots,n\}$. Assume that if $ws_j \lessdot w$ then $i_\ell = j$ and let*

$$y_{\overrightarrow{ws_j}}(\vec{c}) = y_{i_1}(c_1)\cdots y_{i_{\ell-1}}(c_{\ell-1}) \quad \text{and} \quad y_{\vec{w}}(\vec{c}) = y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell) = y_{\overrightarrow{ws_j}}(\vec{c})y_j(c_\ell).$$

Then

$$y_{\vec{w}}(\vec{c})I^+ \cdot T_{s_j} = \begin{cases} y_{i_1}(c_1)\cdots y_{i_{\ell-1}}(c_{\ell-1})I^+ + \sum\limits_{\tilde{c}\in\Bbbk^\times} y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell - \tilde{c}^{-1})I^+, & \textit{if } ws_j \mathrel{<_{+}} w, \\ \sum\limits_{\tilde{c}\in\Bbbk} y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)y_j(\tilde{c})I^+, & \textit{if } w \mathrel{<_{+}} ws_j. \end{cases}$$

Proposition 4 *Let $\Bbbk = \mathbb{F}_q$ the finite field with q elements. Let $w \in W^{\mathrm{ad}}$ and $j \in \{0, \ldots, n\}$. Then*

$$T_w T_{s_j} = \begin{cases} T_{ws_j}, & \textit{if } w \mathrel{<_{+}} ws_j, \\ (q-1)T_w + qT_{ws_j}. & \textit{if } ws_j \mathrel{<_{+}} w, \end{cases} \qquad L^w T_{s_j} = \begin{cases} L^{ws_j}, & \textit{if } w \mathrel{<_{-}} ws_j, \\ qL^{ws_j} + (q-1)L^w, & \textit{if } ws_j \mathrel{<_{-}} w, \end{cases}$$

$$\textit{and} \qquad X^w T_{s_j} = \begin{cases} X^{ws_j}, & \textit{if } w \mathrel{<_{0}} ws_j, \\ qX^{ws_j} + (q-1)X^w, & \textit{if } ws_j \mathrel{<_{0}} w. \end{cases}$$

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References

1. Beck, J.: Braid group action and quantum affine algebras. Commun. Math. Phys. **165**, 555–568 (1994). arXiv:hep-th/9404165, MR1301623
2. Beck, J.: Crystal structure of level zero extremal weight modules. Lett. Math. Phys. **61**, 221–229 (2002). arXiv:math/0205095, MR1942360
3. Beck, J., Chari, V., Pressley, A.: An algebraic characterization of the affine canonical basis. Duke Math. J. **99**(3), 455–487 (1999). arXiv: math/9808060, MR1712630
4. Beck, J., Nakajima, H.: Crystal bases and two sided cells of quantum affine algebras. Duke Math. J. **123**(2), 335–402 (2004). arXiv:math/02122253, MR2066942
5. Braverman, A., Finkelberg, M.: Weyl modules and q-Whittaker functions. Math. Ann. **359**, 45–59 (2014). arXiv:1203.1583, MR3201893
6. Chari, V., Ion, B.: BGG reciprocity for current algebras. Compos. Math. **151**, 1265–1287 (2015). arXiv:1307.1440, MR3371494
7. Chari, V., Pressley, A.: A Guide to Quantum Groups. Cambridge University Press, Cambridge (1994). MR1358358
8. Chari, V., Pressley, A.: Quantum affine algebras and their representations. Represent. Groups (Banff 1994) CMS Conf. Proc. **16**, 59–78 (1995). arXiv:hep-th/9411145, MR1357195
9. Chari, V., Pressley, A.: Weyl modules for classical and quantum affine algebras. Represent. Theory **5**, 191–223 (2001). arXiv:math/0004174, MR1850556

10. Cherednik, I.: Intertwining operators of double affine Hecke algebras. Select. Math. (N.S.) **3**, 459–495 (1997). arXiv:9605014, MR1613515
11. Cherednik, I., Orr, D.Nonsymmetric difference Whittaker functions. Math. Zeit. **279**, 879–938 (2015). arXiv:1302.4094v3, MR3318254
12. Drinfeld, V.G.: A new realization of Yangians and quantized affine algebras. Soviet Math. Dokl. **32**, 212–216 (1988). MR0914215
13. Feigin, E., Makedonskyi, I.: Generalized Weyl modules, alcove paths and Macdonald polynomials. Select. Math. (N.S.) **23**, 2863–2897 (2017). arXiv:1512.03254, MR3703467
14. Fourier, G., Littelmann, P.: Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions. Adv. Math. **211**, 566–593 (2007). arXiv:math/0509276, MR2323538
15. Ion, B.: Nonsymmetric Macdonald polynomials and Demazure characters. Duke Math. J. **116**, 299–318 (2003). arXiv:0105061, MR1953294
16. Ishii, M., Naito, S., Sagaki, D.: Semi-infinite Lakshmibai-Seshadri path model for level zero exremal weight modules over quantum affine algebras. Adv. Math. **290**, 967–1009 (2016). arXiv:1402.3884, MR3451944
17. Kac, V.: Infinite dimensional Lie algebras, Third edition, xxii+400 pp. Cambridge University Press, Cambridge (1990). ISBN 0-521-37215-1; 0-521-46693-8, MR1104219
18. Kashiwara, M.: The crystal base and Littelmann's refined Demazure character formula. Duke Math. J. **71**, 839–858 (1993). MR1240605
19. Kashiwara, M.: Crystal bases of modified quantized enveloping algebra. Duke Math. J. **73**, 383–413 (1994). not on arXiv, MR1262212
20. Kashiwara, M.: On level-zero representations of quantized affine algebras. Duke Math. J. **112**, 117–175 (2002). arXiv:math/0010293, MR1890649
21. Kashiwara, M.: Level zero fundamental representations over quantized affine algebras and Demazure modules. Publ. Res. Inst. Math. Sci. **41**, 223–250 (2005). arXiv:math/0309142, MR2115972
22. Kato, S.: Demazure character formula for semi-infinite flag varieties. Math. Ann. **371**, 1769–1801 (2018). arXiv:1605.049532, MR3831285
23. Kato, S., Naito, S., Sagaki, D.: Equivariant K-theory of semi-infinite flag manifolds and Pieri-Chevalley formula. arXiv: 1702.02408
24. Kumar, S.: Kac-Moody Groups, Their Flag Varieties and Representation Theory. Progress in Mathematics, vol. 204. Birkhäuser, Boston (2002). ISBN: 0-8176-4227-7, MR1923198
25. Littelmann, P.: A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. Invent. Math. **116**, 329–346 (1994)
26. Littelmann, P.: Paths and root operators in representation theory. Ann. Math. **142**,499–525 (1995). MR1356780
27. Lusztig, G.: Intersection cohomology methods in representation theory. Proc. Int. Congress Math. 155–174 (1990)
28. Lusztig, G.: Introduction to Quantum Groups. Progress in Mathematics, vol. 110. Birkhäuser (1993)
29. Macdonald, I.G.: Symmetric Functions and Hall Polynomials. Oxford Mathematical Monographs, 2nd edn. Oxford University Press, New York (1995). ISBN 0-19-853489-2, MR1354144
30. Macdonald, I.G.: Affine Hecke algebras and orthogonal polynomials, Séminaire Bourbaki 1994/95, exposés 790–804. Astérisque **237**, 189–207 (1996)
31. Naito, S., Sagaki, D.: Demazure submodules of level-zero extremal weight modules and specializations of nonsymmetric Macdonald polynomials. Math. Zeit. **283**, 937–978 (2016). arXiv:1404.2436, MR3519989
32. Naito, S., Nomoto, F., Sagaki, D.: Specialization of nonsymmetric Macdonald polynomials at $t = \infty$ and Demazure submodules of level zero extremal weight modules. Trans. Am. Math. Soc. **370**, 2739–2783 (2018). arXiv:1511.07005, MR3748584
33. Naito, S., Nomoto, F., Sagaki, D.: Representation-theoretic interpretation of Cherednik-Orr's recursion formula for the specialization of nonsymmetric Macdonald polynomials at $t = \infty$, Transform. Groups **24**, 155–191 (2019). MR3916094

34. Nakajima, H.: Quiver varieties and finite dimensional representations of quantum affine algebras. J. Am. Math. Soc. **14**(1), 145–238 (2001). arXiv: math/9912158, MR1808477
35. Nakajima, H.: Quiver varieties and t-analogs of q-characters of quantum affine algebras. Ann. Math. **160**(2), 1057–1097 (2004). arXiv:math/0105173, MR2144973
36. Nakajima, H.: Extremal weight modules of quantum affine algebras. Representation theory of algebraic groups and quantum groups. Adv. Stud. Pure Math. **40**, 343–369 Math. Soc. Japan, Tokyo 2004. arXiv:math/0204183, MR2074599
37. Peterson, D.: Quantum Cohomology of G/P. Lecture Notes. Massachusetts Institute of Technology, Cambridge (1997)
38. Parkinson, J., Ram, A., Schwer, C.: Combinatorics in affine flag varieties. J. Algebra **321**, 3469–3493 (2009). arXiv:0801.0709, MR2510057
39. Ram, A.: Alcove walks, Hecke algebras, Spherical functions, crystals and column strict tableaux. Pure Appl. Math. Quart. (Special Issue: In honor of Robert MacPherson, Part 2 of 3) **2**(4), 963–1013 (2006). arXiv:0601.343, MR2282411
40. Ram, A., Yip, M.: A combinatorial formula for Macdonald polynomials. Adv. Math. **226**, 309–331 (2011). arXiv:0803.1146, MR2735761
41. Steinberg, R.: Lectures on Chevalley Groups. Notes prepared by John Faulkner and Robert Wilson. Revised and corrected edition of the 1968 original [MR0466335]. University Lecture Series, vol. 66. American Mathematical Society, Providence (2016). ISBN: 978-1-4704-3105-1
42. Tits, J.: Uniqueness and presentation of Kac-Moody groups over fields. J. Algebra **105**, 542–573 (1987). MR0873684
43. Varagnolo, M., Vasserot, E.: Standard modules of quantum affine algebras. Duke Math. J. **111**(3), 509–533 (2002). arXiv:math/0006084, MR1885830

Stable Bases of the Springer Resolution and Representation Theory

Changjian Su and Changlong Zhong

Abstract In this expository note, we review the recent developments about Maulik and Okounkov's stable bases for the Springer resolution $T^*(G/B)$. In the cohomology case, we compute the action of the graded affine Hecke algebra on the stable basis, which is used to obtain the localization formulae. We further identify the stable bases with the Chern–Schwartz–MacPherson classes of the Schubert cells. This relation is used to prove the positivity conjecture of Aluffi and Mihalcea. For the K theory stable basis, we first compute the action of the affine Hecke algebra on it, which is used to deduce the localization formulae via root polynomial method. Similar as the cohomology case, they are also identified with the motivic Chern classes of the Schubert cells. This identification is used to prove the Bump–Nakasuji–Naruse conjecture about the unramified principal series of the Langlands dual group over non-Archimedean local fields. In the end, we study the wall R-matrices, which relate stable bases for different alcoves. As an application, we give a categorification of the stable bases via the localization of Lie algebras over positive characteristic fields.

Keywords Flag variety · Springer resolution · Stable bases · Hecke algebra

1 Introduction

Schubert calculus studies the cohomology and K-theory of the (partial) flag varieties $\mathcal{B}$. In this note, we study certain basis elements of Maulik and Okounkov, called the *stable basis*, in the equivariant cohomology and K-theory of the cotangent bundle $T^*\mathcal{B}$ (the Springer resolution). Pulling back to the zero section $\mathcal{B}$, the stable basis

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becomes some natural classes in the equivariant cohomology and K-theory of the flag varieties (see Sects. 2.5 and 4.2).

Maulik and Okounkov introduced the stable basis in their study of quantum cohomology of quiver varieties [31]. Later, Okounkov and his collaborators introduced K-theory and elliptic cohomology versions of these bases [1, 34, 37]. They turn out to be very useful both in enumerative geometry and geometric representation theory [35, 36].

The Maulik–Okounkov stable bases are constructed for a class of varieties called symplectic resolutions [26]. We will focus on the Springer resolution, which is one of the classical examples of symplectic resolutions. Both the cohomological and K-theoretic stable bases for the Springer resolution are related to standard objects for certain representation categories.

In the cohomology case, the stable basis elements are just some rational combination of the conormal bundles of the Schubert cells, which were shown to coincide with characteristic cycles of certain $\mathcal{D}$-modules on the flag variety. Via the localization theorem of Beilinson and Bernstein [6], these $\mathcal{D}$-modules correspond to the Verma modules of the Lie algebra. Furthermore, the action of the convolution algebra of the Steinberg variety [21], which is isomorphic to the graded affine Hecke algebra [29], is computed under the stable basis [43]. From this, the first author deduced the localization formula for the stable basis, which is a direct generalizaiton of the well-known AJS/Billey formula [13] for the localization of Schubert classes in the equivariant cohomology of flag varieties. The restriction formulae also play a crucial role in determining the quantum connection of the cotangent bundle of partial flag varieties [43].

On the other hand, the graded affine Hecke algebra also appears in the work of Aluffi and Mihalcea [2] about the Chern–Schwartz–MacPherson (CSM) classes [30, 41, 42] of Schubert cells. Comparing the actions, we identify the pullbacks of the stable bases with the CSM classes [3] (see also [40]). The effectivity of the characteristic cycles of $\mathcal{D}$-modules enables the authors in [3] to prove the non-equivariant positivity conjecture of Aluffi and Mihalcea [2] for CSM classes of the Schubert cells.

The K-theoretic case is similar, but with richer structure. By the famous theorem of Kazhdan, Lusztig and Ginzburg [21, 23, 27], the affine Hecke algebra is isomorphic to the convolution algebra of the equivariant K-theory of the Steinberg variety. Hence, the affine Hecke algebra acts on the equivariant K-theory of $T^*\mathcal{B}$. This action on the stable basis is computed in [44]. It roughly says that the stable basis is the standard basis for the regular representation of the finite Hecke algebra. With this action, we can compute the localization of the stable basis via the root polynomial method.

The K-theretic generalization of the CSM classes are motivic Chern classes defined by Brasselet, Schürmann and Yokura [14]. Pulling back the K-theory stable bases to the zero section, we get the motivic Chern classes of the Schubert cells [4, 22]. By definition, the motivic Chern classes enjoy good functorial properties. These facts enable the authors in [4] to use Schubert calculus to prove some conjectures of Bump, Nakasuji and Naruse [17, 32] about unramified principal series of the p-adic Langland dual group.

Recall that the affine Hecke algebra can be realized either as the equivariant K-group of the Steinberg variety, or as the double Iwahori-invariant functions on the p-adic Langland dual group [21, Introduction], [7]. The bridge connecting the Schubert calculus over complex numbers and the representation theory of the p-adic Langland dual group is the shadow of these two geometric realizations. To be more precise, the (specialized) equivariant K-theory of the flag variety and the Iwahori-invariants of an unramified principal series of the p-adic group are two geometric realizations of the regular representation of the finite Hecke algebra. Identifying these two regular representations, the stable basis (or the motivic Chern classes of the Schubert cells) gets identified with the standard basis on the p-adic side, while the fixed point basis maps to the Casselman basis [18]. With this isomorphism, we give an equivariant K-theory interpretation of Macdonald's formula for the spherical function [18] and the Casselman–Shalika formula [19] for the spherical Whittaker function in [44].

The stable basis is also related to representations of Lie algebras over a field of positive characteristic, via the localization theorem of Bezrukavnikov, Mirković and Rumynin [9, 10], which generalizes the famous localization theorem of Beilinson and Bernstein [6] over the complex numbers. This is achieved as follows. The definition of the K-theory stable basis depends on a choice of the alcove. Changing alcoves defines the so-called wall R-matrices [37]. We first recall these wall R-matrices for the Springer resolution, which are computed in [45]. The formulae can be nicely packed using the Hecke algebra actions. It turns out the wall R-matrices coincide with the monodromy matrices of the quantum connection. Bezrukavnikov and Okounkov conjecture that for general symplectic resolutions X, the monodromy representation is isomorphic to the representation coming from derived equivalences [11, 35]. The Springer resolution case is established by Braverman, Maulik and Okounkov in [16]. Thus, the wall R-matrices is also related to the derived equivalences. Finally, we give a categorification of the stable basis via the affine braid group action, constructed by Bezrukavnikov and Riche, on the derived category of coherent sheaves on the Springer resolution [12, 38]. Moreover, the categorified stable bases are identified with the Verma modules for the Lie algebras over positive characteristic fields under the localization equivalence.

This note is structured as follows. In Sect. 2, we review the following aspects of the cohomological stable basis: the action of the affine graded Hecke algebra, restriction formula, and identification with the Chern–Schwartz–MacPherson classes of the Schubert cells and characteristic cycles of holonomic $\mathcal{D}$-modules on the flag variety. The rest of the note is devoted to the study of the K theory stable basis. In Sect. 3, we introduce the K theory stable basis, compute the action of the affine Hecke algebra on it, and obtain the restriction formula. In Sect. 4, the K theory stable bases are identified with the motivic Chern classes of the Schubert cells. In Sect. 5, we further relate the stable basis to the Casselman basis in the Iwahori invariants of an unramified principal series of the Langlands dual group over a non-Archimedean local field, and use it to prove conjectures of Bump, Nakasuji and Naruse. Finally, in Sects. 6 and 7, we review the wall crossing matrix and the categorification of the stable bases.

2 Cohomological Stable Basis

In this section, we first recall the definition of the stable basis of Maulik and Okounkov [31]. Then we compute the action of the graded Hecke algebra on the stable basis, which will have two applications: the localization formulae and the identification of the stable bases with the Chern–Schwartz–MacPherson classes of the Schubert cells. We further relate the stable bases to characteristic cycles of $\mathcal{D}$-modules, which plays an important role in proving the positivity conjecture of Aluffi and Mihalcea.

2.1 Notations

Let G be a complex, semisimple, simply connected linear algebraic group with a Borel subgroup B and a maximal torus A. Let B^- be the opposite Borel subgroup. Let Λ (resp. $\Lambda^\vee$) be the group of characters (resp. cocharacters) of A. Let R^+ denote the roots in B. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. A). Let ρ be the half-sum of the positive roots. Let $\mathfrak{C}_\pm \subset \mathfrak{h}$ be the dominant/anti-dominant Weyl chamber in $\mathfrak{h}$. Let $\mathcal{B} = G/B$ be the variety of all Borel subgroups of G. Denote by $T^*\mathcal{B}$ (resp. $T\mathcal{B}$) the cotangent (tangent) bundle. The cotangent bundle $T^*\mathcal{B}$ is a resolution of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$, which is the so-called *Springer resolution*. The group $\mathbb{C}^*$ acts on $T^*\mathcal{B}$ by $z \cdot (B', x) = (B', z^{-2}x)$ for any $z \in \mathbb{C}^*$, $(B', x) \in T^*\mathcal{B}$. Let $-\hbar$ be the Lie $\mathbb{C}^*$-weight of the cotangent fiber. Let $T = A \times \mathbb{C}^*$. Then $H_T^*(pt) = \mathbb{C}[\mathfrak{h}][\hbar]$, where $\mathbb{C}[\mathfrak{h}]$ denotes the functions on $\mathfrak{h}$. For any $\lambda \in \Lambda$, let $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$ be the line bundle on $\mathcal{B}$. Pulling back to $T^*\mathcal{B}$, we get a line bundle on $T^*\mathcal{B}$, which is still denoted by $\mathcal{L}_\lambda$. Let W be the Weyl group, and let $\leq$ denote the Bruhat order on W. The A-fixed points in $T^*\mathcal{B}$ are indexed by the Weyl group W. For each $w \in W$, the corresponding fixed point is $wB \in \mathcal{B} \subset T^*\mathcal{B}$. Let $\iota_w : wB \hookrightarrow T^*\mathcal{B}$ denote the embedding. For any $\gamma \in H_T^*(T^*\mathcal{B})$, let $\gamma|_w = \iota_w^*\gamma \in H_T^*(wB) = H_T^*(pt)$.

2.2 Definition of the Cohomological Stable Basis

The definition of the stable basis depends on a choice of a Weyl chamber $\mathfrak{C}$ in $\mathfrak{h}_\mathbb{R}$. Pick one parameter $\sigma : \mathbb{C}^* \to A$, such that $d\sigma \in \mathfrak{C}$. Recall the attracting set of the torus fixed point $wB \in T^*\mathcal{B}$

$$\mathrm{Attr}_\mathfrak{C}(w) = \{x \in T^*\mathcal{B} | \lim_{z\to 0} \sigma(z) \cdot x = wB\}.$$

A partial order on W is defined as

$$v \preceq_\mathfrak{C} w \quad \text{if} \quad \overline{\mathrm{Attr}_\mathfrak{C}(w)} \cap vB \neq \emptyset.$$

The order determined by the dominant chamber (resp. anti-dominant chamber) is the usual Bruhat order (resp., the opposite Bruhat order). The full attracting set is

$$\mathrm{FAttr}_{\mathfrak{C}}(w) = \cup_{v \preceq_{\mathfrak{C}} w} \mathrm{Attr}_{\mathfrak{C}}(v).$$

For each $w \in W$, let $\epsilon_w = e^A(T^*_{wB}\mathcal{B}) \in H^*_A(pt) = \mathbb{C}[\mathfrak{h}]$, the A-equivariant Euler class of the A-vector space $T^*_{wB}\mathcal{B}$. That is, ϵ_w is the product of A-weights in the vector space $T^*_{wB}\mathcal{B}$. Let $N_w = T_{wB}(T^*\mathcal{B})$. The chamber $\mathfrak{C}$ gives a decomposition of the tangent bundle

$$N_w = N_{w,+} \oplus N_{w,-}$$

into A-weights which are positive and negative when paired with σ, respectively. Let $e(N_{w,-}) \in H^*_T(pt)$ denote the T-equivariant Euler class, that is, the product of T-weights in the T-equivariant vector space $N_{w,-}$. Since $T = A \times \mathbb{C}^*$, $H^*_T(pt) = H^*_A(pt)[\hbar]$. If we specialize the equivariant parameter $\hbar$ to zero, the T-equivariant Euler class $e(N_{w,-}) \in H^*_T(pt)$ equals $a_w\epsilon_w$, where $a_w \in \{1, -1\}$. See Example 2.3.

Recall the fixed point set $(T^*\mathcal{B})^A$ is a discrete set indexed by the Weyl group W. Hence, $H^*_T((T^*\mathcal{B})^A) = \sqcup_{w\in W} H^*_T(pt)$. Let 1_w denote the unit element in the w-th copy $H^*_T(pt)$. The cohomological stable basis is defined as follows.

Theorem 2.1 ([31, Theorem 3.3.4]) *There exists a unique map of $H^*_T(pt)$-modules*

$$\mathrm{stab}_{\mathfrak{C}} : H^*_T((T^*\mathcal{B})^A) \to H^*_T(T^*\mathcal{B})$$

satisfying the following properties. For any $w \in W$, denote $\mathrm{stab}_{\mathfrak{C}}(w) = \mathrm{stab}_{\mathfrak{C}}(1_w)$. *Then*

(1) (Support) $\mathrm{supp}\,\mathrm{stab}_{\mathfrak{C}}(w) \subset \mathrm{FAttr}_{\mathfrak{C}}(w)$,
(2) (Normalization) $\mathrm{stab}_{\mathfrak{C}}(w)|_w = a_w e(N_{-,w})$,
(3) (Degree) $\mathrm{stab}_{\mathfrak{C}}(w)|_v$ *is divisible by $\hbar$, for any $v \prec_{\mathfrak{C}} w$.*

Remark 2.2 (1) The map is defined by a Lagrangian correspondence between $(T^*\mathcal{B})^A \times T^*\mathcal{B}$, hence mapping middle degree to middle degree. Therefore, the last condition is equivalent to

$$\deg_A \mathrm{stab}_{\mathfrak{C}}(w)|_v < \deg_A \mathrm{stab}_{\mathfrak{C}}(v)|_v.$$

Here, for any polynomial $f \in H_T(pt) = \mathbb{C}[\mathfrak{h}][\hbar] = \mathrm{Sym}(\mathfrak{h}^*)[\hbar]$, $\deg_A f$ denotes the degree of f in the $\mathfrak{h}^*$-variables.

(2) By the first and second conditions above, $\{\mathrm{stab}_{\mathfrak{C}}(w) | w \in W\}$ is a basis for the localized equivariant cohomology $H^*_T(T^*\mathcal{B})_{loc} := H^*_T(T^*\mathcal{B}) \otimes_{H_T(pt)} \mathrm{Frac}\, H_T(pt)$. It is the so-called **stable basis**.

(3) By [31, Theorem 4.4.1], the stable bases for opposite chambers are dual bases. That is,

$$\langle \mathrm{stab}_{\mathfrak{C}}(v), \mathrm{stab}_{-\mathfrak{C}}(w)\rangle = (-1)^{\dim G/B}\delta_{v,w},$$

where the non-degenerate pairing $\langle -, - \rangle$ on $H_T^*(T^*\mathcal{B})$ is defined via localization as follows

$$\langle \gamma_1, \gamma_2 \rangle := \sum_w \frac{\gamma_1|_w \cdot \gamma_2|_w}{e(N_w)} \in \text{Frac}\, H_T^*(pt).$$

Let $\text{stab}_+(w) = \text{stab}_{\mathfrak{C}_+}(w)$ and $\text{stab}_-(w) = \text{stab}_{\mathfrak{C}_-}(w)$.

Example 2.3 Let $G = \text{SL}(2, \mathbb{C})$, B be the upper triangular matrices, and α be the simple root. In this case, the flag variety $\mathcal{B} = \mathbb{P}^1$, the identity element $1 \in W$ corresponds to the fixed point $0 := [1 : 0] \in \mathbb{P}^1$, and $s_\alpha \in W$ corresponds to the other fixed point $\infty := [0 : 1] \in \mathbb{P}^1$. The torus $T = A \times \mathbb{C}^*$ weights of the vector spaces are as follows

$$\text{weight}(T_0\mathbb{P}^1) = -\alpha, \quad \text{weight}(T_0^*\mathbb{P}^1) = \alpha - \hbar,$$

and

$$\text{weight}(T_\infty\mathbb{P}^1) = \alpha, \quad \text{weight}(T_\infty^*\mathbb{P}^1) = -\alpha - \hbar.$$

Let us pick the positive Weyl chamber $\mathfrak{C}_+$ so that α takes non-negative values on it. Then $\text{Attr}_{\mathfrak{C}_+}(0) = T_0^*\mathbb{P}^1$, and $\text{Attr}_{\mathfrak{C}_+}(\infty) = \mathbb{P}^1 \setminus \{0\}$. Hence, $0 \prec_{\mathfrak{C}_+} \infty$. The Euler classes are $\epsilon_0 = e^A(T_0^*\mathbb{P}^1) = \alpha$, $e(N_{0,-}) = -\alpha$, $\epsilon_\infty = e^A(T_\infty^*\mathbb{P}^1) = -\alpha$, and $e(N_{\infty,-}) = -\alpha - \hbar$. Therefore, $a_1 = -1$ and $a_{s_\alpha} = 1$. By the support and normalization conditions, we get

$$\text{stab}_+(0) = -[T_0^*\mathbb{P}^1], \quad \text{and} \quad \text{stab}_+(\infty) = [\mathbb{P}^1] + c[T_0^*\mathbb{P}^1],$$

where $c \in H_T^*(pt)$. Restricting $\text{stab}_{\mathfrak{C}_+}(\infty)$ to the smaller fixed point 0, we get

$$\text{stab}_+(\infty)|_0 = (-\hbar + \alpha) + c(-\alpha).$$

By the degree condition, the above is divisible by $\hbar$. Hence, $c = 1$. So

$$\text{stab}_+(0) = -[T_0^*\mathbb{P}^1], \quad \text{stab}_+(\infty) = [\mathbb{P}^1] + [T_0^*\mathbb{P}^1]. \tag{1}$$

Similarly, for the negative chamber $\mathfrak{C}_-$, we have

$$\text{stab}_-(0) = [T_\infty^*\mathbb{P}^1] + [\mathbb{P}^1], \quad \text{stab}_-(\infty) = -[T_\infty^*\mathbb{P}^1]. \tag{2}$$

2.3 *The Graded Affine Hecke Algebra Action*

The graded affine Hecke algebra $\mathcal{H}_\hbar$ is generated by the elements x_λ of degree 1 for $\lambda \in \mathfrak{h}^*$, $s_i \in W$ of degree 0 and a central element $\hbar$ of degree 1 such that

(1) $x_{\lambda+\mu} = x_\lambda + x_\mu$ for any $\lambda, \mu \in \mathfrak{h}^*$;
(2) $x_\lambda x_\mu = x_\mu x_\lambda$ for any $\lambda, \mu \in \mathfrak{h}^*$;
(3) The s_i's generated the Weyl group inside $\mathcal{H}_\hbar$;
(4) For any simple root α_i and $\lambda \in \mathfrak{h}^*$, we have

$$s_i x_\lambda - x_{s_i\lambda} s_i = \hbar(\lambda, \alpha_i^\vee).$$

According to [29], there is a natural algebra isomorphism

$$\mathcal{H}_\hbar \simeq H_*^{G\times\mathbb{C}^*}(Z),$$

where $Z = T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}$ is the Steinberg variety, $H_*^{G\times\mathbb{C}^*}(Z)$ is the $G \times \mathbb{C}^*$-equivariant Borel–Moore homology of Z, and it is endowed with the convolution algebra structure [21]. The isomorphism is constructed as follows. Recall that for any $\lambda \in \Lambda$, we have the line bundle $\mathcal{L}_\lambda \in \mathrm{Pic}_{A\times\mathbb{C}^*}(T^*\mathcal{B})$. Then the above isomorphism sends x_λ to the push-forward of the first Chern class $c_1(\mathcal{L}_\lambda)$ under the diagonal embedding $T^*\mathcal{B} \hookrightarrow Z$. The image of a simple reflection s_α in the Weyl group is constructed as follows. Let P_α denote the corresponding minimal parabolic subgroup containing B. Let $\mathcal{P}_\alpha = G/P_\alpha$ and $Y_\alpha = \mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B} \subset \mathcal{B} \times \mathcal{B}$. The conormal bundle $T^*_{Y_\alpha}(\mathcal{B} \times \mathcal{B})$ is denoted by $T^*_{Y_\alpha}$, which is a smooth closed $G \times \mathbb{C}^*$-invariant subvariety of Z. Then $s_\alpha - 1$ is sent to the cohomology class $[T^*_{Y_\alpha}] \in H_*^{G\times\mathbb{C}^*}(Z)$.

Via this isomorphism, $\mathcal{H}_\hbar$ acts on $H_T^*(T^*\mathcal{B})$ by convolution [21, Chap. 2]. Let π denote this action. The action on the stable basis is given by

Theorem 2.4 ([43, Lemma 3.2]) *For any $w \in W$ and simple root α,*

$$\pi(s_\alpha)(\mathrm{stab}_\pm(w)) = -\,\mathrm{stab}_\pm(ws_\alpha).$$

2.4 The Restriction Formula

One important corollary of the above theorem is the restriction formula for the stable basis.

Theorem 2.5 ([43, Theorem 1.1]) *Let $y = s_1 s_2 \cdots s_l$ be a reduced expression for $y \in W$. Then*

$$\mathrm{stab}_-(w)|_y = (-1)^{l(y)} \prod_{\alpha\in R^+\setminus R(y)} (\alpha - \hbar) \sum_{\substack{1\le i_1<i_2<\cdots<i_k\le l\\ w=s_{i_1}s_{i_2}\ldots s_{i_k}}} \hbar^{l-k} \prod_{j=1}^{k} \beta_{i_j}, \tag{3}$$

where s_i is the simple reflection associated to a simple root α_i, $\beta_i = s_1 \cdots s_{i-1}\alpha_i$, R^+ is the set of positive roots, and $R(y) = \{\beta_i | 1 \le i \le l\}$. Furthermore, the sum in Eq. (3) does not depend on the reduced expression for y.

From this formula, we can recover the AJS/Billey formula [13]

$$[\overline{B^-wB/B}]|_y = \sum_{\substack{1\le i_1<i_2<\cdots<i_k\le l\\ w=s_{i_1}s_{i_2}\ldots s_{i_k}\text{ reduced}}} \beta_{i_1}\cdots\beta_{i_k},$$

via the following limit formula ([43, Theorem 4.8])

$$[\overline{B^-wB/B}]|_y = (-1)^{\ell(w)} \lim_{\hbar\to\infty} \frac{\text{stab}_-(w)|_y}{(-\hbar)^{\dim \overline{B^-wB/B}}}.$$

Example 2.6 Let us consider the case $G = \text{SL}(3, \mathbb{C})$, $y = s_1s_2s_1$ and $w = s_1$. Then $\text{stab}_-(w)|_y = (-1)^3\hbar^2(\alpha_1 + \alpha_2)$, and $[\overline{B^-wB/B}]|_y = \alpha_1 + \alpha_2$. It is obvious that the above limit formula holds.

We can also compute restriction formula for $\text{stab}_-(w)$ for all $w \in S_3$ (following notations in Sect. 3.4, with $\alpha_{1+2} = \alpha_1 + \alpha_2$, $\tilde{\alpha} = \alpha - \hbar$, $iji = s_is_js_i$, ect):

$$\begin{aligned}
\text{stab}_-(121) &= -\alpha_1\alpha_2\alpha_{1+2}f_{121},\\
\text{stab}_-(12) &= -\hbar\alpha_1\alpha_{1+2}f_{121} + \alpha_1\alpha_{1+2}\tilde{\alpha}_2 f_{12},\\
\text{stab}_-(21) &= -\hbar\alpha_2\alpha_{1+2}f_{121} + \alpha_2\alpha_{1+2}\tilde{\alpha}_1 f_{21},\\
\text{stab}_-(1) &= -\hbar^2\alpha_{1+2}f_{121} + \hbar\alpha_1\tilde{\alpha}_2 f_{12} + \hbar\alpha_{1+2}\tilde{\alpha}_1 f_{21} - \alpha_1\tilde{\alpha}_2\tilde{\alpha}_{1+2}f_1,\\
\text{stab}_-(2) &= -\hbar^2\alpha_{1+2}f_{121} + \hbar\alpha_2\tilde{\alpha}_1 f_{21} + \hbar\alpha_{1+2}\tilde{\alpha}_2 f_{12} - \alpha_2\tilde{\alpha}_1\tilde{\alpha}_{1+2}f_2,\\
\text{stab}_-(e) &= -(\hbar^3 + \hbar\alpha_1\alpha_2)f_{121} + \hbar^2\tilde{\alpha}_2 f_{12} + \hbar^2\tilde{\alpha}_1 f_{21} - \hbar\tilde{\alpha}_2\tilde{\alpha}_{1+2}f_1 - \hbar\tilde{\alpha}_1\tilde{\alpha}_{1+2}f_2 + \tilde{\alpha}_1\tilde{\alpha}_2\tilde{\alpha}_{1+2}f_e.
\end{aligned}$$

In the above formulae, the restriction $\text{stab}_-(w)|_y$ is given by the coefficient of f_y in the identity of $\text{stab}_-(w)$.

2.5 Relation with Chern–Schwartz–MacPherson (CSM) Classes

Let us first recall the definition of Chern–Schwartz–MacPherson classes. For any algebraic variety Y over $\mathbb{C}$, let $\mathcal{F}(Y)$ denote the group of constructible functions on Y. If $f : Y \to X$ is a proper morphism, we can define a pushforward $f_* : \mathcal{F}(Y) \to \mathcal{F}(X)$ by setting $f_*(1_W)(p) = \chi(f^{-1}(p) \cap W)$, where $W \subset Y$ is a locally closed subvariety, 1_W is the characteristic function of W, $p \in X$, and χ is the topological Euler characteristic. According to a conjecture attributed to Deligne and Grothendieck, there is a unique natural transformation $c_* : \mathcal{F} \to H_*$ from the functor $\mathcal{F}$ of constructible functions on a complex algebraic variety to the homology functor, such that if X is smooth then $c_*(1_X) = c(TX) \cap [X]$, where $c(TX)$ is the total Chern class. The naturality of c_* means that it commutes with proper pushforward. This conjecture was proved by MacPherson [30]. The class $c_*(1_X)$ for possibly singular

X was shown to coincide with a class defined earlier by M.-H. Schwartz [41, 42]. For any constructible subset $W \subset X$, we call the class $c_{SM}(W) := c_*(1_W) \in H_*(X)$ the *Chern–Schwartz–MacPherson* (CSM) class of W in X. The theory of CSM classes was later extended to the equivariant setting by Ohmoto [33].

In this section, we identity the equivariant homology $H_*^A(\mathcal{B})$ with the equivairant cohomology $H_A^*(\mathcal{B})$ via the Poincaré duality. Let $X(w)^\circ = BwB/B \subset \mathcal{B}$ and $Y(w)^\circ = B^- wB/B$ be the Schubert cells in $\mathcal{B}$. Let $X(w) = \overline{X(w)^\circ}$ be the Schubert variety. A similar formula to that given in Theorem 2.4 was also obtained by Aluffi and Mihalcea for the CSM classes for the Schubert cells $c_{SM}(X(w)^\circ)$ [2]. Comparing these formulae, it is easy to get the following relation between the stable bases and the CSM classes [3, Corollary 1.2] and [40].

Theorem 2.7 *Let $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ be the inclusion. For any $w \in W$,*

$$i^*(\mathrm{stab}_+(w))|_{\hbar=1} = (-1)^{\dim G/B} c_{SM}(X(w)^\circ) \in H_A^*(\mathcal{B}),$$

and

$$i^*(\mathrm{stab}_-(w))|_{\hbar=1} = (-1)^{\dim G/B} c_{SM}(Y(w)^\circ) \in H_A^*(\mathcal{B}).$$

Example 2.8 Let us consider the case $G = \mathrm{SL}(2, \mathbb{C})$. We follow the notations in Example 2.3. By definition, $c_{SM}(X(id)^\circ) = [X(id)] = [0]$, and $c_{SM}(X(s_\alpha)^\circ) = c_{SM}(X(s_\alpha)) - c_{SM}(X(id)) = c(T\mathbb{P}^1) - [0] = [\mathbb{P}^1] + [\infty]$. Let us check

$$i^*(\mathrm{stab}_+(s_\alpha))|_{\hbar=1} = -c_{SM}(X(s_\alpha)^\circ) \in H_A^*(\mathcal{B}). \tag{4}$$

By localization, it suffices to check the equality after restricting both sides to the fixed points. For the fixed point 0, $i^*(\mathrm{stab}_+(s_\alpha))|_{0,\hbar=1} = -1$, and $c_{SM}(X(s_\alpha)^\circ)|_0 = 1$. For the fixed point ∞, $i^*(\mathrm{stab}_+(s_\alpha))|_{\infty,\hbar=1} = -1 - \alpha$, and $c_{SM}(X(s_\alpha)^\circ)|_\infty = 1 + \alpha$. Thus, Eq. (4) holds.

The equivariant cohomology of the flag variety $\mathcal{B}$ has a natural basis, namely, the Schubert basis $\{[X(w)] | w \in W\}$. Thus we can expand the CSM classes in terms of this basis

$$c_{SM}(X(w)^\circ) = \sum_u c(w,u)[X(u)] \in H_A^*(\mathcal{B}),$$

where $c(w,u) \in H_A^*(pt)$. It is conjectured by Aluffi and Mihalcea that [2]

$$c(w,u) \in \mathbb{Z}_{\geq 0}[\alpha | \alpha > 0].$$

The non-equivariant case of this conjecture is proved in [3], in which the relation between the stable basis and the characteristic cycles of holonomic $\mathcal{D}_{\mathcal{B}}$-modules plays an important role.

2.6 Characteristic Cycles of $\mathcal{D}$-modules

For $w \in W$, let M_w be the Verma module of highest weight $-w\rho - \rho$, a module over the universal enveloping algebra $U(\mathfrak{g})$. Let $\mathfrak{M}_w$ denote the holonomic $\mathcal{D}_{\mathcal{B}}$-module

$$\mathfrak{M}_w = \mathcal{D}_{\mathcal{B}} \otimes_{U(\mathfrak{g})} M_w.$$

The image of this regular holonomic $\mathcal{D}_{\mathcal{B}}$-module under the Riemann–Hilbert correspondence is the constructible complex $\mathbb{C}_{X(w)^\circ}[\ell(w)]$ [6, 15, 24]. The characteristic cycle $\mathrm{Char}(\mathfrak{M}_w)$ of $\mathfrak{M}_w$ is a linear combination of the conormal bundles of the Schubert cells. The relation with the stable basis is given by the following formula. It is claimed in [31, Remark 3.5.3], and later proved in [3, Lemma 6.5].

Theorem 2.9 *For any $w \in W$,*

$$\mathrm{stab}_+(w) = (-1)^{\dim G/B - \ell(w)}[\mathrm{Char}(\mathfrak{M}_w)] \in H_T^*(T^*\mathcal{B}).$$

3 K-Theoretic Stable Bases and the Affine Hecke Algebra Action

From now on, we focus on the K theory stable basis. In this section, we introduce the definitions and compute the action of the affine Hecke algebras on the K theory stable bases. As an application, we compute the restriction formula of stable bases by using the root polynomial method.

3.1 Notations Continued

We will follow the notations in Sect. 2. Let us introduce more notations. Let $H_{\alpha^\vee,n} = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* | (\lambda, \alpha^\vee) = n\}$ be the hyperplanes determined by the coroot $\alpha^\vee$ and integers n. The union of the hyperplanes is a closed subset of $\mathfrak{h}_{\mathbb{R}}^*$, whose complement has connected components called alcoves. The fundamental alcove is $\nabla_+ = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* | 0 < \langle \lambda, \alpha^\vee \rangle < 1$, for any positive coroot $\alpha^\vee\}$. Denote $\nabla_- = -\nabla_+$.

In the remaining parts of this note, we will consider the equivariant K-theory, a good introduction of which can be found in [21]. If a group H acts on an algebraic variety X, then the H-equivariant K-theory of X, which is denoted by $K_H(X)$, is defined to be the Grothendieck group of the H-equivariant coherent sheaves on X. Namely, $K_H(X) := K^0(\mathrm{Coh}_H(X))$. By definition, $K_H(X)$ is a module over $K_H(pt) = K^0(\mathrm{Rep}(H))$.

Recall the group $\mathbb{C}^*$ acts on $T^*\mathcal{B}$ by $z \cdot (B', x) = (B', z^{-2}x)$ for any $z \in \mathbb{C}^*$, $(B', x) \in T^*\mathcal{B}$. Let q^{-1} be the $\mathbb{C}^*$-character of the cotangent fiber under this action, i.e., $q = e^\hbar$. Therefore, the $\mathbb{C}^*$-equivariant K-group of a point is $K_{\mathbb{C}^*}(pt) =$

$K^0(\mathrm{Rep}(\mathbb{C}^*)) = \mathbb{Z}[q^{1/2}, q^{-1/2}]$. Recall $T = A \times \mathbb{C}^*$. Denote $S = K_T(\mathrm{pt}) = \mathbb{Z}[q^{1/2}, q^{-1/2}][\Lambda]$. We consider the T-equivariant K-theory of $T^*\mathcal{B}$. Recall ι_w denotes the inclusion of the fixed point wB into $T^*\mathcal{B}$. For any $\mathcal{F} \in K_T(T^*\mathcal{B})$, let $\mathcal{F}|_w$ denote $\iota_w^*\mathcal{F} \in K_T(pt)$. By localization theorem [21, Chapter 5], the localized K-group $K_T(T^*\mathcal{B})_{loc} := K_T(T^*\mathcal{B}) \otimes_{K_T(pt)} \mathrm{Frac}\, K_T(pt)$ has a fixed point basis $\{\iota_{w*}1 | w \in W\}$.

The non-degenerate pairing on $K_T(T^*\mathcal{B})$ can be defined using localization as follows

$$\langle \mathcal{F}, \mathcal{G} \rangle = \sum_{w \in W} \frac{\mathcal{F}|_w \mathcal{G}|_w}{\bigwedge^\bullet T_w(T^*\mathcal{B})} = \sum_{w \in W} \frac{\mathcal{F}|_w \mathcal{G}|_w}{\prod_{\alpha>0}(1 - e^{w\alpha})(1 - qe^{-w\alpha})}, \quad \mathcal{F}, \mathcal{G} \in K_T(T^*\mathcal{B}).$$

Here for each T-space V, $\bigwedge^\bullet V = \sum_{k \geq 0}(-1)^k \bigwedge^k V^\vee = \prod(1 - e^{-\alpha}) \in K_T(\mathrm{pt})$, where the product runs through all T-weights in V, counted with multiplicities.

3.2 Definition of the Stable Basis

The definition of the K theory stable basis depends on choices of a chamber, an alcove and a polarization.

A polarization $T^{1/2}$ is a solution of the following equation

$$T^{1/2} + q^{-1}(T^{1/2})^\vee = T(T^*\mathcal{B}) \quad \text{in } K_T(T^*\mathcal{B}).$$

Denote $T^{1/2}_{\mathrm{opp}} = q^{-1}(T^{1/2})^\vee$. We will frequently focus on the two mutually opposite polarizations: $T\mathcal{B}$ and $T^*\mathcal{B}$. Recall $N_w = T_{wB}(T^*\mathcal{B})$ is the normal bundle at w, and the chamber $\mathfrak{C}$ determines a decomposition $N_w = N_{w,+} \oplus N_{w,-}$ according to the sign of A-weights with respect to the chamber $\mathfrak{C}$. Let $N_w^{1/2} = N_w \cap (T^{1/2}|_w)$, and similarly define $N^{1/2}_{w,+}$ and $N^{1/2}_{w,-}$. It follows that the square root $(\frac{\det N_{w,-}}{\det N_w^{1/2}})^{1/2}$ exists in $K_T(T^*\mathcal{B})$.

For any Laurent polynomial $f = \sum_{\mu \in \Lambda} f_\mu e^\mu \in K_T(\mathrm{pt})$, with $f_\mu \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$, $e^\mu \in K_A(\mathrm{pt})$, define the A-degree of f to be the Newton polygon

$$\deg_A f = \text{ Convex hull } (\{\mu | f_\mu \neq 0\}) \subset \Lambda \otimes \mathbb{R}.$$

Let 1_w denote the unit in the w-th copy of $K_T(pt)$ in $K_T((T^*\mathcal{B})^A)$. We can now recall the definition of the K-theory stable basis.

Theorem 3.1 *[34] For any chamber $\mathfrak{C}$, polarization $T^{1/2}$, and alcove ∇, there is a unique map of $K_T(\mathrm{pt})$-modules (called the stable envelope):*

$$\mathrm{stab}_{\mathfrak{C}, T^{1/2}, \nabla} : K_T((T^*\mathcal{B})^A) \to K_T(T^*\mathcal{B}),$$

satisfying the following properties. Denote $\mathrm{stab}_w^{\mathfrak{C}, T^{1/2}, \nabla} = \mathrm{stab}_{\mathfrak{C}, T^{1/2}, \nabla}(1_w)$. Then

(1) (Support) $\mathrm{supp}(\mathrm{stab}_w^{\mathfrak{C},T^{1/2},\nabla}) \subset \mathrm{FAttr}_{\mathfrak{C}}(w)$.

(2) (Normalization) $\mathrm{stab}_w^{\mathfrak{C},T^{1/2},\nabla}|_w = (-1)^{\mathrm{rank}\, N_{w,+}^{1/2}} \left(\frac{\det N_{w,-}}{\det N_w^{1/2}}\right)^{1/2} \mathcal{O}_{\mathrm{Attr}_{\mathfrak{C}}(w)}|_w$.

(3) (Degree) $\deg_A(\mathrm{stab}_w^{\mathfrak{C},T^{1/2},\nabla}|_v) \subset \deg_A(\mathrm{stab}_v^{\mathfrak{C},T^{1/2},\nabla}|_v) + v\lambda - w\lambda$ *for any* $v \prec_{\mathfrak{C}} w, \lambda \in \nabla$.

Note that the degree condition depends on the alcove ∇ only, not on a particular $\lambda \in \nabla$. On the other hand, the normalization does not depend on the alcove.

Denote

$$\mathrm{stab}_w^+ = \mathrm{stab}_w^{\mathfrak{C}_+,T\mathcal{B},\nabla_-}, \quad \mathrm{stab}_w^- = \mathrm{stab}_w^{\mathfrak{C}_-,T^*\mathcal{B},\nabla_+}.$$

We list some basic properties of stable bases [37, Proposition 1] and [44, Lemma 2.2]:

(1) The duality: $\langle \mathrm{stab}_w^{\mathfrak{C},T^{1/2},\nabla}, \mathrm{stab}_v^{-\mathfrak{C},T^{1/2}_{\mathrm{opp}},-\nabla}\rangle = \delta_{v,w} \in K_T(T^*\mathcal{B})$.
(2) $\mathrm{stab}_w^+|_w = q^{-\ell(w)/2}\prod_{\beta>0,w\beta<0}(q - e^{w\beta})\prod_{\beta>0,w\beta>0}(1 - e^{w\beta})$.
(3) $\mathrm{stab}_w^-|_w = q^{\ell(w)/2}\prod_{\beta>0,w\beta<0}(1 - e^{-w\beta})\prod_{\beta>0,w\beta>0}(1 - qe^{-w\beta})$.

See Example 3.5 for restrictions of stab_w^- for all $w \in S_3$.

Example 3.2 Let us consider the case $G = \mathrm{SL}(2, \mathbb{C})$. We follow the notations in Example 2.3. The dual Cartan $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}\alpha$ is one dimensional. The alcoves are $\nabla_n := (\frac{n\alpha}{2}, \frac{(n+1)\alpha}{2})$, where $n \in \mathbb{Z}$. The stable basis are given by the following formulae. We refer the readers to [44, Example 1.4] for more details.

For the negative chamber, the cotangent polarization and the fundamental alcove ∇_0, we have

$$\mathrm{stab}_{id}^- = \mathrm{stab}_{id}^{\mathfrak{C}_-,T^*\mathbb{P}^1,\nabla_0} = [\mathcal{O}_{\mathbb{P}^1}] + qe^{\alpha}[\mathcal{O}_{T^*_\infty\mathbb{P}^1}], \quad \textit{and} \quad \mathrm{stab}_{s_\alpha}^- = \mathrm{stab}_{s_\alpha}^{\mathfrak{C}_-,T^*\mathbb{P}^1,\nabla_0} = -q^{\frac{1}{2}}e^{\alpha}[\mathcal{O}_{T^*_\infty\mathbb{P}^1}].$$

For a general alcove ∇_n,

$$\mathrm{stab}_w^{\mathfrak{C}_-,T^*\mathbb{P}^1,\nabla_n} = e^{-\frac{n}{2}w\alpha}\mathcal{L}_{\frac{n}{2}\alpha} \otimes \mathrm{stab}_w^-,$$

where $w = id$ or s_α.

For the opposite choices,

$$\mathrm{stab}_{id}^+ = [\mathcal{O}_{T_0^*\mathbb{P}^1}], \quad \textit{and} \quad \mathrm{stab}_{s_\alpha}^+ = -q^{-\frac{1}{2}}e^{-\alpha}[\mathcal{O}_{\mathbb{P}^1}] + \left(-q^{\frac{1}{2}}e^{-2\alpha} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})e^{-\alpha}\right)[\mathcal{O}_{T_0^*\mathbb{P}^1}].$$

3.3 The Action of the Affine Hecke Algebra

This section can be thought of as a K-theoretic generalization of the results in Sect. 2.3. Recall the affine Hecke algebra $\mathbb{H}$ is a free $\mathbb{Z}[q, q^{-1}]$ module with basis $\{T_w e^{\lambda} | w \in W, \lambda \in \Lambda\}$, such that

- For any $\lambda, \mu \in \Lambda$, $e^\lambda e^\mu = e^{\lambda+\mu}$.
- For any simple root α, $(T_{s_\alpha} + 1)(T_{s_\alpha} - q) = 0$.
- For any $w, y \in W$, such that $\ell(wy) = \ell(w) + \ell(y)$, $T_w T_y = T_{wy}$.
- For any simple root α and $\lambda \in \Lambda$,

$$T_{s_\alpha} e^\lambda - e^{s_\alpha \lambda} T_{s_\alpha} = (1-q)\frac{e^{s_\alpha \lambda} - e^\lambda}{1 - e^{-\alpha}}.$$

Let $H(W)$ denote the finite Hecke algebra, the subalgebra of $\mathbb{H}$ generated by the $T'_w s$. Recall the famous theorem of Kazhdan–Lusztig [27] and Ginzburg [21]

$$\mathbb{H} \simeq K_{G\times\mathbb{C}^*}(Z), \tag{5}$$

where $Z = T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}$ is the Steinberg variety, and the right hand side is endowed with the convolution algebra structure [21, Chap. 5]. The isomorphism is constructed as follows. We use the notations in Sect. 2.3. Let $\pi_i : T^*_{Y_\alpha} \to T^*\mathcal{B}$ be the two projections. Let $\mathcal{O}_\triangle$ be the structure sheaf of the diagonal $\triangle(T^*\mathcal{B}) \subset T^*\mathcal{B} \times T^*\mathcal{B}$. Then the isomorphism in (5) sends the simple generator $T_{s_\alpha} \in H(W)$ to $-[\mathcal{O}_\triangle] - \pi_2^*\mathcal{L}_\alpha \in K_{G\times\mathbb{C}^*}(Z)$ and $e^\lambda \in \mathbb{Z}[\Lambda]$ to $\mathcal{O}_\triangle(\lambda)$ [38, Proposition 6.1.5].

By convolution, $K_{G\times\mathbb{C}^*}(Z)$ acts on $K_T(T^*\mathcal{B})$ [21, Chap. 5]. Since the kernel defining T_α is not symmetric, the left and right convolution actions are different. Following [44], we use T_α (resp. T'_α) to denote the left (resp. right) convolution action of T_α. From [44, Lemma 3.4], these two operators are adjoint to each other:

$$\langle T_\alpha(\mathcal{F}), \mathcal{G}\rangle = \langle \mathcal{F}, T'_\alpha(\mathcal{G})\rangle, \quad \forall \mathcal{F}, \mathcal{G} \in K_T(T^*\mathcal{B}).$$

One of the main results in [44] is the computation of the affine Hecke algebra action on the stable bases. More precisely, we have

Theorem 3.3 ([44, Proposition 3.3, Theorem 3.5]) *Let α be a simple root. Then*

$$T_\alpha(\mathrm{stab}^-_w) = \begin{cases} (q-1)\,\mathrm{stab}^-_w + q^{1/2}\,\mathrm{stab}^-_{ws_\alpha}, & \text{if } ws_\alpha < w; \\ q^{1/2}\,\mathrm{stab}^-_{ws_\alpha}, & \text{if } ws_\alpha > w. \end{cases} \tag{6}$$

$$T'_\alpha(\mathrm{stab}^+_w) = \begin{cases} (q-1)\,\mathrm{stab}^+_w + q^{1/2}\,\mathrm{stab}^+_{ws_\alpha}, & \text{if } ws_\alpha < w; \\ q^{1/2}\,\mathrm{stab}^+_{ws_\alpha}, & \text{if } ws_\alpha > w. \end{cases} \tag{7}$$

In particular,

$$\mathrm{stab}^-_w = q^{\frac{\ell(w_0 w)}{2}} T^{-1}_{w_0 w}(\mathrm{stab}^-_{w_0}), \quad \textit{and} \quad \mathrm{stab}^+_w = q^{-\frac{\ell(w)}{2}} T'_{w^{-1}}(\mathrm{stab}^+_{id}).$$

In the proof of this theorem, an elementary but essential method called rigidity was used. Its simplest form says the following. If $p(z) \in \mathbb{C}[z^{\pm 1}]$, then

$$p(z) \text{ is bounded as } z^{\pm 1} \to \infty \quad \Longleftrightarrow \quad p(z) \text{ is a constant.}$$

More applications of this method can be found in the survey by Okounkov [34]. As a immediate corollary, we obtain recursive formulas for localizations of stable bases [44, Proposition 3.6]. This plays an important role in the identification between the stable basis and motivic Chern classes of Schubert cells (see Sect. 4), which was used to prove some conjectures of Bump, Nakasuji and Naruse in representations of p-adic Langlands dual groups in [4] (see Sect. 5).

3.4 Root Polynomials and Restriction Formulas

Since localization is such a powerful tool in calculations, it is important to study the localization of the stable basis. The cohomology case was done in [43] (see Sect. 2.4). For the K-theoretic case, we use the root polynomial method. The notations defining root polynomials are very technical, so here we just introduce a simplified version without explanation. Please see [44, Sects. 5, 6] for details. For simplicity, denote $S = \mathbb{Z}[q^{1/2}, q^{-1/2}][\Lambda]$ and $Q = \mathrm{Frac}(S)$. Let $Q_W^* = \mathrm{Hom}(W, Q)$, which is the set of all functions from the set W to the set Q. Q_W^* is a Q-module with basis f_w defined by $f_w(v) = \delta_{w,v}$, and it is also a commutative ring with product $f_w f_v = \delta_{w,v} f_v$. Indeed $Q_W^* \cong Q \otimes_S K_T(T^*\mathcal{B})$, and one can view $\mathrm{stab}_w^{\pm}$ inside Q_W^*. For example,

$$\mathrm{stab}_w^{\pm} = \sum_{v\in W} \mathrm{stab}_w^{\pm}|_v f_v, \quad \mathrm{stab}_w^{\pm}|_v \in S \subset Q.$$

We now introduce the root polynomials. For each $w = s_{i_1} \cdots s_{i_l}$, denote $\beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_j$, and define the root polynomials as

$$R_w = \prod_{j=1}^{l} h_{i_j}(\beta_j), \text{ where } h_i(\beta) = \tau_{s_i} - \frac{q-1}{x_{-\beta}} \text{ and } x_\beta = 1 - e^{-\beta}.$$

Here $\{\tau_w | w \in W\}$ are some formal symbols, generating the finite Hecke algebra $H(W)$, and we assume that they commute with the x-variables (so we can forget about the Bernstein relation). See Example 3.5 below. Because of that, when expressing $R_w = \sum_{v\le w} K_{v,w}\tau_v$, $K_{v,w} \in Q$, the coefficients $K_{v,w}$ can be computed easily. To relate with localization formula of the stable basis, we have

Theorem 3.4 ([44, Theorem 6.3, 6.5]) *For any $w \in W$, we have*

$$\mathrm{stab}_w^{-} = q^{\ell(w)/2} \sum_{v\ge w} \prod_{\alpha>0, v^{-1}\alpha<0} (1 - e^{\alpha}) \prod_{\alpha>0, v^{-1}\alpha>0} (1 - qe^{-\alpha}) K_{w,v} f_v \in K_T(T^*\mathcal{B}).$$

Note that similar formula can be obtained for stab_w^{+} by using the operators T_v', but it is a bit more complicated, and was not included in [44].

Example 3.5 We consider the case of SL_3, in which case there are two simple roots α_1, α_2 and $W = S_3$. Let $w = s_1 s_2$ with $\beta_1 = \alpha_1, \beta_2 = \alpha_1 + \alpha_2$. Then

$$\begin{aligned} R_w = h_1(\alpha_1)h_2(\alpha_1+\alpha_2) &= (\tau_{s_1} - \frac{q-1}{x_{-\alpha_1}})(\tau_{s_2} - \frac{q-1}{x_{-\alpha_1-\alpha_2}}) \\ &= \tau_{s_1 s_2} - \frac{q-1}{x_{-\alpha_1}}\tau_{s_2} - \frac{q-1}{x_{-\alpha_1-\alpha_2}}\tau_{s_1} + \frac{(q-1)^2}{x_{-\alpha_1}x_{-\alpha_1-\alpha_2}}. \end{aligned}$$

For instance, we have

$$\begin{aligned} \mathrm{stab}^-_{s_1}|_{s_1s_2} =& q^{\ell(s_1)/2} \prod_{\alpha>0, s_2s_1\alpha<0}(1-e^{\alpha}) \prod_{\alpha>0, s_2s_1\alpha>0}(1-qe^{-\alpha})K_{s_1,s_1s_2} \\ =& q^{1/2}(1-e^{\alpha_1})(1-e^{\alpha_1+\alpha_2})(1-qe^{-\alpha_2})(-\frac{q-1}{1-e^{\alpha_1+\alpha_2}}) \\ =& -q^{1/2}(q-1)(1-e^{\alpha_1})(1-qe^{-\alpha_2}). \end{aligned}$$

Similarly, we can get restriction formula for all stable basis with $w \in S_3$ (with $iji := s_is_js_i$, $x_\alpha = 1 - e^{-\alpha}$, $\hat{x}_\alpha = 1 - qe^{-\alpha}$, $x_{\pm i \pm j} = x_{\pm\alpha_i\pm\alpha_j}$, and $\hat{x}_{\pm i\pm j} = \hat{x}_{\pm\alpha_i\pm\alpha_j}$):

$$\begin{aligned} q^{-3/2}\,\mathrm{stab}^-_{121} &= x_{-1}x_{-2}x_{-1-2}f_{121}, \\ q^{-1}\,\mathrm{stab}^-_{12} &= x_{-1}x_{-1-2}\hat{x}_2 f_{12} - (q-1)x_{-1}x_{-1-2}f_{121}, \\ q^{-1}\,\mathrm{stab}^-_{21} &= x_{-2}x_{-1-2}\hat{x}_1 f_{21} - (q-1)x_{-2}x_{-1-2}f_{121}, \\ q^{-1/2}\,\mathrm{stab}^-_{1} &= \hat{x}_2\hat{x}_{1+2}x_{-1}f_1 - (q-1)\hat{x}_2x_{-1}f_{12} - (q-1)\hat{x}_1x_{-1-2}f_{21} + (q-1)^2x_{-1-2}f_{121}, \\ q^{-1/2}\,\mathrm{stab}^-_{2} &= \hat{x}_1\hat{x}_{1+2}x_{-2}f_2 - (q-1)\hat{x}_1x_{-2}f_{21} - (q-1)\hat{x}_2x_{-1-2}f_{12} + (q-1)^2x_{-1-2}f_{121}, \\ \mathrm{stab}^-_{e} &= \hat{x}_1\hat{x}_2\hat{x}_{1+2}f_e - (q-1)\hat{x}_2\hat{x}_{1+2}f_1 - (q-1)\hat{x}_1\hat{x}_{1+2}f_2 + (q-1)^2\hat{x}_2f_{12}, \\ &+ (q-1)^2\hat{x}_1f_{21} - [(q-1)^3 + q(q-1)x_{-1}x_{-2}]f_{121}. \end{aligned}$$

4 Motivic Chern Classes of the Schubert Cells

In this section, as another application of the relationship between the affine Hecke algebra and the stable bases, we relate the stable bases to motivic Chern classes for the Schubert cells [4, 22].

4.1 Definition of Motivic Chern Classes

The motivic Chern classes are K-theoretic generalizations of the Chern–Schwartz–MacPherson classes (see Sect. 2.5). Let us first recall the definition of the motivic Chern classes, following [4, 14, 22]. Recall A is the maximal torus in G. For any quasi-projective complex smooth A-variety X, let $G_0^A(var/X)$ be the free abelian group generated by the isomorphism classes of algebraic morphisms $[f : Z \to X]$ where Z is a quasi-projective A-variety and f is an A-equivariant morphism modulo the usual additivity relations

$$[f : Z \to X] = [f : U \to X] + [f : Z \setminus U \to X]$$

for $U \subset Z$ an open A-invariant subvariety. Then there exists a unique natural transformation

$$MC_y : G_0^A(var/X) \to K_A(X)[y],$$

satisfying the following properties:

- It is functorial with respect to A-equivariant proper morphisms of smooth, quasi-projective varieties.
- It satisfies the following normalization condition

$$MC_y([id_X : X \to X]) = \lambda_y(T^*X) := \sum y^i[\wedge^i T^*X] \in K_A(X)[y].$$

Here y is a formal variable. The existence in the non-equivariant case was proved in [14]. The equivariant case is established in [4, 22].

4.2 Motivic Chern Classes of the Schubert Cells

Recall $X(w)^\circ = BwB/B \subset \mathcal{B}$ and $Y(w)^\circ = B^- wB/B \subset \mathcal{B}$ are the Schubert cells in $\mathcal{B}$, where B^- is the opposite Borel group. Recall the Serre duality functor on X

$$\mathcal{D}(\mathcal{F}) = R\mathcal{H}om(\mathcal{F}, \omega_{\mathcal{B}}^\bullet),$$

where $\mathcal{F} \in K_A(\mathcal{B})$ and $\omega_{\mathcal{B}}^\bullet = \mathcal{L}_{2\rho}[\dim \mathcal{B}]$ is the dualizing complex of $\mathcal{B}$. We extend it to $K_A(\mathcal{B})[y, y^{-1}]$ by sending y to y^{-1}. Let $i : \mathcal{B} \hookrightarrow T^*\mathcal{B}$ be the inclusion. Then the relation between the motivic Chern classes and the stable bases is following, which is a K-theoretic generalization of Theorem 2.7.

Theorem 4.1 ([4]) *For any $w \in W$, we have*

$$q^{-\frac{\ell(w)}{2}}\mathcal{D}(i^* \operatorname{stab}_w^+) = MC_{-q^{-1}}([X(w)^\circ \hookrightarrow \mathcal{B}]) \in K_A(\mathcal{B})[q, q^{-1}],$$

and

$$q^{\frac{\ell(w)}{2} - \dim \mathcal{B}} i^*(\operatorname{stab}_w^-) \otimes [\omega_X^\bullet] = MC_{-q^{-1}}([Y(w)^\circ \hookrightarrow \mathcal{B}]) \in K_A(\mathcal{B})[q, q^{-1}].$$

Example 4.2 Let $G = \mathrm{SL}(2, \mathbb{C})$. We follow the notations in Example 2.3. By definition,

$$MC_y([Y(s_\alpha)^\circ \hookrightarrow \mathbb{P}^1]) = [\mathcal{O}_\infty],$$

and

$$MC_y([Y(id)^\circ \hookrightarrow \mathbb{P}^1]) = MC_y([id_{\mathcal{B}} : \mathbb{P}^1 \to \mathbb{P}^1]) - MC_y([Y(s_\alpha)^\circ \hookrightarrow \mathbb{P}^1]) = 1 + y[T^*\mathbb{P}^1] - [\mathcal{O}_\infty].$$

Let us check the second equation in Theorem 4.1 for $w = id$. It suffices to check the equality after restricting to the fixed points. From Example 3.2, we have

$$\mathrm{stab}^-_{id} = [\mathcal{O}_{\mathbb{P}^1}] + qe^\alpha[\mathcal{O}_{T^*_\infty\mathbb{P}^1}].$$

Therefore,

$$\left(q^{-1}i^*(\mathrm{stab}^-_{id}) \otimes [\omega^\bullet_X]\right)|_0 = q^{-1}(1 - qe^{-\alpha})(-e^\alpha) = 1 - q^{-1}e^\alpha,$$

and

$$\left(q^{-1}i^*(\mathrm{stab}^-_{id}) \otimes [\omega^\bullet_X]\right)|_\infty = q^{-1}(1 - qe^\alpha + qe^\alpha(1 - e^{-\alpha}))(-e^{-\alpha}) = (1 - q^{-1})e^{-\alpha}.$$

On the other hand,

$$MC_{-q^{-1}}([Y(id)^\circ \hookrightarrow \mathbb{P}^1])|_0 = 1 - q^{-1}e^\alpha,$$

and

$$MC_{-q^{-1}}([Y(id)^\circ \hookrightarrow \mathbb{P}^1])|_\infty = 1 - q^{-1}e^{-\alpha} - (1 - e^{-\alpha}) = (1 - q^{-1})e^{-\alpha}.$$

Thus, the second equation in Theorem 4.1 holds for $w = id$. The other cases can be checked similarly.

It will be proved in [5] that

$$(-q)^{-\dim \mathcal{B}} i^*(gr(i_{w!}\mathbb{Q}^H_{Y(w)^\circ})) \otimes [\omega^\bullet_X] = MC_{-q^{-1}}([Y(w)^\circ \hookrightarrow X]),$$

where $i_w : Y(w)^\circ \hookrightarrow \mathcal{B}$ is the inclusion, $\mathbb{Q}^H_{Y(w)^\circ}$ is the A-equivariant shifted mixed Hodge module on $Y(w)^\circ$, and $gr(i_{w!}\mathbb{Q}^H_{Y(w)^\circ})$ is the associated $A \times \mathbb{C}^*$-equivariant sheaf on the cotangent bundle of $\mathcal{B}$ [46, Sect. 2]. Since i^* is an isomorphism, we get

$$q^{\frac{\ell(w)}{2}} \mathrm{stab}^-_w = (-1)^{\dim \mathcal{B}} gr(i_{w!}\mathbb{Q}^H_{Y(w)^\circ}).$$

This is the K-theoretic analogue of Theorem 2.9.

5 Unramified Principal Series of p-adic Langlands Dual Group

The relation between stable basis and motivic Chern classes is used in [4] to prove some conjectures of Bump, Nakasuji and Naruse [17, 32] about unramified principal series of the p-adic Langlands dual group. In the proof, we need to first build a relation between the stable basis (or motivic Chern classes of Schubert cells) and unramified principal series. This is explained in this section.

5.1 The Two Bases in Iwahori-Invariants of an Unramified Principal Series Representation

Let F be a finite extension of $\mathbb{Q}_p$.[1] Let $\mathcal{O}_F$ be the ring of integers, with a uniformizer $\varpi \in \mathcal{O}_F$, and residue field $\mathbb{F}_q$. Let $\check{G}$ be the split reductive group over F, which is the Langlands dual group of G. Let $\check{A} \subset \check{B}$ be the corresponding dual maximal torus and Borel subgorup. Let I be the Iwahori subgroup.

Let τ be an unramified character of $\check{A}$, i.e., complex characters of $\check{A}(F)$ which are trivial on $\check{A}(\mathcal{O}_F)$. Recall the principal series representation $I(\tau) := \mathrm{Ind}_{\check{B}(F)}^{\check{G}(F)}(\tau)$ is the induced representation, which consists of locally constant functions f on $\check{G}(F)$ such that $f(bg) = \tau(b)\delta^{1/2}(b)f(g), b \in \check{B}(F)$, where $\delta(b) = \prod_{\alpha>0} |\alpha^\vee(b)|_F$ is the modulus function on the Borel subgroup.

It is well known that the affine Hecke algebra in Sect. 3.3 can also be realized as

$$\mathbb{H} = \mathbb{C}_c[I\backslash\check{G}(F)/I],$$

which is the so-called Iwahori–Hecke algebra. The algebra structure on the latter space is induced by convolution. Again by convolution, the Iwahori–Hecke algebra $\mathbb{H}$ acts from the right on the Iwahori-invariants $I(\tau)^I$.

We say an unramified character τ is regular if the stabilizer W_τ in the Weyl group is trivial. In this case, the space $\mathrm{Hom}_{\check{G}(F)}(I(\tau), I(w^{-1}\tau))$ is one-dimensional. It is spanned by the following intertwining operator $\mathcal{A}_w^\tau$:

$$\mathcal{A}_w^\tau(f)(g) := \int_{\check{N}_w} f(wng)dn,$$

where $\check{N}_w = \check{N}(F) \cap w^{-1}\check{N}^-(F)w$ with $\check{N}$ (resp. $\check{N}^-$) being the unipotent radical of the Borel subgroup $\check{B}$ (resp. $\check{B}^-$).

There are two bases of $I(\tau)^I$. The first basis $\{\varphi_w^\tau | w \in W\}$, which is called the standard basis, is induced by the following decomposition

[1] The result also holds for $F = F_q((t))$.

$$\check{G}(F) = \sqcup_{w \in W} \check{B}(F) w I.$$

That is, φ_w^τ is characterized by the following conditions:

(1) φ_w^τ is supported on $\check{B}(F)wI$;
(2) $\varphi_w^\tau(bwg) = \tau(b)\delta^{1/2}(b)$ for any $b \in \check{B}(F)$, $g \in I$.

The other basis $\{f_w^\tau | w \in W\}$, which is the so-called Casselman's basis [18], is characterized by

$$\mathcal{A}_v^\tau(f_w^\tau)(1) = \delta_{v,w}.$$

These bases played an important role in the computation of the Macdonald spherical function [18] and the Casselman–Shalika formula for the spherical Whittaker function [19].

5.2 *The Comparison*

Since τ is an unramified character of $\check{A}$, it is equivalent to a point $\tau \in A$. Therefore, we can evaluate the base ring $K_A(pt)$ at $\tau \in A$. We let $\mathbb{C}_\tau$ denote this evaluation representation of $K_T(pt) = K_{A\times\mathbb{C}^*}(pt)$. For any $\mathcal{F} \in K_{A\times\mathbb{C}^*}(T^*\mathcal{B})$, let $\mathcal{F}_{-\rho}$ denote $\mathcal{F} \otimes \mathcal{L}(-\rho)$.

Now we can state the relation between these bases. The following theorem is a shadow of the two geometric realizations of the affine Hecke algebra

$$K_{G\times\mathbb{C}^*}(Z) \simeq \mathbb{H} \simeq \mathbb{C}_c[I\backslash\check{G}(F)/I].$$

We refer the readers to [44, Sects. 8.2 and 8.3] for a more precise statement of the following theorem.

Theorem 5.1 ([44, Theorem 8.4]) *There is a unique isomorphism of right* $\mathbb{H}$*-modules*

$$\Psi : K_{A\times\mathbb{C}^*}(T^*\mathcal{B}) \otimes_{K_{A\times\mathbb{C}^*}(\mathrm{pt})} \mathbb{C}_\tau \to I(\tau)^I,$$

such that the equivariant parameter q is mapped to the cardinality of the residue field $\mathbb{F}_q$ and

$$\begin{aligned} \frac{q^{\ell(w)}}{\prod_{\beta>0,w\beta>0}(1-e^{w\beta})\prod_{\beta>0,w\beta<0}(q-e^{w\beta})}(\iota_{w*}1)_{-\rho} &\mapsto f_w^\tau, \\ (\mathrm{stab}_w^-)_{-\rho} &\mapsto q^{-\ell(w)/2}\varphi_w^\tau. \end{aligned}$$

As applications of this theorem, in [44], we showed that the left Weyl group action on the K-theory side corresponds to the suitably normalized intertwiners on the representation theory side [44, Corollary 8.6], and provided some K-theoretic interpretations of Macdonald's formula [18] [44, Theorem 8.8] and the Casselman-Shalika formula [19] [44, Theorem 8.11].

It is an interesting question to study the transition matrix between the two bases in $I(\tau)^I$. Define the following transition matrix coefficients $m_{u,w}$ by

$$\sum_{w \geq u} \varphi_w = \sum_{w \in W} m_{u,w} f_w.$$

For instance, the following special case

$$m_{id,w} = \prod_{\alpha>0, w^{-1}\alpha<0} \frac{1 - q^{-1}e^{\alpha}(\tau)}{1 - e^{\alpha}(\tau)}$$

is the Gindikin-Karpelevich formula.

The authors of [4] used Theorems 5.1, 4.1 and some functorial properties for the motivic Chern classes to prove the following conjectures of Bump, Nakasuji and Naruse [17, 32].

Theorem 5.2 ([4]) *For any $u \leq w \in W$,*

(1)

$$m_{u,w} = \prod_{\alpha>0, u \leq s_\alpha w < w} \frac{1 - q^{-1}e^{\alpha}(\tau)}{1 - e^{\alpha}(\tau)},$$

if and only if the Schubert variety $Y(u)$ is smooth at the torus fixed point e_w.

(2) As a rational function of $\tau \in A$, the product

$$\prod_{\alpha>0, u \leq s_\alpha w < w} (1 - e^{\alpha}) m_{u,w}$$

has no poles on the maximal torus A.

6 Wall Crossings for the Stable Bases

Note that the definition of the stable bases depend on the alcoves in $\mathfrak{h}^*_{\mathbb{R}}$. Stable bases for different alcoves are related by the so-called *wall R-matrices*[37]. In this section, we study the wall R-matrices for the Springer resolution, which will be used for the categorification in the next section. The main reference for these two sections is [45].

6.1 Wall Crossing Matrix

By uniqueness of the stable basis, it is immediate to see that for any $\mu \in \Lambda$,

$$\operatorname{stab}_y^{\mathfrak{C},T^{1/2},\nabla+\mu} = e^{-y\mu}\mathcal{L}_\mu \otimes \operatorname{stab}_y^{\mathfrak{C},T^{1/2},\nabla}. \tag{8}$$

Because of this property, instead of crossing all the walls on $H_{\alpha^\vee,n}$, it is enough to just cross the walls on the 0 hyperplanes $H_{\alpha^\vee,0}$. From now on, let ∇_1, ∇_2 be two alcoves sharing a wall on $H_{\alpha^\vee,0}$, and $(\lambda_1, \alpha^\vee) > 0$ for any $\lambda_1 \in \nabla_1$.

Another useful fact is [37, Theorem 1]:

$$\operatorname{stab}_y^{\mathfrak{C},T^{1/2},\nabla_2} = \begin{cases} \operatorname{stab}_y^{\mathfrak{C},T^{1/2},\nabla_1} + f_y^{\nabla_2\leftarrow\nabla_1}\operatorname{stab}_{ys_\alpha}^{\mathfrak{C},T^{1/2},\nabla_1}, & \text{if } ys_\alpha \prec_{\mathfrak{C}} y; \\ \operatorname{stab}_y^{\mathfrak{C},T^{1/2},\nabla_1}, & \text{if } ys_\alpha \succ_{\mathfrak{C}} y, \end{cases}$$

where $f_y^{\nabla_2\leftarrow\nabla_1} \in K_T(\mathrm{pt})$.

Then we have

Theorem 6.1 ([45]) *For any $y \in W$, we have*

$$\operatorname{stab}_y^{\mathfrak{C},T\mathcal{B},\nabla_2} = \begin{cases} \operatorname{stab}_y^{\mathfrak{C},T\mathcal{B},\nabla_1} + (q^{1/2} - q^{-1/2})\operatorname{stab}_y^{\mathfrak{C},T\mathcal{B},\nabla_1}, & \text{if } ys_\alpha \prec_{\mathfrak{C}} y; \\ \operatorname{stab}_y^{\mathfrak{C},T\mathcal{B},\nabla_1}, & \text{if } ys_\alpha \succ_{\mathfrak{C}} y. \end{cases}$$

$$\operatorname{stab}_y^{-\mathfrak{C},T^*\mathcal{B},\nabla_2} = \begin{cases} \operatorname{stab}_y^{-\mathfrak{C},T^*\mathcal{B},\nabla_1} + (q^{1/2} - q^{-1/2})\operatorname{stab}_y^{-\mathfrak{C},T^*\mathcal{B},\nabla_1}, & \text{if } ys_\alpha \succ_{\mathfrak{C}} y; \\ \operatorname{stab}_y^{-\mathfrak{C},T^*\mathcal{B},\nabla_1}, & \text{if } ys_\alpha \prec_{\mathfrak{C}} y. \end{cases}$$

This theorem is firstly proved when α is a simple root, by computing the action of T_α and T'_α on the stable basis $\operatorname{stab}_y^{\mathfrak{C},T\mathcal{B},\nabla}$ and $\operatorname{stab}_y^{-\mathfrak{C},T^*\mathcal{B},\nabla}$ for any alcove ∇. That is, we use rigidity to obtain a general version of Theorem 3.3. For the non-simple root case, we use the following equality [4] to reduce to the simple root case

$$w(\operatorname{stab}_y^{\mathfrak{C},T\mathcal{B},\nabla}) = \operatorname{stab}_{wy}^{w\mathfrak{C},T\mathcal{B},\nabla}, \tag{9}$$

where the LHS of the equation above denotes the left action of $w \in W$ on $\operatorname{stab}_y^{\mathfrak{C},T\mathcal{B},\nabla} \in K_T(T^*\mathcal{B})$.

6.2 Wall Crossing and Affine Hecke Algebra Actions

Combining Theorems 3.3 and 6.1, we get the following general formulae, which show that the wall crossing matrices and the affine Hecke algebra action (see Sect. 3) are compatible.

Theorem 6.2 ([45]) *For any $x, y \in W$, we have*

$$\mathrm{stab}_y^{\mathfrak{C}_-, T^*\mathcal{B}, x\nabla_+} = \quad q_x^{-1/2} T_x (\mathrm{stab}_{yx}^{\mathfrak{C}_-, T^*\mathcal{B}, \nabla_+}); \tag{10}$$
$$\mathrm{stab}_y^{\mathfrak{C}_+, T\mathcal{B}, x\nabla_-} = q_x^{1/2} (T'_{x^{-1}})^{-1} (\mathrm{stab}_{yx}^{\mathfrak{C}_+, T\mathcal{B}, \nabla_-}). \tag{11}$$

Therefore, Theorem 3.3, Eqs. (8) and (11) determine all the stable basis $\mathrm{stab}_y^{\mathfrak{C}_+, T\mathcal{B}, \nabla}$ for the dominant Weyl chamber $\mathfrak{C}_+$. The stable basis for the other chambers can be computed by Eq. (9). Thus all the stable basis element $\mathrm{stab}_y^{\mathfrak{C}, T\mathcal{B}, \nabla}$ can be calculated.

For general symplectic resolutions, Bezrukavnikov and Okounkov [11, 35] conjecture that the representation coming from the derived equivalence is isomorphic to the monodromy representation coming from quantum cohomology. The monodromy matrices of the quantum connection of $T^*\mathcal{B}$ is computed in [16, 20]. The above theorem shows that the monodromy matrices coincide with the wall R-matrices for the K stable bases. The relation to derived equivalences is explained in the next section.

7 Categorification and Localization in Positive Characteristic

In this final section, we give a categorification of the stable bases, and study its relation with representation of $\mathfrak{g}$ over positive characteristic fields under the localization equivalence of Bezrukavnikov, Mirković and Rumynin [9, 10].

7.1 Categorification of the Stable Basis via Affine Braid Group Action

Let B_{aff} (resp. B'_{aff}) be the affine braid group (resp. the extended affine braid group) with generators $\widetilde{s}_\alpha$, $\alpha \in I_{\mathrm{aff}}$.

Let $G_\mathbb{Z}$ be a split $\mathbb{Z}$-form of the complex algebraic group G and $A_\mathbb{Z} \subset B_\mathbb{Z} \subset G_\mathbb{Z}$ be the maximal torus and a Borel subgroup, respectively. Let $T_\mathbb{Z} = A_\mathbb{Z} \times_\mathbb{Z} (\mathbb{G}_m)_\mathbb{Z}$. Bezrukavnikov and Riche constructed an extended affine braid group action on $D^{\mathrm{b}}_{T_\mathbb{Z}}(T^*\mathcal{B}_\mathbb{Z}) := D^{\mathrm{b}}\,\mathrm{Coh}^{T_\mathbb{Z}}(T^*\mathcal{B}_\mathbb{Z})$ [12, 38], denoted by $J^{\mathrm{R}}_{\widetilde{w}}$, $\widetilde{w} \in B'_{\mathrm{aff}}$ (here R denotes the right action). Inspired by Theorem 6.2, we give the following definition.

Definition 7.1 Let $\lambda \in \Lambda_\mathbb{Q}$ be regular. We define $\mathfrak{stab}^\mathbb{Z}_\lambda(y) \in D^{\mathrm{b}}_{T_\mathbb{Z}}(T^*\mathcal{B}_\mathbb{Z})$, $y \in W$ as follows:

$$\mathfrak{stab}^\mathbb{Z}_{\lambda_0}(id) = \mathcal{L}_{-\rho} \otimes \mathcal{O}_{T^*_{id}\mathcal{B}_\mathbb{Z}}, \quad \lambda_0 \in \nabla_-,$$
$$\mathfrak{stab}^\mathbb{Z}_{\lambda_0}(y) = J^{\mathrm{R}}_{\widetilde{y}} \mathfrak{stab}^\mathbb{Z}_{\lambda_0}(id), \quad \lambda_0 \in \nabla_-,$$
$$\mathfrak{stab}^\mathbb{Z}_{\lambda}(y) = (J^{\mathrm{R}}_{\widetilde{x}})^{-1} \mathfrak{stab}^\mathbb{Z}_{\lambda_0}(yx), \quad y, x \in W, x\lambda_0 = \lambda, \lambda_0 \in \nabla_-,$$
$$\mathfrak{stab}^\mathbb{Z}_{\lambda}(y) = e^{-y\mu} J^{\mathrm{R}}_{\mu} \mathfrak{stab}^\mathbb{Z}_{\lambda_1}(y), \quad y \in W, \mu + \lambda_1 = \lambda, \mu \in Q, \lambda_1 \in W\nabla_-.$$

As an immediate corollary of Theorems 3.3, 6.2, Eq. (8) and [12, Theorem 1.3.1, Proposition 1.4.3, Theorem 1.6.1], we have the following theorem, which gives a categorification of the stable basis.

Theorem 7.2 *[45]Applying the derived tensor* $-\otimes^L_{\mathbb{Z}}\mathbb{C}$ *to* $\mathfrak{stab}^{\mathbb{Z}}_\lambda(w)\in D^{\mathrm{b}}_{T_{\mathbb{Z}}}(T^*\mathcal{B}_{\mathbb{Z}})$*, and taking the class in the Grothendieck group, we get* $\mathcal{L}_{-\rho}\otimes \mathrm{stab}^{\mathfrak{C}_+,T\mathcal{B},\nabla}_w\in K_T(T^*\mathcal{B})$*, where* ∇ *is the alcove containing* λ.

7.2 Verma Modules in Positive Characteristic

In this section, we briefly sketch the relation between the K-theory stable bases and Verma modules for Lie algebras over positive characteristic fields. We refer the readers to [45, Sect. 9] for the details.

We consider the level-p configuration of ρ-shifted affine hyperplanes, that is, $H^p_{\alpha^\vee,n}=\{\mu\in\Lambda_{\mathbb{Q}}|\langle\alpha^\vee,\mu+\rho\rangle=np\}$. Let A_0 be the fundamental alcove, i.e., it contains $(\epsilon-1)\rho$ for small $\epsilon>0$. Let W act on Λ via the level-p dot action, $w:\mu\mapsto w\bullet\mu=w(\mu+\rho)-\rho$.

Let k be an algebraically closed field of characteristic p, and p is greater than the Coxeter number. For any k-variety X, let $X^{(1)}$ be the Frobenius twist. Let $U(\mathfrak{g}_k)$ be the universal enveloping algebra of $\mathfrak{g}_k$ with the Frobenius center $\mathcal{O}(\mathfrak{g}_k^{*(1)})$ and the Harish-Chandra center $\mathcal{O}(\mathfrak{h}_k^*/(W,\bullet))$. Let $\lambda\in\mathfrak{h}_k^*$ be regular (i.e., does not lie on any hyperplane $H^p_{\alpha^\vee,n}$) and integral (i.e., belongs to the image of the derivative $d:\Lambda\to\mathfrak{h}_k^*$). Let $U(\mathfrak{g}_k)^\lambda$ be the quotient of $U(\mathfrak{g}_k)$ by the central ideal corresponding to $W\bullet\lambda\in\mathfrak{h}_k^*/(W,\bullet)$. Let $\mathrm{Mod}_\chi(U(\mathfrak{g}_k)^\lambda)$ be the category of finitely generated $U(\mathfrak{g}_k)^\lambda$-modules on which the Frobenius center $\mathcal{O}(\mathfrak{g}_k^{*(1)})$ acts by the generalized character $\chi\in\mathfrak{g}_k^{*(1)}$.

Let $\mathcal{D}^\lambda$ be the ring of $\mathcal{L}_\lambda$-twisted differential operators on $\mathcal{B}$, and $D^{\mathrm{b}}(\mathrm{Coh}_\chi\mathcal{D}^\lambda)$ be the full subcategory of coherent $\mathcal{D}^\lambda$-modules that are set-theoretically supported on $\mathcal{B}^{(1)}_\chi$ as a coherent sheaf on $T^*\mathcal{B}^{(1)}_k$, where $\mathcal{B}^{(1)}_\chi$ is the Springer fiber. Then the global section functor $R\Gamma_{\mathcal{D}^\lambda,\chi}:D^{\mathrm{b}}(\mathrm{Coh}_\chi\mathcal{D}^\lambda)\to D^{\mathrm{b}}(\mathrm{Mod}_\chi U(\mathfrak{g}_k)^\lambda)$ is an equivalence according to [10, Theorem 3.2], whose inverse is denoted by $\mathfrak{L}^{\lambda_0}$. Let $T^*\mathcal{B}^{(1)\wedge}_\chi$ be the completion of $T^*\mathcal{B}^{(1)}$ at $\mathcal{B}^{(1)}_\chi$. Then for all integral $\lambda\in\mathfrak{h}^*$, the Azumaya algebra $\mathcal{D}^\lambda$ splits on $T^*\mathcal{B}^{(1)\wedge}_\chi$ [10, Theorem 5.1.1]. In particular, there is a Morita equivalence

$$\mathrm{Coh}_{\mathcal{B}^{(1)}_\chi}(T^*\mathcal{B}^{(1)})\cong\mathrm{Coh}_\chi\mathcal{D}^\lambda.$$

The equivalence above depends on the choice of a splitting bundle of $\mathcal{D}^\lambda$ [10, Remark 5.2.2]. As has been done in [8, 12], for $\lambda_0\in A_0$, we normalize the choice of the splitting bundle by [9, Remark 1.3.5]. In this section, we denote this splitting bundle by $\mathcal{E}^s$. That is, $\mathcal{E}nd_{T^*\mathcal{B}^{(1)\wedge}_\chi}(\mathcal{E}^s)\cong\mathcal{D}^{\lambda_0}|_{T^*\mathcal{B}^{(1)\wedge}_\chi}$. This bundle fixes the equivalence $\mathrm{Coh}_{\mathcal{B}^{(1)}_\chi}(T^*\mathcal{B}^{(1)})\cong\mathrm{Coh}_\chi\mathcal{D}^{\lambda_0}$. Composing this equivalence with $\mathfrak{L}^{\lambda_0}$, we have

$$\gamma_\chi^{\lambda_0} : D^{\mathrm{b}} \operatorname{Mod}_\chi(U(\mathfrak{g}_k)^{\lambda_0}) \cong D^{\mathrm{b}} \operatorname{Coh}_{\mathcal{B}_\chi^{(1)}}(T^*\mathcal{B}^{(1)}), \tag{12}$$

which is also referred to as the localization equivalence.

Let $U(\mathfrak{g}_k)_\chi^{\lambda_0}$ be the completion of $U(\mathfrak{g}_k)^{\lambda_0}$ at the central character $\chi \in \mathcal{N}_k^{(1)}$. Then, $End_{T^*\mathcal{B}_\chi^{(1)\wedge}}(\mathcal{E}^s) \cong U(\mathfrak{g}_k)_\chi^{\lambda_0}$. The localization functor $\gamma_\chi^{\lambda_0}$ can then be written as $\otimes_{U(\mathfrak{g}_k)_\chi^{\lambda_0}} \mathcal{E}^s$. We also have the completed version of the equivalence [12, Remark 2.5.5]

$$\gamma_\chi^{\lambda_0} : D^{\mathrm{b}} \operatorname{Mod}(U(\mathfrak{g}_k)_\chi^{\lambda_0}) \cong D^{\mathrm{b}} \operatorname{Coh}(T^*\mathcal{B}_\chi^{(1)\wedge}).$$

From now on we consider the case $\chi = 0$. In this case, we abbreviate $\gamma_\chi^{\lambda_0}$ simply as γ^{λ_0}, and the twisted Springer fiber $\mathcal{B}_\chi^{(1)}$ is the zero section $\mathcal{B}^{(1)} \subseteq T^*\mathcal{B}^{(1)}$.

The bundle $\mathcal{E}^s$ on $T^*\mathcal{B}_0^{(1)\wedge}$ has a natural T_k-equivariant structure [8, § 5.2.4], where $T_k = A_k \times (\mathbb{G}_m)_k$. Taking the ring of endomorphisms, we get a T_k-action on $U(\mathfrak{g}_k)_0^{\lambda_0}$ compatible with that on $\mathcal{N}^{(1)}$. In particular, the $(\mathbb{G}_m)_k \subseteq T_k$-action provides a non-negative grading on $U(\mathfrak{g}_k)_0^{\lambda_0}$, referred to as the Koszul grading. The localization equivalences above can be made into equivalences of equivariant categories. Let $\operatorname{Mod}_0^{\mathrm{gr}}(U(\mathfrak{g}_k)^{\lambda_0}, A_k)$ be the category of finite-dimensional Koszul-graded modules of $U(\mathfrak{g}_k)_0^{\lambda_0}$, which are endowed with compatible actions of the subgroup $A_k \subseteq G_k$. The compatibility is in the sense that the action of A_k differentiates to the action of the subalgebra $\mathfrak{h}_k \subseteq \mathfrak{g}_k$. Then, we have [8, Theorem 1.6.7]

$$\gamma^{\lambda_0} : D^{\mathrm{b}} \operatorname{Mod}_0^{\mathrm{gr}}(U(\mathfrak{g}_k)^{\lambda_0}, A_k) \cong D^{\mathrm{b}}_{T_k} \operatorname{Coh}_{\mathcal{B}_k^{(1)}}(T^*\mathcal{B}_k^{(1)}),$$

and the completed version

$$\gamma^{\lambda_0} : D^{\mathrm{b}} \operatorname{Mod}^{\mathrm{gr}}(U(\mathfrak{g}_k)_0^{\lambda_0}, A_k) \cong D^{\mathrm{b}}_{T_k} \operatorname{Coh}(T^*\mathcal{B}_0^{(1)\wedge}).$$

By the functor of taking finite vectors, we get [8, Theorem 5.1.1]

$$\gamma^{\lambda_0} : D^{\mathrm{b}} \operatorname{Mod}^{\mathrm{gr}}(\mathcal{A}^{\lambda_0}, A_k) \cong D^{\mathrm{b}}_{T_k} \operatorname{Coh}(T^*\mathcal{B}_k^{(1)}).$$

Here $\mathcal{A}^{\lambda_0}$ is an $\mathcal{O}_{\mathcal{N}^{(1)}}$-algebra endowed with a compatible $A_k \times (\mathbb{G}_m)_k$-action, and it has the property that $(\mathcal{A}^{\lambda_0})_0^\wedge \cong U(\mathfrak{g}_k)_0^{\lambda_0}$. Using the correspondence given by taking completion and taking finite vectors, we will freely pass between $\operatorname{Mod}^{\mathrm{gr}}(U(\mathfrak{g}_k)_0^{\lambda_0}, A_k)$ and $\operatorname{Mod}^{\mathrm{gr}}(\mathcal{A}^{\lambda_0}, A_k)$; similarly for $D^{\mathrm{b}}_{T_k} \operatorname{Coh}(T^*\mathcal{B}_0^{(1)\wedge})$ and $D^{\mathrm{b}}_{T_k} \operatorname{Coh}(T^*\mathcal{B}_k^{(1)})$.

For any λ in the W'_{aff}-orbit of λ_0, we can define the localization functor

$$\gamma^{\lambda} : D^{\mathrm{b}} \operatorname{Mod}^{\mathrm{gr}}(\mathcal{A}^{\lambda_0}, A_k) \cong D^{\mathrm{b}}_{T_k} \operatorname{Coh}(T^*\mathcal{B}_k^{(1)}).$$

by precomposing with the affine braid group action functors [45, Sect. 9.3].

For the Lie algebra $\mathfrak{b}_k$, recall the Verma module $Z^{\mathfrak{b}}(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$. Then we have

Theorem 7.3 ([45]) *Let k be an algebraically closed field of characteristic p, and p is greater than the Coxeter number. Assume λ to be regular and integral, then in $D^{\mathrm{b}}_{T_k}(T^*\mathcal{B}_k^{(1)})$, we have isomorphisms*

$$e^{\rho}\mathfrak{stab}^k_{-\frac{\lambda+\rho}{p}}(y) \cong \gamma^{\lambda} Z^{\mathfrak{b}}(y \bullet \lambda + 2\rho),$$

where $\mathfrak{stab}^k_{-\frac{\lambda+\rho}{p}}(y) = \mathfrak{stab}^{\mathbb{Z}}_{-\frac{\lambda+\rho}{p}}(y) \otimes^L_{\mathbb{Z}} k$.

To prove this theorem, one needs to use the affine braid group action on $\mathrm{Mod}_{\chi}\, U(\mathfrak{g}_k)^{\lambda}$ constructed in [9], which iteratively produces all the Verma modules [25], and also its compatibility with the localization equivalences γ^{λ}. Together with the iterative definition of $\mathfrak{stab}^k_{\mathbb{Z}}(y)$, the theorem follows.

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References

1. Aganagic, M., Okounkov, A.: Elliptic stable envelope. J Am Math Soc. arXiv:1604.00423
2. Aluffi, P., Mihalcea, L.: Chern-Schwartz-MacPherson classes for Schubert cells in flag manifolds. Compos. Math. **152**(12), 2603–2625 (2016)
3. Aluffi, P., Mihalcea, L., Schürmann, J., Su, C.: Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells. arXiv:1709.08697
4. Aluffi, P., Mihalcea, L., Schürmann J., Su, C.: Motivic Chern classes of Schubert cells with applications to Casselman's problem. arXiv:1902.10101
5. Aluffi, P., Mihalcea, L., Schürmann, J., Su, C.: Equvivariant Motivic Chern classes via cotangent bundles (in preparation)
6. Beilinson, A., Bernstein, J.: Localisation de g-modules. C. R. Acad. Sci. Paris Sér. I Math. **292**(1), 15–18 (1981)
7. Bezrukavnikov, R.: On two geometric realizations of an affine Hecke algebra. Publications mathématiques de l'IHÉS **123**(1), 1–67 (2016)
8. Bezrukavnikov, R., Mirković, I.: Representations of semisimple Lie algebras in prime characteristic and the noncommutative Springer resolution. With an appendix by Eric Sommers. Ann. Math. (2) **178**(3), 835–919 (2013)
9. Bezrukavnikov, R., Mirković, I., Rumynin, D.: Singular localization and intertwining functors for reductive lie algebras in prime characteristic. Nagoya J. Math. **184**, 1–55 (2006)
10. Bezrukavnikov, R., Mirković, I., Rumynin, D.: Localization of modules for a semisimple Lie algebra in prime characteristic, With an appendix by Bezrukavnikov and Simon Riche. Ann. Math. (2) **167**(3), 945–991 (2008)
11. Bezrukavnikov, R., Okounkov, A.: (In preparation)
12. Bezrukavnikov, R., Riche, S.: Affine braid group actions on derived categories of Springer resolutions. Ann. Sci. Éc. Norm. Supér. (4) **45**(4), 535–599 (2012) (2013)
13. Billey, S.: Kostant polynomials and the cohomology ring for G/B. Duke Math. J. **96**(1), 205–224 (1999)

14. Brasselet, J., Schürmann, J., Yokura, S.: Hirzebruch classes and motivic Chern classes for singular spaces. J. Topol. Anal. **2**(01), 1–55 (2010)
15. Brylinski, J., Kashiwara, M.: Kazhdan-Lusztig conjecture and holonomic systems. Inventi. Math. **64**(3), 387–410 (1981)
16. Braverman, A., Maulik, D., Okounkov, A.: Quantum cohomology of the Springer resolution. Adv. Math. **227**(1), 421–458 (2011)
17. Bump, D., Nakasuji, M.: Casselman's basis of Iwahori vectors and Kazhdan-Lusztig polynomials. Can. J. Math. **1–16** (2017)
18. Casselman, W.: The unramified principal series of p-adic groups. I. The spherical function. Compos. Math. **40**, 387–406 (1980)
19. Casselman, W., Shalika, J.: The unramified principal series of p-adic groups. II. The Whittaker function, Compos. Math. **41**(2), 207–231 (1980)
20. Cherednik, I.: Double Affine Hecke Agebras, vol. 319. Cambridge University Press, Cambridge (2005)
21. Chriss, N.: Ginzburg, V.: Representation Theory and Complex Geometry. Birkhäuser, Boston-Basel-Berlin (1997)
22. Feher, L., Rimanyi, R., Weber, A.: Motivic Chern classes and K-theoretic stable envelopes. In Proc. London Math. Soc. (2020) arXiv:1802.01503
23. Ginzburg, V.: Deligne-Langlands conjecture and representations of affine Hecke algebras (1985)
24. Hotta, R., Tanisaki, T.: D-modules, perverse sheaves, and representation theory, vol. 236. Springer Science & Business Media (2007)
25. Janzten, J.C.: Modular representations of reductive Lie algebras. J. Pure Appl. Algebra **152**, 133–185 (2000)
26. Kaledin, D.: Geometry and topology of symplectic resolutions. Algebraic Geom. Seattle **2**(595–628), 2009 (2005)
27. Kazhdan, D., Lusztig, G.: Proof of the Deligne-Langlands conjecture for Hecke algebras. Invent. Math. **87**(1), 153–215 (1987)
28. Lenart, C., Zainoulline, K.: Towards generalized cohomology Schubert calculus via formal root polynomials. Math. Res. Lett. **24**(3), 839–877 (2017)
29. Lusztig, G.: Cuspidal local systems and graded Hecke algebras. I, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **67**, 145–202 (1998)
30. MacPherson, R.: Chern classes for singular algebraic varieties. Ann. Math. **2**(100), 423–432 (1974)
31. Maulik, D., Okounkov, A.: Quantum groups and quantum cohomology. Astérisque **408** (2019). arXiv:1211.1287
32. Naruse, H.: Schubert calculus and hook formula, Slides at 73rd Sém. Lothar. Combin, Strobl, Austria (2014)
33. Ohmoto, T.: Equivariant Chern classes of singular algebraic varieties with group actions. Math. Proc. Camb. Philos. Soc. **140**(1), 115–134 (2006)
34. Okounkov, A.: Lectures on K-theoretic computations in enumerative geometry. Geometry of Moduli Spaces and Representation Theory, IAS/Park City Mathematical Series, vol. 24, pp. 251–80. American Mathematical Society, Providence, RI (2017)
35. Okounkov, A.: Enumerative geometry and geometric representation theory. Algebraic Geom.: Salt Lake City 2015 (Part 1) **97**, 419–458 (2018)
36. Okounkov, A.: On the crossroads of enumerative geometry and geometric representation theory. arXiv:1801.09818
37. Okounkov, A., Smirnov, A.: Quantum difference equation for Nakajima varieties (2016). arXiv:1602.09007
38. Riche, S.: Geometric braid group action on derived categories of coherent sheaves. Represent. Theory Am. Math. Soc. **12**(5), 131–169 (2008)
39. Rimanyi, R., Tarasov, V., Varchenko, A.: Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety. J. Geom. Phys. **94**, 81–19 (2015)

40. Rimanyi, R., Varchenko, A.: In: Buczynski, J., Michalek, M., Postingel E. (eds.) Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae, Schubert Varieties, Equivariant Cohomology and Characteristic Classes, IMPANGA2015, EMS, pp. 225–35 (2018)
41. Schwartz, M.: Classes caractéristiques définies par une stratification d'une variété analytique complexe. I, C. R. Acad. Sci. Paris **260**, 3262–3264 (1965)
42. Schwartz, M.: Classes caractéristiques définies par une stratification d'une variété analytique complexe. II, C. R. Acad. Sci. Paris **260**, 3535–3537 (1965)
43. Su, C.: Restriction formula for stable basis of the Springer resolution. Sel. Math. **23**(1), 497–518 (2017)
44. Su, C., Zhao, G., Zhong, C.: On the *K-theory stable bases of the Springer resolution.* 43 p, to appear in Annales sc. de l'ENS. arXiv:1708.08013
45. Su, C., Zhao, G., Zhong, C.: Wall-crossings and a categorification of the K-theory stable bases of the Springer resolutions. arXiv:1904.03769
46. Tanisaki, T.: Hodge modules, equivariant K-theory and Hecke algebras. Publ. Res. Inst. Math. Sci. **23**(5), 841–879 (1987)

Characteristic Classes of Orbit Stratifications, the Axiomatic Approach

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Abstract Consider a complex algebraic group G acting on a smooth variety M with finitely many orbits, and let Ω be an orbit. The following three invariants of $\Omega \subset M$ can be characterized axiomatically: (1) the equivariant fundamental class $[\overline{\Omega}, M] \in H^*_G(M)$, (2) the equivariant Chern–Schwartz–MacPherson class $c^{sm}(\Omega, M) \in H^*_G(M)$, and (3) the equivariant motivic Chern class $mC(\Omega, M) \in K_G(M)[y]$. The axioms for Chern–Schwartz–MacPherson and motivic Chern classes are motivated by the axioms for cohomological and K-theoretic stable envelopes of Okounkov and his coauthors. For M a flag variety and Ω a Schubert cell—an orbit of the Borel group acting—this implies that CSM and MC classes coincide with the weight functions studied by Rimányi–Tarasov–Varchenko. In this paper we review the general theory and illustrate it with examples.

Keywords Equivariant characteristic classes · Chern–Schwartz–MacPherson class · Motivic Chern class · Schubert classes · Axiomatic characterisation

1 Introduction

An effective way of studying the subvariety X of a smooth complex variety M is assigning a characteristic class to $X \subset M$, living in the cohomology or K-theory of the ambient space M. When M is a G-representation and X is invariant, then the

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characteristic class can be considered in the richer G-equivariant cohomology or K-theory of M.

In this paper we consider three flavors of characteristic classes: the fundamental class (in cohomology), the Chern–Schwartz–MacPherson (CSM) class (in cohomology) and the motivic Chern class (MC class) in K-theory. These characteristic classes encode fine geometric and enumerative properties of $X \subset M$, and their theory overlaps with the recent advances relating geometry with quantum integrable systems.

We will be concerned with the following situation: a complex algebraic group G acts on a smooth variety M with finitely many orbits, and we want to find the characteristic classes listed above associated to the orbits or their closures. This situation is frequent in Schubert calculus and other branches of enumerative geometry. By the nature how characteristic classes are defined, such calculation assumes resolutions of the singularities of the orbit closures. The main message of the theorems we review in Sects. 2, 3, 4, is that such a resolution calculation can be replaced with an axiomatic approach for all three types of classes mentioned above. For the fundamental class this fact has been known for two decades [16, 29], but for the other two mentioned classes this is a very recent development stemming from the notion of Okounkov's stable envelopes [23, 27].

As an application of the axiomatic approach the characteristic classes of ordinary and matrix Schubert cells can be computed (in type A, for arbitrary partial flag manifolds)—we present the formulas in Sect. 5. For the motivic Chern class one of the axioms concerns convex geometry, namely the containment of one convex polytope in another one. Throughout our exposition we try to illustrate this fascinating connection between complex geometry and the theory of convex polytopes with 2- and 3-dimensional pictures.

It is a classical fact that characteristic classes of geometrically relevant varieties often exhibit positivity (or alternating sign) properties. Having computed lots of examples of equivariant motivic Chern classes, we observe these properties for the motivic Chern class as well. In Sect. 6 we collect three different positivity conjectures on MC classes of Schubert cells. One of these conjectures is a K-theoretic counterpart of properties of CSM classes studied by Aluffi–Mihalcea–Schürmann–Su [2, 3]. Another one of our conjectures is a new phenomenon for characteristic classes: not the sign of some coefficients are conjectured, but rather that the coefficient polynomials are log-concave, cf. [21].

We finish the paper with studying the characteristic classes of varieties in the source and in the target of a GIT map. The fundamental class of an invariant subvariety and the fundamental class of its image under the GIT map are essentially the same (more precisely, one lives in a quotient ring, and the other one is a representative of it). A familiar fact illustrating this phenomenon is that the fundamental class of both the matrix Schubert varieties and the ordinary Schubert varieties are the Schur functions. However, for the CSM and the MC classes the situation is different: these classes before and after the GIT map are not the same (or rather, one does not represent the other). Yet, there is direct relation between them, which we prove in Sect. 8.

The goal of most of this paper is to review and illustrate concepts, although in Sect. 4.4 we complement a proof of [19] with an alternative argument. The mathematical novelties are the conjectures in Sect. 6, the careful treatment of the pull-back property of motivic Segre classes in Sect. 8.1, and its consequence, Theorem 8.12 which relates motivic Segre classes of varieties in the source and in the target of a GIT quotient.

2 Fundamental Class in Equivariant Cohomology

By equivariant cohomology we will mean Borel's version, developed in [8]. Our general reference is [9].

Suppose a complex algebraic group G acts on the smooth complex algebraic variety M. Suppose also that the action has finitely many orbits, and let $\mathcal{O}$ be the set of orbits. For an orbit $\Omega \in \mathcal{O}$ let G_Ω denote the stabilizer subgroup of a point in Ω. Then additively we have

$$H_G^*(M;\mathbb{Q}) \simeq \bigoplus_{\Omega\in\mathcal{O}} H_G^{*-2\operatorname{codim}(\Omega)}(\Omega;\mathbb{Q}).$$

The decomposition holds because each cohomology group $H_G^*(\Omega;\mathbb{Q}) \simeq H^*(BG_\Omega;\mathbb{Q})$ is concentrated in even degrees. Suppose further that the normal bundle ν_Ω of each orbit Ω has nonzero equivariant Euler class

$$e(\nu_\Omega) \neq 0 \in H_G^*(\Omega;\mathbb{Q}). \tag{1}$$

Then a class in $H_G^*(M;\mathbb{Q})$ is determined by the restrictions to orbits, cf. [6, Sect. 9]. The topic of this paper is that certain characteristic classes associated with orbits are determined by less information: they are determined by some *properties* of their restrictions to the orbits.

Example 2.1 For $M=\mathbb{C}$, $G=\mathbb{C}^*$ with the natural action on $\mathbb{C}$. We have the short exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_{\mathbb{C}^*}^{*-2}(\{0\},\mathbb{Q}) & \xrightarrow[t\cdot]{} & H_{\mathbb{C}^*}^{*}(\mathbb{C};\mathbb{Q}) & \xrightarrow[ev_{t=0}]{} & H_{\mathbb{C}^*}^{*}(\mathbb{C}^*;\mathbb{Q}) & \longrightarrow 0. \\ & \| & & \| & & \| & \\ & \mathbb{Q}[t] & & \mathbb{Q}[t] & & \mathbb{Q} & \end{array}$$

In this example the restriction to $\{0\}$ is obviously an isomorphism. In general, when M is an equivariantly formal space (see [20]) and G is a torus, then the restriction to the fixed point set is a monomorphism.

The most natural characteristic class associated to an orbit Ω is the fundamental class of its closure, which we will denote by $[\overline{\Omega}]=[\overline{\Omega},M]\in H_G^*(M,\mathbb{Q})$. In an appropriate sense, it is the Poincaré dual of the homology fundamental class of $\overline{\Omega}$. Such classes are studied in Schubert calculus under the name of Schubert classes, in singularity theory under the name of Thom polynomials, and in the theory of quivers under the name of quiver polynomials. Here is an axiomatic characterization of the equivariant fundamental class, which leads to an effective method to calculate them.

Theorem 2.2 ([16, 29]) *Suppose M has finitely many G-orbits and (1) holds. Then the equivariant fundamental classes of orbit closures are determined by the conditions:*

(i) (support condition) the class $[\overline{\Omega}]$ *is supported on* $\overline{\Omega}$*;*
(ii) (normalization condition) $[\overline{\Omega}]_{|\Omega} = e(\nu_\Omega)$*;*
(iii) (degree condition) $\deg([\overline{\Omega}]) = 2\,\mathrm{codim}(\Omega)$.

We obtain an equivalent system of conditions if we replace condition (i) with any of the following two:

(iv) (homogeneous equations, ver1) $[\overline{\Omega}]_{|\Theta} = 0$*, if* $\Theta \not\subset \overline{\Omega}$*,*
(v) (homogeneous equations, ver2) $[\overline{\Omega}]_{|\Theta} = 0$*, if* $\mathrm{codim}(\Theta) \leq \mathrm{codim}(\Omega)$*,* $\Theta \neq \Omega$.

We also obtain an equivalent system of conditions if we replace condition (iii) with

(vi) (modified degree condition) $\deg([\Omega])_{|\Theta} < \deg([\Theta])_{|\Theta}$ *for* $\Theta \neq \Omega$.

□

The advantage of *(iv)* or *(v)* over *(i)* is that they are local conditions, and hence explicitly computable. The modified degree condition is only added to illustrate the similarity with analogous characterizations of other characteristic classes in Sects. 3, 4.

Example 2.3 Let $G = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ act on the vector space $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^3)$ via $(A, B) \cdot X = BXA^{-1}$. The orbits of this action are $\Omega^i = \{X : \dim(\ker(X)) = i\}$ for $i = 0, 1, 2$. The fundamental class $[\overline{\Omega^1}]$ is a homogeneous degree 2 polynomial in $\mathbb{Z}[a_1, a_2, b_1, b_2, b_3]^{S_2 \times S_3}$ (where $\deg a_i = \deg b_i = 1$). The constraints put on this polynomial by the conditions of Theorem 2.2 are $\phi_0([\overline{\Omega^1}]) = 0$ (condition (v)) and $\phi_1([\overline{\Omega^1}]) = (b_2 - a_2)(b_3 - a_2)$ (condition (ii)), where ϕ_0 is the "restriction to Ω^0" map, namely $a_1 \mapsto t_1, a_2 \mapsto t_2, b_1 \mapsto t_1, b_2 \mapsto t_2, b_3 \mapsto b_3$, and ϕ_1 is the "restriction to Ω^1" map, namely $a_1 \mapsto t_1, a_2 \mapsto a_2, b_1 \mapsto t_1, b_2 \mapsto b_2, b_3 \mapsto b_3$. Calculation shows that the only solution to these two constraints is $[\overline{\Omega^1}] = A_1^2 - A_2 + B_2 - A_1 B_1$, where A_i's (B_i's) are the elementary symmetric polynomials of the a_i's (b_i's).

More details on this calculation as well as on analogous calculations for 'similar' representations (e.g. Dynkin quivers, $\Lambda^2\mathbb{C}^n$, $S^2\mathbb{C}^n$) can be found e.g. in [15–17]. However, these fundamental classes can also be computed by other methods (resolutions, degenerations). The real power of Theorem 2.2 is that it is applicable even in situations where deeper geometric information such as resolutions or degenerations are not known—e.g. contact singularities, matroid representations spaces, see [14, 29].

3 Chern–Schwartz–MacPherson Classes of Orbits

Another cohomological characteristic class associated with an invariant subset X of M is the Chern–Schwartz–MacPherson (CSM) class $\mathrm{c}^{\mathrm{sm}}(X, M) \in H_G^*(M; \mathbb{Q})$, see e.g. [2, 22, 25, 39] for foundational literature and [18] for the version we consider. The CSM class is an inhomogeneous class, whose lowest degree component equals the fundamental class $[\overline{X}, M]$.

In general CSM classes are defined for (equivariant) constructible functions on M. Here we will only consider CSM classes of locally closed invariant smooth subvarieties of M (corresponding to the indicator functions of such varieties).

According to Aluffi [1] the CSM classes can be calculated along the following lines: one finds Y a partial completion of X which maps properly to M and such that $Y \setminus X$ is a smooth divisor with simple normal crossings $D = \bigcup_{i=1}^m D_i$. That is, we have a diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "\simeq"'] & Y \arrow[d, "f"] \\ X \arrow[r, hook] & M \end{tikzcd}$$

where f is a proper map. Then

$$\mathrm{c}^{\mathrm{sm}}(X, M) = f_*\left(c(T_Y) - \sum_i c(T_{D_i}) + \sum_{i<j} c(T_{D_i \cap D_j}) - \cdots \right). \tag{2}$$

In our situation, this method demanding knowledge about a resolution, can be replaced with an axiomatic characterization.

For an orbit $\Omega \subset M$ let $x_\Omega \in \Omega$, and let G_Ω be the stabilizer subgroup of x_Ω. Denote $T_\Omega = T_{x_\Omega}\Omega$, and $\nu_\Omega = T_{x_\Omega}M - T_\Omega$ as G_Ω-representations. By the degree of an inhomogeneous cohomology class $a = a_0 + a_1 + \cdots + a_d$ with $a_i \in H_G^i(\Omega)$ and $a_d \neq 0$ we mean d.

Theorem 3.1 ([18, 34]) *Suppose M has finitely many G-orbits and (1) holds. Then the equivariant CSM classes of the orbits are determined by the conditions:*

(i') (divisibility condition) for any Θ the restriction $\mathrm{c}^{\mathrm{sm}}(\Omega, M)_{|\Theta}$ is divisible by $c(T_\Theta)$ in $H_G^*(\Theta; \mathbb{Q})$;
(ii') (normalization condition) $\mathrm{c}^{\mathrm{sm}}(\Omega, M)_{|\Omega} = e(\nu_\Omega)c(T_\Omega)$;
(iii') (smallness condition) $\deg(\mathrm{c}^{\mathrm{sm}}(\Omega, M)_{|\Theta}) < \deg(\mathrm{c}^{\mathrm{sm}}(\Theta, M)_{|\Theta})$ for $\Theta \neq \Omega$.

□

Example 3.2 Continuing Example 2.3, let us calculate $\mathrm{c}^{\mathrm{sm}}(\Omega^1)$. It is an inhomogeneous polynomial in $\mathbb{Z}[a_1, a_2, b_1, b_2, b_3]^{S_2 \times S_3}$. The constraints put on this polynomial by the conditions of Theorem 3.1 are $\phi_0(\mathrm{c}^{\mathrm{sm}}(\Omega^1)) = 0$, $\phi_1(\mathrm{c}^{\mathrm{sm}}(\Omega^1)) = (b_2 - a_2)(b_3 - a_2)(1 + b_2 - t_1)(1 + b_3 - t_1)(1 + t_1 - a_2)$, and $\deg(\phi_2(\mathrm{c}^{\mathrm{sm}}(\Omega^1))) <$

6.
Here ϕ_1, ϕ_2 are as in Example 2.3, and ϕ_3 is the "restriction to Ω^2" map which turns out to be the identity map. Calculation shows that the unique solution to these constraints is (see notations in Example 2.3)

$$\mathrm{c}^{\mathrm{sm}}(\Omega^1) = (A_1 - A_2 + B_2 - A_1B_1) + (-A_1^3 + 2A_1^2B_1 - A_1B_1^2 - A_1B_2 + B_1B_1 - B_3) + \cdots + (-3A_1A_2^2 + \cdots + 2B_2B_3).$$

Details on such calculations, other examples of interpolation calculations of CSM classes, as well as more conceptual ways of presenting the long CSM polynomials are in [18, 28].

4 Characteristic Classes in Equivariant K-Theory

4.1 The Motivic Chern Class

We consider the topological equivariant K-theory constructed by Segal [36] or Thomason's algebraic K-theory, see [13, Sect. 5]. It is compatible with the K-theory built from locally free sheaves for algebraic varieties. As before we have a decomposition

$$K_G(M) \simeq \bigoplus_{\Omega \in \mathcal{O}} K_G(\Omega)$$

and $K_G(\Omega)$ is isomorphic with the representation ring $R(G_\Omega)$.

The analogue of the Euler class of a vector bundle E in K-theory is provided by the λ-operation

$$e^K(E) = \lambda_{-1}(E^*) = 1 - E^* + \Lambda^2 E^* - \Lambda^3 E^* + \cdots .$$

It has the property that for any submanifold $\iota : N \hookrightarrow M$ and an element $\beta \in K_G(N)$ over N we have

$$\iota^* \iota_*(\beta) = e^K(\nu_N) \cdot \beta .$$

The total Chern class in K-theory is defined by

$$c_G^K(E) = \lambda_y(E^*) = 1 + yE^* + y^2\Lambda^2 E^* + y^3 \Lambda^3 E^* + \cdots \in K_G(M)[y] .$$

Note that, both in cohomology and K-theory, the total Chern class can be interpreted as the Euler class of the bundle E which is equivariant with respect to $G \times \mathbb{C}^*$, where $\mathbb{C}^*$ acts on the base trivially, but acts on the bundle by scalar multiplication. Identifying $K_{G\times\mathbb{C}^*}(M)$ with $K_G(M)[h, h^{-1}]$ and setting $y = -h^{-1}$ we obtain

$$c_G^K(E) = e_{G\times\mathbb{C}^*}^K(E)\,.$$

We will drop the subscript G in the notation, as we do for cohomological Chern classes.

Our next characteristic class defined for a locally closed subvariety X in the smooth ambient space M is the motivic Chern class (MC class) $\mathrm{mC}(X, M) \in K_G(M)[y]$, see [10]. In fact this class is defined more generally for maps $X \to M$ but we will not need this generality here. The G-equivariant MC class

$$\mathrm{mC}(X, M) \in K_G(M)[y]$$

is a straightforward generalization, see details in [4, 19].

A natural method to calculate the motivic Chern class of a locally closed subvariety $X \subset M$ is through the K-theoretic analogue

$$\mathrm{mC}(X, M) = f_*\left(c^K(T_Y) - \sum_i c^K(T_{D_i}) + \sum_{i<j} c^K(T_{D_i\cap D_j}) - \cdots\right) \tag{3}$$

of the formula (2).

4.2 Axiomatic Characterization of MC Classes

In certain situations the resolution method (3) can be replaced with an axiomatic characterization, which we will now explain in several steps.

Theorem 4.1 ([19, Corollary 4.5, Lemma 5.1]) *Suppose M has finitely many G-orbits. Then the equivariant MC classes of the orbits satisfy the conditions:*

- *(i") (support condition) the class $\mathrm{mC}(\Omega, M)$ is supported on $\overline{\Omega}$;*
- *(ii") (normalization condition) $\mathrm{mC}(\Omega, M)_{|\Omega} = e^K(\nu_\Omega)c^K(T_\Omega)$;*
- *(iii") (smallness condition) $\mathcal{N}(\mathrm{mC}(\Omega, M)_{|\Theta}) \subseteq \mathcal{N}(\mathrm{mC}(\Theta, M)_{|\Theta})$ [see explanation below];*
- *(iv") (the divisibility condition) for any Θ the restriction $\mathrm{mC}(\Omega, M)_{|\Theta}$ is divisible by $c^K(T_\Theta)$ in $K_G(\Theta)[y]$.*

$\square$

In fact, the local condition *(iv")* implies the condition *(i")*, and it also implies the obvious condition

(v") $\mathrm{mC}(\Omega, M)_{|\Theta} = 0$ *if* $\Theta \not\subset \overline{\Omega}$.

The K-theoretic version of the condition (1) follows from (iii').

Let us give a precise formulation of the smallness condition *(iii")*. The classes $\mathrm{mC}(\Omega, M)_{|\Theta}$ and $\mathrm{mC}(\Theta, M)_{|\Theta}$ belong to $K_G(\Theta)[y] = R(G_\Theta)[y]$. Restrict these classes to the representation ring $R(\mathbb{T}_\Theta)[y]$, where $\mathbb{T}_\Theta$ is a maximal torus of G_Θ. For a chosen isomorphism $\mathbb{T}_\Theta = (\mathbb{C}^*)^r$ we have $R(\mathbb{T}_\Theta)[y] = \mathbb{Z}[\alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}, y]$ where α_i are the K-theoretic Chern roots corresponding to the factors of $(\mathbb{C}^*)^r$. The Newton polygon of $\beta = \sum_I a_I(y) z^I \in R(\mathbb{T}_\Theta)[y]$ (multiindex notation) is

$$\mathcal{N}(\beta) = \text{convex hull } \{I \in \mathbb{R}^r : a_I(y) \neq 0\},$$

in particular, for this notion the parameter y is considered a constant, not a variable. The definition of $\mathcal{N}$ depends on the chosen isomorphism between $R(\mathbb{T}_\Theta)[y]$ and $\mathbb{Z}[z_1^{\pm 1}, \dots, z_r^{\pm 1}, y]$. Different such choices result in linearly equivalent convex polygons. The smallness condition compares two such Newton polygons, of course the same isomorphism need to be chosen for the two sides.

Example 4.2 If $M = \mathbb{C}$, $G = \mathbb{C}^*$ with the natural action on $\mathbb{C}$ we have a short exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & K_{\mathbb{C}^*}(\{0\}) & \xrightarrow[(1-\xi^{-1})\cdot]{} & K_{\mathbb{C}^*}(\mathbb{C}) & \xrightarrow[ev_{\xi=1}]{} & K_{\mathbb{C}^*}(\mathbb{C}^*) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & \mathbb{Z}[\xi, \xi^{-1}] & & \mathbb{Z}[\xi, \xi^{-1}] & & \mathbb{Z} & \end{array}$$

In this example the restriction to $\{0\}$ is an isomorphism. We have

$$\mathrm{mC}(\mathbb{C}^*, \mathbb{C}) = (1+y)\xi^{-1}, \qquad \mathrm{mC}(\{0\}, \mathbb{C}) = 1 - \xi^{-1}$$

and

$$\mathcal{N}(\mathrm{mC}(\mathbb{C}^*, \mathbb{C}))_{|0} = \{-1\}, \qquad \mathcal{N}(\mathrm{mC}(\{0\}, \mathbb{C}))_{|0} = [-1, 0].$$

Example 4.3 Consider the standard action of $G = \mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$. Then we have

$$\mathcal{N}(\mathrm{mC}(\mathbb{C}^2 \setminus \{0\}, \mathbb{C}^2))_{|0} = \mathcal{N}(\mathrm{mC}(\{0\}, \mathbb{C}^2))_{|0} = [-1, 1].$$

Examples 4.2, 4.3 show that the inclusion of Newton polygons in *(iii")* may or may not be strict. However for an interesting class of actions the inclusion is necessarily strict.

Property 4.4 (cf. [19, Definition 4.4]) *We say that the action is positive, if for each orbit Ω, $x \in \Omega$ there exists a one dimensional torus $\mathbb{C}^* \hookrightarrow G_x$ which acts on the normal space $(\nu_\Omega)_x$ with positive weights.*

Note that Property 4.4 implies that 0 is a vertex of the Newton polygon $\mathcal{N}(e^K(\nu_\Omega))$ because $e^K(\nu_\Omega)_x = \prod_{i=1}^{\mathrm{codim}(\Omega)}(1 - \chi_i^{-1})$, where χ_i are the weights of $\mathbb{T}_\Omega$ acting on $(\nu_\Omega)_x$. In particular $e^K(\nu_\Omega) \neq 0$. Let "+" denote the Minkowski sum of polygons.

Theorem 4.5 ([19, Theorem 5.3]) *Suppose that the action of G on M has Property 4.4. Then the inclusion in (iii") is strict. Moreover, for any pair of different orbits Θ, Ω we have*

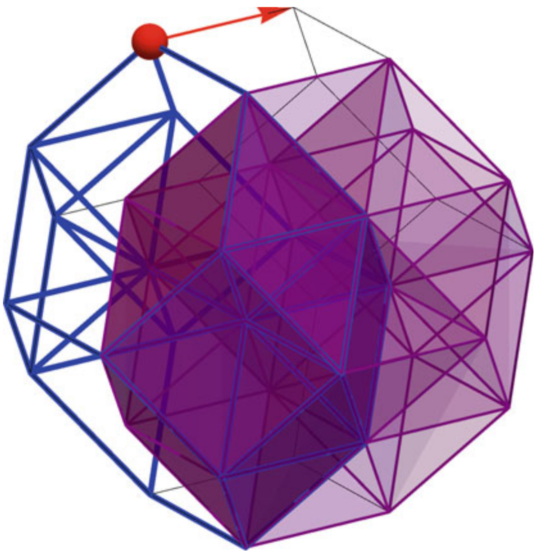
$$\text{(vi")} \quad \mathcal{N}(\mathrm{mC}(\Omega, M)_{|\Theta}) \subseteq \mathcal{N}(e^K(\nu_\Theta) - 1) + \mathcal{N}(c^K(T_\Theta)) \subsetneq \mathcal{N}(\mathrm{mC}(\Theta, M)_{|\Theta}).$$

A reformulation of condition *(vi")* is that $\mathcal{N}(\mathrm{mC}(\Omega, M)_{|\Theta})$ is contained in $\mathcal{N}(\mathrm{mC}(\Theta, M)_{|\Theta})$ in such a way that the origin is a vertex of $\mathcal{N}(\mathrm{mC}(\Theta, M)_{|\Theta})$, but it is not contained in $\mathcal{N}(\mathrm{mC}(\Omega, M)_{|\Theta})$.

Additionally assuming that the stabilizers G_Ω are connected guarantees that an element $\beta \in K_{\mathbb{T}}(M)$ is determined by the restrictions to orbits, and we obtain

Theorem 4.6 ([19, Theorem 5.5]) *If the action of G on M has finitely many orbits, it is positive, and the stabilizers are connected, then the conditions (ii'), (iv"), (vi") determine* $\mathrm{mC}(\Omega, M)$ *uniquely.*

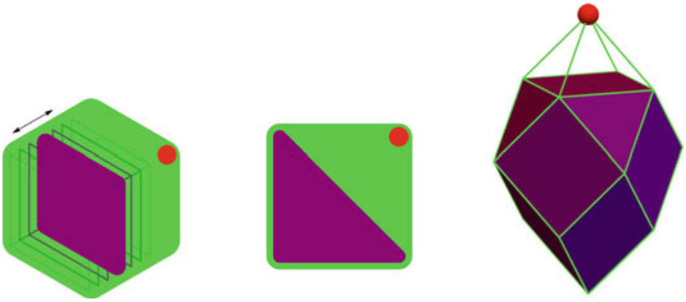
Example 4.7 The Borel subgroup B_n of $\mathrm{GL}_n(\mathbb{C})$ acts on the full flag variety Fl(4) with finitely many orbits. The Newton polygon containment (ie. smallness condition *(iii")*) for two of the orbits, $\Omega = \Omega_{(\{3\},\{4\},\{1\},\{2\})}$ and $\Theta = \Omega_{(\{3\},\{4\},\{2\},\{1\})}$, is illustrated below. For more details about the orbits of this action, their combinatorial codes, and their MC classes see Sect. 7. The description of the Borel-equivariant K-theory of flag varieties can be found in [13, Sect. 6] or in [38]. The figure below illustrates (Newton polygons of) the two MC classes, restricted to $K_{B_n}(\Theta)[y] = K_{\mathbb{T}_\Theta}(\mathrm{pt})[y]$. It turns out that $\mathbb{T}_\Theta = (\mathbb{C}^*)^4$ and hence the two Newton polygons live in $\mathbb{R}^4$. Moreover, both Newton polygons turn out to be contained in a 3-dimensional subspace (sum of coordinates= 0). Hence the picture shows 3-dimensional convex polytopes.



In blue—Newton polygon of $e^K(\nu_\Theta) = \lambda_{-1}(\nu_\Theta^*)$.
Arrow in red—cotangent weight T_Θ^*.
In solid violet—Newton polygon of $\mathrm{mC}(\Omega \cap S_\Theta)$, where S_Θ is a slice to Θ.
Edges in violet—Newton polygon of $\mathrm{mC}(\Omega)$.

Remark 4.8 Our smallness conditions *(iii")* and *(vi")* are motivated by the analogous smallness condition for *K-theoretic stable envelopes* invented by Okounkov, see

[27, (9.1.10)]. There is, however, a difference. In Okounkov's smallness condition the strictness of the inclusion of Newton polygons is guaranteed by the fact that the small Newton polygon can be shifted slightly within the large Newton polygon, see the first picture below. In fact, in the terminology of [27], the inclusion holds for an open set (interior of an alcove) of "fractional shifts". However, such a "wiggle room" for the inclusions does not necessarily exist for MC classes, see the second and third pictures below. The second picture illustrates the smallness condition *(iii")* (or *(vi")*) for the standard representation $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$ with $\Omega = \mathbb{C}^2 - \{0\}$, $\Theta = \{0\}$. The third picture illustrates smallness condition *(iii")* (or *(vi")*) for the natural $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$-representation on $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2)$ (a.k.a. A_2 quiver representation with dimension vector $(2, 2)$) with $\Omega = \{\text{rank 1 maps}\}$ and, $\Theta = \{0\}$.



4.3 The Key Idea of the Proof of Theorem 4.1

At the heart of the arguments proving the statements of Sect. 4.2 is showing that conditions *(iii")* and *(vi")* hold. The crucial step in the proof is the special case when Θ is a point. To test an inclusion of Newton polygons it is enough to restrict an action to each one dimensional torus $\mathbb{T}_0 = \mathbb{C}^* \hookrightarrow \mathbb{T}$, with $K_{\mathbb{T}_0}(pt) = \mathbb{Z}[\xi, \xi^{-1}]$. The argument reduces to examining the limit

$$\lim_{\xi \to \infty} \frac{\mathrm{mC}(\Omega, M)_\Theta}{e^K(\nu_\Theta)} \,. \tag{4}$$

We prove that the limit is equal to

$$\chi_y(\Omega \cap M_\Theta^-)\,,$$

where M_Θ^- is the Białynicki-Birula minus-cell associated to Θ, (see [7] or [12, Sect. 4.1])

$$M_\Theta^- = \{x \in M \mid \lim_{t \to \infty} tx \in \Theta\}\,, \tag{5}$$

and χ_y is the Hirzebruch χ_y-genus. The proof of this statement in full generality is given in [41] and in the particular cases needed in the proof of Theorem 4.6 in [19, Theorem 5.3].

Since the limit (4) is finite the degree of the denominator is at least the degree of the numerator. This observation leads to the proof of *(iii")*. Moreover, if the action is positive then the limit is equal to $\chi_y(\emptyset) = 0$, thus the degree of the denominator must be strictly larger than the degree of the numerator. This observation leads to the proof of *(vi")*.

4.4 Strict Inclusion for Homogeneous Singularities

In this section we give a rigorous alternative argument proving the containment property *(iii")* of Newton polygons of mC classes. We believe that it sheds more light on this intriguing connection between algebraic and convex geometry.

Suppose $\mathbb{T} = \mathbb{C}^*$ acts on a vector space $M = \mathbb{C}^n$ via scalar multiplication. Let $\Omega \subset \mathbb{C}^n$ be an invariant subvariety, where $0 \notin \Omega$. Denote by $Z \subset \mathbb{P}^n$ the projectivization of Ω. We have a diagram

$$\begin{tikzcd}
\Omega' \arrow[r, hook] \arrow[d, "\simeq"'] & \widetilde{\mathbb{C}^n} \arrow[r, "q"] \arrow[d, "p"] & \mathbb{P}^{n-1} \arrow[l, bend left, "\iota"] & Z \arrow[l, hook'] \\
\Omega \arrow[r, hook] & \mathbb{C}^n & &
\end{tikzcd}$$

where $\widetilde{\mathbb{C}^n}$ is the blow-up of $\mathbb{C}^n$ at 0 and $\iota : \mathbb{P}^{n-1} \to \widetilde{\mathbb{C}^n}$ is the inclusion of the special fiber, $\Omega' = q^{-1}(Z) \setminus \iota(Z)$. Let $h = [\mathcal{O}(1)] \in K(\mathbb{P}^{n-1})$. Then

$$\frac{\mathrm{mC}(\Omega', \widetilde{\mathbb{C}^n})_{|\mathbb{P}^{n-1}}}{\lambda_{-1}(\nu^*_{\mathbb{P}^{n-1}/\widetilde{\mathbb{C}^n}})} = \left(\frac{1+y\xi^{-1}h}{1-\xi^{-1}h} - 1\right) \mathrm{mC}(Z, \mathbb{P}^{n-1}) = \frac{(1+y)\xi^{-1}h}{1-\xi^{-1}h}\ \mathrm{mC}(Z, \mathbb{P}^{n-1}) .$$

Applying the localization formula for the map p we obtain

$$\frac{\mathrm{mC}(\Omega, \mathbb{C}^n)}{e^K(\mathbb{C}^n)} = p_*\left(\frac{(1+y)\xi^{-1}h}{1-\xi^{-1}h}\ \mathrm{mC}(Z, \mathbb{P}^{n-1})\right) . \tag{6}$$

Let $u = h^{-1} - 1 \in K(\mathbb{P}^{n-1})$. Note that $u^n = 0$. We have the expansion

$$\begin{aligned}\frac{(1+y)\xi^{-1}h}{1-\xi^{-1}h} &= \frac{(1+y)\xi^{-1}}{1+u-\xi^{-1}} = \frac{(1+y)\xi^{-1}}{(1-\xi^{-1})(1+\frac{u}{1-\xi^{-1}})} \\ &= \frac{(1+y)\xi^{-1}}{1-\xi^{-1}}\left(1 - \frac{u}{1-\xi^{-1}} + \cdots + (-1)^{n-1}\frac{u^{n-1}}{(1-\xi^{-1})^{n-1}}\right) .\end{aligned}$$

The expression under the push-forward p_* in (6) is of the form

$$\frac{1}{(1-\xi^{-1})^n} P(u, \xi^{-1}, y)\,.$$

The polynomial P is of degree at most n in ξ^{-1}, at most $n-1$ in u and it is divisible by $(y+1)\xi^{-1}$. We have[1] $p_*(u^k) = 1$ for $k < n$. After the push-forward we obtain

$$\frac{\mathrm{mC}(\Omega, \mathbb{C}^n)}{e^K(\mathbb{C}^n)} = \frac{Q(\xi^{-1}, y)}{(1-\xi^{-1})^n}\,,$$

with the polynomial $Q(\xi^{-1}, y) = P(1, \xi^{-1}, y)$ again divisible by $(y+1)\xi^{-1}$ and of degree at most n in ξ^{-1}.

Remark 4.9 The argument above shows that $\mathrm{mC}(\Omega, \mathbb{C}^n)$ is divisible by $(y+1)$. In a similar way this divisibility can be proven for any quasihomogenous subvariety. Such variety can be presented as a quotient (by a finite group) of homogeneous one. Application of the Lefschetz–Riemann–Roch formula [11, Theorem 5.1] gives an explicit formula for $\mathrm{mC}(\Omega, \mathbb{C}^n)$.

4.5 An Example: Quadratic Cone—limits with Different Choices of 1-Parameter Subgroup

This example is based on the computations made in [24]. Let $\Omega \subset \mathbb{C}^4$ be the open set given by the inequality $z_1 z_2 - z_3 z_4 \neq 0$. This is an open orbit of the natural action of $\mathbb{C}^* \times O(4, \mathbb{C})$ on $\mathbb{C}^4$.

On the other hand, identifying $\mathbb{C}^4$ with 2×2 matrices $\begin{pmatrix} z_1 & z_3 \\ z_4 & z_2 \end{pmatrix}$ the set Ω is equal to the set of nondegenerate matrices. Its motivic Chern class was computed in [19, Sect. 8].

Denote the basis characters of the torus $\mathbb{T} = (\mathbb{C}^*)^3$ by $\alpha,\ \beta,\ \gamma$. Suppose $\mathbb{T}$ acts on $\mathbb{C}^4$ with characters

$$\alpha\beta\ ,\ \alpha/\beta\ ,\ \alpha\gamma\ ,\ \alpha/\gamma\,.$$

The action preserves the variety Ω. The action has a unique fixed point at 0. We study $\Omega \cap (\mathbb{C}^4)^-_{\{0\}}$, the Białynicki-Birula minus-cell (defined by (5)) intersected with the orbit, depending on the choice of the one parameter subgroup. We apply the formula [24, Formula 3] for $n = 4$, which allows us to compute the motivic Chern class of Ω:

$$\mathrm{mC}(\Omega, \mathbb{C}^4) = (1+y)^2 \left(\frac{1}{\alpha^4} y^2 + \left(\frac{\beta}{\alpha^3} + \frac{\gamma}{\alpha^3} + \frac{1}{\alpha^3 \beta} + \frac{1}{\alpha^3 \gamma} - \frac{1}{\alpha^2} - \frac{1}{\alpha^4} \right) y + \frac{1}{\alpha^2} \right).$$

[1] Since $u^k = [\mathcal{O}_{\mathbb{P}^{n-1-k}}] \in K(\mathbb{P}^{n-1})$ thus $p_*(u^k) = \chi(\mathbb{P}^{n-1-k}; \mathcal{O}) = 1$.

Below we present a sample of choices of one parameter subgroups and computed limits of motivic Chern classes:

(α, β, γ)	Minus–cell $(\mathbb{C}^4)_{\{0\}-}$	Inequality	$\Omega \cap (\mathbb{C}^4)_{\{0\}-}$	limit of $\frac{\mathrm{mC}(\Omega \cap (\mathbb{C}^4)^-_{\{0\}})_0}{e^K(\nu_0)}$
$(\xi, 1, 1)$	$\{0\}$	$0 \neq 0$	$\emptyset$	0
$(\xi^{-1}, 1, 1)$	$\mathbb{C}^4$	$z_1z_2 - z_3z_4 \neq 0$	cone complement $[\mathbb{C}^4] - [\mathbb{C}^*][\mathbb{P}^1]^2 - [0]$	$y^4 + (y+1)(1-y)^2 - 1$
$(\xi^{-1}, \xi^2, 1)$	$z_1 = 0$	$z_3z_4 \neq 0$	$\mathbb{C} \times (\mathbb{C}^*)^2$	$-y(y+1)^2$

5 Flag Manifolds and Quiver Representation Spaces

After fixing some combinatorial codes we will define two related geometric objects: a flag variety and a quiver representation space, together with the description of the Borel orbits in these spaces. Then we will present formulas for the motivic Chern classes of the orbits.

5.1 Combinatorial Codes

Let n, N be non-negative integers, and let $\mu = (\mu_1, \dots, \mu_N) \in \mathbb{N}^N$ with $\sum_{i=1}^N \mu_i = n$. Denote $\mu^{(i)} = \sum_{j=1}^i \mu_j$.

Let $\mathcal{I}_\mu$ denote the collection of N-tuples $(I_1, \dots, I_N)$ with $I_j \subset \{1, \dots, n\}$, $I_i \cap I_j = \emptyset$ unless $i = j$, $|I_i| = \mu_i$. For $I \in \mathcal{I}_\mu$ we will use the following notation: $I^{(j)} = \cup_{i=1}^j I_i = \{i_1^{(j)} < \dots < i_{\mu^{(j)}}^{(j)}\}$.

For $I \in \mathcal{I}_\mu$ define $\ell(I) = |\{(a, b) \in \{1, \dots, n\}^2 : a > b, a \in I_j, b \in I_k, j < k\}|$.

5.2 Flag Variety

Let Fl_μ denote the partial flag variety parameterizing chains of subspaces $V_\bullet = (0 \subset V_1 \subset V_2 \subset \dots \subset V_{N-1} \subset V_N = \mathbb{C}^n)$ with $\dim(V_i) = \mu^{(i)}$. Consider the natural action of the Borel subgroup $B_n \subset \mathrm{GL}_n(\mathbb{C})$ on Fl_μ. Let $\mathcal{F}_i$ be the tautological bundle over Fl_μ whose fiber over the point $(V_\bullet)$ is V_i. Let the B_n-equivariant K-theoretic Chern roots of $\mathcal{F}_i$ be $\alpha_j^{(i)}$, for $i = 1, \dots, N$, $j = 1, \dots, \mu^{(i)}$, that is, in $K_{B_n}(\mathrm{Fl}_\mu)$ we have $\sum_{j=1}^{\mu^{(i)}} \alpha_j^{(i)} = \mathcal{F}_i$. Then we have

$$K_{B_n}(\mathrm{Fl}_\mu)[y] = \mathbb{Z}[(\alpha_j^{(i)})^{\pm 1}, y]^{S_{\mu(1)} \times \dots \times S_{\mu(N-1)}} / (\text{certain ideal}). \qquad (7)$$

Here the symmetric group $S_{\mu^{(k)}}$ permutes the variables $\alpha_1^{(k)}, \ldots, \alpha_{\mu^{(k)}}^{(k)}$.

Definition 5.1 For $I \in \mathcal{I}_\mu$ define the Schubert cell

$$\Omega_I = \left\{(V_i) \in \mathrm{Fl}_\mu : \dim(V_p \cap \mathbb{C}^q_{last}) = \#\{i \in I^{(p)} : i > n-q\}, \forall p, q\right\} \subset \mathrm{Fl}_\mu,$$

where $\mathbb{C}^q_{last}$ is the span of the last q standard basis vectors in $\mathbb{C}^n$. Its codimension in Fl_μ is $\ell(I)$.

5.3 *Quiver Representation Space*

Consider $\mathrm{Rep}_\mu = \oplus_{j=1}^{N-1} \mathrm{Hom}(\mathbb{C}^{\mu^{(j)}}, \mathbb{C}^{\mu^{(j+1)}})$ with the action of

$$\mathrm{GL} \times B_n = \prod_{i=1}^{N-1} \mathrm{GL}_{\mu^{(i)}}(\mathbb{C}) \times B_n$$

given by

$$(g_i, \ldots, g_N) \cdot (a_j)_{j=1,\ldots,N-1} = (g_{j+1} \circ a_j \circ g_j^{-1})_{j=1,\ldots,N-1}.$$

For $i = 1, \ldots, N$ let $\mathcal{F}_i$ be the tautological rank $\mu^{(i)}$ bundle over the classifying space of the i'th component of $\mathrm{GL} \times B_n$. Let $\alpha_j^{(i)}$ be the K-theoretic Chern roots of $\mathcal{F}_i$, for $i = 1, \ldots, N$, $j = 1, \ldots, \mu^{(i)}$. Then we have

$$K_{\mathrm{GL} \times B_n}(\mathrm{Rep}_\mu)[y] = \mathbb{Z}[(\alpha_j^{(i)})^{\pm 1}, y]^{S_{\mu^{(1)}} \times \cdots \times S_{\mu^{(N-1)}}}. \tag{8}$$

Notice that the K-theory algebra in (7) is a quotient of the K-theory algebra in (8).

Definition 5.2 For $I \in \mathcal{I}_\mu$ define the matrix Schubert cell $M\Omega_I \subset \mathrm{Rep}_\mu$ by

$$\begin{aligned} M\Omega_I = \{(a_i) \in \mathrm{Rep}_\mu &: a_i \text{ is injective } \forall i, \\ &\dim((a_{N-1} \circ \ldots \circ a_p)(\mathbb{C}^{\mu^{(p)}}) \cap \mathbb{C}^q_{last}) = \#\{i \in I^{(p)} : i > n-q\}, \forall p, q\}, \end{aligned}$$

where $\mathbb{C}^q_{last}$ is the span of the last q standard basis vectors in $\mathbb{C}^n$. Its codimension in Rep_μ is $\ell(I)$.

Remark 5.3 Of course we have that $\mathrm{Fl}_\mu = \mathrm{Rep}_\mu /\!\!/ \mathrm{GL}$ as a GIT quotient space. Namely, let Rep_μ^{ss} be the subset consisting only injective maps. Then the natural map

$$\mathrm{Rep}_\mu^{ss} \to \mathrm{Fl}_\mu, \qquad (a_i) \mapsto ((a_{N-1} \circ \ldots \circ a_p)(\mathbb{C}^{\mu^{(p)}}))_p$$

is a topological quotient by GL, in fact $\mathrm{Rep}_\mu^{ss} \to \mathrm{Fl}_\mu$ is a principal bundle, cf. [42]. Under this map, we have $M\Omega_I$ maps to Ω_I. This correspondence between Fl_μ and Rep_μ will be used in Sect. 8.

5.4 Weight Functions

In this section we define some explicit functions that will be used in naming the equivariant motivic Chern classes of Schubert and matrix Schubert cells. For more details on these functions see e.g. [31–33] or references therein.

For $I \in \mathcal{I}_\mu$, $j = 1, \dots, N-1$, $a = 1, \dots, \mu^{(j)}$, $b = 1, \dots, \mu^{(j+1)}$ define

$$\psi_{I,j,a,b}(\xi) = \begin{cases} 1-\xi & \text{if } i_b^{(j+1)} < i_a^{(j)} \\ (1+y)\xi & \text{if } i_b^{(j+1)} = i_a^{(j)} \\ 1+y\xi & \text{if } i_b^{(j+1)} > i_a^{(j)}. \end{cases}$$

Define the *weight function*

$$W_I = \mathrm{Sym}_{S_{\mu^{(1)}} \times \cdots \times S_{\mu^{(N-1)}}} U_I$$

where

$$U_I = \prod_{j=1}^{N-1} \prod_{a=1}^{\mu^{(j)}} \prod_{b=1}^{\mu^{(j+1)}} \psi_{I,j,a,b}(\alpha_a^{(j)}/\alpha_b^{(j+1)}) \cdot \prod_{j=1}^{N-1} \prod_{1 \le a < b \le \mu^{(j)}} \frac{1 + y\alpha_b^{(j)}/\alpha_a^{(j)}}{1 - \alpha_b^{(j)}/\alpha_a^{(j)}}.$$

Here the symmetrizing operator is defined by

$$\mathrm{Sym}_{S_{\mu^{(1)}} \times \cdots \times S_{\mu^{(N-1)}}} U_I = \sum_{\sigma \in S_{\mu^{(1)}} \times \cdots \times S_{\mu^{(N-1)}}} U_I(\sigma(\alpha_a^{(j)}))$$

where the jth component of σ (an element of $S_{\mu^{(j)}}$) permutes the $\alpha^{(j)}$ variables. For

$$c_\mu = \prod_{j=1}^{N-1} \prod_{a=1}^{\mu^{(j)}} \prod_{b=1}^{\mu^{(j)}} (1 + y\alpha_b^{(j)}/\alpha_a^{(j)}), \qquad c'_\mu = \prod_{j=1}^{N-1} \prod_{a=1}^{\mu^{(j+1)}} \prod_{b=1}^{\mu^{(j)}} (1 + y\alpha_b^{(j)}/\alpha_a^{(j+1)})$$

define the *modified weight functions*

$$\widetilde{W}_I = W_I/c_\mu, \qquad \hat{W}_I = W_I/c'_\mu.$$

Observe that $\widetilde{W}_I$, $\hat{W}_I$ are not Laurent polynomials, but rather ratios of two such.

Weight functions were defined by Tarasov and Varchenko in relation with hypergeometric solutions to qKZ differential equations, see e.g. [37].

5.5 Motivic Chern Classes of Schubert Cells Given by Weight Functions

Theorem 5.4 *For the motivic Chern classes of matrix Schubert and ordinary Schubert cells we have*

$$\mathrm{mC}(M\Omega_I, \mathrm{Rep}_\mu) = W_I \qquad \in K_{\mathrm{GL}\times B_n}(\mathrm{Rep}_\mu)[y], \tag{9}$$

$$\mathrm{mC}(\Omega_I, \mathrm{Fl}_\mu) = [\widetilde{W}_I] \qquad \in K_{B_n}(\mathrm{Fl}_\mu)[y]. \tag{10}$$

First let us comment on the two statements of the theorem. As we saw in (8) the ring $K_{\mathrm{GL}\times B_n}(\mathrm{Rep}_\mu)[y]$ is a Laurent polynomial ring, and claim (9) states which Laurent polynomial is the sought MC class. However, the ring $K_{\mathrm{GL}\times B_n}(\mathrm{Fl}_\mu)[y]$ is a quotient ring of that Laurent polynomial ring, cf. (7), hence (10) only names *a representative* of the sought MC class. Moreover, the function $\widetilde{W}_I$ is a rational function, so the equality (10) is meant in the following sense: the restriction to each torus fixed of the two sides of (10) are the same—in particular the fix point restrictions of $\widetilde{W}_I$ are Laurent polynomials.

Remark 5.5 If we define the *motivic Segre class* of $X \subset M$ by

$$\mathrm{mS}(X, M) = \mathrm{mC}(X, M)/c^K(TM),$$

then we can rephrase Theorem 5.4 in the more symmetric form

$$\mathrm{mS}(M\Omega_I, \mathrm{Rep}_\mu) = \hat{W}_I, \qquad \mathrm{mS}(\Omega_I, \mathrm{Fl}_\mu) = [\hat{W}_I].$$

This is due to the fact that $c^K_{\mathrm{GL}\times B_n}(T\,\mathrm{Rep}_\mu) = c'_\mu$, and $[c^K_{B_n}(T\,\mathrm{Fl}_\mu)] = c'_\mu/c_\mu$. That is, c_μ is the K-theoretic total Chern class of the fibers of the GIT quotient map of Remark 5.3.

Now let us review different strategies that can be used to prove Theorem 5.4—some of them already present in the literature. One can

(a) prove (9) by resolution of singularities;
(b) prove (9) by the interpolation Theorem 4.6;
(c) prove (10) by resolution of singularities;
(d) prove (10) by the interpolation Theorem 4.6;
(e) prove that (9) implies (10) via Remark 5.5.

Carrying out (b) and/or (d) has the advantage of not having to construct a resolution with normal crossing divisors. Carrying out (a) and/or (c) has the advantage of not having to prove the sophisticated Newton polygon properties of the weight functions.

Historically, first the Newton polygon properties of weight functions were proved (in the context of K-theoretic stable envelopes), see [33, Sect. 3.5]. Those proofs are complicated algebraic arguments that were unknown before even by the experts of weight functions. Nevertheless, those arguments provide a complete proof of Theorem 5.4, see details in [19].

It is remarkable that resolutions for $\overline{M\Omega_I}$, $\overline{\Omega}_I$ can be constructed from which the motivic Chern classes are calculated easily. More importantly, for well chosen resolutions the formulas obtained for $\mathrm{mC}(M\Omega_I, \mathrm{Rep}_\mu)$, $\mathrm{mC}(\Omega_I, \mathrm{Fl}_\mu)$ are *exactly the defining formulas of weight functions*. As a result, such an argument *reproves* Theorem 5.4, and through that, the Newton polygon properties of weight functions without calculation. In hindsight, such an argument would be a more natural proof of Theorem 5.4. Since formally proving Theorem 5.4 is not needed anymore (as we mentioned above, [33, Sect. 3.5] provides a proof) we will not present a proof based on resolution of singularities in detail. Also, such an argument based on resolution is implicitly present in [30] and an analogous argument is explicitly presented in the elliptic setting in [35]. As a final remark let us mention our appreciation of the creativity of those who defined weight functions without seeing the resolution calculation for $\mathrm{mC}(M\Omega_I, \mathrm{Rep}_\mu)$ and $\mathrm{mC}(\Omega_I, \mathrm{Fl}_\mu)$!

The implication (e) is not formally needed to complete the proof of Theorem 5.4. Yet, we find it important enough, that we include the proof of the general such statement (namely motivic Chern classes before vs after GIT quotient) in Sect. 8.

6 MC Classes of Schubert Cells in Full Flag Manifolds: Positivity and Log-Concavity

Characteristic classes of singularities often display positivity properties. Motivic Chern classes are not exceptions: in this section we present three conjectures on the signs (and concavity) of coefficients of MC classes in certain expansions.

6.1 Positivity

For $\mu = (1, 1, \dots, 1)$ (n times) the space Fl_μ is the full flag variety, let us rename it to $\mathrm{Fl}(n)$. Recall the presentation of its equivariant K-theory algebra from (7). For brevity let us rename the "last" set of variables, the "equivariant variables" to $\tau_i := \alpha_i^{(n)}$.

The traditional geometric basis of $K_{B_n}(\mathrm{Fl}(n))[y]$ consists of the classes of the structure sheaves of the Schubert varieties (the closures of the Schubert cells). For $w = (w(1), \dots, w(n)) \in S_n$ let $[\mathbf{w}]$ denote the class of the structure sheaf of the closure of $\Omega_{(\{w(1)\},\{w(2)\},\dots,\{w(n)\})}$. Also, set $\mathrm{mC}[w] = \mathrm{mC}(\Omega_{(\{w(1)\},\{w(2)\},\dots,\{w(n)\})})$.

Our Theorem 5.4 can be used to expand the MC classes of orbits in the traditional basis of structure sheaves of Schubert varieties. For $n = 3$ we obtain the following expansions.

$$
\begin{aligned}
\mathrm{mC}[1,2,3] = {} & \Big(\tfrac{\tau_1^2}{\tau_3^2}y^3 + \Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2}\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3}\Big)y + 1\Big)\,[\mathbf{1},\mathbf{2},\mathbf{3}] \\
& - \Big(\Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1^2}{\tau_3^2}\Big)y^3 + \Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1}{\tau_2} + \tfrac{2\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2}\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 1\Big)y + 1\Big)\,[\mathbf{1},\mathbf{3},\mathbf{2}] \\
& - \Big(\Big(\tfrac{\tau_1^2}{\tau_3^2} + \tfrac{\tau_2\tau_1}{\tau_3^2}\Big)y^3 + \Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{2\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2} + \tfrac{\tau_2}{\tau_3}\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 1\Big)y + 1\Big)\,[\mathbf{2},\mathbf{1},\mathbf{3}] \\
& + \Big(\Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1^2}{\tau_3^2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2} + \tfrac{\tau_2}{\tau_3}\Big)y^3 + \Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1}{\tau_2} + \tfrac{3\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2} + \tfrac{2\tau_2}{\tau_3} + 1\Big)y^2 \\
& \qquad + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 2\Big)y + 1\Big)\,[\mathbf{2},\mathbf{3},\mathbf{1}] \\
& + \Big(\Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1^2}{\tau_3^2} + \tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2}\Big)y^3 + \Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{2\tau_1}{\tau_2} + \tfrac{3\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2} + \tfrac{\tau_2}{\tau_3} + 1\Big)y^2 \\
& \qquad + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 2\Big)y + 1\Big)\,[\mathbf{3},\mathbf{1},\mathbf{2}] \\
& - \Big(\Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1^2}{\tau_3^2} + \tfrac{\tau_1}{\tau_2} + \tfrac{2\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2} + \tfrac{\tau_2}{\tau_3} + 1\Big)y^3 + \Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{2\tau_1}{\tau_2} + \tfrac{3\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2} + \tfrac{2\tau_2}{\tau_3} + 2\Big)y^2 \\
& \qquad + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 2\Big)y + 1\Big)\,[\mathbf{3},\mathbf{2},\mathbf{1}].
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{mC}[1,3,2] = {} & \Big(\tfrac{\tau_1^2}{\tau_2\tau_3}y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3}\Big)y + 1\Big)\,[\mathbf{1},\mathbf{3},\mathbf{2}] \\
& - \Big(\Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3}\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 1\Big)y + 1\Big)\,[\mathbf{2},\mathbf{3},\mathbf{1}] \\
& - \Big(\Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1}{\tau_2}\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + 1\Big)y + 1\Big)\,[\mathbf{3},\mathbf{1},\mathbf{2}] \\
& + \Big(\Big(\tfrac{\tau_1^2}{\tau_2\tau_3} + \tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 1\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 2\Big)y + 1\Big)\,[\mathbf{3},\mathbf{2},\mathbf{1}].
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{mC}[2,1,3] = {} & \Big(\tfrac{\tau_1\tau_2}{\tau_3^2}y^2 + \Big(\tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3}\Big)y + 1\Big)\,[\mathbf{2},\mathbf{1},\mathbf{3}] \\
& - \Big(\Big(\tfrac{\tau_2}{\tau_3} + \tfrac{\tau_1\tau_2}{\tau_3^2}\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 1\Big)y + 1\Big)\,[\mathbf{2},\mathbf{3},\mathbf{1}] \\
& - \Big(\Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2}\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 1\Big)y + 1\Big)\,[\mathbf{3},\mathbf{1},\mathbf{2}] \\
& + \Big(\Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2\tau_1}{\tau_3^2} + \tfrac{\tau_2}{\tau_3} + 1\Big)y^2 + \Big(\tfrac{\tau_1}{\tau_2} + \tfrac{\tau_1}{\tau_3} + \tfrac{\tau_2}{\tau_3} + 2\Big)y + 1\Big)\,[\mathbf{3},\mathbf{2},\mathbf{1}].
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{mC}[2,3,1] = {} & \Big(\tfrac{\tau_2}{\tau_3}y + 1\Big)\,[\mathbf{2},\mathbf{3},\mathbf{1}] \\
& - \Big(\Big(\tfrac{\tau_2}{\tau_3} + 1\Big)y + 1\Big)\,[\mathbf{3},\mathbf{2},\mathbf{1}]. \\
\mathrm{mC}[3,1,2] = {} & \Big(\tfrac{\tau_1}{\tau_2}y + 1\Big)\,[\mathbf{3},\mathbf{1},\mathbf{2}] \\
& - \Big(\Big(\tfrac{\tau_1}{\tau_2} + 1\Big)y + 1\Big)\,[\mathbf{3},\mathbf{2},\mathbf{1}]. \\
\mathrm{mC}[3,2,1] = {} & [\mathbf{3},\mathbf{2},\mathbf{1}].
\end{aligned}
$$

The following conjecture can be verified in the formulas above, and we also verified it for larger flag varieties ($n \leq 5$).

Conjecture 6.1 Let $p, w \in S_n$. The coefficient of $[\mathbf{w}]$ in the expansion of $\mathrm{mC}[p]$ is a Laurent polynomial in $\tau_1, \ldots, \tau_n, y$ whose terms have sign $(-1)^{\ell(p)-\ell(w)}$.

We have been informed by the authors of [4] that they also observed the sign behavior described in Conjecture 6.1.

6.2 Log Concavity

To illustrate a new feature of the coefficients of the [**w**]-expansions, let us make the substitution $\tau_i = 1$ ($\forall i$), that is, consider the *non-equivariant* motivic Chern classes of the Schubert cells. For $n = 4$ and for the open cell we obtain

$$
\begin{aligned}
\mathrm{mC}[1,2,3,4] = {} & (y+1)^6\,[\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}] \\
& - (y+1)^5(2y+1)\,[\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{3}] \\
& - (y+1)^5(2y+1)\,[\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{4}] \\
& - (y+1)^5(2y+1)\,[\mathbf{2},\mathbf{1},\mathbf{3},\mathbf{4}] \\
& + (y+1)^4(5y^2+4y+1)\,[\mathbf{1},\mathbf{3},\mathbf{4},\mathbf{2}] \\
& + (y+1)^4(5y^2+4y+1)\,[\mathbf{1},\mathbf{4},\mathbf{2},\mathbf{3}] \\
& + (y+1)^4(2y+1)^2\,[\mathbf{2},\mathbf{1},\mathbf{4},\mathbf{3}] \\
& + (y+1)^4(5y^2+4y+1)\,[\mathbf{2},\mathbf{3},\mathbf{1},\mathbf{4}] \\
& + (y+1)^4(5y^2+4y+1)\,[\mathbf{3},\mathbf{1},\mathbf{2},\mathbf{4}] \\
& - (y+1)^3(8y^3+11y^2+5y+1)\,[\mathbf{1},\mathbf{4},\mathbf{3},\mathbf{2}] \\
& - (y+1)^3(14y^3+14y^2+6y+1)\,[\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{1}] \\
& - (y+1)^3(13y^3+13y^2+6y+1)\,[\mathbf{2},\mathbf{4},\mathbf{1},\mathbf{3}] \\
& - (y+1)^3(2y+1)(6y^2+4y+1)\,[\mathbf{3},\mathbf{1},\mathbf{4},\mathbf{2}] \\
& - (y+1)^3(8y^3+11y^2+5y+1)\,[\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{4}] \\
& - (y+1)^3(14y^3+14y^2+6y+1)\,[\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{3}] \\
& + (y+1)^2(24y^4+36y^3+21y^2+7y+1)\,[\mathbf{2},\mathbf{4},\mathbf{3},\mathbf{1}] \\
& + (y+1)^2(23y^4+35y^3+22y^2+7y+1)\,[\mathbf{3},\mathbf{2},\mathbf{4},\mathbf{1}] \\
& + (y+1)^2(3y+1)(10y^3+10y^2+4y+1)\,[\mathbf{3},\mathbf{4},\mathbf{1},\mathbf{2}] \\
& + (y+1)^2(23y^4+35y^3+22y^2+7y+1)\,[\mathbf{4},\mathbf{1},\mathbf{3},\mathbf{2}] \\
& + (y+1)^2(24y^4+36y^3+21y^2+7y+1)\,[\mathbf{4},\mathbf{2},\mathbf{1},\mathbf{3}] \\
& - (y+1)(44y^5+85y^4+66y^3+29y^2+8y+1)\,[\mathbf{3},\mathbf{4},\mathbf{2},\mathbf{1}] \\
& - (y+1)(49y^5+91y^4+69y^3+30y^2+8y+1)\,[\mathbf{4},\mathbf{2},\mathbf{3},\mathbf{1}] \\
& - (y+1)(44y^5+85y^4+66y^3+29y^2+8y+1)\,[\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{2}] \\
& + (64y^6+163y^5+169y^4+98y^3+37y^2+9y+1)\,[\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{1}]
\end{aligned}
$$

In the expression above the permutations are ordered primary by length, secondary by lexicographical order.

The coefficient of $[\mathbf{5},\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{1}]$ in $\mathrm{mC}\,[1,2,3,4,5]$ is

$$1+14y+92y^2+377y^3+1120y^4+2630y^5+4972y^6+7148y^7+7024y^8+4063y^9+1024y^{10}.$$

Definition 6.2 The polynomial $\sum_{k=0}^{d} a_k y^k$ is said to be log-concave if $a_k^2 \geq a_{k-1}a_{k+1}$ for $0 < k < d$. It is strictly log-concave if the equality is strict.

See Huh's survey article [21] for the role of log-concavity in geometry and combinatorics.

Conjecture 6.3 The coefficients in the [**w**]-expansion of the non-equivariant $\mathrm{mC}[p]$ classes are strictly log-concave.

We checked Conjecture 6.3 for $n \leq 6$.

6.3 Positivity in New Variables

There is another kind of positivity which does not follow from the conjectured properties above. Namely, let us substitute $\frac{\tau_i}{\tau_{i+1}} = s_i + 1$ and $y = -1 - \delta$. For example

$$\begin{aligned}\mathrm{mC}[4,3,1,2] &= (1 + y\tfrac{\tau_1}{\tau_2})[\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{2}] - y[\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{1}] \\ &= -(s_1 + \delta(1+s_1))[\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{2}] + (1+\delta)[\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{1}]\,.\end{aligned}$$

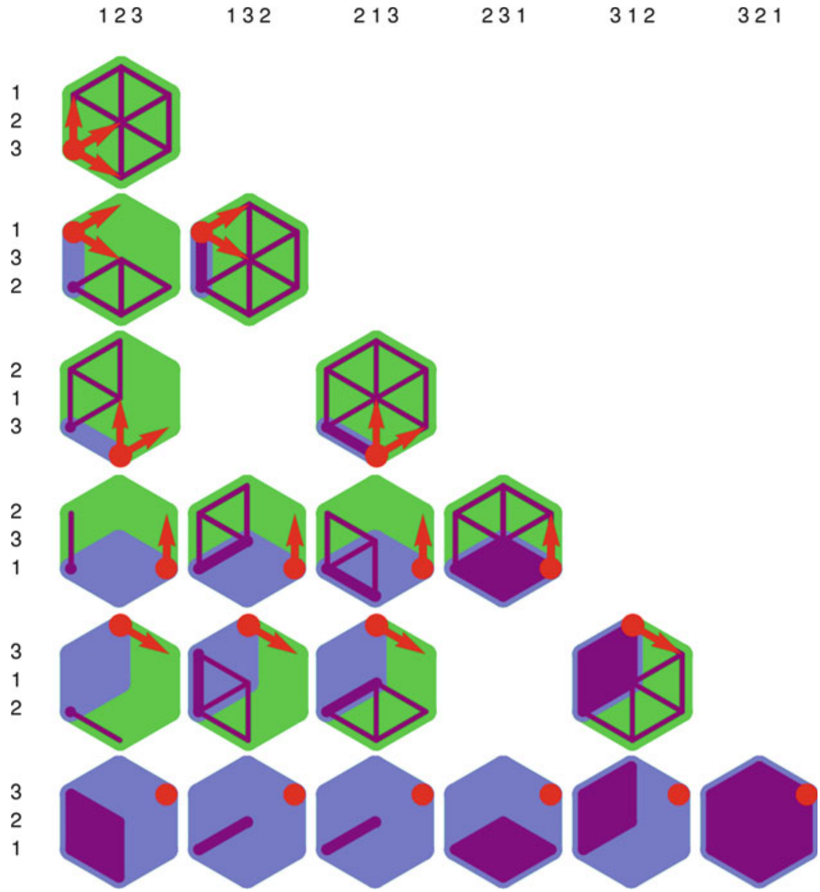
The coefficient of $[\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{1}]$ in $\mathrm{mC}[1,4,3,2]$ in the new variables equals
$(1+s_1)^3(1+s_2)^2(1+s_3)+$
$\delta(9 + 17s_1 + 12s_1^2 + 3s_1^3 + 14s_2 + 28s_1s_2 + 22s_1^2s_2 + 6s_1^3s_2 + 6s_2^2 + 12s_1s_2^2 + 10s_1^2s_2^2 + 3s_1^3s_2^2 + 8s_3 + 15s_1s_3 + 11s_1^2s_3 + 3s_1^3s_3 + 13s_2s_3 + 26s_1s_2s_3 + 21s_1^2s_2s_3 + 6s_1^3s_2s_3 + 6s_2^2s_3 + 12s_1s_2^2s_3 + 10s_1^2s_2^2s_3 + 3s_1^3s_2^2s_3)+$
$\delta^2(21 + 28s_1 + 15s_1^2 + 3s_1^3 + 26s_2 + 40s_1s_2 + 26s_1^2s_2 + 6s_1^3s_2 + 9s_2^2 + 15s_1s_2^2 + 11s_1^2s_2^2 + 3s_1^3s_2^2 + 16s_3 + 22s_1s_3 + 13s_1^2s_3 + 3s_1^3s_3 + 22s_2s_3+35s_1s_2s_3+24s_1^2s_2s_3 + 6s_1^3s_2s_3 + 9s_2^2s_3 + 15s_1s_2^2s_3 + 11s_1^2s_2^2s_3 + 3s_1^3s_2^2s_3)+$
$\delta^3(14 + 14s_1 + 6s_1^2 + s_1^3 + 14s_2 + 18s_1s_2 + 10s_1^2s_2 + 2s_1^3s_2 + 4s_2^2 + 6s_1s_2^2 + 4s_1^2s_2^2 + s_1^3s_2^2 + 9s_3 + 10s_1s_3 + 5s_1^2s_3 + s_1^3s_3 + 11s_2s_3 + 15s_1s_2s_3 + 9s_1^2s_2s_3 + 2s_1^3s_2s_3 + 4s_2^2s_3 + 6s_1s_2^2s_3 + 4s_1^2s_2^2s_3 + s_1^3s_2^2s_3)$.

Conjecture 6.4 The coefficients of the s_i, δ-monomials in the $[\mathbf{w}]$-expansion of the $\mathrm{mC}[p]$ classes have sign $(-1)^{\ell(w)}$. Note that here the sign depends only on the length the permutation w, not on the length of p, as it was in Conjecture 6.1.

Remark 6.5 The positivity properties of Sects. 6.1 and 6.3 imply positivity properties of the restrictions of MC classes to fixed points. These "local" positivity properties are another instances of the positivity in the δ-variables discussed in [40, Sect. 15]. It is implied by the Conjecture 6.4 and [5].

7 Local Picture

To illustrate the pretty convex geometric objects encoded by motivic Chern classes in this section we present some pictures. Consider $\mathrm{Fl}(3)$ and its six Schubert cells parameterized by permutations. According to Theorem 5.4 the six fixed point restrictions of these classes satisfy some strict containment properties. Below we present the Newton polygons (and some related weights) of the six fixed point restrictions (rows) of the six Schubert cells (columns). The Newton polygons live in the $\tau_1 + \tau_2 + \tau_3 = 0$ plane of $\mathbb{R}^3$ (with coordinates τ_i), hence instead of 3D pictures we only draw the mentioned plane.



In blue—Newton polygon of $e^K(\nu_\Theta) = \lambda_{-1}(\nu^*_\Theta)$.
In red—cotangent weights of T^*_Θ.
In violet—Newton polygon of $\mathrm{mC}(\Omega \cap S_\Theta)$, where S_Θ is a slice to the orbit.
Edges in violet—Newton polygon of $\mathrm{mC}(\Omega)$.

8 Transversality and GIT Quotients

Both CSM and MC classes have their Segre version (cf. Remark 5.5): for example for the K-theory case we have

$$\mathrm{mS}(f : Z \to M) := \mathrm{mC}(f)/\lambda_y(T^*M).$$

8.1 Transversality and Motivic Segre Class

Motivic Segre classes behave well for transversal pull-back, for a fine notion of transversality. To formulate this notion let us recall the definition of the motivic Chern class given in [19]. Consider the case of $\mathrm{mC}(f)$ where $f : U \to M$ is an inclusion of smooth varieties, but the inclusion is not necessarily closed (or equivalently, proper).

Definition 8.1 A proper normal crossing extension (PNC) of $f : U \to M$ is a morphism $\bar{f} : Y \to M$ and an inclusion $j : U \hookrightarrow Y$ such that

(1) $f = \bar{f} \circ j$,
(2) Y is smooth
(3) $\bar{f}$ is proper
(4) The exceptional divisor $D := Y \setminus j(U) = \bigcup_{i=1}^{s} D_i$ is a simple normal crossing divisor (i.e. the D_i's are smooth hypersurfaces in transversal position).

If f is G-equivariant for some group G acting on U and M, then we require all maps to be G-equivariant in the definition.

Remark 8.2 Notice that if f is injective then $\bar{f} : Y \to M$ is a resolution of the closure of $f(U)$.

We can use the existence of proper normal crossing extensions to define the (G-equivariant) motivic Chern class:

Definition 8.3 Suppose that $f : U \to M$ is a map of smooth G-varieties and let $\bar{f} : Y \to M$ be a proper normal crossing extension of f. For $I \subset \underline{s} = \{1, 2, \dots, s\}$ let $D_I = \bigcap_{i \in I} D_i$, $f_I = \bar{f}|D_I$, in particular $f_\emptyset = \bar{f}$. Then

$$\mathrm{mC}(f) := \sum_{I \subset \underline{s}} (-1)^{|I|} f_{I*} \lambda_y(T^* D_I) \,. \tag{11}$$

It is explained in [19] that this is a good definition: independent of the PNC chosen.

Definition 8.4 Let N be a smooth variety. Then $g : N \to M$ is motivically transversal [2] to $f : U \to M$ if there is a PNC $\bar{f} : Y \to M$ for f such that g is transversal to all the D_I's.

Theorem 8.5 *If $g : N \to M$ is motivically transversal to $f : U \to M$ then*

$$\mathrm{mS}(\widetilde{f}) = g^* \, \mathrm{mS}(f),$$

where $\widetilde{f} : \widetilde{U} \to N$ is the map in the pull-back diagram

[2] This idea appeared already in an early version of [26].

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "\widetilde{f}"] \arrow[d] & N \arrow[d, "g"] \\
U \arrow[r, "f"] & M.
\end{tikzcd}$$

The proof is a straightforward consequence of the fact that a motivically transversal pull-back of a PNC is PNC. We can extend the notion to non-smooth varieties Z requiring the existence of a stratification of Z such that g is motivically transversal to the restrictions of f to these strata.

Being motivically transversal is a very restrictive condition. However it holds in some special situations:

Proposition 8.6 *Let U, M be smooth and a connected group G is acting on M. Assume that $f : U \to M$ is transversal to all G-orbits. Then f is motivically transversal to all G-invariant subvarieties of M.*

The proof is simple and can be found in the proof of Lemma 5.1 in [19].

8.2 G-equivariant Motivic Segre Class as a Universal Motivic Segre Class for Degeneracy Loci

Now we prove a statement (Corollary 8.8) expressing the fact that the G-equivariant motivic Segre class is a universal formula for motivic Chern classes of degeneracy loci. The analogous statement for the Segre version of the CSM class was proved in [26]. In K-theory the proof is simpler because reference to classifying spaces and maps can be avoided. Suppose that $\pi_P : P \to M$ is a principal G-bundle over the smooth M and A is a smooth G-variety. Then we can define a map

$$a : K_G(A) \to K(P \times_G A) \tag{12}$$

by association: For any G-vector bundle E over A the associated bundle $P \times_G E$ is a vector bundle over $P \times_G A$.

In the rest of the paper we will use the notation $\mathrm{mS}(A, B)$ for $\mathrm{mS}(f : A \to B)$ when the map f is clear from the context. The diagrams

$$\begin{tikzcd}
P \arrow[d, "\pi_P"', "G-principal"] \\
M
\end{tikzcd} \qquad G \curvearrowright (i : Y \subset A) \qquad \begin{tikzcd}
P \times_G Y \arrow[rr, hook, "i_P"] \arrow[dr, "\pi_Y"'] & & P \times_G A \arrow[dl, "\pi_A"] \\
& M &
\end{tikzcd}$$

will be useful when reading the proof of the next Proposition.

Proposition 8.7 *Let $Y \subset A$ be G-invariant. Then*

$$\mathrm{mS}(P \times_G Y, P \times_G A) = a\big(\mathrm{mS}_G(Y, A)\big).$$

Proof To calculate the left hand side we need to calculate $\lambda_y(T^*P \times_G A)$ first. For the tangent space we have

$$T(P \times_G A) = \pi_A^*(TM) \oplus (P \times_G TA),$$

where $\pi_A : P \times_G A \to M$ is the projection. Consequently

$$\lambda_y(T^*(P \times_G A)) = \pi_A^* \lambda_y(T^*M) \cdot a(\lambda_y^G(T^*A)).$$

Assume first that Y is a closed submanifold of A. To ease the notation we will use the $\lambda(M) = \lambda_y(T^*M)$ abbreviation. Then

$$\mathrm{mS}(P \times_G Y, P \times_G A) = \frac{i_*^P \lambda(P \times_G Y)}{\lambda(P \times_G A)} = \frac{i_*\big(\pi_Y^* \lambda(M) a(\lambda^G(Y)\big)}{\pi_A^* \lambda(M) a(\lambda^G(A))},$$

where $\pi_Y : P \times_G Y \to M$ is the projection and $i : Y \to A$, $i_P : P \times_G Y \to P \times_G A$ are the inclusions. Then noticing that $\pi_Y = \pi_A \circ i_P$ and applying the adjunction formula we arrive at the right hand side.

For general Y we can use Definition 8.3 to reduce the calculation to the smooth case. □

Corollary 8.8 *Suppose that $\sigma : M \to P \times_G A$ is a section motivically transversal to $P \times_G Y$. Then*

$$\mathrm{mS}(Y(\sigma), M) = \sigma^* a\big(\mathrm{mS}_G(Y, A)\big), \tag{13}$$

where $Y(\sigma) = \sigma^{-1}(P \times_G Y)$ is the Y-locus of the section σ.

If A is a vector space then σ^* can be identified with the identity map $K(P \times_G A) \simeq K(P/G) = K(M)$.

8.3 GIT Quotients

With the applications in mind we use the following simple version of GIT quotient: Let V be a G-vector space for a connected algebraic group G and assume that $P \subset V$ is an open G-invariant subset such that $\pi : P \to P/G$ is a principal G-bundle over the smooth $M := P/G$. (We want P to be a right G-space so we define $pg := g^{-1}p$.)

To state the main theorem we first introduce the *K-theoretic Kirwan map* $\kappa : K_G(V) \to K(M)$ as the composition of the pull back $K(P \times_G V) \to K(M)$ via the zero section and the association map $K_G(V) \to K(P \times_G V)$ given in Equation (12) (cf. the RHS of (13)). Notice again that this definition is simpler than the cohomological analogue.

Theorem 8.9 *Let V be a G-vector space and assume that $P \subset V$ is an open G-invariant subset such that $\pi : P \to P/G$ is a principal G-bundle over the smooth $M := P/G$. Let $Y \subset M$. Then*

$$\mathrm{mS}(Y, M) = \kappa(\mathrm{mS}_G(\pi^{-1}(Y), V)).$$

To prove this theorem we apply Corollary 8.8 to the *universal section*: The inclusion $j : P \to V$ is obviously G-equivariant therefore induces a section $\sigma_j : M \to P \times_G V$. In other words $\sigma_j(m) = [p, p]$ for any p with $\pi(p) = m$.

Example 8.10 For $V = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ with the $G = \mathrm{GL}_k(\mathbb{C})$-action we can choose $P := \Sigma^0(\mathbb{C}^k, \mathbb{C}^n)$, the set of injective maps, so $P/G = \mathrm{Gr}_k(\mathbb{C}^n)$. Then $P \times_G V$ is the vector bundle $\mathrm{Hom}(\gamma^k, \mathbb{C}^n)$ for the tautological subbundle γ^k and σ_j is the section expressing the fact that γ^k is a subbundle of the trivial bundle of rank n. More generally see Remark 5.3.

The reason we call σ_j universal is the following simple observation: For any $Y \subset M$ the $\pi^{-1}(Y)$-locus of σ_j is Y. Therefore Theorem 8.9 is a consequence of the following:

Proposition 8.11 *The universal section σ_j is motivically transversal to $P \times_G \pi^{-1}(Y)$ for any $Y \subset M$.*

In fact we can replace $\pi^{-1}(Y)$ with any G-invariant (algebraic) subset $Z \subset V$.

Proof The statement is local so it is enough to study the restriction of σ_j to an open subset of M over which $P \times_G V$ is trivial. A local trivialization can be obtained by a $\varphi : W \to P$ local section of P which is transversal to the fibers i.e. to the G-orbits. In this local trivialization the section $\sigma_j|_W$ is the graph of the map $j\varphi : W \to V$, therefore σ_j is motivically transversal to $P \times_G Z$ if φ is motivically transversal to $Z \subset V$, which is implied by Proposition 8.6. □

With the same argument we can obtain an equivariant version of Theorem 8.9:

Theorem 8.12 *Let V be a $G \times H$-vector space and assume that $P \subset V$ is an open $G \times H$-invariant subset such that $\pi : P \to P/G$ is a principal G-bundle over the smooth $M := P/G$. Let $Y \subset M$ be H-invariant. Then*

$$\mathrm{mS}_H(Y, M) = \kappa_H(\mathrm{mS}_{G\times H}(\pi^{-1}(Y), V)),$$

where κ_H is the equivariant Kirwan map: the composition of the pull back $K_H(P \times_G V) \to K_H(M)$ via the zero section and the association map $K_{G\times H}(V) \to K_H(P \times_G V)$.

As a consequence we proved argument (e) in Sect. 5.4 (cf. Remark 5.3): the calculation of motivic Chern classes of matrix Schubert cells leads directly to the calculation of motivic Chern classes of ordinary Schubert cells. The formulas for motivic Chern classes are modified by the respective total Chern classes of the ambient spaces, while the formulas for the motivic Segre classes are identical (cf. Remark 5.5).

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References

1. Aluffi, P.: Chern classes of birational varieties. Int. Math. Res. Not. **63**, 3367–3377 (2004)
2. Aluffi, P., Mihalcea, L.C.: Chern-Schwartz-MacPherson classes for Schubert cells in flag manifolds. Compositio Mathematica **152**(12), 2603–2625 (2016)
3. Aluffi, P., Mihalcea, L.C., Schürmann, J., Su, C.: Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells. arXiv:1709.08697
4. Aluffi, P., Mihalcea, L.C., Schürmann, J., Su, C.: Motivic chern classes of schubert cells, hecke algebras, and applications to casselman's problem. arXiv:1902.10101
5. Anderson, D., Griffeth, S., Miller, E.: Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces. J. Eur. Math. Soc. **13**(1), 57–84 (2011)
6. Atiyah, M.F., Bott, R.: The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A **308**(1505), 523–615 (1983)
7. Białynicki-Birula, A.: Some theorems on actions of algebraic groups. Ann. Math. 98(2), 480–497 (1973)
8. Borel, A.: Seminar on transformation groups, (with contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais). Annals of Mathematics Studies, No. 46, Princeton, UP (1960)
9. Bott, R.: An introduction to equivariant cohomology. Quantum field theory: perspective and prospective (Les Houches, 1998), pp. 35–56, NATO Science Institutes Series C: Mathematical and Physical Sciences, vol. 530. Kluwer Academic Publishers, Dordrecht (1999)
10. Brasselet, J.-P., Schürmann, J., Yokura, S.: Hirzebruch classes and motivic Chern classes for singular spaces. J. Topol. Anal. **2**(1), 1–55 (2010)
11. Cappell, S.E., Maxim, L.G., Schürmann, J., Shaneson, J.L.: Equivariant characteristic classes of singular complex algebraic varieties. Commun. Pure Appl. Math. **65**(12), 1722–1769 (2012)
12. Carrell, J.B.: Torus actions and cohomology. Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action, pp. 83–158, Encyclopaedia of Mathematical Sciences, vol. 131, Invariant Theory and Algebraic Transformation Groups, II. Springer, Berlin (2002)
13. Chriss, N., Ginzburg, V.: Representation Theory and Complex Geometry. Modern Birkhäuser Classics. Birkhäuser, Boston (1997); 2nd edn. (2009)
14. Fehér, L.M., Némethi, A., Rimányi, R.: Equivariant classes of matroid realization spaces. Comment. Math. Helv. **87**, 861–889 (2012)
15. Fehér, L., Rimányi, R.: Schur and Schubert polynomials as Thom polynomials–cohomology of moduli spaces. Cent. European J. Math. **4**, 418–434 (2003)
16. Fehér, L., Rimányi, R.: Calculation of Thom polynomials and other cohomological obstructions for group actions In: Real and Complex Singularities (Sao Carlos 2000) Gaffney, T., Ruas, M. (eds.) Contemp. Math. #354, pp. 69–93. AMS (2004)
17. Fehér, L., Rimányi, R.: Classes of degeneracy loci for quivers–the Thom polynomial point of view. Duke Math. J. **114**(2), 193–213 (2002)
18. Fehér, L.M., Rimányi, R.: Chern-Schwartz-MacPherson classes of degeneracy loci. Geom. Topol. **22**, 3575–3622 (2018)
19. Fehér, L.M., Rimányi, R., Weber, A.: Motivic Chern classes and K-theoretic stable envelopes. arXiv:1802.01503
20. Goresky, M., Kottwitz, R., MacPherson, R.: Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math. **131**(1), 25–83 (1998)

21. Huh, J.: Combinatorial applications of the Hodge-Riemann relations. In: Proceedings of the International Congress of Mathematicians 2018, Rio de Janeiro, vol. 3, pp. 3079–3098
22. MacPherson, R.: Chern classes for singular algebraic varieties. Ann. Math. **100**, 421–432 (1974)
23. Maulik, D., Okounkov, A.: Quantum Groups and Quantum Cohomology, Astérisque, vol. 408. Société Mathématique de France (2019). arXiv:1211.1287
24. Mikosz, M., Weber, A.: Equivariant Hirzebruch class for quadratic cones via degenerations. J. Singul. **12**, 131–140 (2015)
25. Ohmoto, T.: Equivariant Chern classes of singular algebraic varieties with group actions. Math. Proc. Cambridge Phil. Soc. **140**, 115–134 (2006)
26. Ohmoto, T.: Singularities of maps and characteristic classes. Adv. Stud. Pure Math. **68**(Math. Soc. Japan), 171–245 (2016)
27. Okounkov, A.: Lectures on K-theoretic computations in enumerative geometry. Geometry of moduli spaces and representation theory, pp. 251–380. IAS/Park City Mathematics Series, vol. 24. AMS (2017)
28. Promtapan, S., Rimányi, R.: Characteristic classes of symmetric and skew-symmetric degeneracy loci. arXiv:1908.07373
29. Rimányi, R.: Thom polynomials, symmetries and incidences of singularities. Invent. Math. **143**(3), 499–521 (2001)
30. Rimányi, R.: Motivic characteristic classes in cohomological Hall algebras. To appear in Adv. Math. (2019)
31. Rimányi, R., Tarasov, V., Varchenko, A.: Partial flag varieties, stable envelopes and weight functions. Quantum Topol. **6**(2), 333–364 (2015)
32. Rimányi, R., Tarasov, V., Varchenko, A.: Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety. J. Geom. Phys. **94**, 81–119 (2015)
33. Rimányi, R., Tarasov, V., Varchenko, A.: Elliptic and K-theoretic stable envelopes and Newton polytopes. Selecta Math. **25**, 16 (2019)
34. Rimányi, R., Varchenko, A.: Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae. In: Schubert Varieties, Equivariant Cohomology and Characteristic Classes—IMPANGA 15, pp. 225–235. EMS (2018)
35. Rimányi, R., Weber, A.: Elliptic classes of Schubert cells via Bott-Samelson resolution. arXiv:1904.10852
36. Segal, G.: Equivariant K-theory. Institut des Hautes Études Scientifiques Publications Math. No. 34, pp. 129–151 (1968)
37. Tarasov, V., Varchenko, A.: Combinatorial formulae for nested Bethe vectors. SIGMA **9**(048), 1–28 (2013)
38. Uma, V.: Equivariant K-theory of flag varieties revisited and related results. Colloq. Math. **132**(2), 151–175 (2013)
39. Weber, A.: Equivariant Chern classes and localization theorem. J. Singul. **5**, 153–176 (2012)
40. Weber, A.: Equivariant Hirzebruch class for singular varieties. Selecta Math. (N.S.) **22**(3), 1413–1454 (2016)
41. Weber, A.: Hirzebruch class and Białynicki-Birula decomposition. Transform. Groups **22**(2), 537–557 (2017)
42. Zielenkiewicz, M.: The Gysin homomorphism for homogeneous spaces via residues, Ph.D. Thesis (2017). arXiv:1702.04123

A Survey of Recent Developments on Hessenberg Varieties

Hiraku Abe and Tatsuya Horiguchi

Abstract This article surveys recent developments on Hessenberg varieties, emphasizing some of the rich connections of their cohomology and combinatorics. In particular, we will see how hyperplane arrangements, representations of symmetric groups, and Stanley's chromatic symmetric functions are related to the cohomology rings of Hessenberg varieties. We also include several other topics on Hessenberg varieties to cover recent developments.

Keywords Hessenberg varieties · Flag varieties · Cohomology · Hyperplane arrangements · Representations of symmetric groups · Chromatic symmetric functions.

1 Introduction

Hessenberg varieties are subvarieties of the full flag variety which was introduced by F. De Mari, C. Procesi, and M. A. Shayman [21, 22] around 1990. They provide a relatively new research subject, and similarly to Schubert varieties it has been found that geometry, combinatorics, and representation theory interact nicely on Hessenberg varieties. Let X be a complex $n \times n$ matrix considered as a linear map $X : \mathbb{C}^n \to \mathbb{C}^n$ and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ a Hessenberg function, i.e. a non-decreasing function satisfying $h(j) \geq j$ for $1 \leq j \leq n$. The **Hessenberg variety** (in type A_{n-1}) associated to X and h is defined as follows:

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$$\text{Hess}(X, h) := \{V_\bullet \in Fl(\mathbb{C}^n) | XV_j \subseteq V_{h(j)} \text{ for all } 1 \leq j \leq n\},$$

where $Fl(\mathbb{C}^n)$ is the flag variety of $\mathbb{C}^n$ consisting of sequences $V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$ of linear subspaces of $\mathbb{C}^n$ with dim $V_i = i$ for $1 \leq i \leq n$. Particular examples include the full flag variety itself, Springer fibres, the Peterson variety, and the permutohedral variety.

Over the past 20 years, plentiful developments of Hessenberg varieties has been made. For example, it has been discovered that hyperplane arrangements and representations of symmetric groups appear when we deal with the cohomology rings of Hessenberg varieties. Also, these representations are determined by Stanley's chromatic symmetric functions of certain graphs, and is related with the Stanley-Stembridge conjecture in graph theory.

This article is a survey of recent developments on Hessenberg varieties, and it is intended to stimulate future research. We keep our explanations concise and include concrete examples so as to make the important ideas accessible, especially for young mathematicians (e.g. graduate students, postdoctoral fellows, and so on). We also include several other topics on Hessenberg varieties to cover recent developments. For simplicity, we explain most of the results in type A, but we make comments for results which hold in arbitrary Lie type.

2 Background and Notations

In this section, we recall some background, and establish some notations for the rest of the document.

2.1 *Definitions and Basic Properties*

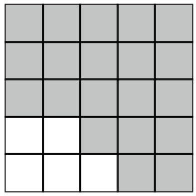
Let n be a positive integer, and we use the notation $[n] := \{1, 2, \ldots, n\}$ throughout this document. A function $h : [n] \to [n]$ is a **Hessenberg function** if it satisfies the following two conditions:

(i) $h(1) \leq h(2) \leq \cdots \leq h(n)$,
(ii) $h(j) \geq j$ for all $j \in [n]$.

Note that $h(n) = n$ by definition. We frequently write a Hessenberg function by listing its values in a sequence, i.e. $h = (h(1), h(2), \ldots, h(n))$. We may identify a Hessenberg function h with a configuration of (shaded) boxes on a square grid of size $n \times n$ which consists of boxes in the i-th row and the j-th column satisfying $i \leq h(j)$ for $i, j \in [n]$, as we illustrate in the following example.

Example 2.1 Let $n = 5$. The Hessenberg function $h = (3, 3, 4, 5, 5)$ corresponds to the configuration of the shaded boxes drawn in Fig. 1.

Fig. 1 The configuration corresponding to $h = (3, 3, 4, 5, 5)$



In particular, this identification implies that the set of Hessenberg functions and the set of Dyck paths are in one-to-one correspondence. That is, the number of Hessenberg functions is the Catalan number:

$$\#\{h : [n] \to [n] : \text{ Hessenberg functions}\} = \frac{1}{n+1}\binom{2n}{n}.$$

There is a natural partial order $\subset$ on Hessenberg functions defined as follows. For any two Hessenberg functions h and h', we define $h \subset h'$ by

$$h \subset h' \iff h(j) \leq h'(j) \quad \forall j \in [n].$$

We use the symbol $\subset$ for this order since it corresponds to the inclusion of the configurations of boxes under the above visualization of Hessenberg functions.

The (full) flag variety $Fl(\mathbb{C}^n)$ of $\mathbb{C}^n$ is the collection of nested linear subspaces $V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$ with $\dim V_i = i$ for $i \in [n]$. For an $n \times n$ matrix X considered as a linear map $X : \mathbb{C}^n \to \mathbb{C}^n$ and a Hessenberg function $h : [n] \to [n]$, the **Hessenberg variety**[1] (in type A_{n-1}) associated with X and h is defined as

$$\mathrm{Hess}(X, h) = \{V_\bullet \in Fl(\mathbb{C}^n) | XV_j \subset V_{h(j)} \text{ for all } j \in [n]\} \tag{2.1}$$

References [21, 22]. If X is the zero matrix or $h = (n, n, \dots, n)$, then it is clear that $\mathrm{Hess}(X, h) = Fl(\mathbb{C}^n)$ is the flag variety itself from the definition (2.1). If X is nilpotent and $h = id = (1, 2, \dots, n)$, then $\mathrm{Hess}(X, h)$ is called a **Springer fiber** which plays an important role in the geometric representation theory of the symmetric group S_n [64, 65].

Remark 2.2 The definition (2.1) can be rephrased in terms of the adjoint representation of $\mathrm{GL}(n, \mathbb{C})$. See [21] for the definition in arbitrary Lie type. Also, Goresky-Kottwitz-MacPherson [33, Sect. 2] considered more general Hessenberg varieties which are defined for arbitrary representations of reductive algebraic groups (cf. Chen-Vilonen-Xue [18, Sect. 2]).

[1] For the origin of the name *Hessenberg varieties*, see [22].

As a general picture of Hessenberg varieties, we remark the following two properties. Suppose that one considers Hessenberg varieties for a fixed matrix X. Then, Hessenberg varieties preserves the inclusions:

$$h \subset h' \;\Rightarrow\; \mathrm{Hess}(X, h) \subset \mathrm{Hess}(X, h'). \tag{2.2}$$

Also, if $g \in \mathrm{GL}(n, \mathbb{C})$, then we have an isomorphism

$$\mathrm{Hess}(X, h) \cong \mathrm{Hess}(gXg^{-1}, h)$$

by sending $V_\bullet$ to $gV_\bullet$. This implies that we may assume that X is in a Jordan canonical form.

J. Tymoczko lays the foundation for the study of Hessenberg varieties as follows:

Theorem 2.3 ([72]) *Every Hessenberg variety* $\mathrm{Hess}(X, h)$ *is paved by affines.*[2] *In particular, the integral cohomology group of* $\mathrm{Hess}(X, h)$ *is torsion-free, and the odd-degree cohomology groups vanish.*

Remark 2.4 This generalizes the work of Spaltenstein [63] for the Springer fibers and De Mari-Procesi-Shayman [21] for the regular semisimple Hessenberg varieties (which we will define below). For generalizations to arbitrary Lie type, see Precup [52] (cf. Tymoczko [73], De Mari-Procesi-Shayman [21], and De Concini-Lusztig-Procesi [20]).

By Theorem 2.3 we may denote the Poincaré polynomial of $\mathrm{Hess}(X, h)$ by

$$\mathrm{Poin}(\mathrm{Hess}(X, h), q) := \sum_{i=0}^{m} \dim H^{2i}(\mathrm{Hess}(X, h); \mathbb{Q})\, q^i \tag{2.3}$$

where $m := \dim_{\mathbb{C}} \mathrm{Hess}(X, h)$ and the variable q stands for the grading with $\deg(q) = 2$.

In this survey, we will focus on so-called regular nilpotent Hessenberg varieties and regular semisimple Hessenberg varieties which are particularly well-studied. As we will see in Sects. 3 and 4, their cohomology rings has an interesting relation between each other, and these are related with other research areas such as hyperplane arrangements, representation theory, and graph theory.

2.2 Regular Nilpotent Hessenberg Varieties

Let N a regular nilpotent matrix of size $n \times n$, i.e. a nilpotent matrix with a single Jordan block. In Jordan canonical form, it is given by

[2] A paving by affines means a "(complex) cellular decomposition" in algebraic geometry. See [72, Definition 2.1] for the details.

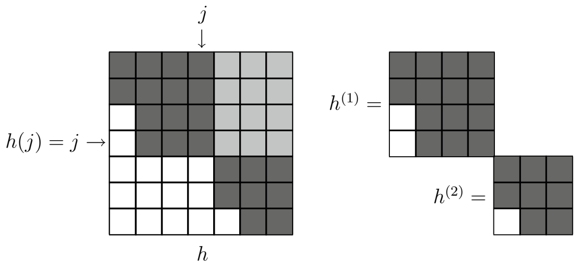


Fig. 2 Decomposition of h into $h^{(1)}$ and $h^{(2)}$

$$N = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

For a Hessenberg function $h : [n] \to [n]$, $\mathrm{Hess}(N, h)$ is called a **regular nilpotent Hessenberg variety**. We have the following two main examples for this class of Hessenberg varieties. For $h = (n, n, \dots, n)$, $\mathrm{Hess}(N, h)$ is the flag variety $Fl(\mathbb{C}^n)$ itself. For $h = (2, 3, 4, \dots, n, n)$, i.e. $h(j) = j + 1$ for $1 \leq j < n$, $\mathrm{Hess}(N, h)$ is called the **Peterson variety** which is related to the quantum cohomology of partial flag varieties (cf. [14, 49, 56]). Here, $h = (n, n, \dots, n)$ is the maximum Hessenberg function, and we may think that $h = (2, 3, 4, \dots, n, n)$ gives the minimum one in the following sense.

If we have $h(j) = j$ for some $j < n$, then the Hessenberg function h can be decomposed into two Hessenberg functions $h^{(1)}$ and $h^{(2)}$ of smaller sizes defined as follows:

$$\begin{aligned} h^{(1)} &:= (h(1), h(2), \dots, h(j)), \\ h^{(2)} &:= (h(j+1) - j, h(j+2) - j, \dots, h(n) - j). \end{aligned}$$

Then, $\mathrm{Hess}(N, h)$ is decomposed as the product of regular nilpotent Hessenberg varieties associated with $h^{(1)} : [j] \to [j]$ and $h^{(2)} : [n-j] \to [n-j]$ of smaller sizes [24, Theorem 4.5]. In this sense, the condition $h(j) \geq j + 1$ for any $j < n$ is essential, and for such Hessenberg functions, $h = (2, 3, 4, \dots, n, n)$ is the minimum one. Hence, from (2.2), we may regard regular nilpotent Hessenberg varieties $\mathrm{Hess}(N, h)$ as a (discrete) family of subvarieties of $Fl(\mathbb{C}^n)$ connecting the Peterson variety and the flag variety itself (Fig. 2).

In the following theorem, we summarize some basic properties of $\mathrm{Hess}(N, h)$. For this purpose, let us prepare some notations. Given a Hessenberg function h, we denote by S_n^h the subset of the n-th symmetric group S_n defined as

$$S_n^h := \{w \in S_n \mid w^{-1}(w(j) - 1) \leq h(j) \ \text{ for all } j \in [n]\}, \tag{2.4}$$

where we take by convention $w^{-1}(w(j)-1)=0$ whenever $w(j)-1=0$. The condition for $w \in S_n^h$ in (2.4) is exactly the condition that the permutation flag[3] associated with w belongs to $\mathrm{Hess}(N,h)$, and this condition is equivalent to the condition that the Schubert cell X_w° intersects with $\mathrm{Hess}(N,h)$ [73]. That is, we have

$$w \in S_n^h \iff X_w^\circ \cap \mathrm{Hess}(N,h) \neq \emptyset.$$

Here, the dimension of the Schubert cell X_w° is equal to $\ell(w)$, which is the length of w. It is known from [73] that the dimension of the intersection $X_w^\circ \cap \mathrm{Hess}(N,h)$ is equal to

$$\ell_h(w) := \#\{(j,i) \mid 1 \le j < i \le n,\ w(j) > w(i),\ i \le h(j)\}, \tag{2.5}$$

and that the intersections $X_w^\circ \cap \mathrm{Hess}(N,h)$ form a paving by affines of $\mathrm{Hess}(N,h)$. Combining the works of D. Anderson, E. Insko, B. Kostant, E. Sommers, J. Tymoczko, and A. Yong, we now give some basic properties of $\mathrm{Hess}(N,h)$.

Theorem 2.5 ([12, 47, 49, 62, 73]) *Let* $\mathrm{Hess}(N,h)$ *be a regular nilpotent Hessenberg variety. Then the following hold.*

(1) $\mathrm{Hess}(N,h)$ *is irreducible, and it is singular in general.*
(2) The (complex) dimension of $\mathrm{Hess}(N,h)$ *is equal to* $\sum_{j=1}^n (h(j)-j)$.
(3) The Poincaré polynomial (2.3) for $X = N$ *has the following two types of expressions:*

$$\begin{aligned} Poin(\mathrm{Hess}(N,h),q) &= \sum_{w \in S_n^h} q^{\ell_h(w)} && (2.6)\\ &= \prod_{j=1}^{n} (1+q+q^2+\cdots+q^{h(j)-j}) && (2.7) \end{aligned}$$

For generalizations of Theorem 2.5 to arbitrary Lie type, see [10, 52, 53, 57, 62, 73].

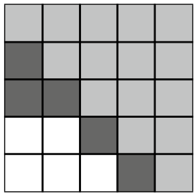
Note that the (complex) dimension of $\mathrm{Hess}(N,h)$ given in Theorem 2.5 (2) is equal to the number of boxes in which lie strictly below the diagonal under the identification of a Hessenberg function h and a configuration of boxes.

Example 2.6 Let $h=(3,3,4,5,5)$. Then we have that $\dim_{\mathbb{C}} \mathrm{Hess}(N,h)=5$, which is the number of boxes which lie strictly below the diagonal (Fig. 3).

Example 2.7 Let $h=(2,3,3)$. Then $S_3^h = \{123, 213, 132, 321\}$ where we use the standard one-line notation $w = w(1)\, w(2)\, \cdots\, w(n)$ for permutations in S_n throughout this document. In particular, we have geometrically that

[3] For $w \in S_n$, the associated permutation flag $V_\bullet$ is given by $V_i := \mathrm{span}_{\mathbb{C}}\{e_{w(1)},\ldots,e_{w(i)}\}$ where $\{e_1,\ldots,e_n\}$ is the standard basis of $\mathbb{C}^n$.

Fig. 3 Boxes which lie strictly below the diagonal



$$\mathrm{Hess}(N,h)\cap X_w^\circ=\emptyset \iff w=312,231.$$

Since $\ell_h(123)=0$, $\ell_h(213)=\ell_h(132)=1$, $\ell_h(321)=2$ for each permutation in S_3^h, we have

$$\mathrm{Poin}(\mathrm{Hess}(N,h),q)=1+2q+q^2=(1+q)^2. \tag{2.8}$$

2.3 *Regular Semisimple Hessenberg Varieties*

Let S be a regular semisimple matrix of size $n\times n$. In Jordan canonical form, it is given by

$$S=\begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix}$$

where $c_1, c_2\dots, c_n$ are mutually distinct complex numbers. For a Hessenberg function $h:[n]\to[n]$, $\mathrm{Hess}(S,h)$ is called a **regular semisimple Hessenberg variety**. It is known that the topology of $\mathrm{Hess}(S,h)$ is independent of the choices of the (distinct) eigenvalues,[4] and hence one may think that the topology of $\mathrm{Hess}(S,h)$ only depends on h. Based on this fact, we have the following two main examples for this class of Hessenberg varieties. For $h=(n,n,\dots,n)$, $\mathrm{Hess}(S,h)$ is the flag variety $Fl(\mathbb{C}^n)$ itself. For $h=(2,3,4,\dots,n,n)$, $\mathrm{Hess}(S,h)$ is called the **permutohedral variety** which is the toric variety associated with the fan consisting of the collection of Weyl chambers of the root system of type A_{n-1} [21, Theorem 11]. Similarly to the case for $\mathrm{Hess}(N,h)$ in Sect. 2.2, the Hessenberg function $h=(2,3,4,\dots,n,n)$ gives the minimum for this class of Hessenberg varieties as well in the following sense.

If we have $h(j)=j$ for some $j<n$, it is known that $\mathrm{Hess}(S,h)$ is not connected but equidimensional. In fact, all the connected components are isomorphic, and each component is decomposed as the product of regular semisimple Hessenberg varieties

[4]For regular semisimple matrices S and S', the associated Hessenberg varieties $\mathrm{Hess}(S,h)$ and $\mathrm{Hess}(S',h)$ with a same Hessenberg function h are diffeomorphic.

of smaller sizes as in Sect. 2.2. See [70] for detail description of the connected components. Hence the condition $h(j) \geq j+1$ for any $j < n$ is essential, and we may regard regular semisimple Hessenberg varieties $\mathrm{Hess}(S, h)$ as a (discrete) family of subvarieties of the flag variety connecting the permutohedral variety and the flag variety itself. De Mari-Procesi-Shayman proved the following properties of $\mathrm{Hess}(S, h)$ (for arbitrary Lie type).

Theorem 2.8 ([21]) *Let* $\mathrm{Hess}(S, h)$ *be a regular semisimple Hessenberg variety. Then the following hold.*

(1) $\mathrm{Hess}(S, h)$ *is smooth, and it is connected if and only if* $h(j) \geq j+1$ *for all* $j < n$.
(2) The (complex) dimension of $\mathrm{Hess}(S, h)$ *is equal to* $\sum_{j=1}^{n}(h(j)-j)$.
(3) The Poincaré polynomial (2.3) *for* $X = S$ *has the following expression:*

$$Poin(\mathrm{Hess}(S, h), q) = \sum_{w \in S_n} q^{\ell_h(w)}$$

where $\ell_h(w)$ *is defined in* (2.5).

From Theorem 2.5 (2) and Theorem 2.8 (2), we see that

$$\dim_{\mathbb{C}} \mathrm{Hess}(N, h) = \dim_{\mathbb{C}} \mathrm{Hess}(S, h) = \sum_{j=1}^{n}(h(j)-j).$$

Unlike the situation for $\mathrm{Hess}(N, h)$, any regular semisimple Hessenberg variety $\mathrm{Hess}(S, h)$ intersects with all the Schubert cells X_w°. It is known that the dimension of the intersection $X_w^\circ \cap \mathrm{Hess}(S, h)$ is equal to $\ell_h(w)$ given in (2.5), and that all of the intersections $X_w^\circ \cap \mathrm{Hess}(S, h)$ form a paving by affines of $\mathrm{Hess}(S, h)$.

Example 2.9 Let $n = 3$ and $h = (2, 3, 3)$. Since $\ell_h(123) = 0, \ell_h(213) = \ell_h(132) = \ell_h(231) = \ell_h(312) = 1, \ell_h(321) = 2$ for each permutation in S_3, the Poincaré polynomial of $\mathrm{Hess}(S, h)$ is given by

$$\mathrm{Poin}(\mathrm{Hess}(S, h), q) = 1 + 4q + q^2.$$

3 Cohomology

In this section, we explain the structures of the cohomology rings of Hessenberg varieties, focusing on regular nilpotent Hessenberg varieties $\mathrm{Hess}(N, h)$ in Sect. 3.1 and regular semisimple Hessenberg varieties $\mathrm{Hess}(S, h)$ in Sect. 3.2. We will also see that these two cohomology rings have an interesting relation in Sect. 3.3.

3.1 Cohomology Rings of Regular Nilpotent Hessenberg Varieties

The cohomology ring of a regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ has been studied from various viewpoints (e.g. [6, 10, 15, 23, 29, 36, 41, 43, 45, 50]).

In this section we explain an explicit presentation of the cohomology ring of a regular nilpotent Hessenberg variety given by [6] due to M. Harada, M. Masuda, and the authors. We also discuss a relation between this presentation and Schubert polynomials along [42].

We first recall an explicit presentation of the cohomology ring of the flag variety $Fl(\mathbb{C}^n)$. Let E_i be the i-th tautological vector bundle over $Fl(\mathbb{C}^n)$; namely, E_i is the subbundle of the trivial vector bundle $Fl(\mathbb{C}^n) \times \mathbb{C}^n$ over $Fl(\mathbb{C}^n)$ whose fiber over a point $V_\bullet = (V_1 \subset \cdots \subset V_n = \mathbb{C}^n) \in Fl(\mathbb{C}^n)$ is exactly V_i. We denote the negative of the first Chern class of the quotient line bundle E_i/E_{i-1} by $\bar{x}_i$, i.e.

$$\bar{x}_i := -c_1(E_i/E_{i-1}) \in H^2(Fl(\mathbb{C}^n); \mathbb{Q}). \tag{3.1}$$

We will also use the same symbol $\bar{x}_i$ for its restrictions to the cohomology of Hessenberg varieties by abuse of notation. It is known that there is a ring isomorphism

$$H^*(Fl(\mathbb{C}^n); \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/(e_1, \ldots, e_n) \tag{3.2}$$

which sends x_i to $\bar{x}_i$ where e_i is the elementary symmetric polynomial of degree i in the variables $x_1, \ldots, x_n$ (cf. [30, p161, Proposition 3]).

In order to describe the cohomology ring of a regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$, we define polynomials $f_{i,j}$ for $1 \leq j \leq i \leq n$ as follows:

$$f_{i,j} := \sum_{k=1}^{j} \Big(\prod_{\ell=j+1}^{i} (x_k - x_\ell) \Big) x_k. \tag{3.3}$$

Here, we take by convention $\prod_{\ell=j+1}^{i}(x_k - x_\ell) = 1$ whenever $i = j$. Note that this definition does not involve n.

Theorem 3.1 ([6]) *The restriction map*

$$H^*(Fl(\mathbb{C}^n); \mathbb{Q}) \to H^*(\mathrm{Hess}(N, h); \mathbb{Q})$$

is surjective, and there is a ring isomorphism

$$H^*(\mathrm{Hess}(N, h); \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/(f_{h(1),1}, f_{h(2),2}, \ldots, f_{h(n),n}) \tag{3.4}$$

which sends x_i to $\bar{x}_i = -c_1(E_i/E_{i-1})|_{\mathrm{Hess}(N,h)}$.

$f_{1,1}$		
$f_{2,1}$	$f_{2,2}$	
$f_{3,1}$	$f_{3,2}$	$f_{3,3}$

$$f_{1,1} = x_1, \quad f_{2,2} = x_1 + x_2, \quad f_{3,3} = x_1 + x_2 + x_3,$$
$$f_{2,1} = (x_1 - x_2)x_1, \quad f_{3,2} = (x_1 - x_3)x_1 + (x_2 - x_3)x_2,$$
$$f_{3,1} = (x_1 - x_2)(x_1 - x_3)x_1.$$

Fig. 4 The polynomials $f_{i,j}$ for $1 \leq i \leq j \leq 3$

Remark 3.2 The presentation (3.4) does not hold for the integral coefficients. See [7, Remark 3].

From the presentation (3.4), we can see that the cohomology ring $H^*(\mathrm{Hess}(N, h); \mathbb{Q})$ is a complete intersection since the number of generators of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ is equal to the number of generators of the ideal $(f_{h(1),1}, \ldots, f_{h(n),n})$. This implies the following corollary.

Corollary 3.3 ([6]) *$H^*(\mathrm{Hess}(N, h); \mathbb{Q})$ is a Poincaré duality algebra.*

Note that a regular nilpotent Hessenberg variety is singular in general, but its cohomology is a Poincaré duality algebra. For arbitrary Lie type, the restriction map is surjective and Corollary 3.3 holds [10] (Fig. 4).

Example 3.4 Let $n = 3$. We first assign polynomials $f_{i,j}$ to each box below the diagonal line.

If we take $h = (3, 3, 3)$, then we obtain from Theorem 3.1 an explicit presentation of the flag variety $\mathrm{Hess}(N, h) = Fl(\mathbb{C}^3)$:

$$H^*(\mathrm{Hess}(N, h); \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2, x_3]/(f_{3,1}, f_{3,2}, f_{3,3}), \tag{3.5}$$

where the polynomials $f_{3,1}, f_{3,2}, f_{3,3}$ in this presentation are obtained by taking the bottom one in each column (See Fig. 5). It is straightforward to verify that the ideals in (3.2) (with $n = 3$) and (3.5) are the same. If we take $h' = (2, 3, 3)$, then we also obtain an explicit presentation of the Peterson variety $\mathrm{Hess}(N, h')$ in $Fl(\mathbb{C}^3)$:

$$H^*(\mathrm{Hess}(N, h'); \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2, x_3]/(f_{2,1}, f_{3,2}, f_{3,3}). \tag{3.6}$$

One may notice that the polynomial $f_{3,1}$ in (3.5) is replaced by $f_{2,1}$ in (3.6) when we changed the Hessenberg function from h to h'. More specifically, the polynomial $f_{2,1}$ does not vanish in $H^*(\mathrm{Hess}(N, h))$, but it does vanish in $H^*(\mathrm{Hess}(N, h'))$. This polynomial $f_{2,1}$ has the following expression as a linear combination of Schubert polynomials $\mathfrak{S}_w$ ($w \in S_n$):

$$f_{2,1} = x_1^2 - x_1x_2 = \mathfrak{S}_{312} - \mathfrak{S}_{231} \tag{3.7}$$

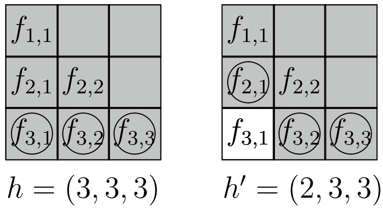


Fig. 5 The bottom $f_{i,j}$'s for $h = (3, 3, 3)$ and $h' = (2, 3, 3)$

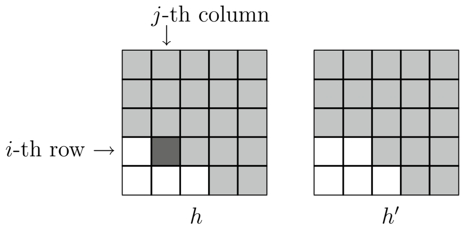


Fig. 6 The pictures of h and h'

since $\mathfrak{S}_{312} = x_1^2$ and $\mathfrak{S}_{231} = x_1 x_2$. As seen in Example 2.7, we also have

$$\mathrm{Hess}(N, h') \cap X_w^\circ = \emptyset \iff w = 312,\ 231.$$

These permutations $w = 312, 231$ are exactly the ones that appeared in (3.7).

In general, we have a similar interpretation of the presentation (3.4) in Theorem 3.1 by considering a smaller Hessenberg variety $\mathrm{Hess}(N, h) \supset \mathrm{Hess}(N, h')$ of codimension 1, as suggested by the above example (Fig. 6).

The unique difference in the generators of the ideal appearing in the presentation (3.4) for $H^*(\mathrm{Hess}(N, h); \mathbb{Q})$ and $H^*(\mathrm{Hess}(N, h'); \mathbb{Q})$ is the polynomial $f_{i-1,j}$. The second author showed that this polynomial can be written as an alternating sum of Schubert polynomials $\mathfrak{S}_w$ where the set of permutations w appearing in this sum coincides with the set of minimal length permutations w in S_n satisfying

$$\mathrm{Hess}(N, h) \cap X_w^\circ \neq \emptyset \quad \text{and} \quad \mathrm{Hess}(N, h') \cap X_w^\circ = \emptyset.$$

See [42] for details.

3.2 *The Cohomology Rings of Regular Semisimple Hessenberg Varieties*

Let $\mathrm{Hess}(S, h)$ be a regular semisimple Hessenberg variety. One of the most interesting feature of $\mathrm{Hess}(S, h)$ is the S_n-representation on its cohomology $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ constructed by J. Tymoczko [74]. She first constructed an S_n-representation on a torus equivariant cohomology $H_T^*(\mathrm{Hess}(S, h); \mathbb{C})$ via a combinatorial description called the GKM-presentation, and she showed that this representation descends to the ordinary cohomology $H^*(\mathrm{Hess}(S, h); \mathbb{C})$. An alternative geometric construction via a monodromy action of the fundamental group of the space of regular semisimple matrices is explained in Brosnan-Chow [16]. Following the construction [74], N. Teff started to analyze this S_n-representation in [70, 71], and Shareshian-Wachs [59, 60] announced a beautiful conjecture on this representation using chromatic quasisymmetric functions. In this manuscript, we explain this representation along the construction due to Tymoczko.

Let $T \subset \mathrm{GL}(n, \mathbb{C})$ be the maximal torus consisting of diagonal elements of $\mathrm{GL}(n, \mathbb{C})$. The flag variety $\mathrm{Flag}(\mathbb{C}^n)$ has a natural action of $\mathrm{GL}(n, \mathbb{C})$, and hence the torus T acts on $\mathrm{Flag}(\mathbb{C}^n)$ via its restriction. This T-action preserves the regular semisimple Hessenberg variety $\mathrm{Hess}(S, h)$ since all the elements of T commute with the diagonal matrix S. It is known that $\mathrm{Hess}(S, h)$ contains all the T-fixed points of the flag variety $\mathrm{Flag}(\mathbb{C}^n)$ [21, Proposition 3] so that $\mathrm{Hess}(S, h)^T = \mathrm{Flag}(\mathbb{C}^n)^T \cong S_n$. Here, the last bijection corresponds $w \in S_n$ and the permutation flag associated with w.

Let $H_T^*(\mathrm{Hess}(S, h); \mathbb{C})$ be the T-equivariant cohomology of $\mathrm{Hess}(S, h)$. Recalling that $\mathrm{Hess}(S, h)$ has no odd-degree cohomology from Theorem 2.3, we can apply localization techniques for T-equivariant cohomology which we refer [32, 74] for details. As a conclusion, we obtain the so-called GKM presentation of $H_T^*(\mathrm{Hess}(S, h); \mathbb{C})$ as a subring of a direct sum of polynomial rings $\bigoplus_{w \in S_n} \mathbb{C}[t_1, \ldots, t_n]$.

Proposition 3.5 ([74, Proposition 4.7]) *The equivariant cohomology $H_T^*(\mathrm{Hess}(S, h); \mathbb{C})$ is isomorphic (as rings) to*

$$\left\{ \alpha \in \bigoplus_{w \in S_n} \mathbb{C}[t_1, \ldots, t_n] \;\middle|\; \begin{array}{l} \alpha(w) - \alpha(w') \textit{ is divisible by } t_{w(i)} - t_{w(j)} \\ \textit{if } w' = w(j\ i) \textit{ for some } j < i \textit{ with } i \le h(j) \end{array} \right\} \tag{3.8}$$

where $\alpha(w)$ is the w-component of α and $(j\ i) \in S_n$ is the transposition of j and i.

We can visualize elements of the subring appearing in (3.8) in terms of the so-called GKM graph whose vertex set is S_n and there is an edge between vertices $w, w' \in S_n$ if there exists $1 \le j < i \le n$ with $i \le h(j)$ satisfying $w' = w(j\, i)$. Additionally, we equip such an edge with the data of the polynomial $\pm(t_{w(i)} - t_{w(j)})$ (up to sign) arising in (3.8). This labeled graph is called **the GKM graph** of $\mathrm{Hess}(S, h)$, and we denote it by $\Gamma(h)$. In this language, the condition in (3.8) says that the collection of polynomials $(\alpha(w))_{w \in S_n}$ satisfies the following: if w and w' are connected in $\Gamma(h)$ by an edge labeled by $t_{w(i)} - t_{w(j)}$, then the difference of the polynomials assigned for w and w' must be divisible by the label.

Example 3.6 Let $n = 3$. For $h = (3, 3, 3)$ and $h' = (2, 3, 3)$, the corresponding GKM graphs are depicted in Fig. 7, where we use the one-line notation for each vertex.

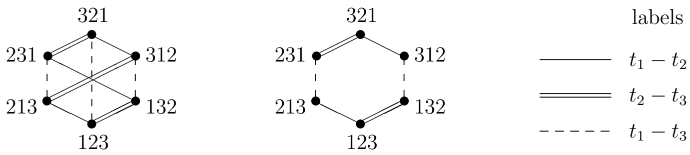


Fig. 7 The GKM graphs $\Gamma(h)$ and $\Gamma(h')$

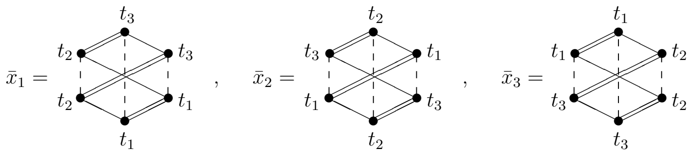


Fig. 8 Some elements of $H_T^*(\mathrm{Hess}(S, h))$

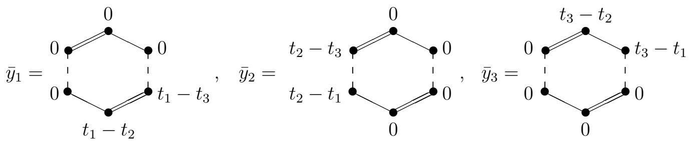


Fig. 9 Some elements of $H_T^*(\mathrm{Hess}(S, h'))$

For example, one can verify that the tuples of polynomials in Fig. 8 are elements of $H_T^*(\mathrm{Hess}(S, h); \mathbb{C}) = H_T^*(Fl(\mathbb{C}^3); \mathbb{C})$ (and hence of $H_T^*(\mathrm{Hess}(S, h'); \mathbb{C})$).

We use the symbols $\bar{x}_1, \bar{x}_2, \bar{x}_3$ by abuse of notation (cf. (3.1)) because of the reason which we will explain soon later (See Example 3.7 below). Also, the elements $\bar{y}_1, \bar{y}_2, \bar{y}_3$ given by the tuples of polynomials in Fig. 9 are elements of $H_T^*(\mathrm{Hess}(S, h'); \mathbb{C})$ but not of $H_T^*(\mathrm{Hess}(S, h); \mathbb{C}) = H_T^*(Fl(\mathbb{C}^3); \mathbb{C})$.

In what follows, we identify $H_T^*(\mathrm{Hess}(S, h); \mathbb{C})$ and the presentation (3.8), and we do not distinguish them. For each $i = 1, \ldots, n$, it is clear that a collection $(t_i)_{w \in S_n}$ lies in (3.8). For simplicity, we also write this element as t_i by abuse of notation. Then the theory of T-equivariant cohomology also shows that there is a ring isomorphism

$$H^*(\mathrm{Hess}(S, h); \mathbb{C}) \cong H_T^*(\mathrm{Hess}(S, h); \mathbb{C})/(t_1, \ldots, t_n). \tag{3.9}$$

This means that one can study the ordinary cohomology ring $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ from the equivariant cohomology ring $H_T^*(\mathrm{Hess}(S, h); \mathbb{C})$.

We now describe the S_n-action on $H_T^*(\mathrm{Hess}(S, h); \mathbb{C})$ constructed by Tymoczko [74]. For $v \in S_n$ and $\alpha = (\alpha(w)) \in \bigoplus_{w \in S_n} \mathbb{C}[t_1, \ldots, t_n]$, we define an element $v \cdot \alpha$ by the formula

$$(v \cdot \alpha)(w) := v \cdot \alpha(v^{-1}w) \ \text{ for all } w \in S_n \tag{3.10}$$

where $v \cdot f(t_1, \ldots, t_n) = f(t_{v(1)}, \ldots, t_{v(n)})$ for $f(t_1, \ldots, t_n) \in \mathbb{C}[t_1, \ldots, t_n]$ in the right-hand side. This S_n-action preserves the subset (3.8), and hence it defines an S_n-action on the equivariant cohomology $H^*_T(\mathrm{Hess}(S, h); \mathbb{C})$ by Proposition 3.5. Since we have $v \cdot t_i = t_{v(i)}$ for the classes $t_i = (t_i)_{w \in S_n}$ defined above, the S_n-action on $H^*_T(\mathrm{Hess}(S, h); \mathbb{C})$ induces an S_n-action on the ordinary cohomology $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ via (3.9). By construction this S_n-representation preserves the cup product of $H^*(\mathrm{Hess}(S, h); \mathbb{C})$.

Example 3.7 Let $n = 3$ and $h = (3, 3, 3)$. For $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in H^2_T(\mathrm{Hess}(S, h); \mathbb{C}) = H^2_T(Fl(\mathbb{C}^3); \mathbb{C})$ given in Fig. 8, one can easily verify from the definition (3.10) that they are invariant under the S_3-action;

$$w \cdot \bar{x}_i = \bar{x}_i \quad (1 \leq i \leq 3)$$

for any $w \in S_3$. Under the isomorphism (3.9), it follows that these equivariant cohomology classes corresponds to $\bar{x}_i = -c_1(E_i/E_{i-1}) \in H^2(Fl(\mathbb{C}^3); \mathbb{Q})$ introduced in (3.1) which gives a justification for our notation. From (3.2) (or (3.5)), this means that the S_3-representation on $H^*(Fl(\mathbb{C}^3); \mathbb{C})$ is trivial, and the same claim holds for the case $H^*(Fl(\mathbb{C}^n); \mathbb{C})$ in general [74, Proposition 4.4].

Example 3.8 Let $n = 3$ and $h = (2, 3, 3)$. For $\bar{y}_1, \bar{y}_2, \bar{y}_3 \in H^2_T(\mathrm{Hess}(S, h); \mathbb{C})$ given in Fig. 9, one can also verify that these classes are naturally permuted by the S_3-action;

$$w \cdot \bar{y}_i = \bar{y}_{w(i)} \quad (1 \leq i \leq 3)$$

for any $w \in S_3$ from the definition (3.10).

Compared to the situation for $\mathrm{Hess}(N, h)$ (e.g. Theorem 3.1), the restriction map $H^*(\mathrm{Flag}(\mathbb{C}^n); \mathbb{C}) \to H^*(\mathrm{Hess}(S, h); \mathbb{C})$ is not surjective in general. Hence, to describe the ring $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ in terms of ring generators and relations among them, we need to find some cohomology classes of $\mathrm{Hess}(S, h)$ which do not come from $H^*(\mathrm{Flag}(\mathbb{C}^n); \mathbb{C})$ by restriction. However, since we have a surjection $H^*_T(\mathrm{Hess}(S, h); \mathbb{C}) \to H^*(\mathrm{Hess}(S, h); \mathbb{C})$ by (3.9), the graphical presentation of the equivariant cohomology ring $H^*_T(\mathrm{Hess}(S, h); \mathbb{C})$ can be used to seek those classes as we saw in Example 3.6. For the case $h = (h(1), n, \ldots, n)$, M. Masuda and the authors showed that we can explicitly describe the integral cohomology ring in terms of ring generators and their relations by this approach. See [8] for details. For the case $h = (2, 3, 4, \ldots, n, n)$, the cohomology ring $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ is well-understood since $\mathrm{Hess}(S, h)$ is a non-singular projective toric variety in this case [1, 48, 55]. However, the ring structure of $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ for general h is not well-understood at this moment.

Example 3.9 Let $n = 3$ and $h = (2, 3, 3)$. Recall from Fig. 9 that we have three classes $\bar{y}_1, \bar{y}_2, \bar{y}_3$ in the equivariant cohomology $H^2_T(\mathrm{Hess}(S, h); \mathbb{C})$. We also denote by the same symbol $\bar{y}_i \in H^2(\mathrm{Hess}(S, h); \mathbb{C})$ the image of $\bar{y}_i$ under the isomorphism (3.9) by abuse of notation. Then the presentation of $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ due to [8] is given by

$$H^*(\mathrm{Hess}(S,h);\mathbb{C}) \cong \mathbb{C}[x_1,x_2,x_3,y_1,y_2,y_3]/J, \tag{3.11}$$

where x_i and y_i correspond to $\bar{x}_i = -c_1(E_i/E_{i-1})|_{\mathrm{Hess}(S,h)}$ and $\bar{y}_i$ respectively, and the ideal J is generated by

$$\begin{aligned}
&y_k y_{k'} \quad \text{for } 1 \le k \neq k' \le 3,\\
&x_1 y_k \quad \text{for } 1 \le k \le 3,\\
&x_3 y_k + x_2 x_3 \quad \text{for } 1 \le k \le 3,\\
&y_1 + y_2 + y_3 - (x_1 - x_2),\\
&x_1 + x_2 + x_3,\; x_1x_2 + x_1x_3 + x_2x_3,\; x_1x_2x_3.
\end{aligned}$$

Also, the S_3-action on $H^*(\mathrm{Hess}(S,h);\mathbb{C})$ is given by $w \cdot x_i = x_i$ and $w \cdot y_i = y_{w(i)}$ for $i = 1, 2, 3$ and $w \in S_3$, and the irreducible decomposition as S_3-representation is given by

$$H^*(\mathrm{Hess}(S,h);\mathbb{C}) \cong S^{(3)} \oplus \left((S^{(3)})^{\oplus 2} \oplus S^{(2,1)}\right) q \oplus S^{(3)}\, q^2, \tag{3.12}$$

where S^{λ} for a partition λ of 3 is the irreducible representation of S_3 corresponding to λ (cf. [30]), and q is a formal symbol standing for the cohomology grading with $\deg(q) = 2$. Note that $S^{(3)}$ is the trivial representation. In particular, this recovers the Poincaré polynomial of $\mathrm{Hess}(S,h)$ in Example 2.9.

3.3 Regular Nilpotent Versus Regular Semisimple

In the last two sections, we have described the cohomology of regular nilpotent Hessenberg varieties $\mathrm{Hess}(N,h)$ and regular semisimple Hessenberg varieties $\mathrm{Hess}(S,h)$, and we saw that the latter cohomology $H^*(\mathrm{Hess}(S,h);\mathbb{C})$ admits the S_n-representation constructed by Tymoczko. Since this representation preserves the cup product, the invariant subgroup $H^*(\mathrm{Hess}(S,h);\mathbb{C})^{S_n}$ in fact forms a subring of $H^*(\mathrm{Hess}(S,h);\mathbb{C})$. M. Harada, M. Masuda, and the authors [6] showed that the S_n-representation provides a connection between the topology of $\mathrm{Hess}(N,h)$ and $\mathrm{Hess}(S,h)$ as follows.

Theorem 3.10 ([6]) *Let* $\mathrm{Hess}(N,h)$ *and* $\mathrm{Hess}(S,h)$ *be a regular nilpotent Hessenberg variety and a regular semisimple Hessenberg variety, respectively. Then, there is a ring isomorphism*

$$H^*(\mathrm{Hess}(N,h);\mathbb{C}) \cong H^*(\mathrm{Hess}(S,h);\mathbb{C})^{S_n}$$

which sends $\bar{x}_i = -c_1(E_i/E_{i-1})|_{\mathrm{Hess}(N,h)}$ *to* $\bar{x}_i = -c_1(E_i/E_{i-1})|_{\mathrm{Hess}(S,h)}$.

For the case $h = (n, n, \dots, n)$, we have $\mathrm{Hess}(N, h) = \mathrm{Hess}(S, h) = Fl(\mathbb{C}^n)$, and the isomorphism in Theorem 3.10 is obvious since the S_n-representation on the cohomology is trivial in this case (See Example 3.7). For the case $h = (2, 3, 4, \dots, n, n)$, explicit presentations for the rings $H^*(\mathrm{Hess}(N, h); \mathbb{C})$ and $H^*(\mathrm{Hess}(S, h); \mathbb{C})^{S_n}$ were given in [29, 36] and [48] respectively, and those presentations are in fact identical although it was not mentioned. In this case, it means that the cohomology ring of the Peterson variety is isomorphic to the S_n-invariant subring of the cohomology of the permutohedral variety. See the work of P. Brosnan and T. Chow [16])for the geometry behind this phenomenon. We also note that Theorem 3.10 holds for arbitrary Lie type [10].

Example 3.11 Let $n = 3$ and $h = (2, 3, 3)$. Then from (2.8) and (3.12), it is clear that we have the following equalities for dimension;

$$\dim H^{2k}(\mathrm{Hess}(N, h); \mathbb{C}) = \dim H^{2k}(\mathrm{Hess}(S, h); \mathbb{C})^{S_3}$$

for all k. However, Theorem 3.10 states more; these are isomorphic as (graded) rings. For this case $h = (2, 3, 3)$, one can directly construct this isomorphism by using the ring presentations (3.6) and (3.11) (see [8, Remark 4.8.]).

4 Combinatorics

In this section, we explain combinatorial objects which are related to Hessenberg varieties. More specifically, we will see how hyperplane arrangements arise to describe the structure of the cohomology rings $H^*(\mathrm{Hess}(N, h); \mathbb{R})$ of regular nilpotent Hessenberg varieties and how Stanley's chromatic symmetric function of graphs determines the S_n-representation on the cohomology rings $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ of regular semisimple Hessenberg varieties.

4.1 Hyperplane Arrangements

In this section we explain a connection between Hessenberg varieties and hyperplane arrangements established in [10]. Originally, E. Sommers and J. Tymoczko pointed out that Hessenberg varieties are related to hyperplane arrangements, and they conjectured that the Ponicaré polynomial of a regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ can be described in terms of certain hyperplane arrangement [62]. This conjecture was verified for some Lie types by Sommers-Tymoczko, G. Röhrle, A. Schauenburg [57, 62]). After this, an explicit presentation of the cohomology ring of $\mathrm{Hess}(N, h)$ was provided by M. Harada, M. Masuda, and the authors [6].

Fig. 10 The Weyl arrangement for $n = 3$

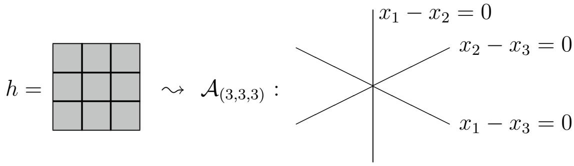
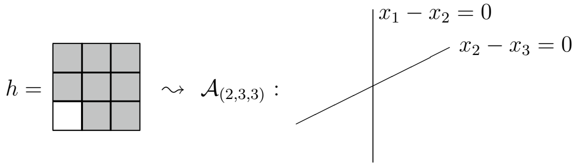


Fig. 11 The ideal arrangement associated with $h = (2, 3, 3)$



Motivated by all these works, T. Abe, M. Masuda, S. Murai, T. Sato, and the second author proved that the cohomology ring of a regular nilpotent Hessenberg variety is isomorphic to a ring coming from those hyperplane arrangements [10]. By taking Hilbert series of both sides of this isomorphism, one sees that the conjecture of Sommers and Tymoczko is true. For general reference about hyperplane arrangements, see [51] (Figs. 10 and 11).

Let V be a real vector space of finite dimension. A **(central) hyperplane arrangement** $\mathcal{A}$ in V is a finite set of linear hyperplanes in V. As we see in the following example, Hessenberg functions naturally determine hyperplane arrangements.

Example 4.1 Let $V = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_1 + \cdots + x_n = 0\}$, and consider hyperplanes in V given by $H_{i,j} = \{x_j - x_i = 0\}$ for $1 \le j < i \le n$. For each Hessenberg function $h : [n] \to [n]$, the set

$$\mathcal{A}_h := \{H_{i,j} | 1 \le j < i \le h(j)\}$$

is called an **ideal arrangement** associated with h. In particular, if $h = (n, n, \dots, n)$, then $\mathcal{A}_h$ is called the **Weyl arrangement** (of type A_{n-1}).

Let $\mathcal{R} = \mathrm{Sym}(V^*)$ be the symmetric algebra of V^*, where V^* is the dual space of V. We regard $\mathcal{R}$ as an algebra of polynomial functions on V. A map $\theta : \mathcal{R} \to \mathcal{R}$ is an $\mathbb{R}$-**derivation** if it satisfies

(1) θ is $\mathbb{R}$-linear,
(2) $\theta(f \cdot g) = \theta(f) \cdot g + f \cdot \theta(g)$ for all $f, g \in \mathcal{R}$.

We denote the module of $\mathbb{R}$-derivations $\theta : \mathcal{R} \to \mathcal{R}$ by $\mathrm{Der}\mathcal{R}$. If one chooses a linear coordinate system $x_1, \dots, x_m$ on V, i.e., $x_1, \dots, x_m$ is a basis for V^*, then the $\mathcal{R}$-module $\mathrm{Der}\mathcal{R}$ can be expressed as $\bigoplus_{i=1}^m \mathcal{R} \frac{\partial}{\partial x_i}$ where $\frac{\partial}{\partial x_i}$ denotes the partial derivative with respect to x_i.

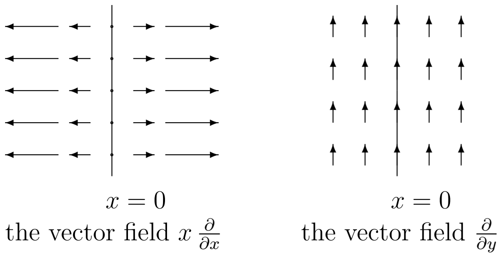


Fig. 12 A basis of the logarithmic derivation module of the hyperplane arrangement $\{x = 0\}$ in $\mathbb{R}^2$

Let $\mathcal{A}$ be a hyperplane arrangement in V. For each $H \in \mathcal{A}$, let $\alpha_H \in V^*$ be the defining linear form of H so that $H = \ker(\alpha_H)$. The **logarithmic derivation module** $D(\mathcal{A})$ of $\mathcal{A}$ is an $\mathcal{R}$-module defined by

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}\mathcal{R} | \theta(\alpha_H) \in \mathcal{R}\alpha_H \text{ for all } H \in \mathcal{A}\}.$$

Geometrically, this consists of polynomial vector fields on V tangent to $\mathcal{A}$.

Example 4.2 Let $V = \mathbb{R}^2$ and $H = \{x = 0\}$ a hyperplane. We consider a hyperplane arrangement $\mathcal{A} = \{H\}$. An element $\theta \in \mathrm{Der}\mathcal{R}$ can be written as

$$\theta = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$$

for some $f, g \in \mathcal{R} = \mathbb{R}[x, y]$. Since we can take $\alpha_H = x$, we have $\theta(\alpha_H) = \theta(x) = f$. Hence, θ belongs to the logarithmic derivation module $D(\mathcal{A})$ if and only if f is divisible by x. From this, it is straightforward to see that $D(\mathcal{A})$ is free module over $\mathbb{R}[x, y]$ with a basis $x\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Hence, $D(\mathcal{A})$ consists of polynomial vector fields on $\mathbb{R}^2$ tangent to $\mathcal{A}$, as desired (Fig. 12).

In Example 4.2, we saw that the logarithmic derivation module $D(\mathcal{A})$ is a free $\mathcal{R}$-module. However, the logarithmic derivation module $D(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ is not free in general (cf. [51, Example 4.34]). A hyperplane arrangement $\mathcal{A}$ is a **free arrangement** if its logarithmic derivation module $D(\mathcal{A})$ is a free module over $\mathcal{R}$. Note that if $\mathcal{A}$ is a free arrangement in V, then $D(\mathcal{A})$ has a basis consisting of m homogeneous elements where $m := \dim V$ (cf. [51, Proposition 4.18]). Here, a nonzero element $\theta \in \mathrm{Der}\mathcal{R} = \mathcal{R} \otimes V$ is homogeneous if $\theta = \sum_{k=1}^{\ell} f_k \otimes v_k$ ($f_k \in \mathcal{R},\ v_k \in V$) and all non-zero f_k's are homogeneous polynomials of the same degree.

We now explain the connection with the cohomology rings of regular nilpotent Hessenberg varieties. Let h be a Hessenberg function and $\mathcal{A}_h$ the ideal arrangement in $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 + \cdots + x_n = 0\}$ given in Example 4.1. First, it is known to be free (for arbitrary Lie type) by T. Abe, M. Barakat, M. Cuntz, T. Hoge, and H. Terao.

Theorem 4.3 ([9]) *Any ideal arrangement $\mathcal{A}_h$ is free.*

We next define an ideal of $\mathcal{R}$ from the logarithmic derivation module $D(\mathcal{A}_h)$ as follows. Let Q be an S_n-invariant non-degenerate quadratic form on V, which is unique up to a non-zero scalar multiple. We may take $Q = x_1^2 + \cdots + x_n^2 \in \mathrm{Sym}^2(V^*)^{S_n}$. We define an ideal $\mathfrak{a}(h)$ of $\mathcal{R}$ by

$$\mathfrak{a}(h) := \{\theta(Q) \in \mathcal{R} | \theta \in D(\mathcal{A}_h)\}.$$

T. Abe, M. Masuda, S. Murai, T. Sato, and the second author proved the following.

Theorem 4.4 ([10]) *There is a ring isomorphism*

$$H^*(\mathrm{Hess}(N, h); \mathbb{R}) \cong \mathcal{R}/\mathfrak{a}(h)$$

which sends x_i to $\bar{x}_i = -c_1(E_i/E_{i-1})|_{\mathrm{Hess}(N,h)}$ of (3.1).

In arbitrary Lie type, Theorem 4.4 is proved in [10]. It is known that the Poincaré polynomial of $\mathrm{Hess}(N, h)$ has a summation formula such as (2.6) [52]. On the other hand, the Hilbert series of the quotient ring $\mathcal{R}/\mathfrak{a}(h)$ has a product formula such as (2.7). Theorem 4.4 gives an affirmative answer to a conjecture of Sommers and Tymoczko in [62] which states that these formulas are equal.

Remark 4.5 The quotient ring in the right-hand side of Theorem 4.4 is an example of Solomon-Terao algebras studied in [28] and [11], where [28] considered more general hypersurface singularities. The Solomon-Terao algebra $ST(\mathcal{A}, \eta)$ is defined by a hyperplane arrangement $\mathcal{A}$ and a homogeneous polynomial η. Motivated by the work of L. Solomon and H. Terao in [61], it is proved in [11] that $ST(\mathcal{A}, \eta)$ for a generic η is a complete intersection if and only if the hyperplane arrangement $\mathcal{A}$ is free. In [28] the same equivalence is proved more generally for hypersurface singularities which are holonomic in the sense of K. Saito [58, (3.8)]. This generalization was obtained independently and published slightly earlier.

Theorem 4.4 also tells us that if we can find an explicit $\mathcal{R}$-basis of $D(\mathcal{A})$, then we obtain an explicit presentation of the cohomology ring of $\mathrm{Hess}(N, h)$. In fact, we can recover the presentation (3.4) as follows. First, we recall a well-known criterion for bases of logarithmic derivation modules.

Theorem 4.6 (Saito's criterion, [58], see also [51]) *Let $\mathcal{A}$ be a hyperplane arrangement in V and let $\theta_1, \ldots, \theta_m \in D(\mathcal{A})$ be homogeneous derivations. Then $\theta_1, \ldots, \theta_m$ form an $\mathcal{R}$-basis of $D(\mathcal{A})$ if and only if $\theta_1, \ldots, \theta_m$ are $\mathcal{R}$-independent and $\sum_{i=1}^m \deg \theta_i = |\mathcal{A}|$.*

To recover the presentation (3.4) by Saito's criterion, we define derivations $\psi_{i,j}$ for $1 \leq j \leq i \leq n$ on the ambient vector space $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 + \cdots + x_n = 0\}$ of ideal arrangements as follows:

$$\psi_{i,j} := \sum_{k=1}^{j} \left(\prod_{\ell=j+1}^{i} (x_k - x_\ell) \right) \left(\frac{\partial}{\partial x_k} - \frac{1}{n} \overline{\partial} \right) \in \mathrm{Der}\mathcal{R}$$

where $\overline{\partial} := \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}$ and we take by convention $\prod_{\ell=j+1}^{i}(x_k - x_\ell) = 1$ whenever $i = j$. Note that $\frac{\partial}{\partial x_k}$ is not an element of Der$\mathcal{R}$ but $(\frac{\partial}{\partial x_k} - \frac{1}{n}\overline{\partial})$ is, since $\mathcal{R} = \mathbb{R}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n)$. Using Theorem 4.6, one can verify that $\{\psi_{h(j),j} | 1 \leq j \leq n-1\}$ form an $\mathcal{R}$-basis of $D(\mathcal{A}_h)$ [10, Proposition 10.3]. Since $\psi_{i,j}(Q) = \psi_{i,j}(x_1^2 + \cdots + x_n^2) = 2f_{i,j}$ in $\mathcal{R}$, we recover the presentation (3.4) from Theorem 4.4.

Abe, Barakat, Cuntz, Hoge, and Terao in [9] gave a theoretical method to construct a basis of $D(\mathcal{A}_h)$ over $\mathcal{R}$ for arbitrary Lie type by classification-free proof. For a construction of an explicit basis for each Lie type, Barakat, Cuntz, and Hoge provided ones for types E and F by computer when the work of [9] was in progress. Also, Terao and Abe worked for types A and B, respectively. In [10], an explicit basis was constructed for types A, B, C, G. Motivated by this, Enokizono, Nagaoka, Tsuchiya, and the second author in [26] introduced and studied uniform bases for the logarithmic derivation modules of the ideal arrangements. In particular, from Theorem 4.4 which is valid for all Lie types, we obtained explicit presentations of the cohomology rings of regular nilpotent Hessenberg varieties in all Lie types.

4.2 Chromatic Symmetric Functions in Graph Theory

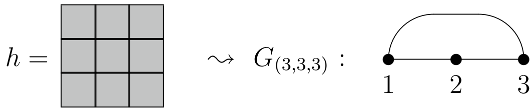
In Sect. 3.2, we explained the representation of S_n on the cohomology ring $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ of regular semisimple Hessenberg varieties constructed by J. Tymoczko. Recall that there is a natural correspondence between representations of S_n and symmetric functions of degree n (cf. [30, Sect. 7]). In this section, we describe the symmetric function corresponding to the S_n-representation on $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ which was conjectured by J. Shareshian and M. Wachs [59, Conjecture 1.2], [60, Conjecture 1.4] in terms of chromatic symmetric functions of a graph determined by h. This conjecture is proved by P. Brosnan and T. Chow in [16] and soon after by M. Guay-Paquet [35]. We refer [30] for notations and elementary knowledge of symmetric functions.

Let $h : [n] \to [n]$ be a Hessenberg function and G_h a graph on the vertex set $[n]$ defined as follows; there is an edge between the vertices $i, j \in [n]$ if $j < i \leq h(j)$. Namely, if we visualize h as a configuration of boxes as in Example 2.1, we have an edge between i and j if and only if we have a box in (i, j)-th position which is strictly below the diagonal (cf. Fig. 3).

Example 4.7 If we take $h = (n, n, \ldots, n)$, then the graph G_h is the complete graph[5] on the vertex set $[n]$. If we take $h = (2, 3, 4, \ldots, n, n)$, then the graph G_h is the path graph on the vertex set $[n]$ (Fig. 13).

[5] A complete graph is a graph in which every pair of distinct vertices is connected by an edge.

Fig. 13 The complete graph on the vertex set [3]



For a graph $G = (V, E)$ on the vertex set $V = [n]$, Shareshian-Wachs [59, 60] introduced the chromatic quasisymmetric function $X_G(\mathbf{x}, q)$ of G as

$$X_G(\mathbf{x}, q) = \sum_{\kappa} \Big(\prod_{i=1}^{n} x_{\kappa(i)} \Big) q^{\mathrm{asc}(\kappa)}, \tag{4.1}$$

where the summation runs over all proper colorings[6] $\kappa : [n] \to \mathbb{N} = \{1, 2, 3, \ldots\}$ of G and $\mathrm{asc}(\kappa) := |\{(j, i) \in E \mid j < i,\ \kappa(j) < \kappa(i)\}|$ is the number of ascents of κ. Here, the variable q stands for the grading, and this is a graded version of Stanley's chromatic symmetric function $X_G(\mathbf{x})$ of G [67]. In general, $X_G(\mathbf{x}, q)$ is quasisymmetric in $\mathbf{x}$-variables but may not be symmetric. However, for our graph G_h, it is known that $X_{G_h}(\mathbf{x}, q)$ is in fact symmetric [60, Theorem 4.5]. In [60, Theroem 6.3], the Schur basis expansion of $X_{G_h}(\mathbf{x}, q)$ is determined in terms of combinatorics, where the non-graded version was originally obtained by V. Gasharov [31].

Example 4.8 Let $n = 3$ and $h = (2, 3, 3)$. Then, one can verify that

$$X_{G_h}(\mathbf{x}, q) = s_{(1,1,1)}(\mathbf{x}) + \big(2s_{(1,1,1)}(\mathbf{x}) + s_{(2,1)}(\mathbf{x})\big)\, q + s_{(1,1,1)}(\mathbf{x})\, q^2. \tag{4.2}$$

Here, $s_\lambda(\mathbf{x})$ is the Schur function corresponding to a partition λ of 3.

The following theorem determines the S_n-representation on $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ in terms of the combinatorics of the graph G_h. This beautiful theorem was conjectured by Shareshian-Wachs [59, Conjecture 1.2], [60, Conjecture 1.4], and it was proved by Brosnan-Chow [16] and soon after by Guay-Paquet [35]. We denote by ch the Frobenius character under which symmetric functions of degree n corresponds to representations of S_n.

Theorem 4.9 ([16, 35]) *Let* $\mathrm{Hess}(S, h)$ *be a regular semisimple Hessenberg variety. Then,*

$$\omega X_{G_h}(\boldsymbol{x}, q) = \sum_{k=0}^{m} ch H^{2k}(\mathrm{Hess}(S, h); \mathbb{C})\, q^k.$$

where $m = \dim_{\mathbb{C}} \mathrm{Hess}(S, h)$ *and* ω *is the involution on the ring of symmetric functions in* $\boldsymbol{x}$*-variables sending each Schur function* $s_\lambda(\boldsymbol{x})$ *to* $s_{\tilde{\lambda}}(\boldsymbol{x})$ *associated with the transpose* $\tilde{\lambda}$.

[6] A map $\kappa : [n] \to \mathbb{N}$ is called a proper coloring of G if $\kappa(i) \neq \kappa(j)$ for all pair of vertices i and j which are connected by an edge.

For the case $h = (n, n, \dots, n)$, recall from Sect. 3.2 that the S_n-representation on $H^*(\mathrm{Hess}(S, h); \mathbb{C}) = H^*(Fl(\mathbb{C}^n); \mathbb{C})$ is trivial, and it is straightforward to see that $X_{G_h}(\mathbf{x}, q) = e_n(\mathbf{x}) \sum_\kappa q^{\mathrm{asc}(\kappa)}$ in this case where the latter sum is equal to the Poincaré polynomial of $Fl(\mathbb{C}^n)$. Since the complete symmetric function $h_n(\mathbf{x}) = \omega(e_n(\mathbf{x}))$ corresponds to the 1-dimensional trivial representation, the equality in Theorem 4.9 holds in this case. For the case $h = (2, 3, 4, \dots, n, n)$, both of $X_{G_h}(\mathbf{x}, q)$ and the S_n-representation on $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ were well-studied in [55, 66, 69], and the equality was known. See [60, Sect. 1] for details. The proof of Theorem 4.9 given by Brosnan-Chow is geometric in the sense that they used the theory of monodromy actions, whereas Guay-Paquet provided a combinatorial proof using the theory of Hopf algebras.

Example 4.10 Let $n = 3$ and $h = (2, 3, 3)$. Then, from (3.12) and (4.2), we see that

$$\begin{aligned}\omega X_{G_h}(\mathbf{x}, q) &= s_{(3)}(\mathbf{x}) + \big(2s_{(3)}(\mathbf{x}) + s_{(2,1)}(\mathbf{x})\big)\, q + s_{(3)}(\mathbf{x})\, q^2 \\ &= \mathrm{ch} H^0(\mathrm{Hess}(S, h); \mathbb{C}) + \mathrm{ch} H^2(\mathrm{Hess}(S, h); \mathbb{C})\, q + \mathrm{ch} H^4(\mathrm{Hess}(S, h); \mathbb{C})\, q^2.\end{aligned}$$

Theorem 4.9 is related with the Stanley-Stembridge conjecture in graph theory as we explain in what follows. Recall that the chromatic quasisymmetric function $X_G(\mathbf{x}, q)$ is a graded version of Stanley's chromatic symmetric function $X_G(\mathbf{x})$ [67] which we obtain by forgetting the grading parameter q in (4.1). Given a poset P, we can construct its incomparability graph which has as its vertices the elements of P and we have an edge between two vertices if they are not comparable in P. The Stanley-Stembridge conjecture is stated as follows.

Conjecture 4.11 *(Stanley-Stembridge conjecture* [68, Conjecture 5.5], [67, Conjecture 5.1]*)* Let G be the incomparability graph of a $(3 + 1)$-free poset. Then $X_G(\mathbf{x})$ is e-positive.

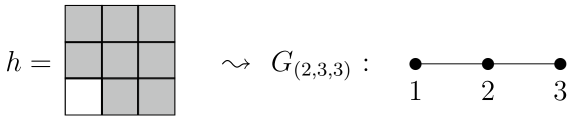
Here, a poset is called $(r + s)$-**free** if the poset does not contain an induced subposet isomorphic to the direct sum of an r element chain and an s element chain, and a symmetric function of degree n is called e-**positive** if it is a non-negative sum of elementary symmetric functions $e_\lambda(\mathbf{x})$ for partition λ of n. See [60], for more information on Stanley-Stembridge conjecture.

Example 4.12 Take a poset on [3]={1,2,3} for which $1 < 3$ is the unique comparable pair. It is obviously (3+1)-free. In this case, the incomparability graph G is the path graph given in Fig. 14. Hence, from (4.2), we see that

$$\begin{aligned}X_G(\mathbf{x}) &= s_{(1,1,1)}(\mathbf{x}) + \big(2s_{(1,1,1)}(\mathbf{x}) + s_{(2,1)}(\mathbf{x})\big) + s_{(1,1,1)}(\mathbf{x}) \\ &= 3e_{(3)}(\mathbf{x}) + e_{(2,1)}(\mathbf{x}),\end{aligned}$$

as desired since we have $e_{(3)}(\mathbf{x}) = s_{(1,1,1)}(\mathbf{x})$ and $e_{(2,1)}(\mathbf{x}) = s_{(2,1)}(\mathbf{x}) + s_{(1,1,1)}(\mathbf{x})$.

Fig. 14 The path graph on the vertex set [3]



Guay-Paquet [34] showed that, to solve the Stanley-Stembridge conjecture, it is enough to solve it for posets which are $(3+1)$-free and $(2+2)$-free, and this is precisely the case that the corresponding incomparability graph can be identified with G_h for some Hessenberg function $h : [n] \to [n]$, where n is the number of the elements of the poset, as we encountered in Example 4.12 (See [59, Sect. 4] for details). Thus, the Stanley-Stembridge conjecture is now reduced to the following conjecture by Theorem 4.9.

Conjecture 4.13 ([59, Conjecture 5.4], [60, Conjecture 10.4]) The S_n-representation on $H^*(\mathrm{Hess}(S, h); \mathbb{C})$ constructed by Tymoczko is a permutation representation, i.e. a direct sum of induced representations of the trivial representation from S_λ to S_n where $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$ for $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.

Motivated by the connection to Stanely-Stembridge conjecture, M. Harada and M. Precup verified Conjecture 4.11 for so-called *abelian* regular semisimple Hessenberg varieties, and they also derived a set of linear relations satisfied by the multiplicities of certain permutation representations. See [39, 40] for details.

5 More Topics

5.1 Semisimple Hessenberg Varieties

E. Insko and M. Precup studied Hessenberg varieties associated with semisimple matrices which may not be regular semisimple [44]. They determined the irreducible components of semisimple Hessenberg varieties for $h = (2, 3, 4, \ldots, n, n)$ in arbitrary Lie type. They also proved that irreducible components are smooth and gave an explicit description of their intersections.

5.2 Hessenberg Varieties for the Minimal Nilpotent Orbit

P. Crooks and the first author studied Hessenberg varieties associated with minimal nilpotent matrices (i.e. nilpotent matrices with Jordan blocks of size 1 and a single Jordan block of size 2). Explicit descriptions of their Poincaré polynomials and irreducible components are described, and a certain presentation of their cohomology rings are also provided in terms of Schubert classes. See [3] for details.

5.3 Regular Hessenberg Varieties

An $n \times n$ matrix R is called **regular** if the Jordan blocks of R have distinct eigenvalues. For a regular matrix R, $\mathrm{Hess}(R, h)$ is called a **regular Hessenberg variety**. This class of Hessenberg varieties contains regular nilpotent Hessenberg varieties and regular semisimple Hessenberg varieties, and $\mathrm{Hess}(R, h)$ plays an important role in the work of P. Brosnan and T. Chow [16] proving Shareshian-Wachs conjecture (Theorem 4.9). They are singular varieties in general, however M. Precup proved that the Betti numbers of $\mathrm{Hess}(R, h)$ are palindromic [53]. N. Fujita, H. Zeng, and the first author proved that higher cohomology groups of their structure sheaves vanish and that they degenerate to the regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ if $h(i) \geq i + 1$ for all $1 \leq i \leq n$ [5], which was motivated by the works of D. Anderson and J. Tymoczo [12] and L. DeDieu, F. Galetto, M. Harada, and the first author [2].

5.4 Poincaré Dual of Hessenberg Varieties

In [5], the Poincaré dual of a regular Hessenberg variety $\mathrm{Hess}(R, h)$ in $H^*(Fl(\mathbb{C}^n))$ was computed in terms of positive roots associated the Hessenberg function h, and E. Insko, J. Tymoczko, and A. Woo gave a combinatorial formula for this class using Schubert polynomials [46]. Also, the cohomology class $[\mathrm{Hess}(R, h)] \in H^*(Fl(\mathbb{C}^n))$ does not depend on a choice of a regular matrix R if $h(i) \geq i + 1$ for all $1 \leq i \leq n$ (See for details [5, 46]).

5.5 Additive Bases of the Cohomology Rings of Regular Nilpotent Hessenberg Varieties

M. Enokizono, T. Nagaoka, A. Tsuchiya, and the second author constructed in [27] an additive basis of the cohomology ring of a regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$. This basis is obtained by extending the Poincaré duals $[\mathrm{Hess}(N, h')] \in H^*(\mathrm{Hess}(N, h))$ of smaller regular nilpotent Hessenberg varieties $\mathrm{Hess}(N, h')$ with $h' \subset h$. In particular, all of the classes $[\mathrm{Hess}(N, h')] \in H^*(\mathrm{Hess}(N, h))$ with $h' \subset h$, are linearly independent. On the other hand, M. Harada, S. Murai, M. Precup, J. Tymoczko, and the second author derive in [38] a filtration on the cohomology ring $H^*(\mathrm{Hess}(N, h))$ of regular nilpotent Hessenberg varieties, from which they obtain a monomial basis for $H^*(\mathrm{Hess}(N, h))$. This basis is different from the one obtained in [27]. From the filtration they additionally obtain an inductive formula for the Poincaré polynomials of $\mathrm{Hess}(N, h)$; moreover, the monomial basis has an interpretation in terms of Schubert calculus.

5.6 The Volume Polynomials of Hessenberg Varieties

Recall from Corollary 3.3 that the cohomology ring $H^*(\mathrm{Hess}(N, h); \mathbb{Q})$ of a regular nilpotent Hessenberg variety is a Poincaré duality algebra. In [10], T. Abe, M. Masuda, S. Murai, T. Sato, and the second author computed the volume polynomial of this ring $H^*(\mathrm{Hess}(N, h); \mathbb{Q})$, and it precisely gives the volume of a certain embedding of any regular Hessenberg variety associated with h into a projective space [2, 5]. M. Harada, M. Masuda, S. Park, and the second author provided a combinatorial formula for this polynomial in terms of the volumes of certain faces of the Gelfand-Zetlin polytope [37].

5.7 Hessenberg Varieties of Parabolic Type

J. Tymoczko and M. Precup showed that the Betti numbers of parabolic Hessenberg varieties decompose into a combination of those of Springer fibers and Schubert varieties associated to the parabolic [54]. As a corollary, they deduced that the Betti numbers of some parabolic Hessenberg varieties in Lie type A are equal to those of a specific union of Schubert varieties.

5.8 Twins for Regular Semisimple Hessenberg Varieties

Given a Hessenberg function $h : [n] \to [n]$, A. Ayzenberg and V. Buchstaber introduced a smooth T-manifold X_h, where T is the maximal torus of $\mathrm{GL}(n, \mathbb{C})$ given in Sect. 3.2. This manifold is similar to the regular semisimple Hessenberg variety $\mathrm{Hess}(S, h)$ in some sense. For example, they have the same Betti numbers and their T-equivariant cohomology rings are isomorphic *as rings*. See [13] for details.

5.9 Integrable Systems and Hessenberg Varieties

For a Hessenberg function $h : [n] \to [n]$, we denote by $\mathcal{X}(h)$ the family of Hessenberg varieties associated with h. Then, $\mathcal{X}(h)$ in fact have a structure of a vector bundle over the flag variety $Fl(\mathbb{C}^n)$. For the case $h = (2, 3, 4, \ldots, n, n)$, the family $\mathcal{X}(h)$ contains the Peterson variety and the permutohedral variety. In this special case, P. Crooks and the first author showed that $\mathcal{X}(h)$ admits a Poisson structure with an open dense symplectic leaf, and that there is a completely integrable system on $\mathcal{X}(h)$ which contains the Toda lattice as a sub-system [4].

5.10 The Poset of Hessenberg Varieties

E. Drellich studied the poset of the Hessenberg varieties $\mathrm{Hess}(X, h)$ in $Fl(\mathbb{C}^n)$ for a given $n \times n$ matrix X. She proved that if X is not a scalar multiple of the identity matrix, then the Hessenberg functions determine distinct Hessenberg varieties. See [25] for details.

5.11 Springer Correspondence for Symmetric Spaces

As mentioned in Remark 2.2, Hessenberg varieties can be defined in a more general setting [33], and those Hessenberg varieties also appear in the works of T. H. Chen, K. Vilonen, and T. Xue [17–19] on Springer correspondence for symmetric spaces.

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References

1. Abe, H.: Young diagrams and intersection numbers for toric manifolds associated with Weyl chambers. Electron. J. Comb. **22(2)**(2), 24 pp (2015). Paper 2.4
2. Abe, H., DeDieu, L., Galetto, F., Harada, M.: Geometry of Hessenberg varieties with applications to Newton-Okounkov bodies. Sel. Math. (N.S.) **24**(3), 2129–2163 (2018)
3. Abe, H., Crooks, P.: Hessenberg varieties for the minimal nilpotent orbit. Pure Appl. Math. Q. **12**(2), 183–223 (2016)
4. Abe, H., Crooks, P.: Hessenberg varieties, Slodowy slices, and integrable systems. Math. Z. **291**(3), 1093–1132 (2019)
5. Abe, H., Fujita, N., Zeng, H.: Geometry of regular Hessenberg varieties, to appear in Transform. Groups
6. Abe, H., Harada, M., Horiguchi, T., Masuda, M.: The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A. Int. Math. Res. Not. **2019**(17), 5316–5388 (2019)
7. Abe, H., Harada, M., Horiguchi, T., Masuda, M.: The equivariant cohomology rings of regular nilpotent Hessenberg varieties in Lie type A: Research Announcement. Morfismos **18**(2), 51–65 (2014)
8. Abe, H., Horiguchi, T., Masuda, M.: The cohomology rings of regular semisimple Hessenberg varieties for $h = (h(1), n, \ldots, n)$. J. Comb. **10**(1), 27–59 (2019)
9. Abe, T., Barakat, M., Cuntz, M., Hoge, T., Terao, H.: The freeness of ideal subarrangements of Weyl arrangements. J. Eur. Math. Soc. **18**, 1339–1348 (2016)
10. Abe, T., Horiguchi, T., Masuda, M., Murai, S., Sato, T.: Hessenberg varieties and hyperplane arrangements, to appear in J. Reine Angew. Math. https://doi.org/10.1515/crelle-2018-0039.
11. Abe, T., Maeno, T., Murai, S., Numata, Y.: Solomon-Terao algebra of hyperplane arrangements. J. Math. Soc. Jpn. **71**(4), 1027–1047 (2019)

12. Anderson, D., Tymoczko, J.: Schubert polynomials and classes of Hessenberg varieties. J. Algebra **323**(10), 2605–2623 (2010)
13. Ayzenberg, A., Buchstaber, V.: Manifolds of isospectral matrices and Hessenberg varieties. arXiv:1803.01132
14. Balibanu, A.: The Peterson variety and the wonderful compactification. Represent. Theory **21**, 132–150 (2017)
15. Brion, M., Carrell, J.: The equivariant cohomology ring of regular varieties. Michigan Math. J. **52**(1), 189–203 (2004)
16. Brosnan, P., Chow, T.: Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. Adv. Math. **329**, 955–1001 (2018)
17. Chen, T.H., Vilonen, K., Xue, T.: Springer Correspondence for Symmetric Spaces. arXiv:1510.05986
18. Chen, T.H., Vilonen, K., Xue, T.: Hessenberg varieties, intersections of quadrics, and the Springer correspondence. arXiv:1511.00617
19. Chen, T.H., Vilonen, K., Xue, T.: Springer correspondence for the split symmetric pair in type A. Compos. Math. **154**(11), 2403–2425 (2018)
20. De Concini, C., Lusztig, G., Procesi, C.: Homology of the zero-set of a nilpotent vector field on a flag manifold. J. Am. Math. Soc. **1**(1), 15–34 (1988)
21. De Mari, F., Procesi, C., Shayman, M.A.: Hessenberg varieties. Trans. Am. Math. Soc. **332**(2), 529–534 (1992)
22. De Mari, F., Shayman, M.A.: Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix. Acta Appl. Math. **12**(3), 213–235 (1988)
23. Drellich, E.: Monk's rule and Giambelli's formula for Peterson varieties of all Lie types. J. Algebraic Comb. **41**(2), 539–575 (2015)
24. Drellich, E.: Combinatorics of equivariant cohomology: Flags and regular nilpotent Hessenberg varieties. Ph.D. thesis, University of Massachusetts (2015)
25. Drellich, E.: The Containment Poset of Type A Hessenberg Varieties. arXiv:1710.05412
26. Enokizono, M., Horiguchi, T., Nagaoka, T., Tsuchiya, A.: Uniform bases for ideal arrangements. arXiv:1912.02448
27. Enokizono, M., Horiguchi, T., Nagaoka, T., Tsuchiya, A.: An additive basis for the cohomology rings of regular nilpotent Hessenberg varieties (in preparation)
28. Epure, R., Schulze, M.: A Saito criterion for holonomic divisors. Manuscripta Math. **160**(1–2), 1–8 (2019)
29. Fukukawa, Y., Harada, M., Masuda, M.: The equivariant cohomology rings of Peterson varieties. J. Math. Soc. Jpn. **67**(3), 1147–1159 (2015)
30. Fulton, W.: Young Tableaux. London Mathematical Society Student Texts, vol. 35. Cambridge University Press, Cambridge
31. Gasharov, V.: Incomparability graphs of (3+1)-free posets are s-positive, Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994). Discrete Math. **157**(1-3), 193–197 (1996)
32. Goresky, M., Kottwitz, R., MacPherson, R.: Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math. **131**(1), 25–83 (1998)
33. Goresky, M., Kottwitz, R., MacPherson, R.: Purity of equivalued affine Springer fibers. Represent. Theory **10**, 130–146 (2006)
34. Guay-Paquet, M.: A modular relation for the chromatic symmetric functions of (3+1)-free posets. arXiv:1306.2400
35. Guay-Paquet, M.: A second proof of the Shareshian-Wachs conjecture, by way of a new Hopf algebra. arXiv:1601.05498
36. Harada, M., Horiguchi, T., Masuda, M.: The equivariant cohomology rings of Peterson varieties in all Lie types. Can. Math. Bull. **58**(1), 80–90 (2015)
37. Harada, M., Horiguchi, T., Masuda, M., Park, S.: The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Zetlin polytope. Proc. Steklov Inst. Math. **305**, 318–44 (2019)

38. Harada, M., Horiguchi, T., Murai, S., Precup, M., Tymoczko, J.: A filtration on the cohomology rings of regular nilpotent Hessenberg varieties (In Preparation)
39. Harada, M., Precup, M.: The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture. arXiv:1709.06736, to be published in J. Alg. Comb
40. Harada, M., Precup, M.: Upper-triangular linear relations on multiplicities and the Stanley-Stembridge conjecture. arXiv:1812.09503
41. Harada, M., Precup, M.: Upper-triangular linear relations on multiplicities and the Stanley-Stembridge conjecture. arXiv:1812.09503
42. Horiguchi, T.: The cohomology rings of regular nilpotent Hessenberg varieties and Schubert polynomials. Proc. Jpn. Acad. Ser. A Math. Sci. **94**, 87–92 (2018)
43. Insko, E.: Schubert calculus and the homology of the Peterson variety. Electron. J. Comb. **22**(2), 12 pp (2015). Paper 2.26
44. Insko, E., Precup, M.: The singular locus of semisimple Hessenberg varieties. J. Algebra **521**, 65–96 (2019)
45. Insko, E., Tymoczko, J.: Intersection theory of the Peterson variety and certain singularities of Schubert varieties. Geom. Dedicata **180**, 95–116 (2016)
46. Insko, E., Tymoczko, J., Woo, A.: A formula for the cohomology and K-class of a regular Hessenberg variety. J. Pure Appl. Algebra. https://doi.org/10.1016/j.jpaa.2019.106230
47. Insko, E., Yong, A.: Patch ideals and Peterson varieties. Transform. Groups **17**(4), 1011–1036 (2012)
48. Klyachko, A.: Orbits of a maximal torus on a flag space. Funct. Anal. Appl. **19**(2), 65–66 (1985)
49. Kostant, B.: Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ. Sel. Math. (N.S.) **2**, 43–91 (1996)
50. Mbirika, A., Tymoczko, J.: Generalizing Tanisaki's ideal via ideals of truncated symmetric functions. J. Algebraic Comb. **37**(1), 167–199 (2013)
51. Orlik, P., Terao, H.: Arrangements of Hyperplanes, Grundlehren der Mathematischen Wissenschaften, vol. 300. Springer, Berlin (1992)
52. Precup, M.: Affine pavings of Hessenberg varieties for semisimple groups. Sel. Math. New Ser. **19**, 903–922 (2013)
53. Precup, M.: The Betti numbers of regular Hessenberg varieties are palindromic. Transform. Groups **23**(2), 491–499 (2018)
54. Precup, M., Tymoczko, J.: Hessenberg varieties of parabolic type. arXiv:1701.04140
55. Procesi, C.: The Toric Variety Associated to Weyl Chambers, Mots, pp. 153–161. Lang. Raison. Calc, Hermés, Paris (1990)
56. Rietsch, K.: Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties. J. Am. Math. Soc. **16**, 363–392 (2003)
57. Röhrle, G.: Arrangements of ideal type. J. Algebra **484**, 126–167 (2017)
58. Saito, K.: Theory of logarithmic differential forms and logarithmic vector fields. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27**, 265–291 (1980)
59. Shareshian, J., Wachs, M.L.: Chromatic quasisymmetric functions and Hessenberg varieties, Configuration spaces, CRM Series. Ed. Norm., Pisa, vol. 14, pp. 433–460 (2012)
60. Shareshian, J., Wachs, M.L.: Chromatic quasisymmetric functions. Adv. Math. **295**, 497–551 (2016)
61. Solomon, L., Terao, H.: A formula for the characteristic polynomial of an arrangement. Adv. Math. **64**(3), 305–325 (1987)
62. Sommers, E., Tymoczko, J.: Exponents for B-stable ideals. Trans. Am. Math. Soc. **358**(8), 3493–3509 (2006)
63. Spaltenstein, N.: The fixed point set of a unipotent transformation on the flag manifold. Nederl. Akad. Wetensch. Proc. Ser. A **79**; Indag. Math. **38**(5), 452–456 (1976)
64. Springer, T.A.: Trigonometric sums, Green functions of finite groups and representations of Weyl groups. Invent. Math. **36**, 173–207 (1976)
65. Springer, T.A.: A construction of representations of Weyl groups. Invent. Math. **44**, 279–293 (1978)

66. Stanley, R.P.: Log-concave and unimodal sequences in algebra, combinatorics, and geometry. Graph Theory and Its applications: East and West (Jinan, 1986), vol. 576, pp. 500–535. Ann. New York Academic Science, New York (1989)
67. Stanley, R.P.: A symmetric function generalization of the chromatic polynomial of a graph. Adv. Math. **111**(1), 166–194 (1995)
68. Stanley, R.P., Stembridge, J.R.: On immanants of Jacobi-Trudi matrices and permutations with restricted position. J. Comb. Theory Ser. A **62**(2), 261–279 (1993)
69. Stembridge, J.: Eulerian numbers, tableaux, and the Betti numbers of a toric variety. Discret. Math. **99**(1–3), 307–320 (1992)
70. Teff, N.: Representations on Hessenberg varieties and Young's rule. In: 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), pp. 903–914, Discrete Math. Theor. Comput. Sci. Proc., AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy (2011)
71. Teff, N.: A divided difference operator for the highest root Hessenberg variety, 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013), 993–1004, Discrete Math. Theor. Comput. Sci. Proc., AS, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2013
72. Tymoczko, J.: Linear conditions imposed on flag varieties. Am. J. Math. **128**(6), 1587–1604 (2006)
73. Tymoczko, J.: Paving Hessenberg varieties by affines. Sel. Math. (N.S.) **13**, 353–367 (2007)
74. Tymoczko, J.: Permutation actions on equivariant cohomology of flag varieties. Toric Topology, Contemporary Mathematics, vol. 460, 365–384. American Mathematical Society, Providence (2008)

Stability of Bott–Samelson Classes in Algebraic Cobordism

Thomas Hudson, Tomoo Matsumura, and Nicolas Perrin

Abstract In this paper, we construct *stable Bott–Samelson classes* in the projective limit of the algebraic cobordism rings of full flag varieties, upon an initial choice of a reduced word in a given dimension. Each stable Bott–Samelson class is represented by a bounded formal power series modulo symmetric functions in positive degree. We make some explicit computations for those power series in the case of infinitesimal cohomology. We also obtain a formula of the restriction of Bott–Samelson classes to smaller flag varieties.

Keywords Schubert calculus · Cobordism · Flag variety · Bott–Samelson resolution

1 Introduction

Let k be an algebraically closed field of characteristic 0. Let Fl_n be the flag variety of complete flags in k^n. It can be identified with the homogeneous space $\mathrm{GL}_n(\mathsf{k})/B$ where B is the Borel subgroup of upper triangular matrices. For each permutation $w \in S_n$, the corresponding Schubert variety $X_w^{(n)} \subset Fl_n$ is defined as $\overline{B_- wB}$, the closure of the orbits of wB by the action of the opposite Borel subgroup B_-. If

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$\iota_n : Fl_n \to Fl_{n+1}$ is the natural embedding, the cohomology fundamental classes of these Schubert varieties have the property that $\iota_n^*[X_w^{(n+1)}] = [X_w^{(n)}]$, i.e., *the Schubert classes are stable under the pullback maps*. The exact analogue of this property also holds in K-theory, in which one defines the Schubert classes as the K-theory classes of the structure sheaves of Schubert varieties.

In this paper, we attempt to generalize the above notion of stability to Bott-Samelson classes in algebraic cobordism. The algebraic cobordism, denoted by Ω^*, was introduced by Levine–Morel in [16] and represents the universal object among oriented cohomology theories, a family of functors which includes both the Chow ring CH^* and $K^0[\beta, \beta^{-1}]$, a graded version of the Grothendieck ring of vector bundles. In recent years a lot of energy has been spent to lift results of Schubert calculus to Ω^*, in the same way in which Bressler–Evens did in [1, 2] for topological cobordism. The first works in this direction were those of Calmés–Petrov–Zanoulline [3] and Hornbostel–Kiritchenko [6] who investigated the algebraic cobordism of flag manifolds. Later, the interest shifted to Grassmann and flag bundles (cf. [4, 7–13]). One of the main difficulties of Schubert calculus in algebraic cobordism is caused by the fact that the fundamental classes of Schubert varieties are not well-defined in general oriented cohomology theories. A candidate for the replacement of Schubert classes is the family of the push-forward classes of Bott–Samelson resolutions of Schubert varieties.

Since a Bott–Samelson variety is defined upon a choice of a reduced word, our stability of Bott–Samelson classes depends on a particular choice of a *sequence of* reduced words. The followings are the main results in this paper: (1) For a given Bott–Samelson variety Y_n in Fl_n, we construct a sequence of Bott-Samelson varieties Y_m over Fl_m for $m \geq n$ such that their push-forward classes in algebraic cobordism are stable under pullbacks, namely, the identity $\iota_m^*[Y_{m+1} \to Fl_{m+1}] = [Y_m \to Fl_m]$ holds in $\Omega^*(Fl_m)$ for all $m \geq n$; (2) For a given Bott–Samelson variety Y_n over Fl_n, we find an explicit formula for the pullback $\iota_{n-1}^*[Y_n \to Fl_n]$ of its push-forward class in $\Omega^*(Fl_{n-1})$.

The pullback maps $\iota_n^* : \Omega^*(Fl_{n+1}) \to \Omega^*(Fl_n)$ give rise to a projective system of graded rings. Based on the ring presentation of $\Omega^*(Fl_n)$ obtained by Hornbostel–Kiritchenko [6], we observe that their graded projective limit, denoted by $\mathcal{R}$, is isomorphic to the graded ring of *bounded formal power series* in an infinite sequence of variables $x = (x_i)_{i \in \mathbb{Z}_{>0}}$ with coefficients in the *Lazard ring* $\mathbb{L}$ modulo the ideal of symmetric functions of positive degrees in x. Our stable sequence of Bott–Samelson classes determine a class in this limit, which we call a *stable Bott–Samelson class*. On each $\Omega^*(Fl_n)$ the divided difference operators commute with the pullback maps and therefore lift to the limit $\mathcal{R}$. This gives a method of computing the power series representing stable Bott–Samelson classes, which we apply to the case of a chosen infinitesimal cohomology theory. In particular, we obtain a formula for the power series representing stable Bott–Samelson classes associated to *dominant permutations*.

In [11, 12], the first and second authors obtained determinant formula of the cobordism push-forward classes of so-called Damon–Kempf–Laksov resolutions, generalizing the classical Damon–Kempf–Laksov determinant formula of Schubert classes.

In [10], more explicit formula of Damon–Kempf–Laksov classes were obtained for infinitesimal cohomology. While these resolutions only exist for Schubert varieties associated to vexillary permutations (like for instance Grassmannian elements), their push-forward classes are stable and so is their determinantal formula. On the other hand, Naruse–Nakagawa [17–20] achieved, by considering a different resolution, a stable generalization of the Hall–Littlewood type formula for Schur polynomials in the context of topological cobordism. The differences among these stable expressions, including the ones obtained in this paper, should reflect the geometric nature of the different resolutions, each of which gives a different class in cobordism.

The paper is organized as follows. In Sect. 2, we recall basic facts about the algebraic cobordism ring of flag varieties and, in particular, we identify their projective limit. In Sect. 3, we review the definition of Bott–Samelson resolutions and show the stability of their push-forward classes in cobordism based on the choice of a sequence of reduced words. We then focus on infinitesimal cohomology theory and compute, using divided difference operators, the power series representing the limits of the classes associated to dominant permutations. In Sect. 4, we prove a formula for the product of any Bott–Samelson class with the class $[Fl_{n-1} \to Fl_n]$, generalizing the restriction formula given in Sect. 3.

2 Preliminary

Let k be an algebraic closed field of characteristic 0.

2.1 Basics on Algebraic Cobordisms

For the reader's convenience, we will briefly recall some basic facts about algebraic cobordism and infinitesimal theories. More details on the construction and the properties of Ω^* can be found in [16], while a more comprehensive description of I_n^* is given in [10].

Both Ω^* and I_n^* are examples of oriented cohomology theories, a family of contravariant functors $A^* : \mathbf{Sm}_{\mathsf{k}} \to \mathcal{R}^*$ from the category of smooth schemes to graded rings, which are furthermore endowed with push-forward maps for projective morphisms. Such functors are required to satisfy, together with some expected functorial compatibilities, the projective bundle formula and the extended homotopy property. These imply that, for every vector bundle $E \to X$, one is able to describe the evaluation of A^* on the associated projective bundle $\mathbb{P}(E) \to X$ as well as on every E-torsor $V \to X$. The Chow ring CH^* is probably the most well-known example of oriented cohomology theory and it should be kept in mind as a first approximation to the general concept.

As a direct consequence of the projective bundle formula one has that every oriented cohomology theory admits a theory of Chern classes, which can be defined using Grothendieck's method. These satisfy most of the expected properties, like for instance the Whitney sum formula, however it is no longer true that the first Chern class behaves linearly with respect to tensor product: this is a key difference with CH^*. For a pair of line bundles L and M defined over the same base, classically one has

$$c_1^{CH}(L \otimes M) = c_1^{CH}(L) + c_1^{CH}(M) \quad \text{and} \quad c_1^{CH}(L^\vee) = -c_1^{CH}(L), \tag{1}$$

but these equalities in general fail for c_1^A. Instead, in order to describe $c_1^A(L \otimes M)$, it becomes necessary to introduce a *formal group law*, a power series in two variables defined over the coefficient ring $F_A \in A^*(\operatorname{Spec} k)[[u, v]]$ satisfying some requirements. Similarly, expressing $c_1^A(L^\vee)$ in terms of $c_1^A(L)$ requires one to consider the *formal inverse* $\chi_A \in A^*(\operatorname{Spec} k)[[u]]$. The analogues of (1) then become

$$c_1^A(L \otimes M) = F_A(c_1^A(L), c_1^A(M)) \quad \text{and} \quad c_1^A(L^\vee) = \chi_A(c_1^A(L)). \tag{2}$$

It is a classical result of Lazard [15] that every formal group law (R, F_R) can be obtained from the universal one $(\mathbb{L}, F_{\mathbb{L}})$, which is defined over a ring later named after him. He also proved that, as a graded ring, $\mathbb{L} = \bigoplus_{m \leq 0} \mathbb{L}^m$ is isomorphic to a polynomial ring in countably many variables y_i, each appearing in degree $-i$ for $i \geq 1$. In the case of a field of characteristic 0, Levine and Morel were able to prove that the coefficient ring of algebraic cobordism is isomorphic to $\mathbb{L}$ and that its formal group law F_Ω coincides with the universal one, which from now on we will simply denote F. The universality of Ω^* does not restrict itself only to its coefficient ring, in fact, Levine and Morel were able to prove the following theorem.

Theorem 2.1 ([16, Theorem 1.2.6]) *Ω^* is universal among oriented cohomology theories on* $\mathbf{Sm}_\mathsf{k}$. *That is, for any other oriented cohomology theory A^* there exists a unique morphism*

$$\vartheta_A : \Omega^* \to A^*$$

of oriented cohomology theories.

It essentially follows formally from this result that for any given formal group law (R, F_R) the functor $\Omega^* \otimes_{\mathbb{L}} R$ is universal among the oriented cohomology theories with R as coefficient ring and F_R as associated law. This procedure can be used to produce functors, like the infinitesimal theories I_n^*, whose formal group laws are far simpler than the universal one and as a consequence more suitable for explicit computations. More precisely the projection $\mathbb{L} \to \mathbb{Z}[y_n]/(y_n^2)$, which maps y_i to 0 unless $i = n$, gives rise to the following formal group law F_{I_n} on $\mathbb{Z}[y_n]/(y_n^2)$:

$$F_{I_n}(u, v) = u + v + y_n \cdot \frac{1}{d_n} \sum_{j=1}^{n} \binom{n+1}{j} u^j v^{n+1-j}. \tag{3}$$

Here one has $d_n = p$, if $n + 1$ is a power of a prime p, and $d_n = 1$ otherwise. In our computations we will only consider the case $n = 2$, for which (3) becomes

$$u \boxplus v := F_{I_2}(u, v) = u + v + y_2(u^2v + uv^2) = (u + v)(1 + y_2uv)$$

with the formal inverse being $\boxminus u := \chi_{I_2}(u) = -u$. For the remainder of the paper we will write γ instead of y_2.

Let us finish this overview by discussing fundamental classes, another aspect in which a general oriented cohomology theory differs from CH^*. While in CH^* it is possible to associate such a class to every equi-dimensional scheme, for a general oriented cohomology theory A^* one has to restrict to schemes whose structure morphism is a local complete intersection. In particular, since not all Schubert varieties satisfy this requirement, it becomes necessary to find an alternative definition for Schubert classes. One possible option is to choose a family of resolutions of singularities and replace the fundamental classes of Schubert varieties with the pushforwards of the associated resolutions.

2.2 Algebraic Cobordism of Flag Varieties and Their Limit

For any integers a, b such that $a \leq b$, let $[a, b] := \{a, a + 1, \ldots, b\}$. Let $\mathsf{k}^{\mathbb{Z}_{>0}}$ be the infinite dimensional vector space generated by a formal basis $(e_i)_{i\in\mathbb{Z}_{>0}}$. For each $m \in \mathbb{Z}_{>0}$, let E_m be the subspace of $\mathsf{k}^{\mathbb{Z}_{>0}}$ generated by $e_1, \ldots, e_m$. We set $E_0 = 0$. We often identify E_m with k^m the space of column vectors.

For each $n \in \mathbb{Z}_{>0}$, the flag variety Fl_n consists of flags $U_\bullet = (U_i)_{i\in[1,n-1]}$ of subspaces in E_n where $U_i \subset U_{i+1}$ and $\dim U_i = i$ for each $i \in [1, n - 1]$. Note that this implies $U_n = E_n$. For a fixed n, let $\mathcal{U}_i^{(n)}$, $i \in [0, n]$ denote the tautological vector bundles of Fl_n and $\mathcal{E}_i$ the trivial bundles of fiber E_i. In particular, $\mathcal{U}_0^{(n)} = 0$ and $\mathcal{U}_n^{(n)} = \mathcal{E}_n$.

Let $\mathrm{GL}_n(\mathsf{k}) = \mathrm{GL}(E_n)$ be the general linear group. We consider the maximal torus $T_n \subset \mathrm{GL}_n(\mathsf{k})$ given by the matrices having $(e_i)_{i\in[1,n]}$ as a basis of eigenvectors and the Borel subgroup $B_n \subset \mathrm{GL}_n(\mathsf{k})$ given by the upper triangular matrices stabilizing the flag $E_\bullet = (E_i)_{i\in[1,n-1]}$ in Fl_n. We can identify Fl_n with the homogeneous space $\mathrm{GL}_n(\mathsf{k})/B_n$ by associating the matrix $M = (u_1, \ldots, u_n)$ to a flag $U_\bullet$ where $\{u_j\}_{j\in[1,i]}$ is a basis of U_i.

There is an isomorphism of graded rings ([6, Theorem 1.1])

$$\Omega^*(Fl_n) \cong \mathbb{L}[x_1, \ldots, x_n]/\mathbb{S}_n \tag{4}$$

sending $c_1((\mathcal{U}_i^{(n)}/\mathcal{U}_{i-1}^{(n)})^\vee)$ to x_i, where $\mathbb{S}_n$ is the ideal generated by the homogeneous symmetric polynomials in $x_1, \ldots, x_n$ of strictly positive degree.

Let $\iota_n : Fl_n \hookrightarrow Fl_{n+1}$ be the embedding induced by the canonical inclusion $E_n \hookrightarrow E_{n+1}$. We have $\iota_n^*\mathcal{U}_i^{(n+1)} = \mathcal{U}_i^{(n)}$ for all $i \in [1, n]$ and $\iota_n^*\mathcal{U}_{n+1}^{(n+1)} = \mathcal{E}_{n+1}$. As a consequence, under the isomorphism (4), the pullback map $\iota_n^* : \Omega^*(Fl_{n+1}) \to \Omega^*(Fl_n)$ is the natural projection given by setting $x_{n+1} = 0$. For each $m \in \mathbb{Z}$, let $\mathcal{R}^m$ be the projective limit of $\Omega^m(Fl_n)$ with respect to ι_n^*. We define the *graded projective limit* of $\Omega^*(Fl_n)$ with respect to ι_n^* to be $\mathcal{R} := \bigoplus_{m\in\mathbb{Z}} \mathcal{R}^m$.

In order to give a ring presentation of $\mathcal{R}$, we introduce the following ring of formal power series. Let $x = (x_i)_{i\in\mathbb{Z}_{>0}}$ be a sequence of infinitely many indeterminates. Let $\mathbb{Z}^\infty$ be the set of infinite sequence $\mathbf{s} = (s_i)_{i\in\mathbb{Z}_{>0}}$ of nonnegative integers such that all but finitely many s_i's are zero. Let $\mathbb{L}[[x]]^{(m)}$ be the space of formal power series of degree $m \in \mathbb{Z}$. An element $f(x)$ of $\mathbb{L}[[x]]^{(m)}$ is uniquely given as

$$f(x) = \sum_{\mathbf{s}\in\mathbb{Z}^\infty} a_{\mathbf{s}} x^{\mathbf{s}}, \quad a_{\mathbf{s}} \in \mathbb{L}, \quad x^{\mathbf{s}} = \prod_{i=1}^{\infty} x_i^{s_i}$$

such that $|\mathbf{s}| + \deg a_{\mathbf{s}} = m$ where $|\mathbf{s}| = \sum_{i=0}^{\infty} s_i$ and $\deg a_{\mathbf{s}}$ is the degree of $a_{\mathbf{s}}$ in $\mathbb{L}$. An element $f(x) \in \mathbb{L}[[x]]^{(m)}$ is *bounded* if $p_n(f(x)) \in \mathbb{L}[x_1, \dots, x_n]^{(m)}$, where p_n is the substitution of $x_k = 0$ for all $k > n$ and $\mathbb{L}[x_1, \dots, x_n]^{(m)}$ is the degree m part of $\mathbb{L}[x_1, \dots, x_n]$. Let $\mathbb{L}[[x]]_{bd}^{(m)}$ be the set of all such bounded elements of $\mathbb{L}[[x]]^m$. We set

$$\mathbb{L}[[x]]_{bd} := \bigoplus_{m\in\mathbb{Z}} \mathbb{L}[[x]]_{bd}^{(m)}.$$

This is a graded sub $\mathbb{L}$-algebra of the ring $\mathbb{L}[[x]]$ of formal power series.

Proposition 2.2 *There is an isomorphism of graded $\mathbb{L}$-algebras*

$$\mathcal{R} \cong \mathbb{L}[[x]]_{bd}/\mathbb{S}_\infty$$

where $\mathbb{S}_\infty$ is the ideal of $\mathbb{L}[[x]]_{bd}$ generated by symmetric functions in x of strictly positive degree.

Proof Let $m \in \mathbb{Z}$. The projections $p_n : \mathbb{L}[[x]]_{bd}^{(m)} \to \mathbb{L}[x_1, \dots, x_n]^{(m)}$ for $n > 0$ induce a surjective homomorphism

$$\Phi : \mathbb{L}[[x]]_{bd}^{(m)} \to \lim_{n\to\infty} \mathbb{L}[x_1, \dots, x_n]^{(m)}$$

sending $f(x)$ to $\{p_n(f(x))\}_{n\in\mathbb{Z}_{>0}}$. It is also easy to see that Φ is injective, and thus an isomorphism. Moreover, p_n's induce surjections

$$\mathbb{L}[[x]]_{bd}^{(m)} \cap \mathbb{S}_\infty \to \mathbb{L}[x_1, \dots, x_n]^{(m)} \cap \mathbb{S}_n, \quad n > 0,$$

inducing a bijection

$$\mathbb{L}[[x]]_{bd}^{(m)} \cap \mathbb{S}_\infty \cong \lim_{n\to\infty} \left(\mathbb{L}[x_1, \dots, x_n]^{(m)} \cap \mathbb{S}_n\right).$$

Thus we obtain the isomorphism

$$\bigoplus_{m \in \mathbb{Z}} \mathbb{L}[[x]]_{bd}^{(m)} / (\mathbb{L}[[x]]_{bd}^{(m)} \cap \mathbb{S}_\infty) \cong \bigoplus_{m \in \mathbb{Z}} \lim_{n \to \infty} \mathbb{L}[x_1, \dots, x_n]^{(m)} / (\mathbb{L}[x_1, \dots, x_n]^{(m)} \cap \mathbb{S}_n),$$

which is the desired one.

Definition 2.3 An element in $\mathcal{R}^i$ is a sequence $(\alpha_n)_{n \in \mathbb{Z}_{>0}}$ such that $\alpha_n \in \Omega^i(Fl_n)$ and $\iota_n^*(\alpha_{n+1}) = \alpha_n$ for all $n > 0$. An element of $\mathcal{R}$ is a finite linear combinations of such sequences and we call it a *stable class*.

Remark 2.4 In order to specify an element of $\mathcal{R}^i$, we only need to provide α_i for all $i \geq N$ for some fixed integer N. In fact, for $i < N$ the elements α_i can be obtained from α_N by applying the projections ι_n^*.

2.3 Divided Difference Operators

Let W_n be the Weyl group of $\mathrm{GL}_n(\mathsf{k})$. The maximal torus T_n and the Borel subgroup B_n define a system of simple reflections $s_1, \dots, s_{n-1} \in W_n$ and we can identify W_n with the symmetric group S_n in n letters, where each s_i corresponds to the transposition of the letters i and $i+1$. We denote the length of w by $\ell(w)$.

For each $i \in [1, n-1]$, the *divided difference operator* ∂_i is an operator on $\Omega^*(Fl_n)$ defined as follows. Let $Fl_n^{(i)}$ be the partial flag variety consisting of flags of the form $U_1 \subset \cdots \subset U_{i-1} \subset U_{i+1} \subset \cdots \subset U_{n-1}$ with $\dim U_k = k$. Denote the canonical projection $Fl_n \to Fl_n^{(i)}$ by p_i. Then define $\partial_i := p_{i*} \circ p_i^*$. It is known from [6] that under the presentation (4), we have

$$\partial_i(f(x)) = (\mathrm{id} + s_i) \frac{f(x)}{F(x_i, \chi(x_{i+1}))} = \frac{f(x)}{F(x_i, \chi(x_{i+1}))} + \frac{s_i f(x)}{F(x_{i+1}, \chi(x_i))}. \quad (5)$$

Lemma 2.5 *The pullback $\iota_n^* : \Omega^*(Fl_{n+1}) \to \Omega^*(Fl_n)$ commutes with ∂_i for all $i \in [1, n-1]$. In particular, this shows that ∂_i can be defined in the projective limit $\mathcal{R}$ and it is given by the formula (5).*

Proof For each $i \in [1, n-1]$, ι_n and p_i form a fiber diagram

$$\begin{tikzcd} Fl_n \arrow[r, "\iota_n"'] \arrow[d, "p_i"'] & Fl_{n+1} \arrow[d, "p_i"'] \\ Fl_n^{(i)} \arrow[r, "\iota_n"'] & Fl_{n+1}^{(i)}, \end{tikzcd}$$

and, since they are transverse, we have $\iota_n^* \circ p_{i*} = p_{i*} \circ \iota_n^*$. Thus $\iota_n^* \circ \partial_i = \iota_n^* \circ p_{i*} \circ p_i^* = p_{i*} \circ \iota_n^* \circ p_i^* = p_{i*} \circ p_i^* \circ \iota_n^* = \partial_i \circ \iota_n^*$. □

For a permutation $w \in W_n$, let $\mathring{X}_w^{(n)} = B_n \cdot w(E_\bullet)$ be the Bruhat cell associated to w in Fl_n, where $w(E_\bullet)$ is the flag consisting of $w(E_i) = \langle e_{w(1)}, \dots, e_{w(i)} \rangle$ for each $i \in [1, n-1]$. The *Schubert varieties* $X_w^{(n)}$ are the closures of the Bruhat cells: $X_w^{(n)} := \overline{B_n \cdot w(E_\bullet)}$. The *opposite Schubert varieties* are defined via $X^w_{(n)} = w_0 \cdot X^{(n)}_{w_0 w}$, where $w_0 = w_0^{(n)}$ is the longest element of W_n. As an orbit closure, we have $X^w_{(n)} = \overline{B_n^- \cdot w(E_\bullet)}$ where $B_n^- := w_0 B_n w_0$ is the opposite Borel subgroup of lower triangular matrices.

Remark 2.6 The fundamental class $[X^w_{(n)}]$ of $X^w_{(n)}$ is well-defined in the Chow ring of Fl_n. Those classes are stable along pullbacks, i.e., $\iota_n^*[X^w_{(n+1)}] = [X^w_{(n)}]$ in $CH^*(Fl_n)$ where $w \in S_n$ is regarded as an element of S_{n+1} under the natural embedding $S_n \subset S_{n+1}$. As it is well-known, its stable limit can be identified with the Schubert polynomial of Lascoux–Schützenberger [14]. It is also worth mentioning that the Schubert classes admit the following compatibility with divided difference operators, reflected on the definition of Schubert polynomials: for each $i \in [1, n-1]$, we have

$$\partial_i [X^w_{(n)}] = \begin{cases} [X^{ws_i}_{(n)}] & \ell(ws_i) = \ell(w) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

2.4 Some Facts on Permutations and Reduced Words

We conclude this section by fixing notations for reduced words and showing a few lemmas and a proposition that will be used in the rest of the paper.

We denote by $\underline{W}_n$ the set of words in $s_1, \ldots, s_{n-1}$: an element of $\underline{W}_n$ will be written as a finite sequence $s_{i_1} \ldots s_{i_r}$, while the empty word is denoted by 1. The *length* of a word $\underline{w} = s_{i_1} \ldots s_{i_r}$ is the number r of the letters s_i's in $\underline{w}$ and we denote it by $\ell(\underline{w})$. For a word $\underline{w} \in \underline{W}_n$, we denote the corresponding permutation by $w \in W_n$. Let W_n^i be the subgroup of W_n generated by all simple reflections s_j with $j \neq i$ and $\underline{W}_n^i$ the corresponding set of words. In particular, we can identify W_n with W_{n+1}^n and $\underline{W}_n$ with $\underline{W}_{n+1}^n$.

We denote the Bruhat order in W_n by $\leq$, i.e., $w \leq v$ if and only if every reduced word for v contains a subword which is a reduced word for w.

We denote by $c^{(n)}$ the *Coxeter element* $s_1 \ldots s_n$ of W_{n+1}. It has a unique reduced word $\underline{c}^{(n)} = s_1 \ldots s_n$. Note that $\underline{c}^{(n)}\underline{c}^{(n-1)} \ldots \underline{c}^{(1)}$ is a reduced word for the longest element $w_0^{(n+1)}$ of W_{n+1}.

Lemma 2.7 *If $\underline{c}^{(n)}\underline{v} \in \underline{W}_{n+1}$ is a reduced word, then $\underline{v}$ is a reduced word in $\underline{W}_n$.*

Proof There exists a reduced word $\underline{u}$ such that $\underline{c}^{(n)}\underline{v}\,\underline{u} = \underline{w}_0^{(n+1)}$ is a reduced word for the longest element $w_0^{(n+1)} \in W_{n+1}$. Since $vu = (c^{(n)})^{-1} w_0^{(n+1)} = w_0^{(n)}$, we have $vu \in W_n$ so that any reduced word of vu lies in $\underline{W}_n$ and in particular $\underline{v}\,\underline{u}$ is a reduced word in $\underline{W}_n$. Thus $\underline{v}$ is a reduced word in $\underline{W}_n$. □

Lemma 2.8 *If $\underline{v} \in \underline{W}_n$ is a reduced word, then $\underline{c}^{(n)}\underline{v} \in \underline{W}_{n+1}$ is a reduced word. In particular, if $v = w_0^{(n)} w$ for some $w \in W_n$, then $\underline{c}^{(n)}\underline{v}$ is a reduced word for $w_0^{(n+1)} w$.*

Proof There exists a reduced word $\underline{u}$ such that $\underline{v}\,\underline{u}$ is a reduced word for $w_0^{(n)}$. Then $\underline{c}^{(n)}\underline{v}\,\underline{u}$ is a reduced word for $w_0^{(n+1)}$. This implies that $\underline{c}^{(n)}\underline{v}$ is a reduced word. □

Proposition 2.9 *Let $w \in W_{n+1}$ such that $c := c^{(n)} \leq w$. Every reduced word $\underline{w} \in \underline{W}_{n+1}$ for w decomposes, modulo commuting relations, as $\underline{w} = \underline{u}\,\underline{c}\,\underline{v}$ with $\underline{u} \in \underline{W}^1_{n+1}$ and $\underline{v} \in \underline{W}_n$.*

Proof In this proof, all the equalities of words are modulo commuting relations. Since $c \leq w$, the definition of the Bruhat order implies that $\underline{w}$ contains as a subword $\underline{c}$, the unique reduced word of c. We choose such a subword by selecting the first occurrence of s_1, the first occurrence of s_2 after the chosen s_1 and so on. We thus have a decomposition

$$\underline{w} = \underline{w}_1 s_1 \underline{w}_2 s_2 \underline{w}_3 \dots s_{n-1} \underline{w}_n s_n \underline{w}_{n+1}$$

with $\underline{w}_i \in \underline{W}^i_{n+1}$ for $i \in [1, n+1]$. We have $\underline{w}_i = \underline{v}_i \underline{u}_i$ for $i \in [2, n]$, where $\underline{v}_i$ is a word in the s_k's for $k \in [1, i-1]$ and $\underline{u}_i$ is a word in the s_k's for $k \in [i+1, n]$. Note in particular that we have $\underline{u}_n = 1$. Observing that $\underline{v}_i \underline{u}_j = \underline{u}_j \underline{v}_i$ and $s_{i-1}\underline{u}_j = \underline{u}_j s_{i-1}$ for $2 \leq i \leq j \leq n$, we thus obtain

$$\underline{w} = \underline{w}_1(\underline{u}_2\underline{u}_3 \dots \underline{u}_n)(s_1\underline{v}_2 s_2 \underline{v}_3 \dots \underline{v}_n s_n)\underline{w}_{n+1}.$$

For each $i \in [2, n]$, we claim that the word $\underline{v}_i$ does not contain s_{i-1}, i.e., it is a word in the s_k's for $1 \leq k \leq i-2$. We prove the claim by induction on i. First of all, it is easy to see that $\underline{v}_2$ is an empty word since it is a word in s_1 only, and there is s_1 on the left of $\underline{v}_2$ in the reduced word $\underline{w}$. Assume that the claim holds for $i \leq k$. We have

$$\underline{w} = \underline{w}_1(\underline{u}_2\underline{u}_3 \dots \underline{u}_n)(s_1 s_2 \dots s_k)(\underline{v}_2\underline{v}_3 \dots \underline{v}_k)(\underline{v}_{k+1} s_{k+1} \dots \underline{v}_n s_n)\underline{w}_{n+1}.$$

Since $s_1 s_2 \dots s_k \underline{v}_1 \dots \underline{v}_k \underline{v}_{k+1} \in \underline{W}_{k+1}$ is reduced, Lemma 2.7 implies that $\underline{v}_1 \dots \underline{v}_{k+1} \in \underline{W}_k$ and, in particular, we find that $\underline{v}_{k+1}$ doesn't contain s_k. Thus the claim holds. Now by moving all $\underline{v}_i$ to the right using commuting relations, we obtain

$$\underline{w} = \underline{w}_1\underline{u}_2\underline{u}_3 \dots \underline{u}_n s_1 s_2 \dots s_n \underline{v}_2 \underline{v}_3 \dots \underline{v}_n \underline{w}_{n+1}.$$

Using Lemma 2.7 again, we obtain $\underline{v}_2\underline{v}_3 \dots \underline{v}_n\underline{w}_{n+1} \in \underline{W}_n$. This proves the proposition since $\underline{w}_1\underline{u}_2\underline{u}_3 \dots \underline{u}_n \in \underline{W}^1_{n+1}$. □

3 Stable Bott–Samelson Classes

In this section, we introduce stable Bott–Samelson classes in the limit $\mathcal{R}$ of $\Omega^*(Fl_n)$. We also compute some of those classes explicitly in the case of infinitesimal cohomology.

3.1 The Stability of Bott–Samelson Classes

A Schubert variety is, in general, normal and Cohen-Macaulay, and has rational singularities. There exists several resolutions of singularities for it. We will be interested in the so-called Bott–Samelson resolutions.

We set $F_i^{(n)} := \langle e_n, \ldots, e_{n+1-i} \rangle$ and denote the trivial bundle with fiber $F_i^{(n)}$ by $\mathcal{F}_i^{(n)}$.

Definition 3.1 For a reduced word $\underline{v} = s_{i_1} \ldots s_{i_r} \in \underline{W}_n$, the *Bott–Samelson variety* $Y_{\underline{v}}^{(n)}$ is a subvariety of $(Fl_n)^r$ defined as follows:

$$Y_{\underline{v}}^{(n)} = \left\{ (U_\bullet^{[0]}, U_\bullet^{[1]}, \ldots, U_\bullet^{[r]}) \in (Fl_n)^r \;\middle|\; U_i^{[k-1]} = U_i^{[k]}, \forall k = [1, r], \forall i \in [1, n-1] \backslash \{i_k\} \right\},$$

where $U_\bullet^{[0]} = F_\bullet^{(n)}$. If there is no confusion, we will sometimes write $Y_{\underline{v}}$ for $Y_{\underline{v}}^{(n)}$.

Remark 3.2 In Definition 4.1 we will give another equivalent construction (denoted $X_{\underline{w}}$) of the Bott–Samelson resolutions.

It is well-known (cf. [5]) that $Y_{\underline{v}}$ is a smooth projective variety of dimension r. Let $\pi_n : (Fl_n)^r \to Fl_n$ be the projection to the r-th component. If $w \in W_n$ and $v = w_0^{(n)} w$, the projection π_n induces a birational map $Y_{\underline{v}} \to X^w$, which we refer to as a *Bott–Samelson resolution* of $X^w \subset Fl_n$.

Theorem 3.3 *Let $\underline{v} \in \underline{W}_n$ be a reduced word. There is a fiber diagram*

$$\begin{tikzcd} Y_{\underline{v}}^{(n)} \arrow[r, "\tilde{\iota}_n"'] \arrow[d, "\pi_n"'] & Y_{\underline{c}^{(n)}\underline{v}}^{(n+1)} \arrow[d, "\pi_{n+1}"] \\ Fl_n \arrow[r, "\iota_n"'] & Fl_{n+1}, \end{tikzcd}$$

and we have $\iota_n^* \left(\left[Y_{\underline{c}^{(n)}\underline{v}} \to Fl_{n+1} \right] \right) = \left[Y_{\underline{v}} \to Fl_n \right]$.

Furthermore, let $\underline{c}^{[n+m]} := \underline{c}^{(n+m-1)} \ldots \underline{c}^{(n+1)} \underline{c}^{(n)}$ *where* $\underline{c}^{[n]} = 1$, *then the sequence*

$$\left[Y_{\underline{c}^{[n+m]}\underline{v}} \to Fl_{n+m} \right], \quad m \geq 0$$

defines a stable class in $\mathcal{R}$, which we call a stable Bott–Samelson class associated to w and denote by $\mathrm{BS}_w^{\underline{v}}$ *if $v = w_0^{(n)} w$.*

Proof First we note that, by definition, an element of $Y_{\underline{v}}^{(n)}$ can be specified by a sequence of subspaces $(V_1, \ldots, V_r)$ where $V_k = U_{i_k}^{[k]}$. We show that the map $\tilde{\iota}_n : Y_{\underline{v}}^{(n)} \to Y_{\underline{c}^{(n)}\underline{v}}^{(n+1)}$ defined by

$$\tilde{\iota}_n(V_1, V_2, \ldots, V_r) := (F_1^{(n)}, \ldots, F_n^{(n)}, V_1, \ldots, V_r)$$

gives the desired fiber diagram. If we write an element of $Y^{(n+1)}_{\underline{c}^{(n)}\underline{v}}$ as

$$(A^{[1]}_{\bullet}, \ldots, A^{[n]}_{\bullet}, B^{[1]}_{\bullet}, \ldots, B^{[r]}_{\bullet}),$$

it suffices to show that $A^{[k]}_k = F^{(n)}_k$ for all $k \in [1, n]$ over the image of Fl_n. Suppose that $B^{[r]}_{\bullet}$ is in the image of Fl_n, then $B^{[r]}_n = E_n$. Since $i_1, \ldots, i_r \in [1, n-1]$, we have $A^{[n]}_n = E_n = F^{(n)}_n$. We use backward induction on k with the base case being $k = n$. Assume $A^{[k+1]}_{k+1} = F^{(n)}_{k+1}$. We then have

$$A^{[k]}_k \subset F^{(n+1)}_{k+1} \cap A^{[k+1]}_{k+1} = F^{(n+1)}_{k+1} \cap F^{(n)}_{k+1} = F^{(n)}_k.$$

For the latter claim, we use the identity $\iota_n^* \pi_{n+1*} = \pi_{n*} \tilde{\iota}_n^*$ (see [16, p.144 (BM2)]). We get

$$\iota_n^* \left[Y_{\underline{c}^{(n)}\underline{v}} \to Fl_{n+1}\right] = \iota_n^* \pi_{n+1*}(1_{Y_{\underline{c}^{(n)}\underline{v}}}) = \pi_{n*} \tilde{\iota}_n^*(1_{Y_{\underline{c}^{(n)}\underline{v}}}) = \pi_{n*}(1_{Y_{\underline{v}}}) = [Y_{\underline{v}} \to Fl_n].$$

This completes the proof of the claim. □

Remark 3.4 We sometimes denote $\mathrm{BS}^{\underline{v}}_w$ by $\mathrm{BS}^{\underline{v}}_w(x)$ in order to stress that we regard it as an element of $\mathbb{L}[x]_{bd}/\mathbb{S}_\infty$ under the identification in Proposition 2.2.

The following compatibility of Bott–Samelson classes with divided difference operators was established in [6].

Lemma 3.5 *For a reduced word $\underline{v} = s_{i_1} \ldots s_{i_r} \in \underline{W}_n$, and $k \in [1, n-1]$, we have*

$$\partial_i \left[Y_{\underline{v}} \to Fl_n\right] = \begin{cases} \left[Y_{\underline{v}s_i} \to Fl_n\right] & \text{if } \underline{v}s_i \text{ is a reduced word} \\ 0 & \text{otherwise} \end{cases}$$

Since, as explained in Sect. 2.2, the divided difference operators commute with the pullbacks ι_n^*, we obtain the next corollary.

Corollary 3.6 *Let $w \in W_n$ and set $v = w_0^{(n)} w$. Let $\underline{v}$ be a reduced word of v. For any $i \in \mathbb{Z}_{>0}$, we have*

$$\partial_i \mathrm{BS}^{\underline{v}}_w = \begin{cases} \mathrm{BS}^{\underline{v}s_i}_{ws_i} & \text{if } \ell(ws_i) < \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.7 In the connective K-theory of Fl_n, the class $[Y_{\underline{v}} \to Fl_n]$ coincides with the class of the opposite Schubert variety X^w, provided that $v = w_0^{(n)} w$. Its associated class in the projective limit is represented by the Grothendieck polynomial $\mathfrak{G}_w(x)$ associated to w.

3.2 Examples in Infinitesimal Cohomology

Throughout this section we will consider infinitesimal cohomology instead of algebraic cobordism. In combination with Proposition 2.2, the use of this simpler theory will allow us to perform an explicit computation of the stable Bott–Samelson classes in terms of power series in x.

As in Sect. 2.1, the formal group law and its formal inverse for the infinitesimal cohomology I_2^* are given by

$$F_{I_2}(x, y) = x \boxplus y = (x + y)(1 + \gamma xy), \qquad \chi_{I_2}(x) = \boxminus x = -x$$

with $\gamma^2 = 0$. We denote $I_2^*(\mathrm{pt}) = \mathbb{Z}[\gamma]/(\gamma^2)$ by $\mathbb{I}$. As explained in Sect. 2.1, we have

$$I_2^*(Fl_n) \cong \mathbb{I}[x_1, \ldots, x_n]/\mathbb{S}_n$$

where $\mathbb{S}_n$ is the ideal generated by the homogeneous symmetric polynomials of strictly positive degree in $x_1, \ldots, x_n$. We set

$$\mathcal{R}_{\mathbb{I}} := \mathcal{R} \otimes_{\mathbb{L}} \mathbb{I} = \mathbb{I}[x]_{bd}/\mathbb{S}_\infty.$$

By specialising (5) to this particular case we obtain that on $\mathcal{R}_{\mathbb{I}}$ the divided difference operator ∂_i is given by

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}} \cdot (1 + \gamma x_i x_{i+1}), \qquad f \in \mathcal{R}_{\mathbb{I}}.$$

Remark 3.8 (1) If f is symmetric in x_i and x_{i+1}, then $\partial_i(fg) = f\partial_i g$ for all $g \in \mathcal{R}_{\mathbb{I}}$.
(2) If $|i - j| \geq 2$, then $\partial_i \partial_j = \partial_j \partial_i$.

For a reduced word $\underline{v} = s_{i_1} \ldots s_{i_r}$, let $\partial_{\underline{v}} := \partial_{i_r} \ldots \partial_{i_1}$. Recall that $\underline{c}^{(n)} = s_1 \ldots s_n$.

Lemma 3.9 *For $n \geq 1$, we have*

$$\partial_{\underline{c}^{(n)}} \left(x_1^n x_2^{n-1} \ldots x_n\right) = \left(x_1^{n-1} x_2^{n-2} \ldots x_{n-1}\right)(1 + \gamma e_2(x_1, \ldots, x_{n+1})).$$

Proof First we observe that

$$\partial_k(x_k(1 + \gamma e_2(x_1, \ldots, x_k))) = 1 + \gamma e_2(x_1, \ldots, x_{k+1}) \tag{6}$$

which can be shown by a straightforward computation using the identities

$$e_2(x_1, \ldots, x_k) = e_2(x_1, \ldots, x_{k-1}) + x_k e_1(x_1, \ldots, x_{k-1})$$
$$e_2(x_1, \ldots, x_{k+1}) = e_2(x_k, x_{k+1}) + e_2(x_1, \ldots, x_{k-1}) + e_1(x_1, \ldots, x_{k-1}) e_1(x_k, x_{k+1}).$$

Now we prove the formula by induction on n. The case $n = 1$ is obvious. If $n > 1$, by induction hypothesis, we have

$$\partial_n \dots \partial_1 \left(x_1^n x_2^{n-1} \dots x_n\right) = (x_1^{n-1} x_2^{n-2} \dots x_{n-1}) \partial_n (x_n (1 + e_2(x_1, \dots, x_n))).$$

Thus the claim follows from Eq. 6. □

Lemma 3.10 *Modulo $\mathbb{S}_N$, we have*

$$\sum_{k=n+1}^{N-1} e_2(x_1, \dots, x_k) = - \sum_{i=n+1}^{N-1} (i-n) x_{i+1} e_1(x_1, \dots, x_i).$$

Proof Let us begin by recalling the following identity of elementary symmetric polynomials

$$e_2(x_1, \dots, x_N) = e_2(x_1, \dots, x_k) + \sum_{i=k}^{N-1} x_{i+1} e_1(x_1, \dots, x_i).$$

Thus modulo $\mathbb{S}_N$, it follows that

$$\sum_{k=n+1}^{N-1} e_2(x_1, \dots, x_k) = - \sum_{k=n+1}^{N-1} \sum_{i=k}^{N-1} x_{i+1} e_1(x_1, \dots, x_i) = - \sum_{i=n+1}^{N-1} \sum_{k=n+1}^{i} x_{i+1} e_1(x_1, \dots, x_i).$$

The right hand side is the desired formula. □

For $w_0^{(n)} \in S_n$ the corresponding Schubert variety $X_{(n)}^{w_0^{(n)}}$ in Fl_n is a point, and so is the unique Bott–Samelson variety $Y_1^{(n)}$. In $I_2^*(Fl_n)$ we have

$$\left[Y_1^{(n)} \to Fl_n\right] = x_1^{n-1} x_2^{n-2} \dots x_{n-1}.$$

The stable Bott–Samelson class $\mathrm{BS}^1_{w_0^{(n)}}$ introduced in Theorem 3.3 is given by the sequence

$$\left[Y^{(N)}_{\underline{c}^{(N-1)} \dots \underline{c}^{(n)}} \to Fl_N\right] \in I_2^*(Fl_N), \quad N \geq n.$$

By Lemmas 3.9 and 3.10, we can identify a formal power series representing this class in the ring $\mathcal{R}_{\mathbb{I}}$ as follows.

Theorem 3.11 *In $\mathcal{R}_{\mathbb{I}}$, we have*

$$\mathrm{BS}^1_{w_0^{(n)}}(x) = \left(\prod_{i=1}^{n-1} x_i^{n-i}\right) \left(1 - \gamma \sum_{i=n+1}^{\infty} (i-n) x_{i+1} e_1(x_1, \dots, x_i)\right). \tag{7}$$

Proof In view of Lemma 3.5, we can compute the class $\left[Y^{(N)}_{\underline{c}^{(N-1)}\ldots\underline{c}^{(n)}} \to Fl_N\right]$ via divided difference operators:

$$\begin{aligned}\left[Y^{(N)}_{\underline{c}^{(N-1)}\ldots\underline{c}^{(n)}} \to Fl_N\right] &= \partial_{\underline{c}^{(n)}}\ldots\partial_{\underline{c}^{(N-1)}}\left[Y^{(N)}_1 \to Fl_N\right]\\ &= \partial_{\underline{c}^{(n)}}\ldots\partial_{\underline{c}^{(N-1)}}\left(x_1^{N-1}x_2^{N-2}\ldots x_{N-1}\right).\end{aligned}$$

Consecutive applications of Lemma 3.9 give

$$\left[Y^{(N)}_{\underline{c}^{(N-1)}\ldots\underline{c}^{(n)}} \to Fl_N\right] = \left(\prod_{i=1}^{n-1} x_i^{n-i}\right)\left(1+\gamma\sum_{k=n+1}^{N-1} e_2(x_1,\ldots,x_k)\right).$$

By Lemma 3.10 we can rewrite this expression (modulo $\mathbb{S}_N$) as:

$$\left[Y^{(N)}_{\underline{c}^{(N-1)}\ldots\underline{c}^{(n)}} \to Fl_N\right] = \left(\prod_{i=1}^{n-1} x_i^{n-i}\right)\left(1-\gamma\sum_{i=n+1}^{N-1} (i-n)x_{i+1}e_1(x_1,\ldots,x_i)\right). \tag{8}$$

The right hand side of (7) is well-defined as an element of $\mathbb{I}[x]_{bd}$ and it projects to (8) for all $N \geq n$. This completes the proof. □

In view of Corollary 3.6, all stable Bott–Samelson classes can be computed from (7) by applying divided difference operators. More precisely, pick $w \in W_n$ and $\underline{v} \in \underline{W}_n$ such that $v = w_0^{(n)}w$. Then

$$\mathrm{BS}^{\underline{v}}_w = \partial_{\underline{v}}\mathrm{BS}^1_{w_0^{(n)}}.$$

Moreover, since the second factor of (7) is symmetric in $x_1,\ldots,x_n$, one simply has to identify $\partial_{\underline{v}}\left(x_1^{n-1}x_2^{n-2}\ldots x_{n-1}\right)$. That is,

$$\mathrm{BS}^{\underline{v}}_w = \mathcal{B}_n(x)\partial_{\underline{v}}\left(x_1^{n-1}x_2^{n-2}\ldots x_{n-1}\right),$$

where we denote

$$\mathcal{B}_n(x) := 1-\gamma\sum_{i=n+1}^{\infty}(i-n)x_{i+1}e_1(x_1,\ldots,x_i).$$

Based on this, we will now obtain explicit closed formulas for the power series representing the stable Bott–Samelson classes associated to dominant permutations.

Definition 3.12 For a permutation $w \in S_n$, consider a $n \times n$ grid with dots in the boxes $(i, w(i))$. The diagram of w is the set of boxes that remain after deleting boxes weakly east and south of each dot. A permutation w is called *dominant* if its diagram is located at the NW corner of the grid, and coincides with a Young diagram of a partition $\lambda = (\lambda_1,\ldots,\lambda_r)$ with $\lambda_i \leq n-i$. For a given such partition λ, there is a unique dominant permutation $w_\lambda \in S_n$. For example, the longest element $w_0^{(n)}$ is dominant and its associated partition is $\rho := (n-1, n-2, \ldots, 2, 1)$.

Let T be the standard tableau of ρ, i.e., the fillings of the boxes of the i-th row of T are all i. One places λ at the NW corner of ρ with its boxes shaded. We order the anti-diagonals starting from the the inner ones to the outer ones, i.e., the i-th anti-diagonal consists of boxes at (a, b) with $a + b = n + 2 - i$. Let m be the biggest number such that the m-th anti-diagonal contains unshaded boxes. Let $\underline{v}^{(i)}$, $1 \leq i \leq m$ be the reduced word obtained by reading the numbers in the i-th anti-diagonal. Then $\underline{v} := \underline{v}^{(1)} \ldots \underline{v}^{(m)}$ is a reduced word of $v = w_0^{(n)} w_\lambda$. For each i, let $\mathbf{x}_k^{(i)}$ $(k = 1, \ldots, a_i)$ be the orbits of $v^{(i)}$ in $\{x_1, \ldots, x_n\}$ with cardinality greater than 1.

Theorem 3.13 *Let $w_\lambda \in S_n$ be the dominant permutation associated to the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$. Let $\underline{v} := \underline{v}^{(1)} \ldots \underline{v}^{(m)}$ be the reduced word of $v = w_0^{(n)} w_\lambda$ constructed in Definition 3.12. We have*

$$\mathrm{BS}_{w_\lambda}^{\underline{v}} = x_1^{\lambda_1} \ldots x_r^{\lambda_r} \left(1 + \gamma \left(\sum_{i=1}^{m} \sum_{k=1}^{a_i} e_2(\mathbf{x}_k^{(i)})\right)\right) \mathcal{B}_n(x).$$

Proof We prove the formula by induction on m. If $m = 1$, then by Lemma 3.9 we have

$$\partial_{\underline{v}^{(1)}} \left(x_1^{n-1} x_2^{n-2} \ldots x_{n-1}\right) = x_1^{\lambda_1} \ldots x_r^{\lambda_r} \left(1 + \gamma \left(\sum_{k=1}^{a_1} e_2(\mathbf{x}_k^{(1)})\right)\right).$$

Now, let $m > 1$. By the induction hypothesis, we have

$$\partial_{\underline{v}^{(m)}} \ldots \partial_{\underline{v}^{(1)}} \left(x_1^{n-1} x_2^{n-2} \ldots x_{n-1}\right) = \partial_{\underline{v}^{(m)}} \left(x_1^{\lambda'_1} \ldots x_r^{\lambda'_r} \left(1 + \gamma \left(\sum_{i=1}^{m-1} \sum_{k=1}^{a_i} e_2(\mathbf{x}_k^{(i)})\right)\right)\right),$$

where $\lambda' = \lambda \cup (n-m, n-m-1, \ldots, 2, 1,)$. Since the unshaded boxes form a skew shape ρ/λ, it follows that $\underline{v}^{(m)}$ stabilizes the second factor, allowing it pass through $\partial_{\underline{v}^{(m)}}$:

$$\partial_{\underline{v}^{(m)}} \ldots \partial_{\underline{v}^{(1)}} \left(x_1^{n-1} x_2^{n-2} \ldots x_{n-1}\right) = \left(1 + \gamma \left(\sum_{i=1}^{m-1} \sum_{k=1}^{a_i} e_2(\mathbf{x}_k^{(i)})\right)\right) \cdot \partial_{\underline{v}^{(m)}} \left(x_1^{\lambda'_1} \ldots x_r^{\lambda'_r}\right).$$

Now the desired formula follows again from Lemma 3.9. □

Example 3.14 Consider $w_\lambda = (53124) \in S_5$ where $\lambda = (4, 2)$.

1	1	1	1
2	2	2	
3	3		
4			

The reduced word $\underline{v}$ of $v = w_0^{(5)} w_\lambda$ is $\underline{v} = (s_2 s_3 s_4)(s_3)$. We have

$$(\partial_3)(\partial_4\partial_3\partial_2)\left(x_1^4 x_2^3 x_3^2 x_4\right) = (1 + \gamma e_2([2,5]) \cdot \partial_3 \left(x_1^4 x_2^2 x_3\right) = x_1^4 x_2^2 \left(1 + \gamma(e_2^{[2,5]} + e_2^{[3,4]})\right).$$

Example 3.15 Consider $w_\lambda = (45123) \in S_5$ where $\lambda = (3, 3)$.

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \cline{1-3} 3 & 3 & & \\ \cline{1-2} 4 & & & \\ \cline{1-1}\end{array}$$

The reduced word of $v = w_0^{(5)} w_\lambda$ is $\underline{v} = (s_1 s_3 s_4)(s_3)$. We have

$$(\partial_3)(\partial_4\partial_3\partial_1)\left(x_1^4 x_2^3 x_3^2 x_4\right) = \left(1 + \gamma(e_2^{[1,2]} + e_2^{[3,5]})\right) \partial_3 \left(x_1^3 x_2^3 x_3\right) = x_1^3 x_2^3 \left(1 + \gamma(e_2^{[1,2]} + e_2^{[3,5]} + e_2^{[3,4]})\right).$$

Example 3.16 Consider $w_\lambda = (563412) \in S_6$ where $\lambda = (4, 4, 2, 2)$.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & \\ \cline{1-4} 3 & 3 & 3 & & \\ \cline{1-3} 4 & 4 & & & \\ \cline{1-2} 5 & & & & \\ \cline{1-1}\end{array}$$

The reduced word of $v = w_0^{(6)} w_\lambda$ is $\underline{v} = s_1 s_3 s_5$. We have

$$\partial_5\partial_3\partial_1\left(x_1^5 x_2^4 x_3^3 x_4^2 x_5\right) = (x_1 x_2)^4 (x_3 x_4)^2 \left(1 + \gamma(e_2^{[1,2]} + e_2^{[3,4]} + e_2^{[5,6]})\right).$$

4 Restriction of Bott–Samelson Classes

In this section we generalise the restriction formula in Theorem 3.3 of the previous section. Namely, we will prove a formula for the product of the cobordism class of any Bott–Samelson resolution with the class $[Fl_{n-1} \to Fl_n]$. In order to simplify the proof we will use another equivalent definition of Bott–Samelson resolutions.

4.1 Bott–Samelson Resolution Revisited

In this section, we provide another construction of the Bott–Samelson variety $X_{\underline{w}}$ associated to a word $\underline{w}$ by viewing it as a configuration space. This description will be better suited for our purposes.

Definition 4.1 Let $\underline{w} = s_{i_1} \dots s_{i_r}$ be a word in $\underline{W}_{n+1}$.

(1) For $a \in [1, n+1]$, define $\mathrm{LO}_{\underline{w}}(a)$, the last occurence of a in $\underline{w}$, by

$$\mathrm{LO}_{\underline{w}}(a) = \sup\{k \in [1, r] \mid i_k = a\}.$$

Note that if the above set is empty, then $\mathrm{LO}_{\underline{w}}(a) = -\infty$.

(2) If $\mathrm{LO}_{\underline{w}}(a) = -\infty$, then we set $V_{\mathrm{LO}_{\underline{w}}(a)} = \langle e_1, \dots, e_a \rangle$.

(3) For $k \in [1, r]$, define $\underline{w}[k] = s_{i_1} \dots s_{i_k}$.

(4) For $k \in [1, r]$, the left and the right predecessors of k in $\underline{w}$, denoted $\mathrm{LP}_{\underline{w}}$ and $\mathrm{RP}_{\underline{w}}$, are defined as:

$$\begin{aligned} \mathrm{LP}_{\underline{w}}(k) &:= \mathrm{LO}_{\underline{w}[k]}(i_k - 1), \\ \mathrm{RP}_{\underline{w}}(k) &:= \mathrm{LO}_{\underline{w}[k]}(i_k + 1). \end{aligned}$$

Remark 4.2 Note that by (2) and (4) in the above definition, we have:

$$\begin{aligned} V_{\mathrm{LP}_{\underline{w}}(k)} &= \langle e_1, \dots, e_{i_k - 1} \rangle \text{ if } \mathrm{LP}_{\underline{w}}(k) = -\infty \text{ and} \\ V_{\mathrm{RP}_{\underline{w}}(k)} &= \langle e_1, \dots, e_{i_k + 1} \rangle \text{ if } \mathrm{RP}_{\underline{w}}(k) = -\infty. \end{aligned}$$

Definition 4.3 Given a word $\underline{w} = s_{i_1} \dots s_{i_r}$, define the Bott–Samelson variety $X_{\underline{w}}$ as follows:

$$X_{\underline{w}} = \left\{ (V_k)_{k \in [1,r]} \mid \dim V_k = i_k \text{ and } V_{\mathrm{LP}_{\underline{w}}(k)} \subset V_k \subset V_{\mathrm{RP}_{\underline{w}}(k)} \right\}.$$

Define a morphism $\pi_{\underline{w}} : X_{\underline{w}} \to Fl_{n+1}$ by $\pi_{\underline{w}}((V_k)_{k \in [1,r]}) = (V_{\mathrm{LO}_{\underline{w}}(a)})_{a \in [1,n]}$. If $\underline{w}$ is reduced, then the map $\pi_{\underline{w}}$ is a proper birational morphism from $X_{\underline{w}}$ onto the Schubert variety X_w. In this reduced case, we often call $X_{\underline{w}}$ together with the map $\pi_{\underline{w}}$ a Bott–Samelson resolution.

Remark 4.4 There is a natural isomorphism between our two construction of the Bott–Samelson variety $X_{\underline{w}}$ and $Y_{\underline{w}}$. It is given by $(V_k)_{k \in [1,r]} \mapsto (U_\bullet^{(k)})_{k \in [1,r]}$ with $U_a^{(k)} = V_{\mathrm{LO}_{\underline{w}[k]}(a)}$ for all $k \in [1, r]$ and $a \in [1, n]$.

Recall the following well known fact on Bott–Samelson varieties.

Lemma 4.5 *The Bott–Samelson variety does not depend on the choice of a word modulo commuting relations. More precisely, if $\underline{w} = \underline{v}$ modulo commuting relations, then there is an isomorphism $f_{\underline{w},\underline{v}} : X_{\underline{w}} \to X_{\underline{v}}$ such that the following diagram is commutative:*

$$\begin{tikzcd} X_{\underline{w}} \arrow[r, "f_{\underline{w},\underline{v}}"] \arrow[d, "\pi_{\underline{w}}"'] & X_{\underline{v}} \arrow[d, "\pi_{\underline{v}}"'] \\ X_w \arrow[r, "\mathrm{Id}"] & X_v. \end{tikzcd}$$

Proof It is enough to prove this result in the case in which $\underline{w}$ and $\underline{v}$ are obtained from each other by a unique commuting relation. The result then follows by induction on the number of commuting relations. Assume therefore that $\underline{w} = \underline{u}_1 s_a s_b \underline{u}_2$ and $\underline{v} = \underline{u}_1 s_b s_a \underline{u}_2$ with $|a-b| \geq 2$. Let $r_i = \ell(\underline{u}_i)$ for $i \in [1,2]$ and set $r = \ell(\underline{w}) = \ell(\underline{v}) = r_1 + r_2 + 2$. Define the map $f_{\underline{w},\underline{v}} : X_{\underline{w}} \to X_{\underline{v}}$ by $f_{\underline{w},\underline{v}}((V_k)_{k\in[1,r]}) = (W_k)_{k\in[1,r]}$ and $f_{\underline{v},\underline{w}}((W_k)_{k\in[1,r]}) = (V_k)_{k\in[1,r]}$ with $W_k = V_k$ for $k \notin \{r_1+1, r_1+2\}$ and $W_{r_1+\epsilon} = V_{r_1+3-\epsilon}$ for $\epsilon \in \{1,2\}$. These maps are inverses of each other and we only need to check that they indeed map $X_{\underline{w}}$ to $X_{\underline{v}}$ and $X_{\underline{v}}$ to $X_{\underline{w}}$ respectively. By symmetry, we only need to check this for $f_{\underline{w},\underline{v}}$.

We prove that given $(V_k)_{k\in[1,r]} \in X_{\underline{w}}$, the condition $W_{\mathrm{LP}_{\underline{v}}(k)} \subset W_k$ is satisfied for all k. The other inclusion $W_k \subset W_{\mathrm{RP}_{\underline{v}}(k)}$ is obtained by similar arguments. First note that we have the following relations:

$$\begin{array}{ll} \mathrm{LP}_{\underline{v}}(k) = \mathrm{LP}_{\underline{w}}(k) & \text{for } k, \mathrm{LP}_{\underline{w}}(k) \notin \{r_1+1, r_1+2\} \\ \mathrm{LP}_{\underline{v}}(k) = r_1 + 3 - \epsilon & \text{for } \mathrm{LP}_{\underline{w}}(k) = r_1 + \epsilon \text{ and } \epsilon \in \{1,2\} \\ \mathrm{LP}_{\underline{v}}(r_1+\epsilon) = \mathrm{LP}_{\underline{w}}(r_1+3-\epsilon) & \text{for } \epsilon \in \{1,2\} \end{array}$$

For k such that $k, \mathrm{LP}_{\underline{w}}(k) \notin \{r_1+1, r_1+2\}$, we have $W_{\mathrm{LP}_{\underline{v}}(k)} = W_{\mathrm{LP}_{\underline{w}}(k)} = V_{\mathrm{LP}_{\underline{w}}(k)} \subset V_k = W_k$. For $\mathrm{LP}_{\underline{w}}(k) = r_1 + \epsilon$ with $\epsilon \in \{1,2\}$, note that $k \notin \{r_1+1, r_1+2\}$ thus we have $W_{\mathrm{LP}_{\underline{v}}(k)} = W_{r_1+3-\epsilon} = V_{r_1+\epsilon} = V_{\mathrm{LP}_{\underline{w}}(k)} \subset V_k = W_k$. Finally, for $k = r_1 + \epsilon$ with $\epsilon \in \{1,2\}$, note that $\mathrm{LP}_{\underline{w}}(r_1+3-\epsilon) \notin \{r_1+1, r_1+2\}$ thus we have $W_{\mathrm{LP}_{\underline{v}}(k)} = W_{\mathrm{LP}_{\underline{v}}(r_1+\epsilon)} = W_{\mathrm{LP}_{\underline{w}}(r_1+3-\epsilon)} = V_{\mathrm{LP}_{\underline{w}}(r_1+3-\epsilon)} \subset V_{r_1+3-\epsilon} = W_{r_1+\epsilon} = W_k$. Furthermore we have:

$$\begin{array}{ll} \mathrm{LO}_{\underline{w}}(a) = \mathrm{LO}_{\underline{v}}(a) & \text{if } \mathrm{LO}_{\underline{w}}(a) \notin \{r_1+1, r_1+2\} \\ \mathrm{LO}_{\underline{w}}(a) = \mathrm{LO}_{\underline{v}}(a) + 3 - \epsilon & \text{if } \mathrm{LO}_{\underline{w}}(a) = r_1 + \epsilon \text{ for } \epsilon \in \{1,2\}, \end{array}$$

so we easily see that we have $\pi_{\underline{v}} \circ f_{\underline{w},\underline{v}} = \pi_{\underline{w}}$ and $\pi_{\underline{w}} \circ f_{\underline{v},\underline{w}} = \pi_{\underline{v}}$. $\square$

4.2 Fiber Product with a Subflag

We now prove a fiber product formula for Bott–Samelson resolutions.

Define

$$\mathbf{F}_n = \left\{ U_\bullet \in Fl_{n+1} \mid U_n = \langle e_2, \ldots, e_{n+1} \rangle = F_n^{(n+1)} \right\}. \tag{9}$$

We can easily see that $\mathbf{F}_n$ coincides with the opposite Schubert variety X^c in Fl_{n+1} where $c := c^{(n)} = s_1 \ldots s_n$ is the Coxeter element. Therefore, from a well-known fact, we have that for $w \in W_{n+1}$

$$X_w \cap \mathbf{F}_n \neq \emptyset \text{ if and only if } c \leq w. \tag{10}$$

Definition 4.6 For $\underline{u} \in \underline{W}^1_{n+1}$ with $\underline{u} = s_{i_1} \dots s_{i_r}$, define $c^{-1}(\underline{u}) \in \underline{W}^n_{n+1}$ by

$$c^{-1}(\underline{u}) = s_{c^{-1}(i_1)} \dots s_{c^{-1}(i_r)} = s_{i_1-1} \dots s_{i_r-1},$$

where we observe that for each $k \in [2, n+1]$ one has $c^{-1}(k) = k - 1 \in [1, n]$.

Definition 4.7 Denote by c the isomorphism $c : \mathsf{k}^{n+1} \to \mathsf{k}^{n+1}$ defined by $c(e_i) = e_{c(i)}$ for all $i \in [1, n+1]$.

1. The map c induces an isomorphism $c : Fl_n \to \mathbf{F}_n \subset Fl_{n+1}$.
2. For $\underline{w} \in W^n_{n+1}$ with $\ell(\underline{w}) = r$, define

$$c(X_{\underline{w}}) = \left\{ (c(V_k))_{k \in [1,r]} \mid (V_k)_{k \in [1,r]} \in X_{\underline{w}} \right\}$$

and $c(\pi_{\underline{w}}) : c(X_{\underline{w}}) \to c(X_w) \subset c(Fl_n) = \mathbf{F}_n$, so that the following diagram is commutative:

$$\begin{tikzcd} X_{\underline{w}} \arrow[r, "c"] \arrow[d, "\pi_{\underline{w}}"'] & c(X_{\underline{w}}) \arrow[d, "c(\pi_{\underline{w}})"] \\ X_w \arrow[r, "c"] & c(X_w). \end{tikzcd}$$

Lemma 4.8 *For $\underline{u} \in \underline{W}^1_{n+1}$ and $\underline{v} \in \underline{W}^n_{n+1}$, define $\underline{w} = \underline{u}\,\underline{c}\,\underline{v}$ and $\underline{w}' = c^{-1}(\underline{u})\underline{v}$. Let $r_1 = \ell(\underline{u})$ and $r_2 = \ell(\underline{v})$. For $\not\succeq \in [1, n-1]$, we have the following equalities:*

(1) If $k \leq r_1$, then $\mathrm{LO}_{\underline{w}'[k]}(\not\succeq) = \mathrm{LO}_{\underline{w}[k]}(\not\succeq + 1)$;
(2) If $k \geq r_1 + 1$ and $\mathrm{LO}_{\underline{v}[k-r_1]}(\not\succeq) \neq -\infty$, then $\mathrm{LO}_{\underline{w}'[k]}(\not\succeq) = \mathrm{LO}_{\underline{w}[k+n]}(\not\succeq) - n = \mathrm{LO}_{\underline{v}[k-r_1]}(\not\succeq) + r_1 \geq r_1 + 1$;
(3) If $k \geq r_1 + 1$ and $\mathrm{LO}_{\underline{v}[k-r_1]}(\not\succeq) = -\infty$, then $\mathrm{LO}_{\underline{w}'[k]}(\not\succeq) = \mathrm{LO}_{\underline{u}}(\not\succeq + 1) = \mathrm{RP}_{\underline{w}[k+n]}(r_1 + \not\succeq) \leq r_1$ and $\mathrm{LO}_{\underline{w}[k+n]}(\not\succeq) = r_1 + \not\succeq$.

Proof If $k \leq r_1$, then $\underline{w}'[k] = c^{-1}(\underline{w}[k])$ and the result follows. Assume now that $k \geq r_1 + 1$. If $\mathrm{LO}_{\underline{v}[k-r_1]}(\not\succeq) \neq -\infty$, then the last occurence of $\not\succeq$ in $\underline{w}'[k]$ is obtained at a letter of $\underline{v}[k - r_1]$ so that $\mathrm{LO}_{wu'[k]}(\not\succeq) = \mathrm{LO}_{\underline{v}[k-r_1]}(\not\succeq) + r_1 = \mathrm{LO}_{\underline{w}[k+n]}(\not\succeq) - n \geq r_1$. If $\mathrm{LO}_{\underline{v}[k-r_1]}(\not\succeq) = -\infty$, the last occurence of $\not\succeq$ in $\underline{w}'[k]$ is the last occurence of $\not\succeq$ in $c^{-1}(\underline{u})$ and we get the equalities $\mathrm{LO}_{\underline{w}'[k]}(\not\succeq) = \mathrm{LO}_{c^{-1}(\underline{u})}(\not\succeq) = \mathrm{LO}_{\underline{u}}(\not\succeq + 1) \leq r_1$. On the other hand, the last occurence of $\not\succeq$ in $\underline{w}[k+n]$ is obtained as the $\not\succeq$-th letter in $\underline{c}$ thus $\mathrm{LO}_{\underline{w}[k+n]}(\not\succeq) = r_1 + \not\succeq$. This also explains the equality $\mathrm{LO}_{\underline{u}}(\not\succeq + 1) = \mathrm{RP}_{\underline{w}[k+n]}(r_1 + \not\succeq)$. $\square$

Corollary 4.9 *For $\underline{u} \in \underline{W}^1_{n+1}$ and $\underline{v} \in \underline{W}^n_{n+1}$, define $\underline{w} = \underline{ucv}$ and $\underline{w}' = c^{-1}(\underline{u})\underline{v}$. Let $r_1 = \ell(\underline{u})$ and $r_2 = \ell(\underline{v})$. We have the following alternatives:*

(1) *If* $\mathrm{LO}_{\underline{v}}(\not\geq) \neq -\infty$, *then* $\mathrm{LO}_{\underline{w}'}(\not\geq) = \mathrm{LO}_{\underline{w}}(\not\geq) - n = \mathrm{LO}_{\underline{v}}(\not\geq) + r_1 \geq r_1 + 1$;
(2) *If* $\mathrm{LO}_{\underline{v}}(\not\geq) = -\infty$, *then* $\mathrm{LO}_{\underline{w}'}(\not\geq) = \mathrm{LO}_{\underline{u}}(\not\geq + 1) = \mathrm{RP}_{\underline{w}}(r_1 + \not\geq)$ *and* $\mathrm{LO}_{\underline{w}}(\not\geq) = r_1 + \not\geq$.

Proof Apply the previous lemma with $k = r_1 + r_2$. □

Corollary 4.10 *For* $\underline{u} \in \underline{W}^1_{n+1}$ *and* $\underline{v} \in \underline{W}^n_{n+1}$, *define* $\underline{w} = \underline{ucv}$ *and* $\underline{w}' = c^{-1}(\underline{u})\underline{v}$. *Let* $r_1 = \ell(\underline{u})$ *and* $r_2 = \ell(\underline{v})$. *We set* $\mathrm{LP}_{\underline{x}}(-\infty) = -\infty$ *and* $\mathrm{RP}_{\underline{x}}(-\infty) = -\infty$ *for any word* $\underline{x}$.

A. We have the following formulas for $\mathrm{LP}_{\underline{w}}$ *and* $\mathrm{LP}_{\underline{w}'}$:

1. *If* $a \in [1, r_1]$, *then* $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{LP}_{\underline{w}}(a) \leq r_1$.
2. *If* $a > r_1$ *and* $\mathrm{LP}_{\underline{v}}(a - r_1) \neq \infty$, *then* $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{LP}_{\underline{w}}(a + n) - n > r_1$.
3. *If* $a > r_1$ *and* $\mathrm{LP}_{\underline{v}}(a - r_1) = -\infty$, *then* $\mathrm{LP}_{\underline{w}}(a + n) \in [r_1 + 1, r_1 + n] \cup \{-\infty\}$ *and* $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(a + n)) \leq r_1$.

B. We have the following formulas for $\mathrm{RP}_{\underline{w}}$ *and* $\mathrm{RP}_{\underline{w}'}$:

1. *If* $a \in [1, r_1]$, *then* $\mathrm{RP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}}(a) \leq r_1$.
2. *If* $a > r_1$ *and* $\mathrm{RP}_{\underline{v}}(a - r_1) \neq \infty$, *then* $\mathrm{RP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}}(a + n) - n > r_1$.
3. *If* $a > r_1$ *and* $\mathrm{RP}_{\underline{v}}(a - r_1) = -\infty$, *then* $\mathrm{RP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(a + n)) \leq r_1$ *and* $\mathrm{RP}_{\underline{w}}(a + n) \in [r_1 + 1, r_1 + n] \cup \{-\infty\}$.

C. If $k \in [r_1 + 1, r_1 + n]$, *then* $\mathrm{LP}_{\underline{w}}(k) \in [r_1 + 1, r_1 + n] \cup \{-\infty\}$ *Furthermore, for* $k \in [r_1 + 1, r_1 + n]$, *we have the following formulas for* $\mathrm{LP}_{\underline{w}}$ *and* $\mathrm{RP}_{\underline{w}}$:

1. *If* $\mathrm{RP}_{\underline{w}}(k) > \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$, *then* $\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k)) = \mathrm{LP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(k))$.
2. *If* $\mathrm{RP}_{\underline{w}}(k) < \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$, *then* $\mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))) = \mathrm{RP}_{\underline{w}}(k)$.

Proof Write $\underline{u} = s_{i_1} \ldots s_{i_{r_1}}$ and $\underline{v} = s_{i_{r_1+n+1}} \ldots s_{i_{r_1+n+r_2}}$ so that $w = s_{i_1} \ldots s_{i_r}$ with $r = r_1 + r_2 + n$ and $s_{i_{r_1+k}} = s_k$ for $k \in [1, n]$. We have $\underline{w}' = s_{j_1} \ldots s_{j_{r_1+r+2}}$ with

$$j_k = \begin{cases} i_k - 1 & \text{for } k \in [1, r_1] \\ i_{k+n} & \text{for } k \in [r_1 + 1, r_1 + r_2]. \end{cases}$$

A.1. We have $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{LO}_{\underline{w}'[a]}(j_a - 1) = \mathrm{LO}_{\underline{w}[a]}((j_a - 1) + 1) = \mathrm{LO}_{\underline{w}[a]}(j_a) = \mathrm{LO}_{\underline{w}[a]}(i_a - 1) = \mathrm{LP}_{\underline{w}}(a)$. By definition $\mathrm{LP}_{\underline{w}'}(a) < a \leq r_1$.

A.2. We have $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{LO}_{\underline{w}'[a]}(j_a - 1) = \mathrm{LO}_{\underline{w}[a+n]}(i_{a+n} - 1) - n = \mathrm{LP}_{\underline{w}}(a + n) - n \geq r_1 + 1$.

A.3. We have $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{LO}_{\underline{w}'[a]}(j_a - 1) = \mathrm{RP}_{\underline{w}[a+n]}(r_1 + j_a - 1) \leq r_1$ and $r_1 + j_a - 1 = \mathrm{LO}_{\underline{w}[a+n]}(j_a - 1)$. We get $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}[a+n]}(\mathrm{LO}_{\underline{w}[a+n]}(j_a - 1)) = \mathrm{RP}_{\underline{w}[a+n]}(\mathrm{LO}_{\underline{w}[a+n]}(i_{a+n} - 1)) = \mathrm{RP}_{\underline{w}[a+n]}(\mathrm{LP}_{\underline{w}}(a + n))$.

For B.1. and B.2. use the proof of A.1. and A.2 with RP in place of LP.

B.3. We have $\mathrm{RP}_{\underline{w}'}(a) = \mathrm{LO}_{\underline{w}'[a]}(j_a + 1) = \mathrm{RP}_{\underline{w}[a+n]}(r_1 + j_a + 1) \leq r_1$ and $r_1 + j_a + 1 = \mathrm{LO}_{\underline{w}[a+n]}(j_a + 1)$. We get $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}[a+n]}(\mathrm{LO}_{\underline{w}[a+n]}(j_a + 1)) = \mathrm{RP}_{\underline{w}[a+n]}(\mathrm{LO}_{\underline{w}[a+n]}(i_{a+n} + 1)) = \mathrm{RP}_{\underline{w}[a+n]}(\mathrm{RP}_{\underline{w}}(a + n))$.

C. If $k \in [r_1 + 1, r_1 + n]$, then the k-th letter of $\underline{w}$ is the $(k - r_1)$-th letter of $\underline{c}$ and we have $\mathrm{LP}_{\underline{w}}(k) = \mathrm{LP}_{\underline{c}}(k - r_1) = k - r_1 - 1$ for $k > r_1 + 1$ and $\mathrm{LP}_{\underline{w}}(r_1 + 1) =$

$-\infty$. Note that, if any of the two quatities $\mathrm{RP}_{\underline{w}}(k)$ or $\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$ is finite, we have $\mathrm{RP}_{\underline{w}}(k) \neq \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$.

C.1. Assume $\mathrm{RP}_{\underline{w}}(k) > \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$. If $k = r_1 + 1$, then $i_k = 1$ and $\mathrm{LP}_{\underline{w}}(k) = -\infty$ thus $\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k)) = -\infty$. Furthermore, $i_{\mathrm{RP}_{\underline{w}}(k)} = i_k + 1 = 2$ and since $\underline{u} \in \underline{W}^1_{n+1}$ we have $\mathrm{LP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(k)) = \mathrm{LO}_{\underline{u}}(2) = -\infty$. Assume now $k > r_1 + 1$, then $\mathrm{LP}_{\underline{w}}(k) = k - 1$, thus $\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k)) = \mathrm{RP}_{\underline{w}}(k-1) = \mathrm{LO}_{\underline{u}}(i_{k-1} + 1) = \mathrm{LO}_{\underline{u}}(i_k)$. We have $i_{\mathrm{RP}_{\underline{w}}(k)} = i_k + 1$ thus $\mathrm{LP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(k)) = \mathrm{LO}_{\underline{w}[\mathrm{RP}_{\underline{w}}(k)]}(i_{\mathrm{RP}_{\underline{w}}(k)} - 1) = \mathrm{LO}_{\underline{w}[\mathrm{RP}_{\underline{w}}(k)]}(i_k)$. Since $\mathrm{RP}_{\underline{w}}(k) < k$, we have that $\underline{w}[\mathrm{RP}_{\underline{w}}(k)]$ is a subword of $\underline{u}$ thus $\mathrm{LO}_{\underline{w}[\mathrm{RP}_{\underline{w}}(k)]}(i_k) \leq \mathrm{LO}_{\underline{u}}(i_k) = \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$. On the other hand, since $\mathrm{RP}_{\underline{w}}(k) > \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$, we have $\mathrm{LO}_{\underline{w}[\mathrm{RP}_{\underline{w}}(k)]}(i_k) \geq \mathrm{LO}_{\underline{w}[\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))]}(i_k) = \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$. The last equality holds since $i_{\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))} = i_k$.

C.2. Assume $\mathrm{RP}_{\underline{w}}(k) < \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$. This implies $\mathrm{LP}_{\underline{w}}(k) \neq -\infty$ thus $k > r_1 + 1$. We have $\mathrm{RP}_{\underline{w}}(k) = \mathrm{LO}_{\underline{w}[k]}(i_k + 1)$ and $\mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))) = \mathrm{LO}_{\underline{w}[\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))]}(i_k + 1)$. Since $\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k)) < k$, then $\underline{w}[\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))]$ is a subword of $\underline{w}[k]$ thus $\mathrm{LO}_{\underline{w}[\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))]}(i_k + 1) \leq \mathrm{LO}_{\underline{w}[k]}(i_k + 1)$. On the other hand, since $\mathrm{RP}_{\underline{w}}(k) < \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$, we have $\mathrm{LO}_{\underline{w}[\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))]}(i_k + 1) \geq \mathrm{LO}_{\mathrm{RP}_{\underline{w}}[k]}(i_k + 1)$. $\square$

Theorem 4.11 *Let $\underline{w}$ be a reduced word and $w \in W$ the associated element.*

(1) If $w \not\geq c$, then $X_{\underline{w}} \times_{Fl_{n+1}} \mathbf{F}_n$ is empty.
(2) If $w \geq c$, there exist $\underline{u} \in \underline{W}^1_{n+1}$ and $\underline{v} \in \underline{W}^n_{n+1}$ such that $\underline{w} = \underline{u}\,\underline{c}\,\underline{v}$ modulo commuting relations and we have an isomorphism of Fl_{n+1}-varieties $X_{\underline{w}} \times_{Fl_{n+1}} \mathbf{F}_n \simeq c(X_{c^{-1}(\underline{u})\underline{v}})$.

Proof (1) is clear since the condition implies $X^w \cap \mathbf{F}_n = \emptyset$. We prove (2). By Proposition 2.9, we can write $\underline{w} = \underline{u}\,\underline{c}\,\underline{v}$. Let $\ell(w) = r$, and $\ell(u) = r_1$, $\ell(v) = r_2$ so that $r = r_1 + r_2 + n$. Since the obvious inclusion $\mathbf{F}_n \hookrightarrow Fl_{n+1}$ is a closed embedding, we can view $X_{\underline{w}} \times_{Fl_{n+1}} \mathbf{F}_n$ as the closed subvariety of $X_{\underline{w}}$ given as follows:

$$X_{\underline{w}} \times_{Fl_{n+1}} \mathbf{F}_n = \Big\{(V_k)_{k\in[1,r]} \in X_{\underline{w}} \mid V_{r_1+n} = \langle e_2, \ldots, e_{n+1}\rangle\Big\}.$$

Define the map $f : X_{\underline{w}} \times_{Fl_{n+1}} \mathbf{F}_n \to c(X_{c^{-1}(\underline{u})\underline{v}})$ by $f((V_k)_{k\in[1,r]}) = (H_a)_{a\in[r_1+r_2]}$ with

$$H_a = \begin{cases} V_a \cap \langle e_2, \ldots, e_{n+1}\rangle & \text{for } a \in [1, r_1] \\ V_{a+n} & \text{for } a \in [r_1 + 1, r_1 + r_2]. \end{cases}$$

Define the map $g : c(X_{c^{-1}(\underline{u})\underline{v}}) \to X_{\underline{w}} \times_{Fl_{n+1}} \mathbf{F}_n$ by $g((H_a)_{a\in[r_1+r_2]}) = (V_k)_{k\in[1,r]}$ with

$$V_k = \begin{cases} H_k + \langle e_1\rangle & \text{for } k \in [1, r_1] \\ V_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \ldots, e_{n+1}\rangle & \text{for } k \in [r_1 + 1, r_1 + n] \\ H_{k-n} & \text{for } k \in [r_1 + n + 1, r]. \end{cases}$$

Set $\underline{w}' = c^{-1}(\underline{u})\underline{v} \in \underline{W}^n_{n+1}$. Write $\underline{u} = s_{i_1} \ldots s_{i_{r_1}}$ and $\underline{v} = s_{i_{r_1+n+1}} \ldots s_{i_{r_1+n+r_2}}$ so that $w = s_{i_1} \ldots s_{i_r}$ with $s_{i_{r_1+k}} = s_k$ for $k \in [1, n]$. We have $\underline{w}' = s_{j_1} \ldots s_{j_{r_1+r+2}}$ with

$$j_k = \begin{cases} i_k - 1 & \text{for } k \in [1, r_1] \\ i_{k+n} & \text{for } k \in [r_1+1, r_1+r_2]. \end{cases}$$

We first prove that these maps are well defined. We start with f. Note that if $(V_k)_{k\in[1,r]} \in X_{\underline{w}} \times_{Fl_{n+1}} \mathbf{F}_n$, then

$$\begin{array}{ll} V_k \supset \langle e_1 \rangle & \text{for } k \in [1, r_1] \\ V_k \subset \langle e_2, \dots, e_{n+1} \rangle & \text{for } k \in [r_1+1, r_1+r_2+n] \\ V_k = V_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \dots, e_{n+1} \rangle & \text{for } k \in [r_1+1, r_1+n]. \end{array}$$

Indeed, since $\underline{u} \in \underline{W}_1$, we have $\mathrm{LP}_{\underline{w}}(k) = -\infty$ for any $k \in [1, r_1]$ with $i_k = 2$ implying our first claim. Furthermore, since $\underline{v} \in \underline{W}_{n+1}$, by the same type of arguments we have the equality $V_{r_1+n} = \langle e_2, \dots, e_{n+1} \rangle$ proving the second claim. In particular for $k \in [r_1+1, r_1+n]$, we have $V_k \subset V_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \dots, e_{n+1} \rangle$ and $\langle e_1 \rangle \subset V_{\mathrm{RP}_{\underline{w}}(k)}$. Since $\dim V_k = \dim V_{\mathrm{RP}_{\underline{w}}(k)} - 1$ this proves the last equality.

We check that $\dim H_a = j_a$. For $a \in [1, r_1]$, we have $\dim H_a = \dim(V_a \cap \langle e_2, \dots, e_{n+1} \rangle) = \dim V_a - 1 = i_a - 1 = j_a$, where the second equality holds since $\underline{u} \in \underline{W}^1_{n+1}$, therefore $\langle e_1 \rangle \subset V_a$. For $a \in [r_1+1, r_1+r_2]$, we have $\dim H_a = \dim V_{a+n} = i_{a+n} = j_a$.

We now check the inclusions $H_{\mathrm{LP}_{\underline{w}'}(a)} \subset H_a \subset H_{\mathrm{RP}_{\underline{w}'}(a)}$. We start with the inclusions $H_{\mathrm{LP}_{\underline{w}'}(a)} \subset H_a$. For $a \in [1, r_1]$, then $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{LP}_{\underline{w}}(a) \le r_1$ and we have $H_{\mathrm{LP}_{\underline{w}'}(a)} = H_{\mathrm{LP}_{\underline{w}}(a)} = V_{\mathrm{LP}_{\underline{w}}(a)} \cap \langle e_2, \dots, e_{n+1} \rangle \subset V_a \cap \langle e_2, \dots, e_{n+1} \rangle = H_a$. For $a \in [r_1+1, r_1+r_2]$ and $\mathrm{LP}_{\underline{v}}(a - r_1) \ne -\infty$, then $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{LP}_{\underline{w}}(a+n) - n \ge r_1 + 1$ and we have $H_{\mathrm{LP}_{\underline{w}'}(a)} = H_{\mathrm{LP}_{\underline{w}}(a+n)-n} = V_{\mathrm{LP}_{\underline{w}}(a+n)} \subset V_{a+n} = H_a$. For $a \in [r_1+1, r_1+r_2]$ and $\mathrm{LP}_{\underline{v}}(a) = -\infty$, then $\mathrm{LP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(a+n)) \le r_1$ and $\mathrm{LP}_{\underline{w}}(a+n) \in [r_1+1, r_1+n] \cup \{-\infty\}$. If $\mathrm{LP}_{\underline{w}}(a+n) = -\infty$, then $\mathrm{LP}_{\underline{w}'}(a) = -\infty$ and $H_{\mathrm{LP}_{\underline{w}'}(a)} = 0$, so the inclusion holds. Otherwise, we have $H_{\mathrm{LP}_{\underline{w}'}(a)} = H_{\mathrm{RP}(\mathrm{LP}_{\underline{w}}(a))} = V_{\mathrm{RP}(\mathrm{LP}_{\underline{w}}(a+n))} \cap \langle e_2, \dots, e_{n+1} \rangle = V_{\mathrm{LP}_{\underline{w}}(a+n)} \subset V_a = H_a$.

We prove the inclusions $H_k \subset H_{\mathrm{RP}_{\underline{w}'}(a)}$. For $a \in [1, r_1]$, then $\mathrm{RP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}}(a) \le r_1$ and we have $H_a = V_a \cap \langle e_2, \dots, e_{n+1} \rangle \subset V_{\mathrm{RP}_{\underline{w}}(a)} \cap \langle e_2, \dots, e_{n+1} \rangle = H_{\mathrm{RP}_{\underline{w}}(a)} = H_{\mathrm{RP}_{\underline{w}'}(a)}$. For $a \in [r_1+1, r_1+r_2]$ and $\mathrm{RP}_{\underline{v}}(a - r_1) \ne -\infty$, then $\mathrm{RP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}}(a+n) - n \ge r_1 + 1$ and we have $H_a = V_{a+n} \subset V_{\mathrm{RP}_{\underline{w}}(a+n)} = H_{\mathrm{RP}_{\underline{w}}(a+n)-n} = H_{\mathrm{RP}_{\underline{w}'}(a)}$. For $a \in [r_1+1, r_1+r_2]$ and $\mathrm{RP}_{\underline{v}}(a) = -\infty$, then $\mathrm{RP}_{\underline{w}'}(a) = \mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(a+n)) \le r_1$ and $\mathrm{RP}_{\underline{w}}(a+n) \in [r_1+1, r_1+n] \cup \{-\infty\}$. If $\mathrm{RP}_{\underline{w}}(a+n) = -\infty$, then $\mathrm{RP}_{\underline{w}'}(a) = -\infty$ and $H_{\mathrm{LP}_{\underline{w}'}(a)} = \langle e_1, \dots, e_{n+1} \rangle$, so the inclusion holds. Otherwise, we have $H_a = V_{a+n} \subset V_{\mathrm{RP}_{\underline{w}}(a+n)} = V_{\mathrm{RP}(\mathrm{RP}_{\underline{w}}(a+n))} \cap \langle e_2, \dots, e_{n+1} \rangle = H_{\mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(a+n))} = H_{\mathrm{RP}_{\underline{w}'}(a)}$.

We now prove that g is well defined. Note that for all a, we have $H_a \subset \langle e_2, \dots, e_{n+1} \rangle$. We first check the equalities $\dim V_k = i_k$ for all $k \in [1, r]$. For $k \in [1, r_1]$, we have $\dim V_k = \dim(H_k + \langle e_1 \rangle) = \dim H_k + 1 = j_k + 1 = i_k$, where the second equality holds since $H_k \subset \langle e_2, \dots, e_{n+1} \rangle$. For $k \in [r_1+1, r_1+n]$, we have $\dim V_k = \dim(V_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \dots, e_{n+1} \rangle) = \dim V_{\mathrm{RP}_{\underline{w}}(k)} - 1 = (i_k + 1) - 1 = i_k$, where the second equality holds since $\mathrm{RP}_{\underline{w}}(k) \le r_1$ for $k \in [r_1+1, r_1+n]$ thus $\langle e_1 \rangle \subset V_{\mathrm{RP}_{\underline{w}}(k)}$. For $k \in [r_1+n+1, r]$, we have $\dim V_k = \dim H_{k-n} = j_{k-n} = i_k$.

We now check, for $k \in [1, r]$, the inclusions $V_{\mathrm{LP}_{\underline{w}}(k)} \subset V_k \subset V_{\mathrm{RP}_{\underline{w}}(k)}$. We start with the inclusions $V_{\mathrm{LP}_{\underline{w}}(k)} \subset V_k$. For $k \in [1, r_1]$, we have $\mathrm{LP}_{\underline{w}}(k) = \mathrm{LP}_{\underline{w}'}(k) \leq r_1$ thus $V_{\mathrm{LP}_{\underline{w}}(k)} = V_{\mathrm{LP}_{\underline{w}'}(k)} = H_{\mathrm{LP}_{\underline{w}'}(k)} + \langle e_1 \rangle \subset H_k + \langle e_1 \rangle = V_k$. For $k \in [r_1 + 1, r_1 + n]$ and $\mathrm{RP}_{\underline{w}}(k) > \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$, we have $\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k)) = \mathrm{LP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(k))$ and $\mathrm{LP}_{\underline{w}}(k) \in [r_1 + 1, r_1 + n] \cup \{-\infty\}$. If $\mathrm{LP}_{\underline{w}}(k) = -\infty$, then $V_{\mathrm{LP}_{\underline{w}}(k)} = 0$ and the inclusion holds. Otherwise, we have $V_{\mathrm{LP}_{\underline{w}}(k)} = V_{\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))} \cap \langle e_2, \ldots, e_{n+1} \rangle = V_{\mathrm{LP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(k))} \cap \langle e_2, \ldots, e_{n+1} \rangle \subset V_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \ldots, e_{n+1} \rangle = V_k$. For $k \in [r_1 + 1, r_1 + n]$ and $\mathrm{RP}_{\underline{w}}(k) < \mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))$, we have $\mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))) = \mathrm{RP}_{\underline{w}}(k) \leq r_1$. We have $V_{\mathrm{LP}_{\underline{w}}(k)} = V_{\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))} \cap \langle e_2, \ldots, e_{n+1} \rangle \subset V_{\mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k)))} \cap \langle e_2, \ldots, e_{n+1} \rangle \subset V_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \ldots, e_{n+1} \rangle = V_K$. For $k \geq r_1 + n$ and $\mathrm{LP}_{\underline{v}}(k - n - r_1) \neq -\infty$, we have $\mathrm{LP}_{\underline{w}}(k) - n = \mathrm{LP}_{\underline{w}'}(k - n) \geq r_1 + n + 1$ thus $V_{\mathrm{LP}_{\underline{w}}(k)} = H_{\mathrm{LP}_{\underline{w}}(k)-n} = H_{\mathrm{LP}_{\underline{w}'}(k-n)} \subset H_{k-n} = V_k$. For $k \geq r_1 + n$ and $\mathrm{LP}_{\underline{v}}(k - n - r_1) = -\infty$, we have $\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k)) = \mathrm{LP}_{\underline{w}'}(k - n) \leq r_1$ and $\mathrm{LP}_{\underline{w}}(k) \in [r_1 + 1, r_1 + n] \cup \{-\infty\}$. If $\mathrm{LP}_{\underline{w}}(k) = -\infty$, then $V_{\mathrm{LP}_{\underline{w}}(k)} = 0$ and the inclusion holds. Otherwise, we have $V_{\mathrm{LP}_{\underline{w}}(k)} = V_{\mathrm{RP}_{\underline{w}}(\mathrm{LP}_{\underline{w}}(k))} \cap \langle e_2, \ldots, e_{n+1} \rangle = V_{\mathrm{LP}_{\underline{w}'}(k-n)} \cap \langle e_2, \ldots, e_{n+1} \rangle = (H_{\mathrm{LP}_{\underline{w}'}(k-n)} + \langle e_1 \rangle) \cap \langle e_2, \ldots, e_{n+1} \rangle = H_{\mathrm{LP}_{\underline{w}'}(k-n)} \subset H_{k-n} = V_k$.

We finish with the inclusions $V_k \subset V_{\mathrm{RP}_{\underline{w}}(k)}$. For $k \in [1, r_1]$, we have $\mathrm{RP}_{\underline{w}}(k) = \mathrm{RP}_{\underline{w}'}(k) \leq r_1$ thus $V_k = H_k + \langle e_1 \rangle \subset H_{\mathrm{RP}_{\underline{w}'}(k)} + \langle e_1 \rangle = V_{\mathrm{RP}_{\underline{w}'}(k)} = V_{\mathrm{RP}_{\underline{w}}(k)}$. For $k \in [r_1 + 1, r_1 + n]$, we have $V_k = V_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \ldots, e_{n+1} \rangle \subset V_{\mathrm{RP}_{\underline{w}}(k)}$. For $k \geq r_1 + n$ and $\mathrm{RP}_{\underline{v}}(k - n - r_1) \neq -\infty$, we have $\mathrm{RP}_{\underline{w}}(k) - n = \mathrm{RP}_{\underline{w}'}(k - n) \geq r_1 + n + 1$ thus $V_k = H_{k-n} \subset H_{\mathrm{RP}_{\underline{w}'}(k-n)} = H_{\mathrm{RP}_{\underline{w}}(k)-n} = V_{\mathrm{RP}_{\underline{w}}(k)}$. For $k \geq r_1 + n$ and $\mathrm{RP}_{\underline{v}}(k - n - r_1) = -\infty$, we have $\mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(k)) = \mathrm{RP}_{\underline{w}'}(k - n) \leq r_1$ and $\mathrm{RP}_{\underline{w}}(k) \in [r_1 + 1, r_1 + n] \cup \{-\infty\}$. If $\mathrm{RP}_{\underline{w}}(k) = -\infty$, then $V_{\mathrm{LP}_{\underline{w}}(k)} = \langle e_1, \ldots, e_{n+1} \rangle$ and the inclusion holds. Otherwise, we have $V_k = H_{k-n} \subset H_{\mathrm{RP}_{\underline{w}'}(k-n)} = V_{\mathrm{RP}_{\underline{w}'}(k-n)} = V_{\mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(k))}$ But since $V_k = H_{k-n} \subset \langle e_2, \ldots, e_{n+1} \rangle$, we get $V_k \subset V_{\mathrm{RP}_{\underline{w}}(\mathrm{RP}_{\underline{w}}(k))} \cap \langle e_2, \ldots, e_{n+1} \rangle = V_{\mathrm{RP}_{\underline{w}}(k)}$ where the last equality holds since $\mathrm{RP}_{\underline{w}}(k) \in [r_1 + 1, r_1 + n]$.

Now we prove that f and g are inverse to each other and that $\pi_{\underline{w}} = \pi_{\underline{w}'} \circ c^{-1} \circ f$. We first prove that $g \circ f$ is the identity. Write $g \circ f((V_k)_{k\in[1,r]}) = (V'_k)_{k\in[1,r]}$ and $f((V_k)_{k\in[1,r]}) = (H_a)_{a\in[1,r_1+r_2]}$. For $k \in [1, r_1]$, we have $V'_k = H_k + \langle e_1 \rangle = (V_k \cap \langle e_2, \ldots, e_{n+1} \rangle) + \langle e_1 \rangle$. But for such k, we have $\langle e_1 \rangle \subset V_k$, this implies $(V_k \cap \langle e_2, \ldots, e_{n+1} \rangle) + \langle e_1 \rangle = V_k$. For $k \in [r_1 + 1, r_1 + n]$, we proceed by induction on k and remark that $\mathrm{RP}_{\underline{w}}(k) < k$. We have $V'_k = V'_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \ldots, e_{n+1} \rangle = V_{\mathrm{RP}_{\underline{w}}(k)} \cap \langle e_2, \ldots, e_{n+1} \rangle = V_k$. For $k \geq r_1 + n + 1$, we have $V'_k = H_{k-n} = V_k$.

Next we prove that $f \circ g$ is the identity. Write $f \circ g((H_a)_{a\in[1,r_1+r_2]}) = (H'_a)_{a\in[1,r_1+r_2]}$ and $g((H_a)_{a\in[1,r_1+r_2]}) = (V_k)_{k\in[1,r]}$. For $a \in [1, r_1]$, we have $H'_a = V_a \cap \langle e_2, \ldots, e_{n+1} \rangle = (H_a + \langle e_1 \rangle) \cap \langle e_2, \ldots, e_{n+1} \rangle$. But for such a, we have $H_a \subset \langle e_2, \ldots, e_{n+1} \rangle$ and this implies $(H_a + \langle e_1 \rangle) \cap \langle e_2, \ldots, e_{n+1} \rangle = H_a$. For $a \in [r_1 + 1, r_1 + r_2]$, we have $H'_a = V_{a+n} = H_a$.

Finally we check that $\pi_{\underline{w}} = \pi_{\underline{w}'} \circ c^{-1} \circ f$. Write $f((V_k)_{k\in[1,r]}) = (H_a)_{a\in[r_1+r_2]}$, $\pi_{\underline{w}}((V_k)_{k\in[1,r]}) = (U_{\not\succcurlyeq})_{\not\succcurlyeq\in[1,n]}$ and $\pi_{\underline{w}'}((H_a)_{a\in[1,r_1+r_2]}) = (U'_{\not\succcurlyeq})_{\not\succcurlyeq\in[1,n]}$. We need to prove that $U_{\not\succcurlyeq} = U'_{\not\succcurlyeq}$ for all $\not\succcurlyeq \in [1, n]$. If $\mathrm{LO}_{\underline{v}}(\not\succcurlyeq) \neq -\infty$, we have $\mathrm{LO}_{\underline{w}'}(\not\succcurlyeq) = \mathrm{LO}_{\underline{w}}(\not\succcurlyeq) - n \geq r_1 + 1$. We get $U'_{\not\succcurlyeq} = H_{\mathrm{LO}_{\underline{w}'}(\not\succcurlyeq)} = V_{\mathrm{LO}_{\underline{w}'}(\not\succcurlyeq)+n} = V_{\mathrm{LO}_{\underline{w}}(\not\succcurlyeq)} = U_{\not\succcurlyeq}$. If

$\mathrm{LO}_{\underline{v}}(\not\geq) = -\infty$, we have $\mathrm{LO}_{\underline{w}'}(\not\geq) = \mathrm{LO}_{\underline{u}}(\not\geq + 1) = \mathrm{RP}_{\underline{w}}(r_1 + \not\geq) \leq r_1$ and $\mathrm{LO}_{\underline{w}}(\not\geq) = r_1 + \not\geq$. We get $U'_{\not\geq} = H_{\mathrm{LO}_{\underline{w}'}(\not\geq)} = H_{\mathrm{RP}_{\underline{w}}(r_1+\not\geq)} = V_{\mathrm{RP}_{\underline{w}}(r_1+\not\geq)} \cap \langle e_2; \ldots, e_{n+1}\rangle = V_{r_1+\not\geq} = V_{\mathrm{LO}_{\underline{w}}(\not\geq)} = U_{\not\geq}$. $\square$

4.3 A Product Formula in Cobordism

As a consequence of Theorem 4.11 we prove a product formula in the algebraic cobordism $\Omega^*(Fl_{n+1})$.

Corollary 4.12 *Let $\underline{w}$ be a reduced word and $w \in W$ the associated element.*

1. *If $w \not\geq c$, then $[X_{\underline{w}}] \cdot [\mathbf{F}_n] = 0$ in $\Omega^*(Fl_{n+1})$.*
2. *If $w \geq c$, there exist $\underline{u} \in \underline{W}^1_{n+1}$ and $\underline{v} \in \underline{W}^n_{n+1}$ such that $\underline{w} = \underline{u}\,\underline{c}\,\underline{v}$ modulo commuting relations and we have*

$$[X_{\underline{w}}] \cdot [\mathbf{F}_n] = [X_{c^{-1}(\underline{u})\underline{v}}].$$

in $\Omega^*(Fl_{n+1})$.

Proof The product $[X_{\underline{w}}] \cdot [\mathbf{F}_n]$ is given by pulling back the exterior product $X_{\underline{w}} \times \mathbf{F}_n \to Fl_{n+1} \times Fl_{n+1}$ along the diagonal map $\Delta : Fl_{n+1} \to Fl_{n+1} \times Fl_{n+1}$, see [16, Remark 4.1.14]. We thus have $[X_{\underline{w}}] \cdot [\mathbf{F}_n] = \Delta^*[X_{\underline{w}} \times \mathbf{F}_n \to Fl_{n+1} \times Fl_{n+1}]$. Applying [16, Corollary 6.5.5.1], we get $\Delta^*[X_{\underline{w}} \times \mathbf{F}_n \to Fl_{n+1} \times Fl_{n+1}] = [X_{\underline{w}} \times_{Fl_{n+1}} \mathbf{F}_n]$ in $\Omega^*(X)$. The result follows since $[c(X_{c^{-1}(\underline{u})\underline{v}})] = [X_{c^{-1}(\underline{u})\underline{v}}]$. $\square$

As a special case, we recover the restriction formula in Theorem 3.3 as a product formula.

Corollary 4.13 *Let $\underline{w} = \underline{c}\,\underline{v}$ be a reduced word with $\underline{v} \in \underline{W}^n_{n+1}$. Then we have the following formula in $\Omega^*(Fl_{n+1})$:*

$$[X_{\underline{w}}] \cdot [\mathbf{F}_n] = [X_{\underline{v}}].$$

By reversing the order of the simple reflection $s_1, \ldots, s_n$ (or equivalently by conjugating with the element w_0) we also obtain the following results in $\Omega^*(Fl_{n+1})$:

Proposition 4.14 *Let $\underline{w} \in \underline{W}_{n+1}$ be a reduced word and $\underline{c}' := s_n \ldots s_1$ a Coxeter element. Let $\mathbf{F}'_n = \{U_\bullet \in Fl_{n+1} \mid U_1 = \langle e_{n+1}\rangle\}$. Then $w \not\geq c'$ is equivalent to $X_w \cap \mathbf{F}'_n = \emptyset$ and we have the following alternatives:*

(1) If $w \not\geq c'$, then $[X_{\underline{w}}] \cdot [\mathbf{F}'_n] = 0$ in $\Omega^*(Fl_{n+1})$.
(2) If $w \geq c'$, then, modulo commuting relations, we have $\underline{w} = \underline{u}\,\underline{c}'\,\underline{v}$ with $\underline{u} \in \underline{W}^1_{n+1}$ and $\underline{v} \in \underline{W}^n_{n+1}$. Furthermore, we have

$$[X_{\underline{w}}] \cdot [\mathbf{F}'_n] = [X_{c'^{-1}(\underline{u})\underline{v}}]$$

in $\Omega^*(Fl_{n+1})$.

In particular, if $\underline{w} = \underline{c}'\underline{v}$ *is reduced with* $\underline{v} \in W^1_{n+1}$, *then we have the following formula in* $\Omega^*(Fl_{n+1})$:

$$[X_{\underline{w}}] \cdot [\mathbf{F}'_n] = [X_{\underline{v}}].$$

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References

1. Bressler, P., Evens, S.: The Schubert calculus, braid relations, and generalized cohomology. Trans. Amer. Math. Soc. **317**(2), 799–811 (1990)
2. Bressler, P., Evens, S.: Schubert calculus in complex cobordism. Trans. Amer. Math. Soc. **331**(2), 799–813 (1992)
3. Calmès, B., Petrov, V., and Zainoulline, K.: Invariants, torsion indices and oriented cohomology of complete flags. Ann. Sci. Éc. Norm. Supér. (4) **46**(3), 405–448 (2013)
4. Calmès, B., Zainoulline, K., and Zhong, C.: Equivariant oriented cohomology of flag varieties. Doc. Math. 113–144 (2015)(Extra vol.: Alexander S. Merkurjev's sixtieth birthday)
5. Demazure, M.: Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup. (4) **7**, 53–88 (1974) (Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I)
6. Hornbostel, J., Kiritchenko, V.: Schubert calculus for algebraic cobordism. J. Reine Angew. Math. **656**, 59–85 (2011)
7. Hornbostel, J., Perrin, N.: Smooth Schubert varieties and generalized Schubert polynomials in algebraic cobordism of Grassmannians. Pacific J. Math. **294**(2), 401–422 (2018)
8. Hudson, T.: A Thom-Porteous formula for connective K-theory using algebraic cobordism. J. K-Theory **14**(2), 343–369 (2014)
9. Hudson, T.: Generalised symplectic Schubert classes (2015). arXiv:1504.07924
10. Hudson, T., Matsumura, T.: Kempf-Laksov Schubert classes for even infinitesimal cohomology theories. In: Schubert varieties, equivariant cohomology and characteristic classes—IMPANGA 15, EMS Series of Congress Reports European Mathematical Society, Zürich, , pp. 127–151 (2018)
11. Hudson, T., Matsumura, T.: Vexillary degeneracy loci classes in K-theory and algebraic cobordism. European J. Combin. **70**, 190–201 (2018)
12. Hudson, T., Matsumura, T.: Segre classes and Damon-Kempf-Laksov formula in algebraic cobordism. Math. Ann. **374**(3–4), 1439–1457 (2019)
13. Kiritchenko, V., Krishna, A.: Equivariant cobordism of flag varieties and of symmetric varieties. Transform. Groups **18**(2), 391–413 (2013)
14. Lascoux, A., Schützenberger, M.-P.: Géométric algébrique-polynômes de schubert. C. R. Acad. Sci. Paris Sér. I Math. **294** (1982)
15. Lazard, M.: Sur les groupes de Lie formels à un paramètre. Bull. Soc. Math. France **83**, 251–274 (1955)
16. Levine, M., Morel, F.: Algebraic cobordism. Springer Monographs in Mathematics. Springer, Berlin (2007)

17. Nakagawa, M., Naruse, H.: Generalized (co)homology of the loop spaces of classical groups and the universal factorial Schur P- and Q-functions. In: Schubert calculus—Osaka 2012, Advanced Studies in Pure Mathematics. Mathematical Society of Japan [Tokyo], vol. 71, pp. 337–417 (2016)
18. Nakagawa, M., Naruse, H.: Generating functions for the universal Hall-Littlewood P- and Q-functions (2017). arXiv:1705.04791
19. Nakagawa, M., Naruse, H.: Universal factorial Schur P, Q-functions and their duals (2018). arXiv:1812.03328
20. Nakagawa, M., Naruse, H.: Universal Gysin formulas for the universal Hall-Littlewood functions. In: An Alpine Bouquet of Algebraic Topology, Contemporary Mathematics, vol. 708, pp. 201–244. American Mathematical Society, Providence (2018)

Residue Mirror Symmetry for Grassmannians

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Abstract Motivated by recent works on localizations in A-twisted gauged linear sigma models, we discuss a generalization of toric residue mirror symmetry to complete intersections in Grassmannians.

Keywords A-twisted gauged linear sigma models · Quasimaps · Toric residue mirror symmetry

1 Introduction

A-twisted gauged linear sigma models are 2-dimensional topological field theories introduced by Witten [69]. An A-twisted gauged linear sigma model is specified by a reductive algebraic group G (or its compact real form) called the *gauge group*, an affine space W with a linear action of $G \times \mathbb{G}_m$ called the *matter*, and an element ξ of the dual $\mathfrak{z}^*$ of the center of the Lie algebra of G called the *Fayet–Iliopoulos param-*

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eter. The weights of the $\mathbb{G}_m$-actions are called *R-charges*. One can also introduce a *superpotential* in the theory, which is a G-invariant function on W of R-charge 2. The *correlation functions*, which are quantities of primary interest, do not depend on the potential.

An A-twisted gauged linear sigma model with a suitable Fayet–Iliopoulos parameter is expected to be equivalent to the topological sigma model whose target is the classical vacuum subspace of the symplectic reduction $W /\!\!/_\xi G$. This comes from a stronger expectation that the low-energy limit of a gauged linear sigma model should give the non-linear sigma model whose target is the symplectic reduction $W /\!\!/_\xi G$.

A prototypical example is the case $G = \mathbb{G}_m$ and $W = \mathbb{A}^6$, with the action

$$G \times \mathbb{G}_m \ni (\alpha, \beta) \colon (z_1, \dots, z_5, P) \mapsto (\alpha z_1, \dots, \alpha z_5, \alpha^{-5} \beta^2 P) \tag{1}$$

and a potential Pf, which is the product of the variable P and a homogeneous polynomial f in $z_1, \dots, z_5$ of degree 5. The symplectic reduction $W /\!\!/_\xi G$ for the positive ξ gives the total space of the bundle $\mathcal{O}_{\mathbb{P}^4}(-5)$. The R-charge of the P-field indicates that the target space should be considered not as a manifold but as a supermanifold, where the parity of the fiber is odd.

One candidate for a mathematical theory of A-twisted gauged linear sigma models is symplectic vortex invariants [19, 20, 35, 59, 60, 74] and their generalizations incorporating potentials [28, 65]. Another candidate is quasimap theory, which is an intersection theory on moduli spaces of maps to the quotient stacks $[W/G]$. A review of the latter theory, with historical remarks and extensive references, can be found in [21]. These two approaches should be related by Hitchin–Kobayashi correspondence for vortices [13, 58, 68].

When the gauge group is abelian, quasimap theory as a mathematical theory of A-twisted gauged linear sigma models goes back to [52]. The relation with the Yukawa coupling of the mirror is formulated as *toric residue mirror conjecture* in [3, 4] and proved in [11, 46, 63, 64].

Quasimap theory in the special case of projective hypersurfaces is also studied in the insightful paper [34], where a heuristic relation with semi-infinite homologies of loop spaces is discussed. This eventually leads to Givental's proof [31] of classical mirror symmetry [16] for the quintic 3-fold. This has been extended to toric complete intersections in [32].

The correlation functions of A-twisted gauged linear sigma models in the cases when gauge groups are not necessarily abelian are computed in [9, 24] using supersymmetric localization of path integrals. The result is given in terms of Jeffrey–Kirwan residues, and reproduces the results of [52] in abelian cases.

The aim of this paper is twofold. One is to give an expository account of quasimap theory and its relation to other subjects such as instantons and integrable systems. The other is to formulate Conjecture 5, which states that the correlation function defined in (5) in terms of residues coincides with the generating function of quasimap invariants defined in (190), and prove it for Grassmannians in Sect. 12. This can be considered as a generalization of toric residue mirror symmetry to Grassmannians. We also show in Sect. 8 that a slightly weakened version of toric residue mirror conjecture follows from Givental's mirror theorem. Nothing else in this paper is new.

This paper is organized as follows: In Sect. 2, we recall the description of correlation functions of A-twisted gauged linear sigma models given in [9, 24]. In Sect. 3, we recall the definition of the quasimap spaces $\mathbf{Q}(\mathbb{P}^{n-1}; d)$. They are compactifications of the spaces of holomorphic maps of degrees d from $\mathbb{P}^1$ to $\mathbb{P}^{n-1}$, and play an essential role in Givental's homological geometry [33, 34]. In Sect. 4, we recall toric residue mirror symmetry for Calabi–Yau complete intersections in projective spaces. In Sect. 5, we discuss quasimap invariants of concave bundles. In Sect. 6, we recall classical mirror symmetry for toric hypersurfaces proved in [32]. The exposition in Sect. 6 follows [43] closely. In Sect. 7, we briefly recall the definition of quasimap spaces for toric varieties due to [52]. In Sect. 8, we show that a slightly weakened version of toric residue mirror conjecture for CY hypersurfaces follows from classical mirror symmetry. In Sect. 9, we recall a theorem of Martin which relates integration on a symplectic quotient by a compact Lie group to that on the quotient by a maximal torus. In Sect. 10, we recall the definition of quasimap spaces to GIT quotients, which are called *quasimap graph spaces* in [22]. The quasimap spaces come with the universal G-bundle and the canonical virtual fundamental classes, which allow us to define numerical invariants. We formulate Conjecture 5, which states that correlation functions of A-twisted gauged linear sigma models given in (5) are generating functions of quasimap invariants. There is a natural $\mathbb{G}_m$-action on the quasimap graph space coming from the $\mathbb{G}_m$-action on the domain curve. There is a distinguished connected component of the fixed locus of this action, which is used to define the *I-function*. In Sect. 11, the quasimap spaces and the I-functions for Grassmannians are recalled from [10]. In Sect. 12, we prove Conjecture 5 for Grassmannians. For this purpose, we introduce *abelianized quasimap spaces* for Grassmannians, which allows us to relate quasimap invariants for Grassmannians with correlations function in (5). In Sect. 13, we discuss the relation between gauged linear sigma models and Bethe ansatz following [56]. In Sects. 14–16, we recall the relations of quasimaps with instantons, monopoles, and vortices respectively.

2 Correlation Functions of A-Twisted Gauged Linear Sigma Models

2.1

Let G be a reductive algebraic group of rank r, and W be a representation of $G \times \mathbb{G}_m$. The center of G and its Lie algebra will be denoted by $Z(G)$ and $\mathfrak{z}$. Fix a maximal torus T of G, and let $\mathfrak{t}$ be its Lie algebra. The set of roots, its subset of positive roots, and the Weyl group will be denoted by Δ, Δ_+, and $\mathscr{W} := N(T)/T$. Let $W = \bigoplus_{i=1}^N W_i$ be the weight space decomposition of W with respect to the action of $T \times \mathbb{G}_m$. The weight of W_i will be denoted by $(\rho_i, r_i) \in \mathfrak{t}^\vee \oplus \mathbb{Z}$, and r_i will be called the *R-charge*. If W admits an action of another torus H commuting with the action of $G \times \mathbb{G}_m$, then one can introduce the *twisted mass* $\lambda \in \mathfrak{h}$ in the theory, which corresponds to the equivariant parameter for the H-action. The $T \times \mathbb{G}_m \times H$-weight of W_i will be denoted by $(\rho_i, r_i, \nu_i) \in \mathfrak{t}^\vee \oplus \mathbb{Z} \oplus \mathfrak{h}^\vee$. We also introduce the *complexified Fayet–Illiopoulos parameter* $t' \in \mathfrak{z}^\vee \otimes_{\mathbb{R}} \mathbb{C}$, which corresponds to the complexified Kähler form of the symplectic quotient. Here, we save the unprimed symbol t for the indeterminate in the generating function of quasimap invariants (see (22), (60), (85) and (231)).

For $d \in \mathfrak{t}$ and $t' \in \mathfrak{z}^\vee$, the composition of the surjection $\mathfrak{t}^\vee \twoheadrightarrow \mathfrak{z}^\vee$ dual to the inclusion $\mathfrak{z} \hookrightarrow \mathfrak{t}$ and the evaluation $\mathfrak{t}^\vee \times \mathfrak{t} \to \mathbb{C}$ will be denoted by $t' \cdot d$ or $t'(d)$.

2.2

For $d \in \mathfrak{t}$ and $x \in \mathfrak{t}$, let

$$Z_d(x) := Z_d^{\text{vec}}(x) Z_d^{\text{mat}}(x) \tag{2}$$

be the product of

$$Z_d^{\text{vec}}(x) := \prod_{\alpha \in \Delta_+} (-1)^{\alpha(d)+1} \alpha^2(x) \tag{3}$$

and

$$Z_d^{\text{mat}}(x) := \prod_{i=1}^{N} (\rho_i(x) + \nu_i(\lambda))^{r_i - \rho_i(d) - 1}. \tag{4}$$

Here the superscripts ‘vec’ and ‘mat’ stands for the vector multiplet and the matter chiral multiplet respectively. According to [9, 24], the correlation function of a $\mathscr{W}$-invariant polynomial $P(x) \in \mathbb{C}[\mathfrak{t}]^{\mathscr{W}}$ on a 2-sphere is given, up to sign introduced by hand, by

$$\langle P(x) \rangle_{\text{GLSM}} = \frac{1}{|\mathscr{W}|} \sum_{d \in \mathsf{P}^\vee} e^{t' \cdot d} \, \mathrm{JK}_{\mathfrak{c}}(Z_d(x) P(x)). \tag{5}$$

Here $\mathsf{P}^\vee$ is the coweight lattice of G and $\mathrm{JK}_{\mathfrak{c}}$ is the Jeffrey–Kirwan residue defined in [63, Sect. 2] (cf. also [15]). The cone $\mathfrak{c} \subset \mathfrak{z}^\vee$ is the ample cone of the GIT quotient determined by the Fayet–Iliopoulos parameter η.

2.3

One can introduce a variable z associated with the background value of an auxiliary gauge field in the gravity multiplet. This corresponds to the equivariant parameter for the $\mathbb{G}_m$-action on the domain curve. This turns (2) into the product

$$Z_d(x;\mathsf{z}) := Z_d^{\mathrm{vec}}(x;\mathsf{z}) Z_d^{\mathrm{mat}}(x;\mathsf{z}) \tag{6}$$

of

$$Z_d^{\mathrm{vec}}(x;\mathsf{z}) := \prod_{\alpha\in\Delta_+} (-1)^{\alpha(d)+1}\alpha(x)\alpha(x+d\mathsf{z}) \tag{7}$$

and

$$Z_d^{\mathrm{mat}}(x;\mathsf{z}) := \prod_{i=1}^{N} \frac{\prod_{l=-\infty}^{-1}\left(\rho_i(x)+\nu_i(\lambda)-\left(l+\frac{r_i}{2}\right)\mathsf{z}\right)}{\prod_{l=-\infty}^{\rho_i(d)-r_i}\left(\rho_i(x)+\nu_i(\lambda)-\left(l+\frac{r_i}{2}\right)\mathsf{z}\right)}, \tag{8}$$

and the correlation function of $P(x)\in\mathbb{C}[\mathfrak{t}]^{\mathscr{W}}$ is given by

$$\langle P(x)\rangle_{\mathrm{GLSM}}^{H\times\mathbb{G}_m} = \frac{1}{|\mathscr{W}|}\sum_{d\in\mathsf{P}^\vee} e^{t'\cdot d}\,\mathrm{JK}_{\mathfrak{c}}(Z_d(x;\mathsf{z})P(x)). \tag{9}$$

2.4

Another quantity of interest is the *effective twisted superpotential* on the Coulomb branch, or the *effective potential* for short. It is defined as the sum

$$W_{\mathrm{eff}}(x;t') := W_{\mathrm{FI}}(x;t') + W_{\mathrm{vec}}(x) + W_{\mathrm{mat}}(x) \tag{10}$$

of the Fayet–Illiopoulos term

$$W_{\mathrm{FI}}(x;t') := t'\cdot x, \tag{11}$$

the vector multiplet term

$$W_{\mathrm{vec}}(x) := -\pi\sqrt{-1}\sum_{\alpha\in\Delta^+}\alpha(x), \tag{12}$$

and the matter term

$$W_{\mathrm{mat}}(x) := -\sum_{i=1}^{N} (\rho_i(x) + \nu_i(\lambda)) \left(\log(\rho_i(x) + \nu_i(\lambda)) - 1\right). \tag{13}$$

3 Quasimap Spaces for Projective Spaces

3.1

A holomorphic map $u\colon \mathbb{P}^1 \to \mathbb{P}^{n-1}$ of degree d is given by a collection $(u_i(z_1, z_2))_{i=1}^n$ of n homogeneous polynomials of degree d satisfying the following condition:

$$\text{There exists no } (z_1, z_2) \in \mathbb{A}^2 \setminus \{0\} \text{ such that } u(z_1, z_2) = 0 \in \mathbb{A}^n. \tag{14}$$

Two collections $(u_i(z_1, z_2))_{i=1}^n$ and $\left(u_i'(z_1, z_2)\right)_{i=1}^n$ define the same map if and only if there exists $\alpha \in \mathbb{G}_m$ such that $u_i(z_1, z_2) = \alpha u_i'(z_1, z_2)$ for all $i \in \{1, \dots, n\}$. It follows that the space

$$\mathcal{M}(\mathbb{P}^{n-1}; d) := \left\{u\colon \mathbb{P}^1 \to \mathbb{P}^{n-1} \;\middle|\; \deg u = d\right\} \tag{15}$$

of holomorphic maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^{n-1}$ can be compactified to the projective space of dimension $n(d+1) - 1$, whose homogeneous coordinate is given by the coefficients $\left(a_{ij}\right)_{i,j}$ of the collection $(u_i(z_1, z_2))_{i=1}^n$ of homogeneous polynomials of degree d;

$$u_i(z_1, z_2) = \sum_{j=0}^{d} a_{ij} z_1^j z_2^{d-j}, \qquad i = 1, \dots, n. \tag{16}$$

This compactification is called the *quasimap space* and denoted by $\mathbf{Q}(\mathbb{P}^{n-1}; d)$. An element of the quasimap space is called a *quasimap*.

3.2

A point $[z_1 : z_2] \in \mathbb{P}^1$ is a *base point* (or *singularity*) of a quasimap u if $u(z_1, z_2) = 0$. A quasimap is a genuine map outside of the base locus. If the degree of the base locus is d', then a quasimap can be considered as a genuine map of degree $d - d'$. However, it is more convenient to think of a quasimap as a morphism to the quotient stack $[\mathbb{A}^n / \mathbb{G}_m]$. By definition, a morphism from $\mathbb{P}^1$ to $[\mathbb{A}^n / \mathbb{G}_m]$ is a principal $\mathbb{G}_m$-bundle P over $\mathbb{P}^1$ and a $\mathbb{G}_m$-equivariant morphism $\widetilde{u}\colon P \to \mathbb{A}^n$. It is a quasimap if the generic point of P is mapped to the semi-stable locus $\mathbb{A}^n \setminus \{0\}$.

3.3

Let $x \in H^2(\mathbf{Q}(\mathbb{P}^{n-1}; d); \mathbb{Z})$ be the ample generator of the cohomology ring of $\mathbf{Q}(\mathbb{P}^{n-1}; d) \cong \mathbb{P}^{n(d+1)-1}$, so that

$$H^*(\mathbf{Q}(\mathbb{P}^{n-1}; d); \mathbb{Z}) \cong \mathbb{Z}[x]/\left(x^{n(d+1)}\right). \tag{17}$$

Given a polynomial $P(x) \in \mathbb{C}[x]$, we set

$$\langle P(x) \rangle_{\mathbb{P}^{n-1}} := \sum_{d=0}^{\infty} q^d \langle P(x) \rangle_{\mathbb{P}^{n-1},d} \in \mathbb{C}[\![q]\!], \tag{18}$$

where

$$\langle P(x) \rangle_{\mathbb{P}^{n-1},d} := \int_{\mathbf{Q}(\mathbb{P}^{n-1};d)} P(x) \tag{19}$$

is the integration over the quasimap space. It follows from

$$\left\langle x^k \right\rangle_{\mathbb{P}^{n-1},d} = \begin{cases} 1 & k = n(d+1) - 1, \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

that

$$\left\langle x^k \right\rangle_{\mathbb{P}^{n-1}} = \begin{cases} q^d & k = n(d+1) - 1 \text{ for some } d \in \mathbb{Z}^{\geq 0}, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

3.4

If we set $G := \mathbb{G}_m$ and $W := \mathbb{C}^n$ with the action $G \times \mathbb{G}_m \ni (\alpha, \beta) \colon (w_1, \ldots, w_n) \mapsto (\alpha w_1, \ldots, \alpha w_n)$, then we have $Z_d^{\mathrm{vec}}(x) = 1$ and $Z_d^{\mathrm{mat}}(x) = \left(x^{-d-1}\right)^n$, so that (5) gives the same result as (21) under the identification

$$q = e^{t'}. \tag{22}$$

3.5

The *small quantum cohomology* of $\mathbb{P}^{n-1}$ is the free $\mathbb{C}[\![q]\!]$-module

$$\mathrm{QH}\left(\mathbb{P}^{n-1}\right) := H^*\left(\mathbb{P}^{n-1}; \mathbb{C}[\![q]\!]\right) \tag{23}$$

equipped with multiplication given by

$$x^i \circ x^j := \sum_{k=0}^{n} \sum_{d=0}^{\infty} q^d \left\langle I_{0,3,d} \right\rangle (x^i, x^j, x^k)\, x^{n-k-1}. \tag{24}$$

Here

$$\left\langle I_{0,3,d} \right\rangle (a, b, c) := \int_{\left[\overline{\mathcal{M}}_{0,3}\left(\mathbb{P}^{n-1}; d\right)\right]^{\mathrm{virt}}} \mathrm{ev}_1^* a \cup \mathrm{ev}_2^* b \cup \mathrm{ev}_3^* c \tag{25}$$

is the 3-point Gromov–Witten invariant. It is an associative commutative deformation of the classical cohomology ring;

$$\mathrm{QH}\left(\mathbb{P}^{n-1}\right) / \left(q\right) \cong H^*\left(\mathbb{P}^{n-1}; \mathbb{C}\right). \tag{26}$$

Since the virtual dimension of the moduli space of stable maps is given by

$$\mathrm{virt.dim}\overline{\mathcal{M}}_{g,k}(X; d) = (1-g)(\dim X - 3) + \langle c_1(X), d \rangle + k \tag{27}$$

in general, one has

$$\mathrm{virt.dim}\overline{\mathcal{M}}_{0,3}\left(\mathbb{P}^{n-1}; d\right) = nd + n - 1. \tag{28}$$

The 3-point Gromov–Witten invariant in (24) is non-zero only if

$$\mathrm{virt.dim}\overline{\mathcal{M}}_{0,3}\left(\mathbb{P}^{n-1}; d\right) = i + j + k. \tag{29}$$

Since $0 \le i, j, k \le n-1$, one has (29) only if $d = 0$, $i + j + k = n - 1$ or $d = 1$, $i + j + k = 2n - 1$. This shows that $x^i \circ x^j = x^{i+j}$ for $i + j \le n - 1$. Since there is a unique line passing through two points on $\mathbb{P}^{n-1}$ in general position, and this line intersects a hyperplane at one point, one has $x \circ x^{n-1} = q$. Hence the ring structure of the quantum cohomology of $\mathbb{P}^{n-1}$ is given by

$$\mathrm{QH}\left(\mathbb{P}^{n-1}\right) \cong \left(\mathbb{C}[\![q]\!]\right)[x] / \left(x^n - q\right). \tag{30}$$

We write the ring homomorphism $\mathbb{C}[x] \to \mathrm{QH}(\mathbb{P}^{n-1})$ sending x to x as $P(x) \mapsto \mathring{P}(x)$.

Theorem 1 *For any $P(x) \in \mathbb{C}[x]$, one has*

$$\langle P(x) \rangle_{\mathbb{P}^{n-1}} = \int_{\mathbb{P}^{n-1}} \mathring{P}(x). \tag{31}$$

Proof Since both sides of (31) are linear in $P(x) \in \mathbb{C}[x]$, it suffices to show

$$\left\langle x^k \right\rangle_{\mathbb{P}^{n-1}} = \int_{\mathbb{P}^{n-1}} x^{\circ k} \tag{32}$$

for any $k \in \mathbb{N}$, which is obvious from (21) and (30). □

Theorem 1 is equivalent to the Vafa–Intriligator formula [42, 67]:

Corollary 2 (Vafa–Intriligator formula for projective spaces) *For any* $P(x) \in \mathbb{C}[x]$, *one has*

$$\int_{\mathbb{P}^{n-1}} \mathring{P}(x) = \frac{1}{n} \sum_{\lambda^n = q} \frac{P(\lambda)}{\lambda^{n-1}}, \tag{33}$$

where the sum is over $\lambda \in \mathbb{C}\left[\!\left[q^{1/n} \right]\!\right]$ *satisfying* $\lambda^n = q$.

Proof Since the integration over the projective space can be written by residue as

$$\int_{\mathbb{P}^{r-1}} x^k = \delta_{r-1,k} = \operatorname{Res} \frac{x^k dx}{x^r}, \tag{34}$$

one has

$$\begin{aligned}
\int_{\mathbb{P}^{n-1}} \mathring{P}(x) &= \langle P(x) \rangle_{\mathbb{P}^{n-1}} && (35)\\
&= \sum_{d=0}^{\infty} q^d \int_{\mathbf{Q}(\mathbb{P}^{n-1};d)} P(x) && (36)\\
&= \sum_{d=0}^{\infty} q^d \operatorname{Res} \frac{P(x) dx}{x^{n(d+1)}} && (37)\\
&= \operatorname{Res} \frac{x^{-n} P(x)}{1 - q x^{-n}} && (38)\\
&= \operatorname{Res} \frac{P(x)}{x^n - q} && (39)\\
&= \frac{1}{n} \sum_{\lambda^n = q} \frac{P(\lambda)}{\lambda^{n-1}}, && (40)
\end{aligned}$$

and (33) is proved. □

3.6

The projective space $\mathbb{P}^{n-1}$ has a natural action of GL_n, which restricts to the action of the diagonal maximal torus H. The *equivariant cohomology* is defined as the ordinary cohomology $H^*_H(\mathbb{P}^{n-1}) := H^*(\mathbb{P}^{n-1}_H)$ of the *Borel construction* $\mathbb{P}^{n-1}_H := \mathbb{P}^{n-1} \times_H EH$, where EH is the product of n copies of the total space of the tautological bundle $\mathcal{O}_{\mathbb{P}^\infty}(-1)$ over $B\mathbb{G}_m = \mathbb{P}^\infty$. It follows that $\mathbb{P}^{n-1}_H$ is the projectivization $\mathbb{P}(\mathcal{E})$ of the vector bundle $\mathcal{E} := \bigoplus_{i=1}^n \pi_i^* \mathcal{O}_{\mathbb{P}^\infty}(-1)$ of rank n over $(\mathbb{P}^\infty)^n$. A standard result on the cohomology of a projective bundle (see e.g. [37, p. 606]) shows that $H^*(\mathbb{P}^{n-1}_H)$ is generated over $H^*_H(\mathrm{pt}) = H^*((\mathbb{P}^\infty)^n) \cong \mathbb{C}[\lambda_1, \ldots, \lambda_n]$ by $x := -c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))$ with one relation

$$(-x)^n - c_1(\mathcal{E})(-x)^{n-1} + c_2(\mathcal{E})(-x)^{n-2} + \cdots + (-1)^n c_n(\mathcal{E}) = 0. \tag{41}$$

Since $c_i(\mathcal{E}) = (-1)^i \sigma_i(\lambda_1, \ldots, \lambda_n)$, one obtains

$$H^*_H(\mathbb{P}^{n-1}) \cong \mathbb{C}[x, \lambda_1, \ldots, \lambda_n] \Big/ \prod_{i=1}^n (x - \lambda_i)\ . \tag{42}$$

The H-fixed locus $(\mathbb{P}^{n-1})^H$ consists of n points $\{p_i\}_{i=1}$, where p_i is the point $[z_1 : \cdots : z_n] \in \mathbb{P}^{n-1}$ with $z_i = 1$ and $z_j = 0$ for $i \neq j$. Since the tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$ restricts to $\pi_i^* \mathcal{O}_{\mathbb{P}^\infty}(-1)$ on $(p_i)_T = (\mathbb{P}^\infty)^n$, one has

$$\iota_i^* x = \lambda_i. \tag{43}$$

The push-forward

$$\int_{\mathbb{P}^{n-1}}^H : H^*_H(\mathbb{P}^{n-1}) \to H^*_H(\mathrm{pt}) \cong \mathbb{C}[\lambda_1, \ldots, \lambda_n] \tag{44}$$

along the natural map $(\mathbb{P}^{n-1})_H \to (\mathrm{pt})_H \cong BH$ is called the *equivariant integration*. The localization theorem [2] shows

$$\begin{aligned} \int_{\mathbb{P}^{n-1}}^H P(x) &= \sum_{i=1}^n \frac{\iota_i^* P(x)}{\mathrm{Eul}^H(N_{p_i/\mathbb{P}^{n-1}})} \\ &= \sum_{i=1}^n \frac{P(\lambda_i)}{\prod_{j \neq i}(\lambda_i - \lambda_j)} \\ &= \mathrm{Res} \frac{P(x)dx}{\prod_{i=1}^n (x - \lambda_i)} \end{aligned} \tag{45}$$

for any $P(x) \in H^*_H(\mathbb{P}^{n-1})$.

3.7

The quasimap space $\mathbf{Q}(\mathbb{P}^{n-1}; d)$ has a natural action of $H \times \mathbb{G}_m$ given by

$$H \times \mathbb{G}_m \ni (\alpha_1, \ldots, \alpha_n, \beta) \colon \; (u_i(z_1, z_2))_{i=1}^n \mapsto (\alpha_i u_i(z_1, \beta z_2))_{i=1}^n \,. \tag{46}$$

The equivariant cohomology of $\mathbf{Q}(\mathbb{P}^{n-1}; d)$ with respect to this torus action is given by

$$H^*_{H \times \mathbb{G}_m}(\mathbf{Q}(\mathbb{P}^{n-1}; d); \mathbb{C}) \cong \mathbb{C}[x, \lambda_1, \ldots, \lambda_n, \mathsf{z}] \Bigg/ \left(\prod_{i=1}^n \prod_{j=0}^d (x - \lambda_i - j\mathsf{z}) \right) . \tag{47}$$

The $H \times \mathbb{G}_m$-equivariant integration

$$\langle - \rangle^{H \times \mathbb{G}_m}_{\mathbb{P}^{n-1}, d} \, : \, H^*_{H \times \mathbb{G}_m}(\mathbf{Q}(\mathbb{P}^{n-1}; d); \mathbb{C}) \to H^*(B(H \times \mathbb{G}_m); \mathbb{C}) \tag{48}$$

is given by

$$\langle P(x) \rangle^{H \times \mathbb{G}_m}_{\mathbb{P}^{n-1}, d} = \operatorname{Res} \frac{P(x)dx}{\prod_{i=1}^n \prod_{j=0}^d (x - \lambda_i - jz)} \tag{49}$$

$$= \sum_{i=1}^n \sum_{j=0}^d \frac{P(\lambda_i + j\mathsf{z})}{\prod_{(k,l) \neq (i,j)} ((\lambda_i + j\mathsf{z}) - (\lambda_k + l\mathsf{z}))}. \tag{50}$$

The $H \times \mathbb{G}_m$-equivariant correlator is given by

$$\langle P(x) \rangle^{H \times \mathbb{G}_m}_{\mathbb{P}^{n-1}} := \sum_{d=0}^{\infty} q^d \, \langle P(x) \rangle^{H \times \mathbb{G}_m}_{\mathbb{P}^{n-1}, d} \, . \tag{51}$$

The H-equivariant correlator $\langle P(x) \rangle^H_{\mathbb{P}^{n-1}}$ and the $\mathbb{G}_m$-equivariant correlator $\langle P(x) \rangle^{\mathbb{G}_m}_{\mathbb{P}^{n-1}}$ are obtained by setting $\mathsf{z} = 0$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) = 0$ respectively.

3.8

The fixed point of the $\mathbb{G}_m$-action on $\mathbf{Q}(\mathbb{P}^{n-1}; d)$ is the disjoint union

$$\mathbf{Q}(\mathbb{P}^{n-1}; d)^{\mathbb{G}_m} = \coprod_{i=0}^d \mathbf{Q}(\mathbb{P}^{n-1}; d)_i^{\mathbb{G}_m} \tag{52}$$

of $d + 1$ connected components

$$\mathbf{Q}(\mathbb{P}^{n-1}; d)_i^{\mathbb{G}_m} := \left\{ [a_1 z_1^i z_2^{d-i}, \ldots, a_n z_1^i z_2^{d-i}] \in \mathbf{Q}(\mathbb{P}^{n-1}; d) \;\middle|\; [a_1, \ldots, a_n] \in \mathbb{P}^{n-1} \right\}. \tag{53}$$

Each of these connected components is isomorphic to $\mathbb{P}^{n-1}$, and the base locus is $i0 + (d-i)\infty$. The connected component $\mathbf{Q}(\mathbb{P}^{n-1}; d)_0^{\mathbb{G}_m}$ will be denoted by $\mathbf{Q}_\bullet(\mathbb{P}^{n-1}; d)$. There is a natural map $\mathrm{ev}\colon \mathbf{Q}_\bullet(\mathbb{P}^{n-1}; d) \to \mathbb{P}^{n-1}$ called the *evaluation map*, and one has

$$\mathbf{Q}(\mathbb{P}^{n-1}; d)^{\mathbb{G}_m} \cong \coprod_{d_1+d_2=d} \mathbf{Q}_\bullet(\mathbb{P}^{n-1}; d_1) \times_{\mathbb{P}^{n-1}} \mathbf{Q}_\bullet(\mathbb{P}^{n-1}; d_2). \tag{54}$$

The normal bundle of $\mathbf{Q}_\bullet(\mathbb{P}^{n-1}; d)$ in $\mathbf{Q}(\mathbb{P}^{n-1}; d)$ is given by $\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus nd}$, whose equivariant Euler class is given by

$$\mathrm{Eul}^{H\times\mathbb{G}_m}\left(N_{\mathbf{Q}_\bullet(\mathbb{P}^{n-1};d)/\mathbf{Q}(\mathbb{P}^{n-1};d)}\right) = \prod_{i=1}^{n}\prod_{l=1}^{d}(x - \lambda_i + l\mathsf{z}). \tag{55}$$

The *equivariant I-function* is defined by

$$I_{\mathbb{P}^{n-1}}^H(t; \mathsf{z}) := e^{tx/z}\sum_{d=0}^{\infty} e^{dt} I_d^H \tag{56}$$

where

$$I_d^H(\mathsf{z}) := \mathrm{ev}_*\left(\frac{1}{\mathrm{Eul}^{H\times\mathbb{G}_m}\left(N_{\mathbf{Q}_\bullet(\mathbb{P}^{n-1};d)/\mathbf{Q}(\mathbb{P}^{n-1};d)}\right)}\right) \tag{57}$$

$$= \frac{1}{\prod_{i=1}^n \prod_{l=1}^d (x - \lambda_i + l\mathsf{z})}. \tag{58}$$

The non-equivariant I-function is defined similarly, and given by setting $\boldsymbol{\lambda} = 0$ in (56);

$$I_{\mathbb{P}^{n-1}}(t; \mathsf{z}) := e^{tx/z}\sum_{d=0}^{\infty}\frac{e^{dt}}{\prod_{l=1}^d (x + l\mathsf{z})^n}. \tag{59}$$

3.9

Let $\left(\mathbb{C}\left[\!\left[e^t\right]\!\right]\right)[t]$ be the polynomial ring in t with the ring $\mathbb{C}\left[\!\left[e^t\right]\!\right]$ of formal power series in e^t as a coefficient. The equivariant I-function in (56) is an element of $H_H^*\left(\mathbb{P}^{n-1}; \mathbb{C}\right) \otimes_{\mathbb{C}} \left(\mathbb{C}\left[\!\left[e^t\right]\!\right]\right)[t]$, and the variable t is related to the variable q appearing in the correlator by

$$q = e^t. \tag{60}$$

The equivariant I-function can also be considered as a $H_H^*\left(\mathbb{P}^{n-1}; \mathbb{C}\right)$-valued analytic function, which is multi-valued as a function of q and single-valued as a function of $t = \log q$.

3.10

There is a $\mathbb{G}_m$-equivariant evaluation map $\mathrm{ev}_0 \colon \mathbf{Q}(\mathbb{P}^{n-1}; d) \to [\mathbb{C}^n/\mathbb{G}_m]$ at the point $0 \in \mathbb{P}^1$. By abuse of notation, we also let x denote the $\mathbb{G}_m$-equivariant Euler class of the line bundle $\mathrm{ev}_0^*(\mathcal{O}_{[\mathbb{C}^n/\mathbb{G}_m]}(1))$. Here $\mathcal{O}_{[\mathbb{C}^n/\mathbb{G}_m]}(1)$ is the line bundle $[(\mathbb{C}^n \times \mathbb{C})/\mathbb{G}_m]$ on the quotient stack with weights $((1, \ldots, 1), 1)$.

Let $\iota_i \colon \mathbf{Q}(\mathbb{P}^{n-1}; d)_i^{\mathbb{G}_m} \to \mathbf{Q}(\mathbb{P}^{n-1}; d)$ be the inclusion of the ith connected component (53). Since $\iota_i^*(x) = x + i\mathsf{z}$ (under the identification $\mathbf{Q}(\mathbb{P}^{n-1}; d)_i^{\mathbb{G}_m} = \mathbb{P}^{n-1}$) and

$$\frac{1}{\mathrm{Eul}^{\mathbb{G}_m}\left(N_{\mathbf{Q}(\mathbb{P}^{n-1};d)_i^{\mathbb{G}_m} / \mathbf{Q}(\mathbb{P}^{n-1};d)}\right)} = I_i(\mathsf{z}) \cup I_{d-i}(-\mathsf{z}), \tag{61}$$

localization with respect to the $\mathbb{G}_m$-action shows that

$$\begin{aligned}
\sum_{d=0}^{\infty} e^{d\tau} \left\langle e^{(t-\tau)x/\mathsf{z}} \right\rangle_{\mathbb{P}^{n-1},d}^{\mathbb{G}_m} &= \sum_{d=0}^{\infty} e^{d\tau} \sum_{i=0}^{d} \int_{\mathbf{Q}(\mathbb{P}^{n-1};d)_i^{\mathbb{G}_m}} \frac{\iota_i^*\left(e^{(t-\tau)x/\mathsf{z}}\right)}{\mathrm{Eul}^{\mathbb{G}_m}\left(N_{\mathbf{Q}(\mathbb{P}^{n-1};d)_i^{\mathbb{G}_m} / \mathbf{Q}(\mathbb{P}^{n-1};d)}\right)} \\
&= \sum_{d=0}^{\infty} e^{d\tau} \sum_{i=0}^{d} \int_{\mathbb{P}^{n-1}} e^{(t-\tau)(x+i\mathsf{z})/\mathsf{z}} \cup I_i(\mathsf{z}) \cup I_{d-i}(-\mathsf{z}) \\
&= \sum_{d=0}^{\infty} \sum_{i=0}^{d} \int_{\mathbb{P}^{n-1}} e^{tx/\mathsf{z}} e^{ti} I_i(\mathsf{z}) \cup e^{-\tau x/\mathsf{z}} e^{(d-i)\tau} I_{d-i}(-\mathsf{z}) \\
&= \int_{\mathbb{P}^{n-1}} I_{\mathbb{P}^{n-1}}(t; \mathsf{z}) \cup I_{\mathbb{P}^{n-1}}(\tau; -\mathsf{z}).
\end{aligned} \tag{62}$$

The factorization of the $H \times \mathbb{G}_m$-equivariant correlator is proved similarly as

$$\begin{aligned}
&\sum_{d=0}^{\infty} e^{d\tau} \left\langle e^{(t-\tau)x/\mathsf{z}} \right\rangle_{\mathbb{P}^{n-1},d}^{H\times\mathbb{G}_m} \\
&\quad= \sum_{d=0}^{\infty} \mathrm{Res} \frac{e^{d\tau} e^{(t-\tau)x/\mathsf{z}} dx}{\prod_{i=1}^{n} \prod_{l=0}^{d} (x - \lambda_i - l\mathsf{z})} \\
&\quad= \sum_{d=0}^{\infty} \sum_{m=0}^{d} \sum_{j=1}^{n} \mathrm{Res}_{x=\lambda_j+m\mathsf{z}} \frac{e^{d\tau} e^{(t-\tau)x/\mathsf{z}} dx}{\prod_{i=1}^{n} \prod_{l=0}^{d} (x - \lambda_i - l\mathsf{z})}
\end{aligned}$$

$$= \sum_{d=0}^{\infty}\sum_{m=0}^{d}\sum_{j=1}^{n} \mathrm{Res}_{x=\lambda_j} \frac{e^{d\tau}e^{(t-\tau)x/\mathsf{z}}e^{(t-\tau)m}dx}{\prod_{i=1}^{n}\prod_{l=0}^{d}(x-\lambda_i-(l-m)\mathsf{z})}$$

$$= \sum_{d=0}^{\infty}\sum_{m=0}^{d}\sum_{j=1}^{n} \mathrm{Res}_{x=\lambda_j} \frac{e^{tx/\mathsf{z}}e^{mt}}{\prod_{i=1}^{n}\prod_{l=1}^{m}(x-\lambda_i+l\mathsf{z})} \frac{e^{-\tau x/\mathsf{z}}e^{(d-m)\tau}}{\prod_{i=1}^{n}\prod_{l=1}^{d-m}(x-\lambda_i-l\mathsf{z})} \frac{dx}{\prod_{i=1}^{n}(x-\lambda_i)}$$

$$= \sum_{d=0}^{\infty}\sum_{d'=0}^{\infty}\sum_{j=1}^{n} \mathrm{Res}_{x=\lambda_j} \frac{e^{tx/\mathsf{z}}e^{dt}}{\prod_{i=1}^{n}\prod_{l=1}^{d}(x-\lambda_i+l\mathsf{z})} \frac{e^{-\tau x/\mathsf{z}}e^{d'\tau}}{\prod_{i=1}^{n}\prod_{l=1}^{d'}(x-\lambda_i-l\mathsf{z})} \frac{dx}{\prod_{i=1}^{n}(x-\lambda_i)}$$

$$= \int_{\mathbb{P}^{n-1}}^{H} I^{H}_{\mathbb{P}^{n-1}}(t;\mathsf{z}) \cup I^{H}_{\mathbb{P}^{n-1}}(\tau;-\mathsf{z}).$$

This can also be regarded as a purely combinatorial proof.

3.11

Let $\mathrm{ev}\colon \overline{\mathcal{M}}_{0,1}(\mathbb{P}^{n-1};d) \to \mathbb{P}^{n-1}$ be the evaluation map from the moduli space of stable maps of genus 0 and degree d with 1 marked point, and ψ be the first Chern class of the line bundle over $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^{n-1};d)$ whose fiber at a stable map $\varphi\colon (C,x) \to \mathbb{P}^{n-1}$ is the cotangent line $T^*_x C$ at the marked point. The *equivariant* J*-function* [31] is a $H^*(\mathbb{P}^{n-1};\mathbb{C})$-valued hypergeometric series given by

$$J^{H}_{\mathbb{P}^{n-1}}(t;\mathsf{z}) := e^{tx/\mathsf{z}} \sum_{d=0}^{\infty} e^{dt} J_d \tag{63}$$

where

$$J_d := \mathrm{ev}_*\left(\frac{1}{\mathsf{z}(\mathsf{z}-\psi)}\right). \tag{64}$$

3.12

The *graph space* is defined by $G(\mathbb{P}^{n-1};d) := \overline{\mathcal{M}}_{0,0}(\mathbb{P}^{n-1}\times\mathbb{P}^1;(d,1))$. The source of any map $\varphi\colon C \to \mathbb{P}^{n-1}\times\mathbb{P}^1$ in $G(\mathbb{P}^{n-1};d)$ has a distinguished irreducible component C_1 which maps isomorphically to $\mathbb{P}^1$. Let $G(\mathbb{P}^{n-1};d)_0$ be the open subspace of $G(\mathbb{P}^{n-1};d)$ consisting of stable maps without irreducible components mapping constantly to $0 \in \mathbb{P}^1$. There is a map $\mathrm{ev}_0\colon G(\mathbb{P}^{n-1};d)_0 \to \mathbb{P}^{n-1}$ sending $\varphi\colon C \to \mathbb{P}^{n-1}\times\mathbb{P}^1$ to $\mathrm{pr}_1\circ\varphi\left(\left(\mathrm{pr}_2\circ\varphi\right)^{-1}(0)\right)$. The fixed locus of the natural $\mathbb{G}_m$-action on $G(\mathbb{P}^{n-1};d)_0$ can be identified with $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^{n-1};d)$. Since the natural morphism $G(\mathbb{P}^{n-1};d)_0 \to \mathbf{Q}(\mathbb{P}^{n-1};d)_0$ is a $\mathbb{G}_m$-equivariant birational morphism which commutes with the evaluation maps, the push-forwards J_d and I_d of 1 by $\mathrm{ev}_0\colon G(\mathbb{P}^{n-1};d)_0 \to \mathbb{P}^{n-1}$ and $\mathrm{ev}\colon \mathbf{Q}(\mathbb{P}^{n-1};d)_0 \to \mathbb{P}^{n-1}$ are equal, and hence $I_{\mathbb{P}^{n-1}}(t;\mathsf{z}) = J_{\mathbb{P}^{n-1}}(t;\mathsf{z})$.

3.13

The effective potential (10) is given by

$$W_{\mathrm{eff}}(x;t) = tx - \sum_{i=1}^{n} (x-\lambda_i)\left(\log(x-\lambda_i) - 1\right). \tag{65}$$

One can easily see

$$\frac{\partial W_{\mathrm{eff}}}{\partial x} = t - \sum_{i=1}^{n} \log(x-\lambda_i), \tag{66}$$

$$e^{\partial_x W_{\mathrm{eff}}} = q \prod_{i=1}^{n} (x-\lambda_i)^{-1}, \tag{67}$$

so that

$$\langle P(x)\rangle^{H}_{\mathbb{P}^{n-1}} = \operatorname{Res} \frac{P(x)dx}{\prod_{i=1}^{n}(x-\lambda_i)(1-e^{\partial_x W_{\mathrm{eff}}})}. \tag{68}$$

Note that the equation

$$e^{\partial_x W_{\mathrm{eff}}} = 1 \tag{69}$$

gives the relation

$$\prod_{i=1}^{n}(x-\lambda_i) = q \tag{70}$$

in the equivariant quantum cohomology of $\mathbb{P}^{n-1}$.

4 Projective Complete Intersections

4.1

Let $f_1(w_1,\dots,w_n),\dots,f_r(w_1,\dots,w_n) \in \mathbb{C}[w_1,\dots,w_n]$ be homogeneous polynomials of degrees $l_1,\dots,l_r$ satisfying the Calabi–Yau condition

$$l_1 + \cdots + l_r = n. \tag{71}$$

Assume that $f_1,\dots,f_r$ are sufficiently general so that

$$Y := \left\{[w_1, \dots, w_n] \in \mathbb{P}^{n-1} \;\middle|\; f_1(w_1, \dots, w_n) = \cdots = f_r(w_1, \dots, w_n) = 0\right\} \tag{72}$$

is a smooth complete intersection of dimension $n - r - 1$, whose Poincaré dual is

$$v := \prod_{i=1}^{r} (l_i x). \tag{73}$$

Define the quasimap space $\mathbf{Q}(Y; d)$ as the subset of $\mathbf{Q}(\mathbb{P}^{n-1}; d)$ consisting of $[\varphi_1(z_1, z_2), \dots, \varphi_n(z_1, z_2)]$ satisfying

$$f_i(\varphi_1(z_1, z_2), \dots, \varphi_n(z_1, z_2)) = 0 \in \mathbb{C}[z_1, z_2] \text{ for any } i \in \{1, \dots, r\}. \tag{74}$$

Since $f_i(\varphi_1(z_1, z_2), \dots, \varphi_n(z_1, z_2)) \in \mathbb{C}[z_1, z_2]$ is a homogeneous polynomial of degree dl_i in z_1 and z_2, it contains $dl_i + 1$ terms, each of which is a homogeneous polynomial of degree l_i in $(a_{kl})_{k,l}$. With this in mind, the *Morrison–Plesser class* is defined by

$$\Phi(Y; d) := \prod_{i=1}^{r} (l_i x)^{l_i d} \in H^*(\mathbf{Q}(\mathbb{P}^{n-1}; d); \mathbb{Z}), \tag{75}$$

so that $\Phi(Y; d) \cup v$ is the Poincaré dual of $[\mathbf{Q}(Y; d)]^{\text{virt}} \in H_*(\mathbf{Q}(\mathbb{P}^{n-1}; d); \mathbb{Z})$. If we set

$$\langle P(x) \rangle_{Y,d} := \int_{\mathbf{Q}(\mathbb{P}^{n-1}; d)} P(x) \cup \Phi(Y; d) \cup v \tag{76}$$

and

$$\langle P(x) \rangle_Y := \sum_{d=0}^{\infty} q^d \, \langle P(x) \rangle_{Y,d} \tag{77}$$

for $P(x) \in \mathbb{C}[x]$, then we have

$$\left\langle x^{n-r-1} \right\rangle_Y = \sum_{d=0}^{\infty} q^d \operatorname{Res} \frac{x^{n-r-1} \Phi(Y, d) v dx}{x^{n(d+1)}} \tag{78}$$

$$= \sum_{d=0}^{\infty} q^d \operatorname{Res} \frac{x^{n-r-1} \prod_{i=1}^{r} (l_i x)^{l_i d + 1} dx}{x^{n(d+1)}} \tag{79}$$

$$= \sum_{d=0}^{\infty} q^d \prod_{i=1}^{r} (l_i)^{l_i d + 1} \tag{80}$$

$$= \frac{\prod_{i=1}^{r} l_i}{1 - q \prod_{i=1}^{r} (l_i)^{l_i}}. \tag{81}$$

4.2

The gauged linear sigma model for Y is obtained from the gauged linear sigma model for $\mathbb{P}^{n-1}$ by adding r fields of $G = \mathbb{G}_m$-charge $-l_1, \dots, -l_r$ and R-charge 2. One has $Z_d^{\text{vec}}(x) = 1$ and $Z_d^{\text{mat}}(x) = \left(x^{-d-1}\right)^n \cdot \prod_{i=1}^{r} (-l_i x)^{l_i d+1}$ in this case, so that (5) gives

$$\sum_{d=0}^{\infty} e^{t'd} \operatorname{Res} \left(x^{-d-1}\right)^n \prod_{i=1}^{r} (-l_i x)^{l_i d+1} x^{n-r-1} = \sum_{d=0}^{\infty} e^{t'd} \prod_{i=1}^{r} (-l_i)^{l_i d+1} \tag{82}$$

$$= \sum_{d=0}^{\infty} (-1)^{\sum_{i=1}^{r} l_i d} e^{t'd} \prod_{i=1}^{r} (l_i)^{l_i d+1} \tag{83}$$

$$= \sum_{d=0}^{\infty} \left((-1)^n e^{t'}\right)^d \prod_{i=1}^{r} (l_i)^{l_i d+1}, \tag{84}$$

which coincides with (78) under the identification

$$q = (-1)^n e^{t'}. \tag{85}$$

4.3

The mirror $\check{Y}$ of Y is a compactification of a complete intersection in $\mathbb{C}^n$ defined by

$$\check{f}_1 := 1 - (a_1 \check{y}_1 + \cdots + a_{l_1} \check{y}_{l_1}), \tag{86}$$

$$\check{f}_2 := 1 - (a_{l_1+1} \check{y}_{l_1+1} + \cdots + a_{l_1+l_2} \check{y}_{l_1+l_2}), \tag{87}$$

$$\vdots \tag{88}$$

$$\check{f}_r := 1 - (a_{l_1+\cdots+l_{i-1}+1} \check{y}_{l_1+\cdots+l_{i-1}+1} + \cdots + a_n \check{y}_n), \tag{89}$$

$$\check{f}_0 := \check{y}_1 \cdots \check{y}_n - 1. \tag{90}$$

The complex structure of $\check{Y}$ depends not on the individual a_i but only on $\alpha = a_1 \cdots a_n$. The *Yukawa (n-2)-point function* is defined by

$$\mathcal{Y}(\alpha) := \frac{(-1)^{(n-1)(n-2)/2}}{\left(2\pi\sqrt{-1}\right)^{n-1}} \int_{\check{Y}} \Omega \wedge \left(\alpha \frac{\partial}{\partial \alpha}\right)^{n-2} \Omega, \tag{91}$$

where

$$\Omega := \operatorname{Res}\left(\frac{d\check{y}_1 \wedge \cdots \wedge d\check{y}_n}{\check{f}_0 \check{f}_1 \cdots \check{f}_r}\right) \tag{92}$$

is the holomorphic volume form on $\check{Y}$. The computation in [5, Proposition 5.1.2] shows

$$\mathcal{Y}(\alpha) = \frac{\prod_{i=1}^r l_i}{1 - \alpha \prod_{i=1}^r (l_i)^{l_i}}, \tag{93}$$

which coincides with (81) under the identification $q = \alpha$ of variables;

$$\mathcal{Y}(\alpha) = \left\langle x^{n-r-1} \right\rangle_Y \Big|_{q=\alpha}. \tag{94}$$

A generalization of (94) to toric complete intersections is *toric residue mirror symmetry* conjectured in [3, 4] and proved in [11, 46, 63, 64].

5 Concave Bundles on Projective Spaces

5.1

Let $l_1, l_2, \cdots, l_r$ be positive integers and

$$Y := \mathcal{S}pec_{\mathbb{P}^{n-1}}\left(\mathcal{S}ym^* \, \mathcal{E}^\vee\right) \tag{95}$$

be the total space of the vector bundle associated with the locally free sheaf

$$\mathcal{E} := \mathcal{O}_{\mathbb{P}^{n-1}}(-l_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-l_r) \tag{96}$$

on $\mathbb{P}^{n-1}$. Since any holomorphic map from $\mathbb{P}^1$ to Y of positive degree d factors through the zero-section $\mathbb{P}^{n-1} \to Y$, we define the quasimap space to Y as

$$\mathbf{Q}(Y; d) := \mathbf{Q}(\mathbb{P}^{n-1}; d). \tag{97}$$

5.2

To equip $\mathbf{Q}(Y; d)$ with a natural obstruction theory, we identify $\mathbf{Q}(Y; d)$ with an open substack of the mapping stack $\mathrm{Map}(\mathbb{P}^1, \mathcal{Y})$ to the quotient stack

$$\mathcal{Y} := \left[\left(\mathbb{A}^n \times \mathbb{A}^r\right)/\mathbb{G}_m\right] \tag{98}$$

of $\mathbb{A}^n \times \mathbb{A}^r$ by the $\mathbb{G}_m$-action given by

$$\mathbb{G}_m \ni \alpha \colon (x_1, \ldots, x_n, z_1, \ldots, z_r) \mapsto (\alpha x_1, \ldots, \alpha x_n, \alpha^{-l_1} z_1, \ldots, \alpha^{-l_r} z_r). \tag{99}$$

A morphism $\mathbb{P}^1 \to \mathcal{Y}$ consists of a line bundle $\mathcal{L}$ on $\mathbb{P}^1$ and sections

$$\left((\varphi_i)_{i=1}^n, (\psi_j)_{j=1}^r\right) \in \left((H^0(\mathcal{L}))^n \times \prod_{j=1}^r H^0\left(\mathcal{L}^{\otimes(-l_j)}\right)\right), \tag{100}$$

whose degree is defined as the degree of $\mathcal{L}$.

5.3

Recall from [8, Definition 4.4] that an *obstruction theory* for a Deligne–Mumford stack $\mathcal{X}$ is a morphism $\phi\colon E \to L_{\mathcal{X}}$ from an object E of the derived category of quasicoherent sheaves on $\mathcal{X}$ satisfying

(1) $h^i(E) \cong 0$ for $i > 0$, and
(2) $h^i(E)$ is coherent for $i = 0, -1$

to the cotangent complex $L_{\mathcal{X}}$ such that

(1) $h^0(\phi)$ is an isomorphism, and
(2) $h^{-1}(\phi)$ is an epimorphism.

It is said to be *perfect* if E is locally isomorphic to a two-term complex of locally free sheaves of finite rank [8, Definition 5.1].

5.4

A perfect obstruction theory produces the *virtual fundamental cycle* $[\mathcal{X}]^{\mathrm{virt}}$ in the Chow group $A_{\mathrm{virt.dim}\mathcal{X}}(\mathcal{X})$ of degree

$$\mathrm{virt.dim}\mathcal{X} = \mathrm{rank}\, h^0(E) - \mathrm{rank}\, h^{-1}(E). \tag{101}$$

When $\mathcal{X}$ is a smooth scheme, then the cotangent complex $L_{\mathcal{X}}$ is isomorphic to the sheaf $\Omega_{\mathcal{X}}$ of Kähler differentials, and the virtual fundamental cycle is the Euler class of $h^{-1}(E)$.

5.5

The derived mapping stack $\mathbb{R}\mathrm{Map}(\mathcal{S}, \mathcal{T})$ from a proper scheme $\mathcal{S}$ to a derived Artin stack $\mathcal{T}$ is a derived Artin stack (see e.g. [66, Corollary 3.3]) whose tangent complex is given by

$$T_{\mathbb{R}\mathrm{Map}(\mathcal{S},\mathcal{T})} \cong \mathbb{R}\pi_* \left(\mathbb{L}\mathrm{ev}^* T_{\mathcal{T}}\right), \tag{102}$$

where

$$\pi\colon\ \mathbb{R}\mathrm{Map}(\mathcal{S}, \mathcal{T}) \times \mathcal{S} \to \mathbb{R}\mathrm{Map}(\mathcal{S}, \mathcal{T}) \tag{103}$$

is the first projection and

$$\mathrm{ev}\colon\ \mathbb{R}\mathrm{Map}(\mathcal{S}, \mathcal{T}) \times \mathcal{S} \to \mathcal{T} \tag{104}$$

is the evaluation morphism. It is a derived thickening of the mapping stack $\mathrm{Map}(\mathcal{S}, \mathcal{T})$, and the pull-back

$$j^*\colon\ j^* L_{\mathbb{R}\mathrm{Map}(\mathcal{S},\mathcal{T})} \to L_{\mathrm{Map}(\mathcal{S},\mathcal{T})} \tag{105}$$

by the structure morphism

$$j\colon\ \mathrm{Map}\,(\mathcal{S}, \mathcal{T}) \to \mathbb{R}\mathrm{Map}\,(\mathcal{S}, \mathcal{T}) \tag{106}$$

gives an obstruction theory on $\mathrm{Map}\,(\mathcal{S}, \mathcal{T})$.

5.6

The restriction of the natural obstruction theory for $\mathrm{Map}\left(\mathbb{P}^1, \mathcal{Y}\right)$ to the open substack $\mathbf{Q}(Y; d)$ gives an obstruction theory for $\mathbf{Q}(Y; d)$ with $E = j^* L_{\mathbb{R}\mathrm{Map}(\mathbb{P}^1,\mathcal{Y})}\big|_{\mathbf{Q}(Y;d)}$ and $\phi = j^*|_{\mathbf{Q}(Y;d)}$.

5.7

Since $\operatorname{Pic}(\mathbb{A}^n \times \mathbb{A}^r)$ is trivial, the Picard group $\operatorname{Pic}\mathcal{Y} \cong \operatorname{Pic}^{\mathbb{G}_m}(\mathbb{A}^n \times \mathbb{A}^r)$ can be identified with the group of characters of $\mathbb{G}_m$, which is non-canonically isomorphic to $\mathbb{Z}$. We fix an isomorphism in such a way that $\bigoplus_{a=0}^{\infty} H^0(\mathcal{O}_{\mathcal{Y}}(a))$ is the coordinate ring of $\mathbb{A}^n$, where $\mathcal{O}_{\mathcal{Y}}(a)$ is the line bundle associated with $a \in \mathbb{Z} \cong \operatorname{Pic}\mathcal{Y}$. Since $\mathcal{Y}$ is the quotient stack of $\mathbb{A}^n \times \mathbb{A}^r$ by the action of $\mathbb{G}_m$, the tangent complex $T_{\mathcal{Y}}$ satisfies

$$\varpi^* T_{\mathcal{Y}} \cong \operatorname{Cone}\left(\mathcal{O}_{\mathbb{A}^n \times \mathbb{A}^r} \otimes \operatorname{Lie}(\mathbb{G}_m) \to T_{\mathbb{A}^n \times \mathbb{A}^r}\right) \tag{107}$$

where $\varpi \colon \mathbb{A}^n \times \mathbb{A}^r \to \mathcal{Y}$ is the quotient morphism. This in turn implies that

$$T_{\mathcal{Y}} \cong \operatorname{Cone}\left(\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}(1)^{\oplus n} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathcal{Y}}(-l_i)\right). \tag{108}$$

5.8

We write the restriction of the evaluation morphism $\operatorname{Map}(\mathbb{P}^1, \mathcal{Y}) \times \mathbb{P}^1 \to \mathcal{Y}$ to the open substack $\mathbf{Q}(Y; d) \subset \operatorname{Map}(\mathbb{P}^1, \mathcal{Y})$ as

$$\operatorname{ev} \colon \mathbf{Q}(Y; d) \times \mathbb{P}^1 \to \mathcal{Y} \tag{109}$$

again by abuse of notation. We have

$$\operatorname{ev}^* \mathcal{O}_{\mathcal{Y}}(1) \cong \mathcal{O}_{\mathbf{Q}(Y;d)}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d) \tag{110}$$

essentially by definition, where $\mathcal{O}_{\mathbf{Q}(Y;d)}(1)$ is the ample generator of the Picard group of $\mathbf{Q}(Y; d) \cong \mathbb{P}^{n(d+1)-1}$. The dual of the natural obstruction theory is given by

$$\phi^\vee \colon T_{\mathbf{Q}(Y;d)} \to E^\vee := \mathbb{R}\pi_* \operatorname{ev}^* T_{\mathcal{Y}}. \tag{111}$$

Note that (the inverse of) the isomorphism $h^0(\phi^\vee)$ from $T_{\mathbf{Q}(Y;d)}$ to

$$h^0(E^\vee) \cong R^0\pi_* \operatorname{Cone}\left(\mathcal{O}_{\mathbf{Q}(Y;d)\times\mathbb{P}^1} \to \left(\mathcal{O}_{\mathbf{Q}(Y;d)}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d)\right)^{\oplus n}\right) \tag{112}$$

$$\cong \operatorname{Cone}\left(\mathcal{O}_{\mathbf{Q}(Y;d)} \to \mathcal{O}_{\mathbf{Q}(Y;d)}(1)^{\oplus n(d+1)}\right) \tag{113}$$

gives the Euler sequence

$$0 \to \mathcal{O}_{\mathbf{Q}(Y;d)} \to \mathcal{O}_{\mathbf{Q}(Y;d)}(1)^{\oplus n(d+1)} \to T_{\mathcal{Q}(Y;d)} \to 0 \tag{114}$$

on $\mathbf{Q}(Y; d) \cong \mathbb{P}^{n(d+1)-1}$. One has

$$h^1(E^\vee) \cong R^1\pi_* \left(\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{Q}(Y;d)}(-l_i) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-l_i d) \right) \tag{115}$$

$$\cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{Q}(Y;d)}(-l_i) \otimes H^1\left(\mathcal{O}_{\mathbb{P}^1}(-l_i d)\right) \tag{116}$$

$$\cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{Q}(Y;d)}(-l_i)^{\oplus(l_i d-1)} \tag{117}$$

and $h^i(E^\vee) \cong 0$ for $i \neq 0, 1$, so that this obstruction theory is perfect. By [8, Proposition 5.6], the resulting virtual fundamental class is given by

$$[\mathbf{Q}(Y; d)]^{\mathrm{virt}} = [\mathbf{Q}(Y; d)] \cap \mathrm{Eul}\left(h^1(E^\vee)\right) = [\mathbf{Q}(Y; d)] \cap \prod_{i=1}^{r} (-l_i x)^{l_i d-1}. \tag{118}$$

5.9

When the degree is zero, the quasimap space $\mathbf{Q}(Y; 0)$ is naturally isomorphic to Y equipped with the trivial perfect obstruction theory, so that

$$[\mathbf{Q}(Y; 0)]^{\mathrm{virt}} = [Y]. \tag{119}$$

5.10

For any $P(x) \in \mathbb{C}[x]$, we define

$$\langle P(x) \rangle_{Y,d} := \int_{[\mathbf{Q}(Y;d)]^{\mathrm{virt}}} P(x) \tag{120}$$

and

$$\langle P(x) \rangle_Y := \sum_{d=0}^{\infty} q^d \langle P(x) \rangle_{Y,d}. \tag{121}$$

It follows that

$$\langle P(x) \rangle_Y = \sum_{d=0}^{\infty} q^k \int_{\mathbb{P}^{n(d+1)-1}} P(x) \prod_{i=1}^{r} (-l_i x)^{l_i d - 1} \tag{122}$$

$$= \sum_{d=0}^{\infty} q^k \operatorname{Res} \frac{P(x) \prod_{i=1}^{r} (-l_i x)^{l_i d}}{x^{n(d+1)} \prod_{i=1}^{r} (-l_i x)}. \tag{123}$$

5.11

The gauged linear sigma model for Y is obtained from the gauged linear sigma model for $\mathbb{P}^{n-1}$ by adding r fields of $G = \mathbb{G}_m$-charge $-l_1, \ldots, -l_r$ and R-charge 0. One has $Z_d^{\mathrm{vec}}(x) = 1$ and $Z_d^{\mathrm{mat}}(x) = \left(x^{-d-1}\right)^n \cdot \prod_{i=1}^{r} (-l_i x)^{l_i d - 1}$ in this case, so that (5) gives

$$\langle P(x) \rangle_{\mathrm{GLSM}} = \sum_{d=0}^{\infty} e^{t' d} \operatorname{Res} \left(x^{-d-1}\right)^n \prod_{i=1}^{r} (-l_i x)^{l_i d - 1} P(x), \tag{124}$$

which coincides with (123) under the identification

$$q = e^{t'}. \tag{125}$$

5.12

If $(l_1, \ldots, l_r)$ satisfies the Calabi–Yau condition

$$l_1 + \cdots + l_r = n, \tag{126}$$

then (123) gives

$$\left\langle x^k \right\rangle_Y = \begin{cases} \dfrac{1}{\left(\prod_{i=1}^{r} (-l_i)\right) \left(1 - q \prod_{i=1}^{r} (-l_i)^{l_i}\right)} & k = n + r, \\ 0 & \text{otherwise}, \end{cases} \tag{127}$$

which matches the Yukawa coupling of the mirror (see e.g. [47, Example 6.15]).

6 Classical Mirror Symmetry for Toric Hypersurfaces

6.1

Let $\boldsymbol{N} := \mathbb{Z}^n$ be a free abelian group of rank n and $\boldsymbol{M} := \check{\boldsymbol{N}} := \mathrm{Hom}(\boldsymbol{N}, \mathbb{Z})$ be the dual group. Let further $(\Delta, \check{\Delta})$ be a polar dual pair of reflexive polytopes in $\boldsymbol{M}$ and $\boldsymbol{N}$.

6.2

Recall that the *fan polytope* of a fan is defined as the convex hull of primitive generators of one-dimensional cones. Let $(\Sigma, \check{\Sigma})$ be a pair of smooth projective fans whose fan polytopes are $\check{\Delta}$ and Δ. The associated toric varieties will be denoted by $X := X_\Sigma$ and $\check{X} := X_{\check{\Sigma}}$.

6.3

The set of primitive generators of one-dimensional cones of the fan Σ will be denoted by

$$B := \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\} \subset \boldsymbol{N}. \tag{128}$$

Assume that B generates $\boldsymbol{N}$. One has the *fan sequence*

$$0 \to \boldsymbol{L} \to \mathbb{Z}^m \xrightarrow{\boldsymbol{b}} \boldsymbol{N} \to 0 \tag{129}$$

and the *divisor sequence*

$$0 \to \boldsymbol{M} \xrightarrow{\boldsymbol{b}^\vee} \mathbb{Z}^m \to \check{\boldsymbol{L}} \to 0, \tag{130}$$

where $\boldsymbol{b}$ sends the ith coordinate vector $e_i \in \mathbb{Z}^m$ to $\boldsymbol{b}_i$. Recall that

$$\check{\boldsymbol{L}} \cong \mathrm{Pic}(X) \cong H^2(X; \mathbb{Z}), \quad \mathrm{Eff}(X) \subset \boldsymbol{L} \subset \mathbb{Z}^m, \tag{131}$$

where $\mathrm{Eff}(X)$ denotes the semigroup of the effective curves (see [3, Sect. 3]). We write the group ring of $\boldsymbol{M}$ as $\mathbb{C}[\boldsymbol{M}]$ and define $\mathbb{T} := \boldsymbol{N}_{\mathbb{G}_m} := \mathrm{Spec}\, \mathbb{C}[\boldsymbol{M}]$. We also set $\check{\mathbb{T}} := \mathrm{Spec}\, \mathbb{C}[\boldsymbol{N}]$, $\check{\mathbb{L}} := \mathrm{Spec}\, \mathbb{C}[\boldsymbol{L}]$, and $\mathbb{L} := \mathrm{Spec}\, \mathbb{C}[\check{\boldsymbol{L}}]$. The fan sequence induces the exact sequences

$$1 \to \mathbb{L} \xrightarrow{\chi} (\mathbb{G}_m)^m \to \mathbb{T} \to 1 \tag{132}$$

and

$$1 \to \check{\mathbb{T}} \to (\mathbb{G}_m)^m \to \check{\mathbb{L}} \to 1 \tag{133}$$

of algebraic tori. We write the ith components of the map $\boldsymbol{\chi} \colon \mathbb{L} \to (\mathbb{G}_m)^m$ in (132) as χ_i, and the affine line $\mathbb{A}^1$ equipped with the action of $\mathbb{L}$ through χ_i as $\mathbb{A}_i$. Then one has

$$X \cong \left(\prod_{i=1}^m \mathbb{A}_i \right) /\!\!/_\theta \, \mathbb{L} \tag{134}$$

for a suitable choice of a character $\theta \in \check{\boldsymbol{L}} \cong \operatorname{Hom}(\mathbb{L}, \mathbb{G}_m)$. The right-hand side of (134) denotes the GIT quotient with respect to the linearization determined by θ.

6.4

We define a graded ring $S_\Delta := \bigoplus_{k=0}^\infty S_\Delta^k$ by

$$S_\Delta^k := \bigoplus_{\boldsymbol{m} \in \boldsymbol{M} \cap (k\Delta)} \mathbb{C} \cdot y_0^k \boldsymbol{y}^{\boldsymbol{m}}, \tag{135}$$

which is a subalgebra of the semigroup ring

$$\mathbb{C}[\mathbb{N} \times \boldsymbol{M}] = \mathbb{C}[y_0, \boldsymbol{y}^{\pm 1}] := \mathbb{C}[y_0, y_1^{\pm 1}, \dots, y_n^{\pm 1}] \tag{136}$$

of $\mathbb{N} \times \boldsymbol{M}$. It is the anti-canonical ring of X, so that one has $X \cong \operatorname{Proj} S_\Delta$ if and only if X is Fano. The ring S_Δ is Cohen–Macaulay with the dualizing module $I_\Delta := \bigoplus_{k=0}^\infty I_\Delta^k$ given by

$$I_\Delta^k := \bigoplus_{\boldsymbol{m} \in \boldsymbol{M} \cap \operatorname{Int}(k\Delta)} \mathbb{C} \cdot y_0^k \boldsymbol{y}^{\boldsymbol{m}}, \tag{137}$$

where $\operatorname{Int}(k\Delta)$ is the interior of $k\Delta$.

6.5

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in (\mathbb{G}_m)^m$ (this $(\mathbb{G}_m)^m$ can be naturally considered as the dual torus of the big torus of X_Σ), we define an element of the group ring $\mathbb{C}[\boldsymbol{N}]$ by

$$\check{W}_{\boldsymbol{\alpha}}(\check{\boldsymbol{y}}) := \sum_{i=1}^{m} \alpha_i \check{\boldsymbol{y}}^{\boldsymbol{b}_i} \in \mathbb{C}[\boldsymbol{N}]. \tag{138}$$

An element $\check{f} \in \mathbb{C}[\boldsymbol{N}]$ is said to be $\check{\Delta}$-*regular* if

$$\check{F} := (\check{F}_0, \check{F}_1, \dots, \check{F}_n) := \left(\check{y}_0 \check{f}, \check{y}_0 \check{y}_1 \partial_{\check{y}_1} \check{f}, \dots, \check{y}_0 \check{y}_n \partial_{\check{y}_n} \check{f}\right) \tag{139}$$

is a regular sequence in $S_{\check{\Delta}}$. We write

$$\left((\mathbb{G}_m)^m\right)^{\mathrm{reg}} := \left\{\boldsymbol{\alpha} \in (\mathbb{G}_m)^m \;\middle|\; \check{f}_{\boldsymbol{\alpha}} := 1 - \check{W}_{\boldsymbol{\alpha}}(\check{\boldsymbol{y}}) \text{ is } \check{\Delta}\text{-regular}\right\}. \tag{140}$$

6.6

Let $\tilde{\check{\varphi}} : \tilde{\check{\mathfrak{Y}}} \to ((\mathbb{G}_m)^m)^{\mathrm{reg}}$ be the second projection from

$$\tilde{\check{\mathfrak{Y}}} = \left\{(\check{\boldsymbol{y}}, \boldsymbol{\alpha}) \in \check{\mathbb{T}} \times \left((\mathbb{G}_m)^m\right)^{\mathrm{reg}} \mid \check{W}_{\boldsymbol{\alpha}}(\check{\boldsymbol{y}}) = 1\right\}. \tag{141}$$

Assume that X is Fano. Any fiber $\check{Y}_{\boldsymbol{\alpha}} := \tilde{\check{\varphi}}^{-1}(\boldsymbol{\alpha})$ is an uncompactified mirror of a general anti-canonical hypersurface $Y \subset X$. The closure of $\check{Y}_{\boldsymbol{\alpha}}$ in $\check{X}$ is a smooth anti-canonical Calabi–Yau hypersurface, which is the compact mirror of Y. The quotient of the family $\tilde{\check{\varphi}} \colon \tilde{\check{\mathfrak{Y}}} \to ((\mathbb{G}_m)^m)^{\mathrm{reg}}$ by the free $\check{\mathbb{T}}$-action

$$\check{\mathbb{T}} \ni \check{\boldsymbol{y}} \colon \left(\check{\boldsymbol{y}}', (\alpha_1, \dots, \alpha_m)\right) \mapsto \left(\check{\boldsymbol{y}}^{-1} \check{\boldsymbol{y}}', \left(\check{\boldsymbol{y}}^{\boldsymbol{b}_1} \alpha_1, \dots, \check{\boldsymbol{y}}^{\boldsymbol{b}_m} \alpha_m\right)\right) \tag{142}$$

will be denoted by $\check{\varphi} \colon \check{\mathfrak{Y}} \to \check{\mathbb{L}}^{\mathrm{reg}}$, where $\check{\mathfrak{Y}} := \tilde{\check{\mathfrak{Y}}}/\check{\mathbb{T}}$ and $\check{\mathbb{L}}^{\mathrm{reg}} := ((\mathbb{G}_m)^m)^{\mathrm{reg}}/\check{\mathbb{T}}$.

6.7

Choose an integral basis $\boldsymbol{p}_1, \dots, \boldsymbol{p}_r$ of $\check{\boldsymbol{L}} \cong \operatorname{Pic} X$ such that each $\boldsymbol{p}_i$ is nef. This gives the corresponding coordinate $\boldsymbol{q} = (q_1, \dots, q_r)$ on $\check{\mathbb{L}}$. Let $\check{U}' \subset \check{\mathbb{L}}^{\mathrm{reg}}$ be a neighborhood of $q_1 = \cdots = q_r = 0$, and $\check{U}$ be the universal cover of $\check{U}'$.

6.8

We write the image of the Poincaré residue as

$$H^{n-1}_{\mathrm{res}}(\check{Y}_\alpha) := \operatorname{Im}\left(\operatorname{Res}\colon H^0\left(\check{X}, \Omega^n_{\check{X}}\left(*\check{Y}_\alpha\right)\right) \to H^{n-1}\left(\check{Y}_\alpha\right)\right). \tag{143}$$

Let H_{B} be the pull-back to $\check{U}$ of the local system $\mathrm{gr}^W_{n-1} R^{n-1}\check{\varphi}_! \mathbb{C}_{\check{\mathfrak{Y}}}$ on $\check{U}'$, and $H^{\mathrm{res}}_{\mathrm{B}}$ be the sub-system with stalks $H^{n-1}_{\mathrm{res}}(\check{Y}_\alpha)$. Here gr^W_{n-1} is the weight $n-1$ piece of Deligne's mixed Hodge structure. The *residual B-model VHS* $(\mathcal{H}_{\mathrm{B}}, \nabla_{\mathrm{B}}, \mathscr{F}^\bullet_{\mathrm{B}}, Q_{\mathrm{B}})$ on $\check{U}$ consists of the locally free sheaf $\mathcal{H}_{\mathrm{B}} := H^{\mathrm{res}}_{\mathrm{B}} \otimes_{\mathbb{C}} \mathcal{O}_{\check{U}}$, the Gauss–Manin connection ∇_B, the Hodge filtration $\mathscr{F}^\bullet_B$, and the polarization $Q_B \colon \mathcal{H}_{\mathrm{B}} \otimes_{\mathcal{O}_{\check{U}}} \mathcal{H}_{\mathrm{B}} \to \mathcal{O}_{\check{U}}$ given by

$$Q_{\mathrm{B}}(\omega_1, \omega_2) := (-1)^{(n-1)(n-2)/2} \int_{\check{Y}_\alpha} \omega_1 \cup \omega_2. \tag{144}$$

6.9

On the A-model side, let

$$H^\bullet_{\mathrm{amb}}(Y;\mathbb{C}) := \operatorname{Im}(\iota^*\colon H^\bullet(X;\mathbb{C}) \to H^\bullet(Y;\mathbb{C})) \tag{145}$$

be the subspace of $H^\bullet(Y;\mathbb{C})$ coming from the cohomology classes of the ambient toric variety, and set

$$U := \left\{\boldsymbol{\tau} = \boldsymbol{\beta} + \sqrt{-1}\boldsymbol{\omega} \in H^2_{\mathrm{amb}}(Y;\mathbb{C}) \;\middle|\; \langle \boldsymbol{\omega}, \boldsymbol{d}\rangle \gg 0 \text{ for any non-zero } \boldsymbol{d} \in \operatorname{Eff}(Y)\right\}. \tag{146}$$

This open subset U is considered as a neighborhood of the large radius limit point. Let $(\tau_i)_{i=1}^r$ be the coordinate on $H^2_{\mathrm{amb}}(Y;\mathbb{C})$ dual to the basis $\{\boldsymbol{p}_i\}_{i=1}^r$ so that $\boldsymbol{\tau} = \sum_{i=1}^r \tau_i \boldsymbol{p}_i$.

6.10

The *ambient A-model VHS* $(\mathcal{H}_{\mathrm{A}}, \nabla_{\mathrm{A}}, \mathscr{F}^\bullet_{\mathrm{A}}, Q_{\mathrm{A}})$ consists ([43, Definition 6.2], cf. also [26, Sect. 8.5]) of the locally free sheaf $\mathcal{H}_{\mathrm{A}} = H^\bullet_{\mathrm{amb}}(Y;\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_U$, the connection

$$\nabla_{\mathrm{A}} = d + \sum_{i=1}^r (\boldsymbol{p}_i \circ_{\boldsymbol{\tau}})\, d\tau_i \colon \mathcal{H}_{\mathrm{A}} \to \mathcal{H}_{\mathrm{A}} \otimes \Omega^1_U, \tag{147}$$

the Hodge filtration

$$\mathscr{F}^p_{\mathrm{A}} := H^{\le 2(n-1-p)}_{\mathrm{amb}}(Y;\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_U, \tag{148}$$

and the pairing

$$Q_A\colon \mathcal{H}_\mathrm{A} \otimes_{\mathcal{O}_U} \mathcal{H}_\mathrm{A} \to \mathcal{O}_U, \qquad (\alpha, \beta) \mapsto (2\pi\sqrt{-1})^{n-1} \int_Y (-1)^{\deg \alpha/2} \alpha \cup \beta, \tag{149}$$

which is $(-1)^{n-1}$-symmetric and ∇_A-flat.

6.11

Let $\boldsymbol{u}_i \in H^2_\mathrm{amb}(Y;\mathbb{Z})$ be the first Chern class of the line bundle on Y corresponding to the one-dimensional cone $\mathbb{R}_{\geq 0} \cdot \boldsymbol{b}_i \in \Sigma$ and $\boldsymbol{v} = \boldsymbol{u}_1 + \cdots + \boldsymbol{u}_m$ be the restriction of the anti-canonical class of X. Denote $\boldsymbol{t} := \sum_{i=1}^r t_i \boldsymbol{p}_i$. *Givental's I-function* is defined as the series

$$I_Y(\boldsymbol{t};\mathsf{z}) = e^{\boldsymbol{t}/\mathsf{z}} \sum_{\boldsymbol{d} \in \mathrm{Eff}(X)} e^{\boldsymbol{d}\cdot\boldsymbol{t}} \frac{\prod_{k=-\infty}^{\langle \boldsymbol{d}, \boldsymbol{v}\rangle} (\boldsymbol{v} + k\mathsf{z}) \prod_{j=1}^m \prod_{k=-\infty}^0 (\boldsymbol{u}_j + k\mathsf{z})}{\prod_{k=-\infty}^0 (\boldsymbol{v} + k\mathsf{z}) \prod_{j=1}^m \prod_{k=-\infty}^{\langle \boldsymbol{d}, \boldsymbol{u}_j\rangle} (\boldsymbol{u}_j + k\mathsf{z})}, \tag{150}$$

which is a multi-valued map from $\check{U}'$ (or a single-valued map from $\check{U}$) to the classical cohomology group $H^\bullet_\mathrm{amb}(Y;\mathbb{C}[\mathsf{z}^{-1}])$. The *$J$-function* is defined by

$$J_Y(\boldsymbol{\tau};\mathsf{z}) = L_Y(\boldsymbol{\tau},\mathsf{z})^{-1}(1), \tag{151}$$

where $L_Y(\boldsymbol{\tau},\mathsf{z})$ is the fundamental solution of the quantum differential equation defined explicitly by using the Gromov–Witten invariants as in [43, Eq. (2.3)] with $\mathbf{c}$ set to 1. If we write

$$I_Y(\boldsymbol{t};\mathsf{z}) = F(\boldsymbol{t})\mathbf{1} + \frac{G(\boldsymbol{t})}{\mathsf{z}} + O(\mathsf{z}^{-2}), \tag{152}$$

then Givental's mirror theorem [32] states that

$$I_Y(\boldsymbol{t};\mathsf{z}) = F(\boldsymbol{t}) \cdot J_Y(\varsigma(\boldsymbol{t});\mathsf{z}), \tag{153}$$

where the *mirror map* $\varsigma : \check{U} \to H^2_\mathrm{amb}(Y;\mathbb{C})$ is defined by

$$\varsigma(\boldsymbol{t}) = \iota^* \left(\frac{G(\boldsymbol{t})}{F(\boldsymbol{t})} \right). \tag{154}$$

The relation between $\boldsymbol{\tau} = \varsigma(\boldsymbol{t})$ and $\boldsymbol{\sigma} = \boldsymbol{\beta} + \sqrt{-1}\boldsymbol{\omega}$ is given by $\boldsymbol{\tau} = 2\pi\sqrt{-1}\boldsymbol{\sigma}$, so that $\mathfrak{Im}(\boldsymbol{\sigma}) \gg 0$ corresponds to $\exp(\boldsymbol{\tau}) \sim 0$. The functions $F(\boldsymbol{t})$ and $G(\boldsymbol{t})$ satisfy the Picard–Fuchs equations, and give periods for the B-model VHS $(\mathcal{H}_\mathrm{B}, \nabla^B, \mathscr{F}^\bullet_B, Q_B)$.

6.12

Equation (153) implies the existence of an isomorphism

$$\mathrm{Mir}_{\mathcal{Y}} : \varsigma^*(\mathcal{H}_{\mathrm{A}}, \nabla_{\mathrm{A}}, \mathscr{F}^{\bullet}_{\mathrm{A}}, Q_A) \xrightarrow{\sim} (\mathcal{H}_{\mathrm{B}}, \nabla_{\mathrm{B}}, \mathscr{F}^{\bullet}_{\mathrm{B}}, Q_B) \tag{155}$$

of variations of polarized Hodge structures, which sends $F(\boldsymbol{t})\mathbf{1}$ on the left-hand side to

$$\Omega := \operatorname{Res}\left(\frac{1}{\check{f}_{\alpha}} \frac{d\check{y}_1}{\check{y}_1} \wedge \cdots \wedge \frac{d\check{y}_n}{\check{y}_n}\right) \tag{156}$$

on the right-hand side. A stronger statement, which gives an isomorphism of the $\widehat{\Gamma}$-integral structure on the A-side and the natural integral structure on the B-side, is proved in [43, Theorem 6.9].

7 Quasimap Correlation Functions for Anti-canonical Hypersurfaces in Toric Varieties

7.1

For $\boldsymbol{d} \in \mathrm{Eff}(X)$ and $i \in \{1, \ldots, m\}$, we set

$$k_i := \begin{cases} \langle \boldsymbol{u}_i, \boldsymbol{d}\rangle & \langle \boldsymbol{u}_i, \boldsymbol{d}\rangle \geq 0, \\ -1 & \langle \boldsymbol{u}_i, \boldsymbol{d}\rangle < 0. \end{cases} \tag{157}$$

and define the *quasimap space* of degree $\boldsymbol{d}$ by

$$\mathbf{Q}(X; \boldsymbol{d}) := \left(\prod_{i=1}^{m} \mathbb{A}_i^{k_i+1}\right) /\!\!/_{\theta}\, \mathbb{L} \tag{158}$$

with (134) in mind. An argument parallel to that in Sect. 3.1 shows that $\mathbf{Q}(X; \boldsymbol{d})$ is a compactification of the space of holomorphic maps $\mathbb{P}^1 \to X$ of degree $\boldsymbol{d}$. Later in Sect. 10.4, we will introduce the moduli spaces $\mathbf{Q}(W /\!\!/ G; \boldsymbol{d})$ of degree $\boldsymbol{d}$ quasimaps from $\mathbb{P}^1$ to more general GIT quotients $W /\!\!/ G$, and $\mathbf{Q}(X; \boldsymbol{d})$ here is the special case for $W = \prod_{i=1}^m \mathbb{A}_i^{k_i+1}$ and $G = \mathbb{L}$. The first Chern class of the line bundle on $\mathbf{Q}(X; \boldsymbol{d})$ associated with the character χ_i of $\mathbb{L}$ will also be denoted by $\boldsymbol{u}_i$ by abuse of notation. The *Morrison–Plesser class* is defined by

$$\Phi_{\boldsymbol{d}} := (\boldsymbol{u}_1 + \cdots + \boldsymbol{u}_m)^{\langle \boldsymbol{u}_1+\cdots+\boldsymbol{u}_m, \boldsymbol{d}\rangle} \prod_{\langle \boldsymbol{u}_i, \boldsymbol{d}\rangle < 0} \boldsymbol{u}_i^{-\langle \boldsymbol{u}_i, \boldsymbol{d}\rangle - 1}. \tag{159}$$

Here, the latter part $\prod_{\langle \boldsymbol{u}_i, \boldsymbol{d}\rangle < 0} \boldsymbol{u}_i^{-\langle \boldsymbol{u}_i, \boldsymbol{d}\rangle - 1}$ is the Euler class of $h^1(E^\vee)$ where E is the canonical obstruction theory for $\mathbf{Q}(X; \boldsymbol{d})$ defined in Sect. 10.7. The first part $(\boldsymbol{u}_1 + \cdots + \boldsymbol{u}_m)^{\langle \boldsymbol{u}_1+\cdots+\boldsymbol{u}_m, \boldsymbol{d}\rangle}$ is the Euler class of the vector bundle formed by the obstruction spaces to being a quasimap to an anti-canonical hypersurface $Y \subset X$ for each element in $\mathbf{Q}(X; \boldsymbol{d})$. For a polynomial $P(\alpha_1, \ldots, \alpha_m) \in \mathbb{C}[\alpha_1, \ldots, \alpha_m]$, we set

$$\langle P(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)\rangle_{X,Y,\boldsymbol{d}} := \int_{\mathbf{Q}(X;\boldsymbol{d})} P(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)\Phi_{\boldsymbol{d}} \tag{160}$$

and

$$\langle P(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)\rangle_{X,Y} := \sum_{\boldsymbol{d}\in \mathrm{Eff}(X)} \boldsymbol{\alpha}^{\boldsymbol{d}} \langle P(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)\rangle_{X,Y,\boldsymbol{d}} \in \mathbb{Z}\left[\!\left[\alpha^{\boldsymbol{d}} : \boldsymbol{d} \in \mathrm{Eff}(X)\right]\!\right], \tag{161}$$

where the completion is taken with respect to the ideal generated by $\mathrm{Eff}(X) \setminus \{0\}$. Here $\boldsymbol{\alpha}^{\boldsymbol{d}}$ is defined by (131).

8 Toric Residue Mirror Symmetry

8.1

Let $\check{G} = (\check{G}_0, \ldots, \check{G}_n)$ be a regular sequence in $S_{\check{\Delta}}$. If we set $I_{\check{G}} := I_{\check{\Delta}}/(\check{G}_0, \ldots, \check{G}_n) I_{\check{\Delta}}$, then the graded piece $I_{\check{G}}^{n+1}$ is one-dimensional and spanned by $J_{\check{G}} := \det \left(\check{y}_i \partial_{\check{y}_i} \check{G}_j\right)_{i,j=0}^{n}$. The *toric residue* [25] is the map $\mathrm{Res}_{\check{G}} \colon I_{\check{\Delta}}^{n+1} \to \mathbb{C}$ sending $(\check{G}_0, \ldots, \check{G}_n) I_{\check{\Delta}}$ to zero and $J_{\check{G}}$ to the normalized volume $\mathrm{vol}(\check{\Delta})$, i.e., $n!$ times the Euclidean volume of $\check{\Delta}$. For $\boldsymbol{\alpha} \in \check{\mathbb{L}}^{\mathrm{reg}}$, we define $\check{F}_{\boldsymbol{\alpha}}$ as in (139) and write $\mathrm{Res}_{\check{f}_\alpha} := \mathrm{Res}_{\check{F}_\alpha}$. Theorem 3 below is introduced in [3, Conjecture 4.6] and proved in [11, 63].

Theorem 3 *For any homogeneous polynomial $P(\alpha_1, \ldots, \alpha_m) \in \mathbb{C}[\alpha_1, \ldots, \alpha_m]$ of degree n, the generating function (161) gives the Laurent expansion of the toric residue*

$$\langle P(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)\rangle_{X,Y} = (-1)^n \mathrm{Res}_{\check{f}_\alpha} \left(\check{y}_0^{n+1} P(\alpha_1 \check{\boldsymbol{y}}^{\boldsymbol{b}_1}, \ldots, \alpha_m \check{\boldsymbol{y}}^{\boldsymbol{b}_m})\right) \tag{162}$$

around the large radius limit point associated with the fan Σ.

Reference [3, Conjecture 4.6] is generalized to toric complete intersections in [4, Conjecture 4.6] and proved in [46, 64].

8.2

The family $\varphi\colon \check{\mathcal{Y}} \to \check{\mathbb{L}}^{\mathrm{reg}}$ of Calabi–Yau manifolds comes with the holomorphic volume form

$$\Omega := \operatorname{Res}\left(\frac{1}{\check{f}_{\boldsymbol{\alpha}}}\frac{d\check{y}_1}{\check{y}_1} \wedge \cdots \wedge \frac{d\check{y}_n}{\check{y}_n}\right) \in H^0(\mathcal{H}_{\mathrm{B}}). \tag{163}$$

For a homogeneous polynomial $Q(\alpha_1, \ldots, \alpha_m) \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m]$ of degree $n-1$, the Q-Yukawa $(n-1)$-point function is defined in [3, Definition 9.1] by

$$Y_Q(\boldsymbol{\alpha}) := (-1)^{(n-1)(n-2)/2} \frac{1}{\left(2\pi\sqrt{-1}\right)^{n-1}} \int_{\check{Y}_\alpha} \Omega \wedge Q\left(\alpha_1 \frac{\partial}{\partial \alpha_1}, \ldots, \alpha_m \frac{\partial}{\partial \alpha_m}\right)\Omega, \tag{164}$$

where the differential operators $\alpha_1\partial/\partial\alpha_1, \ldots, \alpha_m\partial/\partial\alpha_m$ act on $\mathcal{H}_{\mathrm{B}}$ by the Gauss–Manin connection.

8.3

For $Q(\alpha_1, \ldots, \alpha_m) \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m]$, we set

$$P(\alpha_1, \ldots, \alpha_m) := (\alpha_1 + \cdots + \alpha_m) Q(\alpha_1, \ldots, \alpha_m) \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m]. \tag{165}$$

By [3, Theorem 9.7], which is attributed to [50], one has an equality

$$Y_Q(\boldsymbol{\alpha}) = (-1)^n \operatorname{Res}_{\check{f}_\alpha}\left(\check{y}_0^n P(\alpha_1 \check{\boldsymbol{y}}^{\boldsymbol{b}_1}, \ldots, \alpha_m \check{\boldsymbol{y}}^{\boldsymbol{b}_m})\right) \tag{166}$$

of the Yukawa $(n-1)$-point function and the toric residue.

8.4

Assume that the unstable locus of the $\mathbb{L}$-action on $\mathbb{A}^m$ with respect to θ has codimension strictly greater than 1. Then one has $H^2(X_\Sigma) = \operatorname{Pic}(X_\Sigma) = \operatorname{Pic}^{\mathbb{L}}(\mathbb{A}^m)$ so that the class $\boldsymbol{p}_i$ corresponds to a one-dimensional representation $\mathbb{C}_{\boldsymbol{p}_i}$ of $\mathbb{L}$. By abuse of notation, we let $\boldsymbol{p}_i$ denote the $\mathbb{G}_m$-equivariant Euler class of the pull-back of the line bundle $[\mathbb{A}^m \times \mathbb{C}_{\boldsymbol{p}_i}/\mathbb{G}_m]$ by the evaluation map $\mathrm{ev}_0 \colon X_{\boldsymbol{d}} \to [\mathbb{A}^m/\mathbb{G}_m]$ at $0 \in \mathbb{P}^1$. Denote $\boldsymbol{v} := \sum_{i=1}^m \boldsymbol{u}_i$.

If we set

$$\Phi(\boldsymbol{t}, \boldsymbol{\tau}; \mathsf{z}) := \sum_{\boldsymbol{d} \in \mathrm{Eff}(X)} e^{\boldsymbol{\tau} \cdot \boldsymbol{d}} \int_{\mathbf{Q}(X;\boldsymbol{d})}^{\mathbb{G}_m} e^{(\boldsymbol{t}-\boldsymbol{\tau})/\mathsf{z}} \Phi_{\boldsymbol{d}} \boldsymbol{v}, \tag{167}$$

then for any polynomial $R(t_1, \ldots, t_r) \in \mathbb{Q}[t_1, \ldots, t_r]$, one has

$$R\left(\mathsf{z}\frac{\partial}{\partial t_1}, \ldots, \mathsf{z}\frac{\partial}{\partial t_r}\right) \Phi(\boldsymbol{t}, \boldsymbol{\tau}; \mathsf{z})\Big|_{\boldsymbol{\tau}=\boldsymbol{t}} = \sum_{\boldsymbol{d} \in \mathrm{Eff}(X)} e^{\boldsymbol{t} \cdot \boldsymbol{d}} \int_{\mathbf{Q}(X;\boldsymbol{d})}^{\mathbb{G}_m} R(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_r) \Phi_{\boldsymbol{d}} \boldsymbol{v}. \tag{168}$$

In addition, one has

$$\Phi(\boldsymbol{t}, \boldsymbol{\tau}; \mathsf{z}) = \int_Y I(\boldsymbol{t}; -\mathsf{z}) \cup I(\boldsymbol{\tau}; \mathsf{z}), \tag{169}$$

by [32, Proposition 6.2]. This is the toric hypersurface version of (62). Note that $I(\boldsymbol{t}; 1)$ is convergent for large enough $-\mathrm{Re}\, \boldsymbol{t}$ by ratio test on the series (150) without the prefactor. By specializing to $\mathsf{z} = 1$ and using the definition of Q_{A}, one obtains

$$\Phi(\boldsymbol{t}, \boldsymbol{\tau}; 1) = Q_{\mathrm{A}}\left(I(\boldsymbol{t}; 1), I(\boldsymbol{\tau}; 1)\right). \tag{170}$$

By combining (170) with (153), one obtains

$$\Phi(\boldsymbol{t}, \boldsymbol{\tau}; 1) = Q_{\mathrm{A}}\left(L^{-1}(\boldsymbol{t}; 1) F(\boldsymbol{t}) \mathbf{1}, L^{-1}(\boldsymbol{\tau}; 1) F(\boldsymbol{\tau}) \mathbf{1}\right). \tag{171}$$

Since L is the fundamental solution for the flat connection ∇_{B}, the function $\Phi(\boldsymbol{t}, \boldsymbol{\tau}; 1)$ is obtained by parallel-transporting $F(\boldsymbol{t})\mathbf{1} \in (H_{\mathrm{B}})_{\boldsymbol{t}}$ and $F(\boldsymbol{\tau})\mathbf{1} \in (H_{\mathrm{B}})_{\boldsymbol{\tau}}$ to the fiber at the same point and taking the pairing Q_{B} at that point (the result does not depend on the choice of the point since Q_{B} is ∇_{B}-parallel). By sending (171) by (155), one obtains

$$(2\pi\sqrt{-1})^{n-1} \int_Y I(\boldsymbol{t}; -1) I(\boldsymbol{\tau}; 1) = (-1)^{(n-1)(n-2)/2} \int_{\check{Y}} \Omega_{\boldsymbol{t}} \wedge \Omega_{\boldsymbol{\tau}}. \tag{172}$$

Assume that $P(\alpha_1, \ldots, \alpha_m) = (\alpha_1 + \cdots + \alpha_m) Q(\alpha_1, \ldots, \alpha_m)$ for a polynomial Q and take $R(t_1, \ldots, t_r) := Q(\sum_{i=1}^r a_{i,1} t_i, \ldots, \sum_{i=1}^r a_{i,m} t_i))$ where $a_{i,j}$ are integers uniquely satisfying $\chi_j = \sum_{i=1}^r a_{i,j} \boldsymbol{p}_i$. By differentiating (172) by $R(\partial_{t_1}, \ldots, \partial_{t_r})$ and setting $\boldsymbol{\tau} = \boldsymbol{t}$, we obtain toric residue mirror symmetry for polynomials of the form $P(\alpha_1, \ldots, \alpha_m) = (\alpha_1 + \cdots + \alpha_m) Q(\alpha_1, \ldots, \alpha_m)$.

9 Martin's Formula

9.1

We use the same notations $G, T, \mathscr{W}$, and Δ for a reductive algebraic group, a maximal torus, the Weyl group, and the set of roots as in Sect. 2. Let W be an affine scheme with G-action, and fix a character θ of G. We write the line bundle on $W /\!\!/ T$ associated with $\alpha \in \Delta$ as L_α, and set

$$e := \prod_{\alpha \in \Delta} c_1(L_\alpha) \in H^{2|\Delta|}(W /\!\!/ T; \mathbb{Z}). \tag{173}$$

We write the natural projection and inclusion as and say that $\widetilde{a} \in H^*(W /\!\!/ T)$ is a *lift* of $a \in H^*(W /\!\!/ G)$ if $\pi^* a = \iota^* \widetilde{a}$.

Theorem 4 (Martin [48]) *If $\widetilde{a}$ is a lift of a, then one has*

$$\int_{W /\!\!/ G} a = \frac{1}{|\mathscr{W}|} \int_{W /\!\!/ T} \widetilde{a} \cup e. \tag{174}$$

9.2

Let $\mathrm{Mat}(r, n) \cong \mathbb{A}^{r \times n}$ be the space of $n \times r$ matrices, which is considered as the space of linear maps from an r-dimensional vector space to an n-dimensional vector space. It has a natural action of GL_r, and the GIT quotient $\mathrm{Gr}(r, n) := \mathrm{Mat}(r, n) /\!\!/ \mathrm{GL}_r$ is the Grassmannian of r-spaces in an n-space.

9.3

When $W = \mathrm{Mat}(r, n)$ and $G = \mathrm{GL}_r$, one has

$$W /\!\!/ G \cong \mathrm{Gr}(r, n), \tag{175}$$

$$W /\!\!/ T \cong (\mathbb{P}^{n-1})^r \tag{176}$$

and

$$H^*(\mathrm{Gr}(r, n)) \cong \mathbb{C}[\sigma_1, \ldots, \sigma_r]/(h_{n-r+1}, \ldots, h_n), \tag{177}$$

$$H^*((\mathbb{P}^{n-1})^r) \cong \mathbb{C}[x_1, \dots, x_r]/(x_1^n, \dots, x_r^n), \tag{178}$$

where $\sigma_i = \sigma_i(x_1, \dots, x_r) \in \mathbb{C}[x_1, \dots, x_r]^{\mathfrak{S}_r}$ are elementary symmetric functions and $h_i = h_i(x_1, \dots, x_r) \in \mathbb{C}[x_1, \dots, x_r]^{\mathfrak{S}_r} = \mathbb{C}[\sigma_1, \dots, \sigma_r]$ are complete symmetric functions. Martin's formula in this case gives

$$\int_{\mathrm{Gr}(r,n)} P(x_1, \dots, x_r) = \frac{1}{r!} \int_{(\mathbb{P}^{n-1})^r} \prod_{i \neq j} (x_i - x_j) P(x_1, \dots, x_r) \tag{179}$$

$$= \frac{(-1)^{r(r-1)/2}}{r!} \int_{(\mathbb{P}^{n-1})^r} \mathbf{\Delta}^2 \cup P(x_1, \dots, x_r) \tag{180}$$

for any $P(x_1, \dots, x_r) \in \mathbb{C}[x_1, \dots, x_r]^{\mathfrak{S}_r}$ where $\mathbf{\Delta} := \prod_{1 \le i < j \le r}(x_i - x_j)$.

9.4

The equivariant cohomology ring of $\mathrm{Gr}(r, n)$ with respect to the natural action of the diagonal maximal abelian subgroup $H \subset \mathrm{GL}_n$ is presented as

$$H_H^\bullet(\mathrm{Gr}(r, n); \mathbb{C}) \cong \mathbb{C}[\sigma_1, \dots, \sigma_r, \lambda_1, \dots, \lambda_n] \big/ (h_{n-r+1}(\sigma, \lambda), \dots, h_n(\sigma, \lambda)) \ , \tag{181}$$

where h_i is the degree $2i$ part of $c^H(\mathcal{S})c^H(\mathcal{Q}) - \prod_{i=1}^n (1 + \lambda_i)$. Here $\mathcal{S}$ and $\mathcal{Q}$ are the tautological bundle and the universal quotient bundle respectively, and $c^H(-)$ stands for the H-equivariant total Chern class. Note that $\sigma_i := c_i^H(\mathcal{S})$ is the elementary symmetric function of the H-equivariant Chern roots $x_1, \dots, x_r$ of $\mathcal{S}$, and $c_i^H(\mathcal{Q})$ for $i = 1, \dots, n - r$ are expressed in terms of $\sigma_1, \dots, \sigma_r$ and $\lambda_1, \dots, \lambda_n$ by the condition $h_1 = \cdots = h_{n-r} = 0$. Martin's formula gives

$$\int_{\mathrm{Gr}(r,n)}^H P(\sigma_1, \dots, \sigma_r) \tag{182}$$

$$= \sum_{1 \le i_1 < i_2 < \cdots i_r \le n} \mathrm{Res}_{\boldsymbol{x} = (\lambda_{i_1}, \dots, \lambda_{i_r})} P(\sigma_1, \dots, \sigma_r) \prod_{i \neq j} (x_i - x_j) \frac{dx_1 \wedge \cdots \wedge dx_r}{\prod_{i=1}^r \prod_{j=1}^n (x_i - \lambda_j)}.$$

10 Quasimap Spaces for GIT Quotients

10.1

Let G be a reductive algebraic group acting on an affine variety W and fix a character θ of G. In this paper, we will always assume the following:

(1) Semi-stability implies stability.
(2) The semi-stable locus W^{ss} is smooth and non-empty.
(3) The G-action on W^{ss} is free (however, see [17] for allowing finite non-trivial stabilizers).
(4) The codimension of the unstable locus $W \setminus W^{\mathrm{ss}}$ is greater than one.

The GIT quotient is defined by $W /\!\!/ G := W^{\mathrm{ss}}/G$, which is an open substack of $[W/G]$.

10.2

A map $u \colon \mathbb{P}^1 \to [W/G]$ to the quotient stack $[W/G]$ is pair $(P, \widetilde{u})$ of a principal G-bundle $P \to \mathbb{P}^1$ and a G-equivariant map $\widetilde{u} \colon P \to W$. It is called a *quasimap* if the generic point of $\mathbb{P}^1$ is mapped to $W /\!\!/ G \subset [W/G]$. A point in the inverse image of the unstable locus will be called a *base point*.

10.3

For a quasimap $u \colon \mathbb{P}^1 \to [W/G]$ and a G-equivariant line bundle L on W, the pull-back $\widetilde{u}^* L$ is a G-equivariant line bundle on P, which descends to a line bundle $u^* L$ on $\mathbb{P}^1$. The *degree* of a quasimap $u \colon \mathbb{P}^1 \to [W/G]$ is the map $\boldsymbol{d} \colon \operatorname{Pic}^G W \to \mathbb{Z}$ sending $L \in \operatorname{Pic}^G W$ to $\deg u^* L$.

10.4

An *isomorphism* of quasimaps $u = (P, \widetilde{u})$ and $u' = (P', \widetilde{u}')$ is an isomorphism $\varphi \colon P \to P'$ of principal G-bundles such that $\widetilde{u} = \widetilde{u}' \circ \varphi$. By [23, Theorem 7.1.6], the moduli functor for quasimaps of degree $\boldsymbol{d}$ is representable by a Deligne–Mumford stack, which will be denoted by $\mathbf{Q}(W /\!\!/ G; \boldsymbol{d})$. This stack is denoted by $\mathrm{Qmap}_{0,0}(W /\!\!/ G, \boldsymbol{d}; \mathbb{P}^1)$ in [23, Sect. 7.2] and $\mathrm{QG}^{0+}_{0,0,\boldsymbol{d}}(W /\!\!/ G)$ in [22, Sect. 2.6]. Note that $\mathbf{Q}(W /\!\!/ G)$ depends not only on $W /\!\!/ G$ and $\boldsymbol{d}$ but also on W, G, and θ.

10.5

Let $\mathbf{Q}_{\bullet}(W /\!\!/ G; \boldsymbol{d}) \subset \mathbf{Q}(W /\!\!/ G; \boldsymbol{d})$ be the substack parametrizing quasimaps such that $u|_{\mathbb{P}^1 \setminus \{0\}}$ is a constant map to $W /\!\!/ G$. This implies that $0 \in \mathbb{P}^1$ is a base point of length $\boldsymbol{d}(\theta)$ by [23, Lemma 7.1.2]. This stack is denoted by $\mathbf{Q}_{0,0+\bullet}(W /\!\!/ G, \boldsymbol{d})_0$ in [22, Sect.

4.1]. There is a natural map $\mathrm{ev}\colon \mathbf{Q}_\bullet(W/\!\!/G; \boldsymbol{d}) \to W/\!\!/G$, called the *evaluation map*, which sends $u \in \mathbf{Q}_\bullet(W/\!\!/G; \boldsymbol{d})$ to $u(\infty) \in W/\!\!/G$.

10.6

There is a natural $\mathbb{G}_m$-action on $\mathbf{Q}(W/\!\!/G; \boldsymbol{d})$ coming from the standard $\mathbb{G}_m$-action on $\mathbb{P}^1$. As described in [22, Sect. 4.1], the fixed locus of this action is identified with the coproduct

$$\coprod_{\boldsymbol{d}_1+\boldsymbol{d}_2=\boldsymbol{d}} \mathbf{Q}_\bullet(W/\!\!/G; \boldsymbol{d}_1) \times_{W/\!\!/G} \mathbf{Q}_\bullet(W/\!\!/G; \boldsymbol{d}_2) \tag{183}$$

of fiber products with respect to the evaluation map.

10.7

If W has at worst lci singularity, then $\mathbf{Q}(W/\!\!/G; \boldsymbol{d})$ has a canonical perfect obstruction theory, which allows one to define the *virtual fundamental cycle*. The canonical perfect obstruction theory is $\left(\mathbb{R}\pi_* \,\mathrm{ev}^* T_{[W/G]}\right)^\vee$, where $T_{[W/G]}$ is the tangent complex of $[W/G]$, $\mathrm{ev} : \mathbf{Q}(W/\!\!/G; \boldsymbol{d}) \times \mathbb{P}^1 \to [W/G]$ is the evaluation map, and $\pi : \mathbf{Q}(W/\!\!/G; \boldsymbol{d}) \times \mathbb{P}^1 \to \mathbf{Q}(W/\!\!/G; \boldsymbol{d})$ is the first projection; see Theorem 7.2.2 of [22] or Sect. 5. The virtual fundamental cycle is an element of the Chow group of $\mathbf{Q}(W/\!\!/G; \boldsymbol{d})$ whose degree is given by the virtual dimension

$$\mathrm{virt.dim}\, \mathbf{Q}(W/\!\!/G; \boldsymbol{d}) = \langle \boldsymbol{d}, \det T_W \rangle + \dim W/\!\!/G. \tag{184}$$

10.8

Since the stack $\mathbf{Q}_\bullet(W/\!\!/G; \boldsymbol{d})$ is the union of connected components of the fixed locus of the $\mathbb{G}_m$-action, it has a perfect obstruction theory inherited from $\mathbf{Q}(W/\!\!/G; \boldsymbol{d})$. The virtual push-forward

$$\mathrm{ev}_*^{\mathrm{virt}}(-) := \mathrm{PD}\left(\mathrm{ev}_*\left((-) \cap \left[\mathbf{Q}_\bullet(W/\!\!/G; \boldsymbol{d})\right]^{\mathrm{virt}}\right)\right) \tag{185}$$

along the evaluation map $\mathrm{ev}\colon \mathbf{Q}_\bullet(W/\!\!/G; \boldsymbol{d}) \to W/\!\!/G$ allows one to define the *I-function*

$$I(\boldsymbol{t};\mathsf{z}) := e^{\boldsymbol{p}\cdot\boldsymbol{t}/\mathsf{z}} \sum_{\boldsymbol{d}\in\mathrm{Eff}(W/\!\!/G)} e^{\boldsymbol{d}\cdot\boldsymbol{t}} I_{\boldsymbol{d}} \tag{186}$$

by

$$I_{\boldsymbol{d}} := \mathrm{ev}_*^{\mathrm{virt}} \left(\frac{1}{\mathrm{Eul}^{\mathbb{G}_m}\left(N^{\mathrm{virt}}_{\mathbf{Q}_\bullet(W/\!\!/G;\boldsymbol{d})/\mathbf{Q}(W/\!\!/G;\boldsymbol{d})}\right)} \right), \tag{187}$$

where the denominator is the $\mathbb{G}_m$-equivariant Euler class of the virtual normal bundle. Here $\boldsymbol{p}$ is a basis of $H^2(W/\!\!/G)$, and $\boldsymbol{t}$ is the coordinate of $H^2(W/\!\!/G)$ corresponding to $\boldsymbol{p}$.

An H-action on W commuting with the G-action induces an H-action on $\mathbf{Q}_\bullet(W/\!\!/G;\boldsymbol{d})$, which allows one to define the H-equivariant I-function of $W/\!\!/G$.

10.9

There exits a G-space V with a G-equivariant closed embedding $W \hookrightarrow V$. Let $\mathfrak{u}\colon \mathbf{Q}(W/\!\!/G;\boldsymbol{d}) \times \mathbb{P}^1 \to [W/G]$ be the universal quasimap. It consists of a principal G-bundle $\mathcal{P}$ on $\mathbf{Q}(W/\!\!/G;\boldsymbol{d}) \times \mathbb{P}^1$ and a G-equivariant morphism $\widetilde{\mathfrak{u}}\colon \mathcal{P} \to W$. Let $\mathcal{P}' := \mathcal{P}|_{\mathbf{Q}(W/\!\!/G;\boldsymbol{d})\times\{\mathrm{pt}\}}$ be the restriction of $\mathcal{P}$ to a fiber of the second projection $\mathbf{Q}(W/\!\!/G;\boldsymbol{d}) \times \mathbb{P}^1 \to \mathbb{P}^1$. We write the Chern–Weil homomorphism defined by $\mathcal{P}'$ as

$$\mathfrak{CW}\colon \mathbb{C}[\mathfrak{g}]^G \to H^*\left(\mathbf{Q}(W/\!\!/G;\boldsymbol{d})\right). \tag{188}$$

Note that $\mathbb{C}[\mathfrak{g}]^G$ is isomorphic to $\mathbb{C}[\mathfrak{t}]^{\mathscr{W}}$ by Chevalley restriction theorem. For $P \in \mathbb{C}[\mathfrak{t}]^{\mathscr{W}}$, we set

$$\langle P\rangle_{W/\!\!/G,\boldsymbol{d}} := \int_{[\mathbf{Q}(W/\!\!/G;\boldsymbol{d})]^{\mathrm{virt}}} \mathfrak{CW}(P), \tag{189}$$

$$\langle P\rangle_{W/\!\!/G} := \sum_{\boldsymbol{d}\in\mathrm{Eff}(W/\!\!/G)} e^{\langle \boldsymbol{d},\boldsymbol{t}\rangle} \langle P\rangle_{W/\!\!/G,\boldsymbol{d}}\,. \tag{190}$$

Conjecture 5 *Suppose that $W \subset V$ is the zero locus of G semi-invariant polynomials f_i, $i = 1,\ldots,r$. Provided with conditions in Sects. 10.1 and 10.7, for any $P \in \mathbb{C}[\mathfrak{t}]^{\mathscr{W}}$, the generating function* (190) *of quasimap invariants coincides with the correlation function* (5) *of the A-twisted gauged linear sigma model up to an overall sign;*

$$\langle P\rangle_{\mathrm{GLSM}} = \pm \langle P\rangle_{W/\!\!/G}\,. \tag{191}$$

Here the potential of GLSM is given as a G-invariant function $\sum_i f_i p_i$ of $V \times \mathbb{A}^r$ where p_i denotes ith coordinate of $\mathbb{A}^r$ with R-charge 2.

10.10

By taking $\mathcal{P}'$ to be the fiber over a fixed point of the natural $\mathbb{G}_m$-action on the domain curve $\mathbb{P}^1$, one can define $\mathbb{G}_m$-equivariant quasimap invariants $\langle P\rangle^{\mathbb{G}_m}_{W/\!\!/G}$. If W has an action of an algebraic torus H commuting with the action of G, then one can define $H \times \mathbb{G}_m$-equivariant quasimap invariants $\langle P\rangle^{H\times\mathbb{G}_m}_{W/\!\!/G}$.

11 Quasimap Spaces for Grassmannians

11.1

The quasimap space $\mathbf{Q}(\mathrm{Gr}(r, n); d)$ classifies pairs (P, u) of a principal GL_r-bundle P and a GL_r-equivariant map u. The choice of a principal GL_r-bundle P is equivalent to the choice of a vector bundle S of rank r, and the choice of a GL_r-equivariant map u is equivalent to the choice of a map $\mathcal{S} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$, which is a sheaf injection since the generic point must go to the semi-stable locus (but not necessarily a morphism of vector bundles). The choice of a sheaf injection $\mathcal{S} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ is equivalent to the choice of a surjection $\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \to \mathcal{Q}$, where $\mathcal{Q}$ is a coherent sheaf whose Hilbert polynomial is $d + (n - r)(t + 1)$. This is the same as the Hilbert polynomial of a locally free sheaf of rank $n - r$ and degree d, and one has an isomorphism

$$\mathbf{Q}(\mathrm{Gr}(r, n); d) \cong \mathrm{Quot}_{\mathbb{P}^1,d}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, n - r). \tag{192}$$

11.2

It is shown in [10, Lemma 1.2] that the subspace $\mathbf{Q}_\bullet(\mathrm{Gr}(r, n); d)$ of $\mathbf{Q}(\mathrm{Gr}(r, n); d)$ is decomposed into connected components as

$$\mathbf{Q}_\bullet(\mathrm{Gr}(r, n); d) = \coprod_{|\boldsymbol{d}|=d} \mathbf{Q}_\bullet(\mathrm{Gr}(r, n); \boldsymbol{d}), \tag{193}$$

where $\boldsymbol{d} = (d_1, \ldots, d_r)$ runs over elements of $\mathbb{N}^r$ satisfying $|\boldsymbol{d}| := d_1 + \cdots + d_r = d,\ d_1 \le d_2 \le \cdots \le d_r$ and each connected component is isomorphic to the partial flag manifold

$$\mathbf{Q}_\bullet(\mathrm{Gr}(r,n);\boldsymbol{d}) \cong \mathrm{Fl}(m_1,\dots,m_k,r,n), \tag{194}$$

where $1 \le m_1 < m_2 < \cdots < m_k = r$ denote the jumping indices;

$$0 \le d_1 = \cdots = d_{m_1} < d_{m_1+1} = \cdots = d_{m+2} < \cdots. \tag{195}$$

Let $x_1,\dots,x_r$ be the Chern roots of the dual of the universal subbundle on $\mathrm{Gr}(r,n)$. We also define $|\boldsymbol{x}| := \sum_{i=1}^r x_i$ and $|\boldsymbol{d}| := \sum_{i=1}^r d_i$ for $\boldsymbol{d} = (d_1,\dots,d_r)$. The I-function can be computed by localization as

$$I_{\mathrm{Gr}(r,n)}(t;\mathsf{z}) = \sum_{\boldsymbol{d}\in\mathbb{N}^r} (-1)^{(r-1)|\boldsymbol{d}|} e^{(|\boldsymbol{d}|+|\boldsymbol{x}|/\mathsf{z})t} I_{\boldsymbol{d}}(\mathsf{z}) \tag{196}$$

where

$$I_{\boldsymbol{d}}(\mathsf{z}) = \frac{\prod_{1\le i<j\le r}(x_i - x_j + (d_i - d_j)\mathsf{z})}{\prod_{1\le i<j\le r}(x_i - x_j)\prod_{i=1}^r\prod_{j=1}^n\prod_{l=1}^{d_i}(x_i + l\mathsf{z})}. \tag{197}$$

As shown in [10, p. 109], the I-function and the J-function agrees for $\mathrm{Gr}(r,n)$ just as in the case of projective spaces.

11.3

The Hori–Vafa conjecture [41] proved in [10] shows that the I-functions of $(\mathbb{P}^{n-1})^r$ and $\mathrm{Gr}(r,n)$ are related by

$$I_{\mathrm{Gr}(r,n)}(t;\mathsf{z}) = e^{-\sigma_1(r-1)\pi\sqrt{-1}/\mathsf{z}} \left.\frac{\mathcal{D}I_{(\mathbb{P}^{n-1})^r}(\boldsymbol{t};\mathsf{z})}{\boldsymbol{\Delta}}\right|_{t_i = t+(r-1)\pi\sqrt{-1}} \tag{198}$$

where

$$\mathcal{D} := \prod_{1\le i<j\le r}\left(\mathsf{z}\frac{\partial}{\partial t_i} - \mathsf{z}\frac{\partial}{\partial t_j}\right). \tag{199}$$

11.4

As shown in [10], the equivariant I-function with respect to the natural action of $H = (\mathbb{G}_m)^n$ on $\mathrm{Mat}(r,n)$ is given by

$$I^H_{\mathrm{Gr}(r,n)}(t;\mathsf{z}) = e^{t\sigma_1/z}\sum_{\boldsymbol{d}\in\mathbb{N}^r}(-1)^{(r-1)|\boldsymbol{d}|}e^{|\boldsymbol{d}|t}\frac{\prod_{1\le i<j\le r}(x_i-x_j+(d_i-d_j)\mathsf{z})}{\prod_{1\le i<j\le r}(x_i-x_j)\prod_{i=1}^r\prod_{j=1}^n\prod_{l=1}^{d_i}(x_i-\lambda_j+l\mathsf{z})}, \tag{200}$$

and the factorization gives

$$\sum_{d=0}^{\infty}e^{d\tau}\left\langle e^{(t-\tau)\sigma_1/\mathsf{z}}\right\rangle^{H\times\mathbb{G}_m}_{\mathrm{Gr}(r,n),d} = \int^H_{\mathrm{Gr}(r,n)} I^H_{\mathrm{Gr}(r,n)}(t;\mathsf{z})\cup I^H_{\mathrm{Gr}(r,n)}(\tau;-\mathsf{z}). \tag{201}$$

Here $\sigma_1=\sum_{i=1}^r x_i$ is the H-equivariant first Chern class of the vector bundle

$$\mathcal{S}^\vee = \left(\mathrm{Mat}(r,n)\times\mathbb{C}^r\right)/\!\!/G \tag{202}$$

on $\mathrm{Gr}(r,n)$, where the G-action on $\mathbb{C}^r$ is the defining representation.

11.5

Let $\mathcal{V}$ be an equivariant vector bundle on $\mathrm{Gr}(r,n)$ associated with a representation V of GL_r. If $\mathcal{V}$ is globally generated and $\det\mathcal{V}\cong\omega^\vee_{\mathrm{Gr}(r,n)}$, then the zero $Y:=s^{-1}(0)$ of a general section $s\in H^0(\mathcal{V})$ is a smooth Calabi–Yau manifold by a generalization of the theorem of Bertini [53, Theorem 1.10].

11.6

Let $[\mathrm{Mat}(r,n)/\mathrm{GL}_r]$ be the quotient stack containing $\mathrm{Gr}(r,n)$ as an open substack. The complete intersection $Y\subset\mathrm{Gr}(r,n)$ is an open substack of $\mathcal{Y}:=[Z/\mathrm{GL}_r]$, where $Z\subset\mathrm{Mat}(r,n)$ is the zero of the map $\widetilde{s}\colon\ \mathrm{Mat}(r,n)\to V$ underlying s. Indeed, Y has a GIT quotient description $Y=Z/\!\!/\mathrm{GL}_r$, which allows us to define $\mathbf{Q}(Y;d)$ and its virtual fundamental cycle as in Sect. 10. Let $\mathcal{S}^\vee_{\mathcal{Y}}$ be the vector bundle on $\mathcal{Y}$ associated with the defining representation of GL_r. Any point $p\in\mathbb{P}^1$ determines a map $\mathrm{ev}_p\colon\ \mathbf{Q}(Y;d)\to\mathcal{Y}$ sending $f\colon\mathbb{P}^1\to\mathcal{Y}$ to $f(p)\in\mathcal{Y}$, and the Chern classes

$$\sigma_i := c_i\left(\mathrm{ev}_p^*\,\mathcal{S}^\vee_{\mathcal{Y}}\right),\quad i=1,\dots,r \tag{203}$$

does not depend on the choice of $p\in\mathbb{P}^1$. For $P(\sigma_1,\dots,\sigma_r)\in\mathbb{C}[\sigma_1,\dots,\sigma_r]$, we set

$$\langle P(\sigma_1,\dots,\sigma_r)\rangle_{Y,d} := \int_{[\mathbf{Q}(Y;\boldsymbol{d})]^{\mathrm{virt}}}P(\sigma_1,\dots,\sigma_r) \tag{204}$$

and

$$\langle P(\sigma_1, \ldots, \sigma_r) \rangle_Y := \sum_{d=0}^{\infty} e^{dt} \langle P(\sigma_1, \ldots, \sigma_r) \rangle_{Y,d} \,. \tag{205}$$

11.7

The equivariant I-function of Y is given by

$$I_Y^H(t; \mathsf{z}) = \sum_{\boldsymbol{d} \in \mathbb{N}^r} e^{(\boldsymbol{d}+\boldsymbol{x}/\mathsf{z})\cdot \boldsymbol{t}} I_{\boldsymbol{d}}(\boldsymbol{t}; \mathsf{z}) \bigg|_{t_i = t + (r-1)\pi\sqrt{-1}}, \tag{206}$$

where

$$I_{\boldsymbol{d}}(\boldsymbol{t}; \mathsf{z}) := \frac{\prod_{\delta \in \Delta(V)} \prod_{l=1}^{\langle \boldsymbol{\delta}, \boldsymbol{d} \rangle} (\langle \boldsymbol{\delta}, \boldsymbol{x} \rangle + l\mathsf{z}) \prod_{1 \le i < j \le r} (x_i - x_j + (d_i - d_j)\mathsf{z})}{\prod_{1 \le i < j \le r} (x_i - x_j) \prod_{i=1}^{r} \prod_{j=1}^{n} \prod_{l=1}^{d_i} (x_i - \lambda_j + l\mathsf{z})}, \tag{207}$$

where $\Delta(V)$ denotes the set of weights of V and $\langle \boldsymbol{\delta}, \boldsymbol{x} \rangle$ denotes the first Chern class associated to the weight $\boldsymbol{\delta}$ (expressed in terms of the fundamental weights $x_1, \ldots, x_r$ of the maximal diagonal torus of G). Localization with respect to the natural $\mathbb{G}_m$-action on $\mathbf{Q}(\mathrm{Gr}(r, n); d)$ shows

$$\left\langle e^{(t-\tau)\sigma_1/\mathsf{z}} \right\rangle_Y^{H \times \mathbb{G}_m} = \int_Y^H I^H(t; \mathsf{z}) \cup I^H(\tau; -\mathsf{z}) \tag{208}$$

just as in (169).

12 Residue Mirror Symmetry for Grassmannians

12.1

We define the *abelianized quasimap space* for $\mathrm{Gr}(r, n)$ by

$$\mathbf{Q}^{\mathrm{ab}}(\mathrm{Gr}(r, n); d) := \coprod_{|\boldsymbol{d}|=d} \mathbf{Q}^{\mathrm{ab}}(\mathrm{Gr}(r, n); \boldsymbol{d}), \tag{209}$$

$$\mathbf{Q}^{\mathrm{ab}}(\mathrm{Gr}(r, n); \boldsymbol{d}) := \mathbf{Q}(\mathbb{P}^{n-1}; d_1) \times \cdots \times \mathbf{Q}(\mathbb{P}^{n-1}; d_r), \tag{210}$$

where $\boldsymbol{d}$ runs over $\boldsymbol{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$ such that $|\boldsymbol{d}| := d_1 + \cdots + d_r = d$. An abelianized quasimap

$$\varphi(z_1, z_2) = \left(\left(\varphi_{i1}(z_1, z_2), \dots, \varphi_{in}(z_1, z_2)\right) \in \mathbf{Q}(\mathbb{P}^{n-1}; d_i)\right)_{i=1}^{r} \tag{211}$$

defines a genuine map of degree d if the matrix $(\varphi_{ij}(z_1, z_2))_{i,j}$ has rank r for any $(z_1, z_2) \neq 0$. For $P(\sigma_1, \dots, \sigma_r) \in \mathbb{C}[\sigma_1, \dots, \sigma_r]$, we set

$$\langle P(\sigma_1, \dots, \sigma_r)\rangle^{\mathrm{ab}}_{\mathrm{Gr}(r,n),\boldsymbol{d}} := \frac{1}{r!}\int_{\mathbf{Q}^{\mathrm{ab}}(\mathrm{Gr}(r,n);\boldsymbol{d})} \prod_{i \neq j}(x_i - x_j) P(\sigma_1(x_1, \dots, x_r), \dots, \sigma_r(x_1, \dots, x_r)), \tag{212}$$

$$\langle P(\sigma_1, \dots, \sigma_r)\rangle^{\mathrm{ab}}_{\mathrm{Gr}(r,n),d} := \sum_{|\boldsymbol{d}|=d} \langle P(\sigma_1, \dots, \sigma_r)\rangle^{\mathrm{ab}}_{\mathrm{Gr}(r,n),\boldsymbol{d}}\,, \tag{213}$$

$$\langle P(\sigma_1, \dots, \sigma_r)\rangle^{\mathrm{ab}}_{\mathrm{Gr}(r,n)} := \sum_{d=0}^{\infty}(-1)^{(r-1)d} q^d \,\langle P(\sigma_1, \dots, \sigma_r)\rangle^{\mathrm{ab}}_{\mathrm{Gr}(r,n),d}\,. \tag{214}$$

12.2

If we set $G := \mathrm{GL}_r$ and $V := \mathrm{Mat}(r, n)$, where G acts naturally on V and $\mathbb{G}_m$ acts trivially on V, then we have $Z^{\mathrm{vec}}_{\boldsymbol{d}}(x) = \prod_{i \neq j}(x_i - x_j)$ and $Z^{\mathrm{mat}}_{\boldsymbol{d}}(x) = \prod_{i=1}^{r}\left(x_i^{-d_i-1}\right)^n$, so that (5) gives the same result as (212);

$$\langle P(\sigma_1, \dots, \sigma_r)\rangle_{\mathrm{GLSM}} = \langle P(\sigma_1, \dots, \sigma_r)\rangle^{\mathrm{ab}}_{\mathrm{Gr}(r,n)}\,. \tag{215}$$

12.3

We write the ring homomorphism $\mathbb{C}[\sigma_1, \dots, \sigma_r] \to \mathrm{QH}(\mathrm{Gr}(r, n))$ sending $\sigma_i \in \mathbb{C}[\sigma_1, \dots, \sigma_r]$ to $\sigma_i \in H^*(\mathrm{Gr}(r, n); \mathbb{C}) \cong \mathbb{C}[\sigma_1, \dots, \sigma_r]/(h_{n-r+1}, \dots, h_n)$ as $P(\sigma_1, \dots, \sigma_r) \mapsto \mathring{P}(\sigma_1, \dots, \sigma_r)$ just as in the case of $\mathbb{P}^{n-1}$.

Theorem 6 *For any $P(\sigma_1, \dots, \sigma_r) \in \mathbb{C}[\sigma_1, \dots, \sigma_r]$, one has*

$$\langle P(\sigma_1, \dots, \sigma_r)\rangle^{\mathrm{ab}}_{\mathrm{Gr}(r,n)} = \int_{\mathrm{Gr}(r,n)} \mathring{P}(\sigma_1, \dots, \sigma_r). \tag{216}$$

Proof It follows from (34) that

$$
\begin{aligned}
&\langle P(\sigma_1,\ldots,\sigma_r)\rangle^{\mathrm{ab}}_{\mathrm{Gr}(r,n)}\\
&=\frac{1}{r!}\sum_{d_1,\ldots,d_r=0}^{\infty}((-1)^{r-1}q)^{d_1+\cdots+d_r}\operatorname{Res}\prod_{i\neq j}(x_i-x_j)P(\sigma_1,\ldots,\sigma_r)\frac{dx_1}{x_1^{n(d_1+1)}}\wedge\cdots\wedge\frac{dx_r}{x_r^{n(d_r+1)}}\\
&=\frac{1}{r!}\operatorname{Res}\prod_{i\neq j}(x_i-x_j)P(\sigma_1,\ldots,\sigma_r)\frac{dx_1}{x_1^n+(-1)^r q}\wedge\cdots\wedge\frac{dx_r}{x_r^n+(-1)^r q}\\
&=\frac{1}{r!n^r}\sum_{x_1^n=(-1)^{r-1}q}\cdots\sum_{x_r^n=(-1)^{r-1}q}\prod_{i\neq j}(x_i-x_j)P(\sigma_1(x_1,\ldots,x_r),\ldots,\sigma_r(x_1,\ldots,x_r))\\
&=\int_{\mathrm{Gr}(r,n)}\mathring{P}(\sigma_1,\ldots,\sigma_r),
\end{aligned}
$$

where the last equality is the Vafa–Intriligator formula [61, Theorem 4.6]. □

12.4

Theorem 6 is related to intersection theory on the moduli space of vector bundles on a Riemann surface through a theorem of Witten [70], which states the existence of a ring isomorphism $\mathrm{QH}(\mathrm{Gr}(r,n))/(q-1)\xrightarrow{\sim}R(U(r))_{n-r,n}$ from the quantum cohomology of $\mathrm{Gr}(r,n)$ at $q=1$ and the Verlinde algebra of $U(r)$ at $\mathrm{SU}(r)$ level $n-r$ and $U(1)$ level n.

12.5

We define the $\mathbb{G}_m$-equivariant correlator of $P(\sigma_1,\ldots,\sigma_r)\in\mathbb{C}[\sigma_1,\ldots,\sigma_r]$ by

$$
\langle P(\sigma_1,\ldots,\sigma_r)\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n)} := \sum_{\boldsymbol{d}\in\mathbb{N}^r} e^{\boldsymbol{d}\cdot\boldsymbol{t}}\,\langle P(\sigma_1,\ldots,\sigma_r)\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n),\boldsymbol{d}}\Big|_{t_i=t+(r-1)\pi\sqrt{-1}} \tag{217}
$$

where

$$
\begin{aligned}
\langle P(\sigma_1,\ldots,\sigma_r)\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n),\boldsymbol{d}} := \int^{\mathbb{G}_m}_{\mathbf{Q}^{\mathrm{ab}}(\mathrm{Gr}(r,n);\boldsymbol{d})}\prod_{1\le i<j\le n}&(x_i-x_j)(x_j-x_i+(d_j-d_i)\mathsf{z})\\
&P(\sigma_1(x_1,\ldots,x_r),\ldots,\sigma_r(x_1,\ldots,x_r)).
\end{aligned} \tag{218}
$$

Since $\mathbf{Q}((\mathbb{P}^{n-1})^r;\boldsymbol{d})^{\mathbb{G}_m}=\prod_{i=1}^r\mathbf{Q}(\mathbb{P}^{n-1};d_i)^{\mathbb{G}_m}$ under $\mathbf{Q}((\mathbb{P}^{n-1})^r;\boldsymbol{d})=\prod_{i=1}^r\mathbf{Q}(\mathbb{P}^{n-1};d_i)$, we have a straightforward generalization of (62):

$$\sum_{\boldsymbol{d}\in\mathbb{N}^r} e^{\boldsymbol{d}\cdot\boldsymbol{\tau}} \left\langle e^{(\boldsymbol{t}-\boldsymbol{\tau})\cdot\boldsymbol{x}/z}\right\rangle^{\mathbb{G}_m}_{(\mathbb{P}^{n-1})^r,\boldsymbol{d}} = \int_{(\mathbb{P}^{n-1})^r} I_{(\mathbb{P}^{n-1})^r}(\boldsymbol{t};\mathsf{z}) \cup I_{(\mathbb{P}^{n-1})^r}(\boldsymbol{\tau};-\mathsf{z}), \tag{219}$$

By acting $\mathcal{D}_{\boldsymbol{t}} := \prod_{1\le i<j\le r}\left(\mathsf{z}\partial_{t_i} - \mathsf{z}\partial_{t_j}\right)$ and $-\mathcal{D}_{\boldsymbol{\tau}} := \prod_{1\le i<j\le r}\left(-\mathsf{z}\partial_{\tau_i} + \mathsf{z}\partial_{\tau_j}\right)$ on both sides of (219) one obtains

$$\sum_{\boldsymbol{d}\in\mathbb{N}^r} e^{\boldsymbol{d}\cdot\boldsymbol{t}} \left\langle \prod_{1\le i<j\le r}\left(x_i - x_j\right) \prod_{1\le i<j\le r}\left((x_j + d_j\mathsf{z}) - (x_i + d_i\mathsf{z})\right)\cdot e^{(\boldsymbol{t}-\boldsymbol{\tau})\cdot\boldsymbol{x}/z}\right\rangle^{\mathbb{G}_m}_{(\mathbb{P}^{n-1})^r} \tag{220}$$

$$= \left\langle e^{(\boldsymbol{t}-\boldsymbol{\tau})\cdot\boldsymbol{x}/z}\right\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n)}. \tag{221}$$

on the left hand side and

$$\int_{(\mathbb{P}^{n-1})^r} \mathcal{D}_{\boldsymbol{t}} I_{(\mathbb{P}^{n-1})^r}(\boldsymbol{t};\mathsf{z}) \cup (-\mathcal{D}_{\boldsymbol{\tau}}) I_{(\mathbb{P}^{n-1})^r}(\boldsymbol{\tau};-\mathsf{z}) \tag{222}$$

on the right hand side. By setting $t_i = t + (r-1)\pi\sqrt{-1}$, $\tau_i = \tau + (r-1)\pi\sqrt{-1}$ and using (198), one obtains

$$\begin{aligned}\left\langle e^{(t-\tau)\sigma_1/\mathsf{z}}\right\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n)} &= \frac{1}{r!}\int_{(\mathbb{P}^{n-1})^r} \Delta \cup I_{\mathrm{Gr}(r,n)}(t;\mathsf{z}) \cup \Delta \cup I_{\mathrm{Gr}(r,n)}(\tau;-\mathsf{z}) \\ &= \int_{\mathrm{Gr}(r,n)} I_{\mathrm{Gr}(r,n)}(t;\mathsf{z}) \cup I_{\mathrm{Gr}(r,n)}(\tau;-\mathsf{z}),\end{aligned} \tag{223}$$

where the last equality is Martin's formula (182). On the other hand, localization with respect to the natural $\mathbb{G}_m$-action on the domain curve gives the factorization

$$\left\langle e^{(t-\tau)\sigma_1/\mathsf{z}}\right\rangle^{\mathbb{G}_m}_{\mathrm{Gr}(r,n)} = \int_{\mathrm{Gr}(r,n)} I_{\mathrm{Gr}(r,n)}(t;\mathsf{z}) \cup I_{\mathrm{Gr}(r,n)}(\tau;-\mathsf{z}). \tag{224}$$

Together with (223), this gives the equality

$$\left\langle e^{(t-\tau)\sigma_1/\mathsf{z}}\right\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n)} = \left\langle e^{(t-\tau)\sigma_1/\mathsf{z}}\right\rangle^{\mathbb{G}_m}_{\mathrm{Gr}(r,n)} \tag{225}$$

of the abelianized correlator and the ordinary correlator.

For any $P(\boldsymbol{x}) \in \mathbb{C}[x_1,\ldots,x_r]^{\mathfrak{S}_r}$, the same argument gives

$$
\begin{aligned}
&\left\langle P(\boldsymbol{x}) e^{(\boldsymbol{t}-\boldsymbol{\tau})\cdot \boldsymbol{x}/z}\right\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n)} \\
&= \int_{\mathrm{Gr}(r,n)} \left(\sum_{\boldsymbol{d}\in\mathbb{N}^r} P(\boldsymbol{x}+\boldsymbol{d}\mathsf{z}) I_{\mathrm{Gr}(r,n),\boldsymbol{d}}(\boldsymbol{t};\mathsf{z}) \right) \cup \left(\sum_{\boldsymbol{d}\in\mathbb{N}^r} I_{\mathrm{Gr}(r,n),\boldsymbol{d}}(\boldsymbol{\tau};-\mathsf{z}) \right) \\
&= \left\langle P(\boldsymbol{x}) e^{(\boldsymbol{t}-\boldsymbol{\tau})\cdot \boldsymbol{x}/z}\right\rangle^{\mathbb{G}_m}_{\mathrm{Gr}(r,n)}
\end{aligned}
\tag{226}
$$

where $t_i = t + (r-1)\pi\sqrt{-1}$ and $\tau_i = \tau + (r-1)\pi\sqrt{-1}$. By setting $t = \tau$ in (226), one obtains

$$
\langle P(\boldsymbol{x})\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n)} = \langle P(\boldsymbol{x})\rangle^{\mathbb{G}_m}_{\mathrm{Gr}(r,n)}\,. \tag{227}
$$

Together with (215), this proves Conjecture 5 for Grassmannians.

12.6

Let $Y \subset \mathrm{Gr}(r,n)$ be the zero locus of a general section of a globally-generated vector bundle $\mathcal{V}$ on $\mathrm{Gr}(r,n)$ associated with a representation V of GL_r. We define the *abelianized* $\mathbb{G}_m$*-equivariant Morrison–Plesser class* of Y by

$$
\Phi^{\mathrm{ab},\mathbb{G}_m}_{\boldsymbol{d}}(Y;\mathsf{z}) := \prod_{\boldsymbol{\delta}\in\Delta(V)} \prod_{l=1}^{\langle\boldsymbol{\delta},\boldsymbol{d}\rangle} (\langle\boldsymbol{\delta},\boldsymbol{x}\rangle + l\mathsf{z})\,. \tag{228}
$$

For $P \in \mathbb{C}[\sigma_1,\ldots,\sigma_r]$, we set

$$
\langle P(\sigma_1,\ldots,\sigma_r)\rangle^{\mathrm{ab},\mathbb{G}_m}_{Y} := \sum_{\boldsymbol{d}\in\mathbb{N}^r} q^{|\boldsymbol{d}|} \left\langle P(\sigma_1,\ldots,\sigma_r)\Phi^{\mathrm{ab},\mathbb{G}_m}_{\boldsymbol{d}}(Y;\mathsf{z}) v\right\rangle^{\mathrm{ab},\mathbb{G}_m}_{\mathrm{Gr}(r,n),\boldsymbol{d}}, \tag{229}
$$

where $v := \prod_{\boldsymbol{\delta}\in\Delta(V)} \langle\boldsymbol{\delta},\boldsymbol{x}\rangle$ is the Euler class of the normal bundle of Y in $\mathrm{Gr}(r,n)$. By the same reasoning as in Sect. 12.5 with the insertion of the abelizanized Morrison–Plesser class, one obtains

$$
\langle P(\sigma_1,\ldots,\sigma_r)\rangle_Y = (-1)^{|\Delta(V)|}\langle P(\sigma_1,\ldots,\sigma_r)\rangle_{\mathrm{GLSM}}\,. \tag{230}
$$

Here, the identification between q and the Fayet–Illiopoulos parameter t' is given by

$$
q = (-1)^{\sum_{\boldsymbol{\delta}\in\Delta(V)}\langle\boldsymbol{\delta},\mathbf{1}\rangle} e^{t'} \tag{231}
$$

where $\mathbf{1} := (1,\cdots,1) \in \mathbb{N}^r$.

12.7

As an example, consider the vector bundle of rank 3 on $\mathrm{Gr}(3, 5) = \mathrm{Mat}(3, 5)/\!\!/U(3)$ associated with the representation of $U(3)$ determined by the Young diagram

$$\lambda = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}. \tag{232}$$

This vector bundle is the tensor product $\wedge^2 \mathcal{Q}(1)$ of the second exterior power $\wedge^2 \mathcal{Q}$ of the universal quotient bundle $\mathcal{Q}$ on $\mathrm{Gr}(2, 5) \cong \mathrm{Gr}(3, 5)$ and the ample generator $\mathcal{O}(1)$ of the Picard group. One can immediately see from the Young diagram that the restriction of the representation of $U(3)$ associated with λ to the diagonal maximal torus $T \cong (\mathbb{G}_m)^3$ is the direct sum $\rho_{1,2,2} \oplus \rho_{2,1,2} \oplus \rho_{2,2,1}$. The associated line bundle on the abelian quotient $(\mathbb{P}^4)^3$ is given by $\mathcal{O}(1, 2, 2) \oplus \mathcal{O}(2, 1, 2) \oplus \mathcal{O}(2, 2, 1)$.

The complete intersection in $\mathrm{Gr}(3, 5)$ defined by $\wedge^2 \mathcal{Q}(1)$ is a Calabi–Yau 3-fold of Picard number 1, which will be denoted by Y henceforth. The Euler class of the normal bundle of Y is

$$v := (x_1 + 2x_2 + 2x_3)(2x_1 + x_2 + 2x_3)(2x_1 + 2x_2 + x_3), \tag{233}$$

the abelianized Morrison–Plesser class is

$$\begin{aligned} \Phi^{\mathrm{ab}}(Y; \boldsymbol{d}) &:= (x_1 + 2x_2 + 2x_3)^{d_1+2d_2+2d_3} \\ &(2x_1 + x_2 + 2x_3)^{2d_1+d_2+2d_3}(2x_1 + 2x_2 + x_3)^{2d_1+2d_2+d_3}, \end{aligned} \tag{234}$$

and the generating function for σ_1^3 is

$$\begin{aligned} \left\langle \sigma_1^3 \right\rangle_Y^{\mathrm{ab}} &= -\frac{1}{6} \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{\infty} \sum_{d_3=0}^{\infty} q^{d_1+d_2+d_3} \operatorname{Res}(x_1 + x_2 + x_3)^3 \\ &\quad (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2 \Phi^{\mathrm{ab}}(Y; \boldsymbol{d}) v \frac{dx_1}{x_1^{n(d_1+1)}} \wedge \frac{dx_2}{x_2^{n(d_2+1)}} \wedge \frac{dx_3}{x_3^{n(d_3+1)}} \end{aligned} \tag{235}$$

$$= \frac{25(1-q)}{(1+q)(1-123q+q^2)}. \tag{236}$$

This matches the Yukawa coupling of the mirror computed by Miura [51, Sect. 5.2].

12.8

When $\mathcal{V}$ is a direct sum of line bundles, the mirror of Y is constructed by toric degenerations [6, 7]. It is an interesting problem to compare the generating function (205) with the Yukawa coupling of this mirror.

13 Bethe/Gauge Correspondence

13.1

Let V_1 and W_1 be Hermitian vector spaces of dimensions r and n. The unitary group $U(r)$ acts naturally on V_1 and trivially on W_1, inducing an action on $T^* \operatorname{Hom}(V_1, W_1) \cong \operatorname{Hom}(V_1, W_1) \oplus \operatorname{Hom}(W_1, V_1)$. The real and complex moment maps for this action are given by

$$\mu_{\mathbb{R}} \colon \operatorname{Hom}(W_1, V_1) \oplus \operatorname{Hom}(V_1, W_1) \to \operatorname{End}(V_1), \quad (i_1, j_1) \mapsto \frac{\sqrt{-1}}{2} \left(i_1 i_1^* - j_1^* j_1 \right), \tag{237}$$

$$\mu_{\mathbb{C}} \colon \operatorname{Hom}(W_1, V_1) \oplus \operatorname{Hom}(V_1, W_1) \to \operatorname{End}(V_1), \quad (i_1, j_1) \mapsto i_1 j_1. \tag{238}$$

If $(i_1, j_1) \in \mu_{\mathbb{R}}^{-1} \left(\zeta \sqrt{-1} \operatorname{id}_{V_1} \right)$ for $\zeta < 0$, then j_1 is injective. If $(i_1, j_1) \in \mu_{\mathbb{C}}^{-1}(0)$, then i_1 descends to a map $W_1 / \operatorname{Im} j_1 \to V_1$. It follows that the hyperKähler quotient is isomorphic to $T^* \operatorname{Gr}(r, n)$;

$$\left(\zeta_{\mathbb{R}}^{-1} \left(\zeta \sqrt{-1} \operatorname{id}_{V_1} \right) \cap \mu_{\mathbb{C}}^{-1}(0) \right) \Big/ U(r) \cong T^* \operatorname{Gr}(r, n). \tag{239}$$

This suggests that the gauged linear sigma model with the gauge group $U(r)$ and the representation $V := \operatorname{Hom}(W_1, V_1) \oplus \operatorname{Hom}(V_1, W_1) \oplus \operatorname{End}(V_1)$ describes the quantum cohomology of $T^* \operatorname{Gr}(r, n)$. Here $\operatorname{End}(V_1)$ is the Lagrange multiplier for the complex moment map equation, and the potential is given by

$$V \ni (i_1, j_1, P) \mapsto \operatorname{tr}(P i_1 j_1). \tag{240}$$

Let $H := H_1 \times H_2$ be the product of

- the diagonal maximal torus H_1 of $U(n)$, acting on $\operatorname{Hom}(W_1, V_1)$ and $\operatorname{Hom}(V_1, W_1)$ through the natural action on W_1, and trivially on $\operatorname{End}(V_1)$, and
- the group $H_2 = U(1)$ acting trivially on $\operatorname{Hom}(W_1, V_1)$, by scalar multiplication on $\operatorname{Hom}(V_1, W_1)$, and by inverse scalar multiplication on $\operatorname{End}(V_1)$.

One has

$$Z_d^{\mathrm{vec}}(x) = \prod_{1 \le i \ne j \le r} (x_i - x_j), \tag{241}$$

$$Z_d^{\mathrm{mat}}(x) = \prod_{j=1}^{n} \prod_{i=1}^{r} (x_i - \lambda_j)^{-d_i - 1} \tag{242}$$

$$\times \prod_{j=1}^{n} \prod_{i=1}^{r} (-x_i + \lambda_j - \mu)^{-(-d_i) - 1} \tag{243}$$

$$\times \prod_{1 \le i \ne j \le r} (x_i - x_j + \mu)^{2-(d_i-d_j)-1}, \tag{244}$$

so that the H-equivariant correlator of $P \in \mathbb{C}[x_1, \dots, x_r]^{\mathfrak{S}_r}$ is given by

$$\begin{aligned} \langle P \rangle^H_{\text{GLSM}} = \frac{1}{r!} \sum_{d_1=0}^{\infty} \cdots \sum_{d_r=0}^{\infty} ((-1)^{r-1} e^t)^{d_1+\cdots+d_r} \\ \text{Res} \Bigg[\frac{\prod_{1 \le i \ne j \le r} (x_i - x_j)}{\prod_{1 \le i, j \le r} (x_i - x_j + \mu)^{(d_i-d_j-1)}} \\ \frac{\prod_{j=1}^n \prod_{i=1}^r (-x_i + \lambda_j - \mu)^{d_i-1}}{\prod_{j=1}^n \prod_{i=1}^r (x_i - \lambda_j)^{d_i+1}} P dx_1 \wedge \cdots \wedge dx_r \Bigg], \end{aligned} \tag{245}$$

where Res denotes the sum of residues at the points where x_i is one of λ_j for $i = 1, \dots, r$ and $j = 1, \dots, n$ (there are n^r such points). This can formally be regarded as an equivariant integration over the projective space of dimension $\sum_{i=1}^r (d_i + 1) - 1$, and it is an interesting problem to give a geometric interpretation.

The effective potential (10) of this gauged linear sigma model is given by

$$W_{\text{eff}}(\boldsymbol{x}; t) = W_{\text{FI}}(\boldsymbol{x}; t') + W_{\text{vec}}(\boldsymbol{x}) + W_{\text{mat}}(\boldsymbol{x}), \tag{246}$$

$$W_{\text{FI}}(\boldsymbol{x}; t) = t(x_1 + \cdots + x_r), \tag{247}$$

$$\begin{aligned} W_{\text{vec}}(\boldsymbol{x}) &= -\pi\sqrt{-1} \sum_{1 \le i < j \le r} (x_j - x_i) \\ &= -\pi\sqrt{-1} \sum_{i=1}^r (2i - r - 1) x_i, \end{aligned} \tag{248}$$

$$\begin{aligned} W_{\text{mat}}(\boldsymbol{x}) = &- \sum_{i=1}^r \sum_{j=1}^n \left(x_i - \lambda_j\right) \left(\log\left(x_i - \lambda_j\right) - 1\right) \\ &- \sum_{i=1}^r \sum_{j=1}^n \left(-x_i + \lambda_j - \mu\right) \left(\log\left(-x_i + \lambda_j - \mu\right) - 1\right) \\ &- \sum_{i=1}^r \sum_{j=1}^r \left(x_i - x_j + \mu\right) \left(\log\left(x_i - x_j + \mu\right) - 1\right), \end{aligned} \tag{249}$$

where λ_j and μ are equivariant parameters for the actions of H_1 and H_2 respectively. Note that

$$e^{\partial W_{\text{eff}}/\partial x_i} = e^t \cdot (-1)^{2i-r-1} \cdot \prod_{j=1}^n (x_i - \lambda_j)^{-1} \prod_{j=1}^n (-x_i + \lambda_j - \mu) \prod_{j \ne i} \frac{x_j - x_i + \mu}{x_i - x_j + \mu} \tag{250}$$

$$= e^{t+n\pi\sqrt{-1}} \prod_{j=1}^{n} \frac{x_i - \lambda_j + \mu}{x_i - \lambda_j} \prod_{j \neq i} \frac{x_i - x_j - \mu}{x_i - x_j + \mu}, \tag{251}$$

so that the equations $e^{\partial_{x_i} W_{\text{eff}}} = 1$, $i = 1, \ldots, r$ gives

$$\prod_{j=1}^{n} \frac{x_i - \lambda_j}{x_i - \lambda_j + \mu} = e^{t+n\pi\sqrt{-1}} \prod_{j \neq i} \frac{x_i - x_j - \mu}{x_i - x_j + \mu}. \tag{252}$$

By taking the sum over d_i just as in the proof of Corollary 2, one obtains

$$\langle P \rangle_{\text{GLSM}}^{H} = \frac{1}{r!} \operatorname{Res} \left[\frac{1}{\prod_{i=1}^{r} \left(\left(1 - e^{\partial_{x_i} W_{\text{eff}}}\right) \prod_{j=1}^{n} \left(x_i - \lambda_j\right) \right)} \frac{\prod_{1 \le i \neq j \le r} (x_i - x_j) \prod_{1 \le i, j \le r} (x_i - x_j + \mu)}{\prod_{i=1}^{r} \prod_{j=1}^{n} (-x_i + \lambda_j - \mu)} P dx_1 \wedge \cdots \wedge dx_r \right] \tag{253}$$

where Res denotes the sum of residues at the roots of the Eqs. (252).

13.2

The Heisenberg model, also known as the homogeneous $XXX_{\frac{1}{2}}$ model, is the SU(2) spin chain model with Hamiltonian

$$H = \sum_{i=1}^{n} \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}, \tag{254}$$

where $\boldsymbol{S}_i = (S_i^x, S_i^y, S_i^z) = (\sigma_i^x/2, \sigma_i^y/2, \sigma_i^z/2)$ are halves of Pauli matrices acting on the ith factor of the Hilbert space $\mathcal{H} := (\mathbb{C}^2)^{\otimes n}$ and

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} := S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z. \tag{255}$$

The total spin

$$S^z := \sum_{i=1}^{n} S_i^z \tag{256}$$

clearly commutes with the Hamiltonian, and we restrict to the S^z-eigenspace $\mathcal{H}_r \subset \mathcal{H}$ with eigenvalue $(-n + r)/2$. We impose the quasi-periodicity condition

$$\boldsymbol{S}_{n+1} = e^{\sqrt{-1}\vartheta S_1^z} \boldsymbol{S}_1 e^{-\sqrt{-1}\vartheta S_1^z}. \tag{257}$$

Introduce variables $\boldsymbol{x} = (x_1, \dots, x_r)$ related to quasi-momenta $\boldsymbol{p} = (p_1, \dots, p_r)$ by

$$e^{\sqrt{-1}p_i} = \frac{x_i + \frac{\sqrt{-1}}{2}}{x_i - \frac{\sqrt{-1}}{2}}. \tag{258}$$

Then H-eigenspaces in $\mathcal{H}_r$ correspond bijectively to solutions of the *Bethe equation*

$$\left(\frac{x_i + \frac{\sqrt{-1}}{2}}{x_i - \frac{\sqrt{-1}}{2}}\right)^n = e^{\sqrt{-1}\vartheta} \prod_{j \neq i} \frac{x_i - x_j + \sqrt{-1}}{x_i - x_j - \sqrt{-1}} \tag{259}$$

with eigenvalues $n - 2r + 2\sum_{i=1}^r \cos p_i$. The integrability comes from factorization of many-body S-matrix into the product of the 2-body S-matrix given by

$$S(p_i, p_j) = 1 - 2e^{\sqrt{-1}p_j} + e^{\sqrt{-1}(p_i + p_j)}. \tag{260}$$

See e.g. [62] and references therein for Bethe ansatz for the quasi-periodic Heisenberg model. The Bethe equation (259) coincides with (252) under $\lambda_j = \frac{\sqrt{-1}}{2}$, $j = 1, \dots, n$, $\mu = -\sqrt{-1}$, and $\vartheta = -\sqrt{-1}t + n/2$. This observation and its generalizations is called *Bethe/gauge correspondence* [56]. The relation between classical/quantum cohomology of Grassmannians and integrable systems is studied in [14, 36, 49, 57].

14 Quasimaps and Instantons

14.1

As explained in [29, Sect. 2.3], the moduli space of framed instantons on $\mathbb{C} \times [\mathbb{C}/(\mathbb{Z}/n\mathbb{Z})]$ is isomorphic to the Nakajima quiver variety associated with the *chainsaw quiver* shown in Fig. 1.

14.2

Representations of the chainsaw quiver satisfying $\dim V_n = 0$ are in one-to-one correspondence with representations of the *handsaw quiver* shown in Fig. 2. It is shown in [29, Sect. 2.3] (see also [55, Sect. 3] for an exposition) that the Nakajima quiver variety associated with the handsaw quiver is isomorphic to the *parabolic Laumon space* parametrizing flags

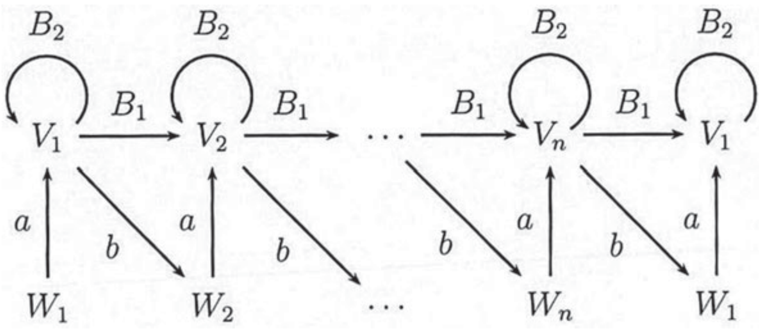

Fig. 1 The chainsaw quiver

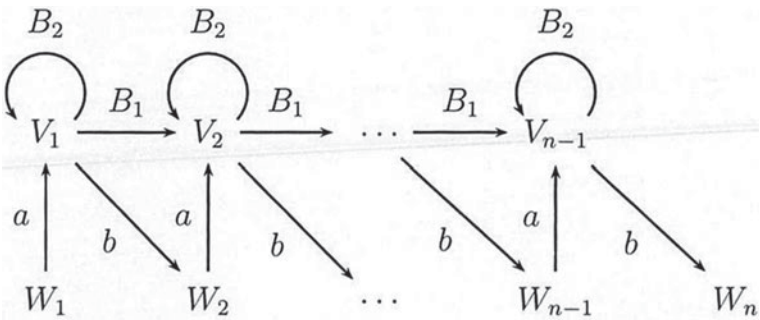

Fig. 2 The handsaw quiver

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \tag{261}$$

of locally free sheaves on $\mathbb{P}^1$ such that rank $E_i = \sum_{j \le i} \dim W_j$, $\deg E_i = -\dim V_i$, and the flag at $\infty \in \mathbb{P}^1$ is equal to the standard flag $0 \subset W_1 \subset W_1 \oplus W_2 \subset \cdots \subset W_1 \oplus W_2 \oplus \cdots \oplus W_{n-1} \subset W$. This coincides with the space of *based quasimaps* to partial flag varieties, i.e., quasimaps with specified value at infinity.

15 Quasimaps and Monopoles

15.1

Let G be a compact Lie group with a maximal torus H. A monopole on $\mathbb{R}^3$ is a pair (A, Φ) of a connection A on a principal G-bundle P and a section Φ of $P \times_G \mathfrak{g}$ satisfying the *Bogomolny equation*

$$F_A = *d_A \Phi. \tag{262}$$

In order for the curvature to have a finite L^2-norm, it is natural to demand that the restriction of Φ to a sphere with large radius tends to a map to a fixed adjoint orbit $\mathcal{O} \cong G/H \cong G_{\mathbb{C}}/P$. The homotopy class $\boldsymbol{k} \in \pi_2(G_{\mathbb{C}}/P)$ of the resulting map is called the *charge* of the monopole.

15.2

A choice of a gauge satisfying a certain boundary condition at infinity is called a *framing* of the monopole. The framed moduli space is a principal H-bundle over the unframed moduli space. The framed moduli space has a natural hyperKähler structure coming from the dimensional reduction of the anti-self-dual equation in dimension 4.

15.3

Monopoles on $\mathbb{R}^3$ are related to

(1) spectral curves on $T\mathbb{P}^1$,
(2) Nahm's equation

$$\frac{dT_i}{ds} = \epsilon_{ijk}[T_j, T_k], \quad i = 1, 2, 3 \tag{263}$$

for $T_i \in C^\infty((0,2), \mathrm{Mat}(k,k;\mathbb{C}))$, and
(3) based quasimaps from $\mathbb{P}^1$ to $G_{\mathbb{C}}/P$ of degree $\boldsymbol{k}$.

(1) comes from the twistor correspondence [38, 39], and (2) comes from Nahm transform [54]. (3) is proved for $\mathrm{SU}(2)$ in [27], and the general case can be found in [44, 45] and references therein.

16 Quasimaps and Vortices

16.1

Let X be a Kähler manifold, (E, h) be a Hermitian vector bundle on X, and τ be a positive real number. The Yang–Mills–Higgs functional sends a pair (A, ϕ) of a unitary connection d_A of (E, h) and a section ϕ of E to

$$\mathscr{YMH}(A, \phi) = \|F_A\|_{L^2}^2 + \|d_A \phi\|_{L^2}^2 + \frac{1}{4} \left\| \phi \otimes \phi^* - \tau \right\|_{L^2}^2 . \tag{264}$$

By [12, Proposition 2.1], one has

$$\begin{aligned} \mathscr{YMH}(A, \phi) = 4 \left\| F^{0,2} \right\|_{L^2}^2 + 2 \left\| \overline{\partial}_A \phi \right\|_{L^2}^2 + \left\| \sqrt{-1} \Lambda F + \frac{1}{2} \phi \otimes \phi^* - \frac{\tau}{2} \right\|_{L^2}^2 \\ + \tau \int_X \sqrt{-1} \operatorname{tr} F \wedge \omega^{[n-1]} + \int_X \operatorname{tr} F \wedge F \wedge \omega^{[n-2]}. \end{aligned} \tag{265}$$

where $\omega^{[k]} := \omega^k/(k!)$ and Λ is the dual Lefschetz operator.

16.2

Assume that X is a projective curve, so that

$$\deg(E) = \frac{\sqrt{-1}}{2\pi} \operatorname{tr} F. \tag{266}$$

Then (265) immediately implies the Bogomolny–Prasad–Sommerfield inequality

$$\mathscr{YMH}(A, \phi) \geq 2\pi\tau \deg(E), \tag{267}$$

and the equality holds if and only if the *vortex equation*

$$F^{0,2} = 0, \tag{268}$$

$$\overline{\partial}_A \phi = 0, \tag{269}$$

$$-\sqrt{-1} \Lambda F = \frac{1}{2} \left(\phi \otimes \phi^* - \tau \operatorname{id}_E \right) \tag{270}$$

is satisfied. Equations (268) and (269) are holomorphicities for E and ϕ, and (270) is a generalization of the constant central curvature equation.

16.3

By taking the trace of (270) and integrating over X, one obtains

$$-2\pi \deg(E) = \frac{1}{2} \|\phi\|_{L^2}^2 - \frac{1}{2} \tau \operatorname{rank}(E) \operatorname{vol}(X), \tag{271}$$

so that the condition

$$\tau \geq \frac{4\pi \deg(E)}{\operatorname{rank}(E)\operatorname{vol}(X)} \tag{272}$$

is necessary for (270) to have a solution.

16.4

The *slope* of a holomorphic vector bundle E is defined by

$$\mu(E) = \frac{\deg(E)}{\operatorname{rank}(E)}. \tag{273}$$

For a holomorphic section ϕ of E, we set

$$\begin{aligned}
\hat{\mu}(E) &:= \sup\left\{\mu(E') \;\middle|\; E' \text{ is a reflexive subsheaf of } E \text{ of rank less than } E\right\},\\
\mu_M(E) &:= \max\left\{\hat{\mu}(E), \mu(E)\right\},\\
\mu_m(E,\phi) &:= \inf\left\{\frac{\operatorname{rank}(E)\mu(E) - \operatorname{rank}(E')\mu(E')}{\operatorname{rank}(E) - \operatorname{rank}(E')} \;\middle|\; \right.\\
&\qquad \left. E' \text{ is a reflexive subsheaf of } E \text{ such that } \operatorname{rank} E' < \operatorname{rank} E \text{ and } \phi \in \Gamma(E')\right\}.
\end{aligned}$$

A pair (E, ϕ) of a holomorphic vector bundle E and its holomorphic section ϕ is said to be *stable* if

$$\mu_M(E) < \mu_m(E, \phi). \tag{274}$$

Theorem 7 ([13, Theorem 2.1.6]) *Let (E, ϕ) be a pair of a holomorphic vector bundle and its holomorphic section. If there exists a Hermitian metric on E satisfying the vortex equation, then one has either of the following:*

(i) *(E, ϕ) is stable and satisfies*

$$\mu_M < \frac{\tau \operatorname{Vol}(X)}{4\pi} < \mu_m(\phi). \tag{275}$$

(ii) *E has a direct sum decomposition $E = E_\phi \oplus E'$, ϕ is an element of $H^0(E_\phi) \subset H^0(E)$, (E_ϕ, ϕ) satisfies (i) above, and E' is the direct sum of stable vector bundles of slope $\tau \operatorname{Vol}(X)/4\pi$.*

Theorem 8 ([13, Theorem 3.1.1]) *Let (E, ϕ) be a stable pair of a holomorphic vector bundle and its holomorphic section. Then for any real number τ satisfying (275), there exists a Hermitian metric on E satisfying (270).*

Bradlow proved these results not only for projective curves but also for compact Kähler manifolds.

16.5

Vortex Eq. (270) admits the following generalization, which also contains Hitchin's self-duality equation [40] as a special case. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $M = (M_a)_{a \in Q_1}$ be a collection of vector bundles on X labeled by Q_1. An *M-twisted Q-sheaf on X* is a pair $R = \left((E_v)_{v \in Q_0}, (\phi_a)_{a \in Q_1}\right)$ of a collection $(E_v)_{v \in Q_0}$ of vector bundles labeled by Q_0 and a collection

$$(\phi_a)_{a \in Q_1} \in \prod_{a \in Q_1} \operatorname{Hom}\left(E_{s(a)} \otimes M_a, E_{t(a)}\right) \tag{276}$$

of morphisms labeled by Q_1.

Given a collection $(E_v)_{v \in Q_0}$ of holomorphic vector bundles on a Kähler manifold X, another collection $(M_a)_{a \in Q_1}$ of holomorphic vector bundles on X, a collection $\sigma = (\sigma_v)_{v \in Q_0}$ of positive real numbers, and a collection $\tau = (\tau_v)_{v \in Q_0}$ of real numbers, the equation

$$\sigma_v \sqrt{-1} \Lambda F_v + \sum_{t(a)=v} \phi_a \circ \phi_a^* - \sum_{s(a)=v} \phi_a^* \circ \phi_a = \tau_v \operatorname{id}_{E_v} \tag{277}$$

for Hermitian metrics on $(E_v)_{v \in Q_0}$ is called the *M-twisted quiver (σ, τ)-vortex equation*.

The (σ, τ)-*degree* and the (σ, τ)-*slope* of an M-twisted Q-sheaf R is defined by

$$\deg_{\sigma,\tau}(R) = \sum_{v \in Q_0} (\sigma_v \deg E_v - \tau_v \operatorname{rank} E_v), \tag{278}$$

$$\mu_{\sigma,\tau}(R) = \frac{\deg_{\sigma,\tau}(R)}{\sum_{v \in Q_0} \sigma_v \operatorname{rank} E_v}. \tag{279}$$

A Q-sheaf is *stable* if one has $\mu_{\sigma,\tau}(R') < \mu_{\sigma,\tau}(R)$ for any proper subsheaf R'. A Q-sheaf is *polystale* if it is the direct sum of stable Q-sheaf of the same slope.

Theorem 9 ([1, Theorem 3.1]) *A Q-sheaf R with* $\deg_{\sigma,\tau}(R) = 0$ *admits a Hermitian metric satisfying the quiver vortex Eq. (277) if and only if R is (σ, τ)-polystable. This Hermitian metric is unique up to a multiplication by a positive constant for each stable summand.*

Quasimaps to $\operatorname{Mat}(r, n) /\!\!/ \operatorname{GL}_r$ corresponds to the case when the quiver $Q = (1 \to 2)$ consists of two vertices and one arrow between them, M_1 and M_2 are the structure sheaves, $\operatorname{rank} E_1 = r$, and E_2 is the trivial bundle of rank n.

16.6

Note that the map $V \to \mathrm{End}(V)$, $\phi \mapsto \phi \otimes \phi^*$ appearing in (270) is the moment map for the natural action of the unitary group $U(V)$ on V. With this in mind, a generalization

$$*F_A + \mu(\Phi) = \tau \, \mathrm{id}_E \tag{280}$$

of the vortex Eq. (270) to the case where one has a Hamiltonian action of a compact group G on a Kähler manifold X is given in [19, 58]. Here A is a connection on a principal G-bundle on a curve C, Φ is a holomorphic section of $P \times_G X$, and $\mu\colon X \to \mathfrak{g}$ is the moment map. They are used to define invariants of a symplectic manifold with a Hamiltonian group action [19, 20, 59], which are closely related to the Gromov–Witten invariants of the symplectic quotient [30, 71–74]. Reference [18] use wall-crossing in vortex invariants to study quantum cohomology of monotone toric varieties with minimal Chern number greater than or equal to 2.

16.7

Let X be a Kähler manifold with a Hamiltonian action of a compact connected Lie group G. We assume that X is either compact or equivariantly convex at infinity with a proper moment map. We fix an invariant inner product to identify $\mathfrak{g}^\vee$ with $\mathfrak{g}$, and write the moment map as $\mu\colon X \to \mathfrak{g}$.

An *affine vortex* is a pair (A, u) of a connection A on the principal bundle $P = \mathbb{C} \times G$ and a holomorphic section $u\colon C \to P \times_G X$ satisfying the *vortex equation*

$$*F_A + \mu(u) = 0. \tag{281}$$

A *gauged holomorphic map* to X with respect to the complex Lie group $G_{\mathbb{C}}$ acting on X is a map to the quotient stack $[X/G_{\mathbb{C}}]$. In other words, a gauged holomorphic map from a scheme C to X is a pair (P, u) of a principal $G_{\mathbb{C}}$-bundle P over C and a $G_{\mathbb{C}}$-equivariant holomorphic map $u\colon P \to X$.

If the $G_{\mathbb{C}}$-action on X^{ss} is free, then by [68, Theorem 1.1], there is a natural bijection between the set of affine K-vortices with target X up to gauge equivalence and the set of pairs gauged holomorphic maps such that $u(\infty) \in X^{\mathrm{ss}}$. This is an open substack of the set of quasimaps such that ∞ is not contained in the base locus.

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References

1. Álvarez-Cónsul, L., García-Prada, O.: Hitchin-Kobayashi correspondence, quivers, and vortices. Commun. Math. Phys. **238**(1–2), 1–33 (2003). MR 1989667 (2005b:32027)
2. Atiyah, M.F., Bott, R.: The moment map and equivariant cohomology. Topology **23**(1), 1–28 (1984). MR 721448
3. Batyrev, V.V., Materov, E.N.: Toric residues and mirror symmetry. Mosc. Math. J. **2**(3), 435–475 (2002). Dedicated to Yuri I. Manin on the occasion of his 65th birthday. MR 1988969 (2005a:14070)
4. Batyrev, V.V., Materov, E.N.: Mixed toric residues and Calabi-Yau complete intersections. Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001). Fields Institute Communications, vol. 38, pp. 3–26. American Mathematical Society, Providence (2003). MR 2019144 (2005b:14088)
5. Batyrev, V.V., van Straten, D.: Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties. Commun. Math. Phys. **168**(3), 493–533 (1995). MR 1328251 (96g:32037)
6. Batyrev, V.V., Ciocan-Fontanine, I., Kim, B., van Straten, D.: Conifold transitions and mirror symmetry for Calabi-Yau complete intersections in Grassmannians. Nucl. Phys. B **514**(3), 640–666 (1998). MR MR1619529 (99m:14074)
7. Batyrev, V.V., Ciocan-Fontanine, I., Kim, B., van Straten, D.: Mirror symmetry and toric degenerations of partial flag manifolds. Acta Math. **184**(1), 1–39 (2000). MR MR1756568 (2001f:14077)
8. Behrend, K., Fantechi, B.: The intrinsic normal cone. Invent. Math. **128**(1), 45–88 (1997). MR MR1437495 (98e:14022)
9. Benini, F., Zaffaroni, A.: A topologically twisted index for three-dimensional supersymmetric theories. J. High Energy Phys. **7**, 127 (2015). Front matter+75. MR 3383085
10. Bertram, A., Ciocan-Fontanine, I., Kim, B.: Two proofs of a conjecture of Hori and Vafa. Duke Math. J. **126**(1), 101–136 (2005). MR MR2110629
11. Borisov, L.A.: Higher-Stanley-Reisner rings and toric residues. Compos. Math. **141**(1), 161–174 (2005). MR 2099774 (2005j:14074)
12. Bradlow, S.B.: Vortices in holomorphic line bundles over closed Kähler manifolds. Commun. Math. Phys. **135**(1), 1–17 (1990). MR 1086749 (92f:32053)
13. Bradlow, S.B.: Special metrics and stability for holomorphic bundles with global sections. J. Differ. Geom. **33**(1), 169–213 (1991). MR 1085139 (91m:32031)
14. Braverman, A., Maulik, D., Okounkov, A.: Quantum cohomology of the Springer resolution. Adv. Math. **227**(1), 421–458 (2011). MR 2782198 (2012h:14133)
15. Brion, M., Vergne, M.: Arrangement of hyperplanes. I. Rational functions and Jeffrey-Kirwan residue. Ann. Sci. École Norm. Sup. (4) **32**(5), 715–741 (1999). MR 1710758
16. Candelas, P., de la Ossa, X.C., Green, P.S., Parkes, L.: A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. Nucl. Phys. B **359**(1), 21–74 (1991). MR MR1115626 (93b:32029)
17. Cheong, D., Ciocan-Fontanine, I., Kim, B.: Orbifold quasimap theory. Math. Ann. **363**(3–4), 777–816 (2015). MR 3412343
18. Cieliebak, K., Salamon, D.: Wall crossing for symplectic vortices and quantum cohomology. Math. Ann. **335**(1), 133–192 (2006). MR 2217687
19. Cieliebak, K., Gaio, A.R., Salamon, D.A.: J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions. Int. Math. Res. Not. **16**, 831–882 (2000). MR 1777853
20. Cieliebak, K., Gaio, A.R., i Riera, I.M., Salamon, D.A.: The symplectic vortex equations and invariants of Hamiltonian group actions. J. Symplectic Geom. **1**(3), 543–645 (2002). MR 1959059 (2004g:53098)
21. Ciocan-Fontanine, I., Kim, B.: Quasimap theory, to appear in Proceedings of the International Congress of Mathematicians, Seoul (2014)
22. Ciocan-Fontanine, I., Kim, B.: Wall-crossing in genus zero quasimap theory and mirror maps. Algebraic Geometry **1**(4), 400–448 (2014). MR 3272909

23. Ciocan-Fontanine, I., Kim, B., Maulik, D.: Stable quasimaps to GIT quotients. J. Geom. Phys. **75**, 17–47 (2014). MR 3126932
24. Closset, C., Cremonesi, S., Park, D.S.: The equivariant A-twist and gauged linear sigma models on the two-sphere. J. High Energy Phys. **6**, 076 (2015). Front matter+110. MR 3370259
25. Cox, D.A.: Toric residues. Ark. Mat. **34**(1), 73–96 (1996). MR 1396624
26. Cox, D.A., Katz, S.: Mirror Symmetry and Algebraic Geometry. Mathematical Surveys and Monographs, vol. 68. American Mathematical Society, Providence (1999). MR MR1677117 (2000d:14048)
27. Donaldson, S.K.: Nahm's equations and the classification of monopoles. Commun. Math. Phys. **96**(3), 387–407 (1984). MR 769355
28. Fan, H., Jarvis, T.J., Ruan, Y.: A mathematical theory of the gauged linear sigma model. Geom. Topol. **22**(1), 235–303 (2018). MR 3720344
29. Finkelberg, M., Rybnikov, L.: Quantization of Drinfeld Zastava in type A. J. Eur. Math. Soc. (JEMS) **16**(2), 235–271 (2014). MR 3161283
30. Gaio, A.R.P., Salamon, D.A.: Gromov-Witten invariants of symplectic quotients and adiabatic limits. J. Symplectic Geom. **3**(1), 55–159 (2005). MR 2198773
31. Givental, A.: Equivariant Gromov-Witten invariants. Int. Math. Res. Not. **13**, 613–663 (1996). MR MR1408320 (97e:14015)
32. Givental, A.: A mirror theorem for toric complete intersections. Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996). Progress in Mathematics, vol. 160, pp. 141–175. Birkhäuser, Boston (1998). MR MR1653024 (2000a:14063)
33. Givental, A.B.: Homological geometry and mirror symmetry. In: Proceedings of the International Congress of Mathematicians, vol. 1, 2, pp. 472–480. (Zürich, 1994) (Basel), Birkhäuser (1995). MR MR1403947 (97j:58013)
34. Givental, A.B.: Homological Geometry. I. Projective Hypersurfaces. Sel. Math. (N.S.) **1**(2), 325–345 (1995). MR MR1354600 (97c:14052)
35. Gonzalez, E., Woodward, C.: Gauged Gromov-Witten theory for small spheres. Math. Z. **273**(1–2), 485–514 (2013). MR 3010172
36. Gorbounov, V., Rimányi, R., Tarasov, V., Varchenko, A.: Quantum cohomology of the cotangent bundle of a flag variety as a Yangian Bethe algebra. J. Geom. Phys. **74**, 56–86 (2013). MR 3118573
37. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Pure and Applied Mathematics. Wiley-Interscience [Wiley], New York (1978). MR 507725 (80b:14001)
38. Hitchin, N.J.: Monopoles and geodesics. Commun. Math. Phys. **83**(4), 579–602 (1982). MR 649818
39. Hitchin, N.J.: On the construction of monopoles. Commun. Math. Phys. **89**(2), 145–190 (1983). MR 709461
40. Hitchin, N.J.: The self-duality equations on a Riemann surface. Proc. Lond. Math. Soc. (3) **55**(1), 59–126 (1987). MR 887284 (89a:32021)
41. Hori, K., Vafa, C.: Mirror symmetry. arXiv:hep-th/0002222
42. Intriligator, K.: Fusion residues. Modern Phys. Lett. A **6**(38), 3543–3556 (1991). MR 1138873 (92k:81180)
43. Iritani, H.: Quantum cohomology and periods. Ann. Inst. Fourier (Grenoble) **61**(7), 2909–2958 (2011). MR 3112512
44. Jarvis, S.: Construction of Euclidean monopoles. Proc. Lond. Math. Soc. (3) **77**(1), 193–214 (1998). MR 1625471
45. Jarvis, S.: Euclidean monopoles and rational maps. Proc. Lond. Math. Soc. (3) **77**(1), 170–192 (1998). MR 1625475
46. Karu, K.: Toric residue mirror conjecture for Calabi-Yau complete intersections. J. Algebr. Geom. **14**(4), 741–760 (2005). MR 2147350 (2006c:14063)
47. Konishi, Y., Minabe, S.: Local B-model and mixed Hodge structure. Adv. Theor. Math. Phys. **14**(4), 1089–1145 (2010). MR 2821394 (2012h:14106)
48. Martin, S.: Symplectic quotients by a nonabelian group and by its maximal torus. arXiv:math.SG/0001002

49. Maulik, D., Okounkov, A.: Quantum groups and quantum cohomology. Astérisque **408**, ix+209 (2019). MR 3951025
50. Mavlyutov, A.R.: Semiample hypersurfaces in toric varieties. Duke Math. J. **101**(1), 85–116 (2000). MR 1733735
51. Miura, M.: Hibi toric varieties and mirror symmetry. Ph.D. thesis, The University of Tokyo (2013)
52. Morrison, D.R., Plesser, M.R.: Summing the instantons: quantum cohomology and mirror symmetry in toric varieties. Nucl. Phys. B **440**(1–2), 279–354 (1995). MR 1336089 (96f:32036)
53. Mukai, S.: Polarized $K3$ surfaces of genus 18 and 20. Complex Projective Geometry (Trieste, 1989/Bergen, 1989). London Mathematical Society Lecture Note Series, vol. 179, pp. 264–276. Cambridge University Press, Cambridge (1992). MR 1201388 (94a:14039)
54. Nahm, W.: The construction of all self-dual multimonopoles by the ADHM method. Monopoles in Quantum Field Theory (Trieste, 1981), pp. 87–94. World Scientific Publishing, Singapore (1982). MR 766754
55. Nakajima, H.: Handsaw quiver varieties and finite W-algebras. Mosc. Math. J. **12**(3), 633–666, 669–670 (2012). MR 3024827
56. Nekrasov, N.A., Shatashvili, S.L.: Supersymmetric vacua and Bethe ansatz. Nucl. Phys. B Proc. Suppl. **192/193**, 91–112 (2009). MR 2570974 (2011i:81217)
57. Okounkov, A.: Lectures on K-theoretic computations in enumerative geometry. Geometry of Moduli Spaces and Representation Theory. IAS/Park City Mathematics Series, vol. 24, pp. 251–380. American Mathematical Society, Providence (2017). MR 3752463
58. i Riera, I.M.: A Hitchin-Kobayashi correspondence for Kähler fibrations. J. Reine Angew. Math. **528**, 41–80 (2000). MR 1801657
59. i Riera, I.M.: Hamiltonian Gromov-Witten invariants. Topology **42**(3), 525–553 (2003). MR 1953239 (2004d:53106)
60. i Riera, I.M., Tian, G.: A compactification of the moduli space of twisted holomorphic maps. Adv. Math. **222**(4), 1117–1196 (2009). MR 2554933
61. Siebert, B., Tian, G.: On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator. Asian J. Math. **1**(4), 679–695 (1997). MR 1621570 (99d:14060)
62. Staudacher, M.: Review of AdS/CFT integrability, Chapter III.1: Bethe ansätze and the R-matrix formalism. Lett. Math. Phys. **99**(1–3), 191–208 (2012). MR 2886419
63. Szenes, A., Vergne, M.: Toric reduction and a conjecture of Batyrev and Materov. Invent. Math. **158**(3), 453–495 (2004). MR 2104791 (2005i:14065)
64. Szenes, A., Vergne, M.: Mixed toric residues and tropical degenerations. Topology **45**(3), 567–599 (2006). MR 2218757
65. Tian, G., Xu, G.: Analysis of gauged Witten equation. arXiv:1405.6352
66. Toën, B.: Derived algebraic geometry. EMS Surv. Math. Sci. **1**(2), 153–240 (2014). MR 3285853
67. Vafa, C.: Topological Landau-Ginzburg models. Modern Phys. Lett. A **6**(4), 337–346 (1991). MR 1093562 (92f:81193)
68. Venugopalan, S., Woodward, C.: Classification of affine vortices. Duke Math. J. **165**(9), 1695–1751 (2016). MR 3513572
69. Witten, E.: Phases of $N = 2$ theories in two dimensions. Nucl. Phys. B **403**(1–2), 159–222 (1993). MR MR1232617 (95a:81261)
70. Witten, E.: The Verlinde algebra and the cohomology of the Grassmannian. In: Geometry, Topology, and Physics, Conference Proceedings. Lecture Notes on Geometry and Topology IV, pp. 357–422. International Press, Cambridge (1995). MR 1358625 (98c:58016)
71. Woodward, C.T.: Quantum Kirwan morphism and Gromov-Witten invariants of quotients III. Transform. Groups **20**(4), 1155–1193 (2015). MR 3416443
72. Woodward, C.T.: Quantum Kirwan morphism and Gromov-Witten invariants of quotients II. Transform. Groups **20**(3), 881–920 (2015).MR 3376153
73. Woodward, C.T.: Quantum Kirwan morphism and Gromov-Witten invariants of quotients I. Transform. Groups **20**(2), 507–556 (2015). MR 3348566
74. Ziltener, F.: A quantum Kirwan map: Bubbling and Fredholm theory for symplectic vortices over the plane. Mem. Am. Math. Soc. **230**(1082), vi+129 (2014). MR 3221852