Rational pull-backs of toric foliations

Ariel Molinuevo

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Joint work with Gargiulo, J. and Velazquez, S.

Codimension 1 foliations in $\mathbb{P}^n$

A **codimension one foliation in $\mathbb{P}^n$** is given by a 1-differential form

$$\omega \in H^0(\Omega^1_{\mathbb{P}^n}(e))$$

that verifies the **Frobenius integrability condition**

$$\omega \wedge d\omega = 0.$$

Such forms define a **projective variety** (the **moduli (or parameter) space of foliations**)

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How do you define this parameter space of foliations?

Just consider the development of ω in terms of its scalar coefficients $a_{i,\alpha}$:

$$\omega = \sum_{i=0}^{n} A_i dx_i = \sum_{i=0,|\alpha|=e-1} a_{i,\alpha} x^{\alpha} dx_i$$

and compute the equation $\omega \wedge d\omega = 0$.

This equation will return many homogeneous (degree two) equations in the coefficients $a_{i,\alpha}$:

$$\omega \wedge d\omega = \sum_{i,j,k} A_i\left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}\right) dx_i \wedge dx_j \wedge dx_k = \sum_{i,j,k} Eq_{ijk}(a_{i,\alpha_i}, a_{j,\alpha_j}, a_{k,\alpha_k})\, dx_i \wedge dx_j \wedge dx_k$$

Then you have that

$$\mathcal{F}^1(\mathbb{P}^n)(e) = \langle Eq_{ijk}(a_{i,\alpha_i}, a_{j,\alpha_j}, a_{k,\alpha_k}) = 0 \rangle \subset \mathbb{P}^N,$$

where N is $N = (n+1)\binom{n+e-1}{e-1}$.

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We are intereseted in knowing how is this variety made: $\mathcal{F}^1(\mathbb{P}^n)(e)$.

Meaning: **what are its irreducible components?**

What do we know? Not much.

- **degree 0 = e-2** : 1 component (of rational type)
- **degree 1** : 2 components, one of rational type and one of logarithmic type
- **degree 2** : 6 componentes, 2 rationals, 2 logarithmic, 1 pull-back form $\mathbb{P}^2$, exceptional component [Cerveau, D. and Lins Neto, A., 1996]
- **degree 3**: a recent article from Jorge Vitorio Pereira, Ruben Lizarbe and Raphael Constant they shows that it has at least 24 components.
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What are the components that appear in degrees ≤ 2?

Rational foliations $\mathcal{R}(n,(r,s)) \subset \mathcal{F}^1(\mathbb{P}^n)(e)$

$$\omega_{\mathcal{R}} = rF\,dG - sG\,dF$$

where F, G are homogeneous polynomials of degrees r and s respectively and $r+s=e$.

Logarithmic foliations $\mathcal{L}(n,(d_1,\ldots,d_s)) \subset \mathcal{F}^1(\mathbb{P}^n)(e)$

$$\omega_{\mathcal{L}} = \left(\prod_{i=1}^{s} f_i\right)\left(\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}\right) = \sum \lambda_i F_i df_i$$

where f_i is homogeneous of degree d_i, $\sum d_i = e$ y $\sum d_i\lambda_i = 0$. We denote $F_i = \prod_{j\neq i} f_j$.

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Exceptional componente $\mathcal{E}(n) \subset \mathcal{F}^1(\mathbb{P}^n)(e)$.

Obtained as the particular action of the affine Lie algebra $\mathbb{C}$ on $\mathbb{P}^3$.

Linear Pullbacks from $\mathbb{P}^2$ $\mathcal{L}(e,n) \subset \mathcal{F}^1(\mathbb{P}^n)(e)$.

Let $\mathcal{F}$ be a foliation of degree e in $\mathbb{P}^2$ and $L : \mathbb{P}^n - - \to \mathbb{P}^2$ a rational map induced by a linear submersion $\mathbb{C}^{n+1} - - \to \mathbb{C}^3$. Then $L^*(\mathcal{F}) \in \mathcal{F}^1(\mathbb{P}^n, e)$.

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How can you prove that a family of foliations define an irreducible component of $\mathcal{F}^1(\mathbb{P}^n, e)$?

One idea is as follows: you consider a generic element of your family and look at it first order deformations. If you can parametrize your family in such a way that the differential of the parametrization is surjective, then you just discover an irreducible component of your space.

For example: a Rational Foliation $\omega_{\mathcal{R}}$ is of the form $\omega_{\mathcal{R}} = r\, F\, dG - s\, G\, dF$ where F, G are homogeneous polynomials of degrees r and s respectively and $r + s = e$.

You can parametrize such foliations as

$$\begin{aligned} H^0(\mathcal{O}_{\mathbb{P}^n}(r) \oplus \mathcal{O}_{\mathbb{P}^n}(s)) &\xrightarrow{\phi} \mathcal{R}(n, (r, s)) \\ (F, G) &\longmapsto rFdG - sGdF \end{aligned}$$

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How is a first order deformation of $\omega_{\mathcal{R}} = r\,F\,dG - s\,G\,dF$? Is of the form

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where F' is a polynomial of degree r and G' a polynomial of degree s.

Since the differential of the parametrization map is surjective, then we have an irreducible component of $\mathcal{F}^1(\mathbb{P}^n, e)$.

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Regarding the component of **Linear Pullbacks**, Cervau, Lins-Neto, Edixhoven, extended the result obtained in the paper of 1996, showing in 2001 that the pullback from $\mathbb{P}^2$ by **any rational map** to $\mathbb{P}^n$ defines an irreducible component of $\mathcal{F}^1(\mathbb{P}^n, e)$.

How did they do this?

The proof use analytic methods (it's not algebraic).

So, we tried to make an algebraic proof of that statement. We couldn't do that, but by taking that path we acquired a lot of insight in what's going on.

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What exactly we were trying to do?

We tried to prove the stability of pullback foliations from $\mathbb{P}^2$ to $\mathbb{P}^n$ by rational maps $\mathbb{P}^n \overset{F}{\dashrightarrow} \mathbb{P}^2$ and also we were considering maps $\mathbb{P}^n \overset{F}{\dashrightarrow} X$ to toric surfaces X to see if we were able to discover some new irreducible component.

How did we thought we could do that?

Let's consider a rational map with a polynomial lifting $F : \mathbb{P}^n \dashrightarrow X$, where X is a toric surface. And consider $\alpha \in H^0(\Omega^1_X(\mathcal{D}))$ where $\mathcal{D}$ is a Weild divisor of X.

If you could prove that an infinitesimal perturbation of the pullback foliation $\omega = F^*(\alpha)$ is given by:

i) a deformation of the map F

ii) a deformation of the differential form α

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- **i)** a deformation of the map F
- **ii)** a deformation of the differential form α

Then we would be ok. Because we were able to classify the deformations of the map F and of the differential form α.

How did we do that?

We did that by considering first order deformations and first order unfoldings of a foliation.

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There are two ways of making a **first order** perturbation of $\omega \in \Omega^1_{\mathbb{P}^n|\mathbb{C}}(e)$ such that $\omega \wedge d\omega = 0$:

Deformations

$$D(\omega) = T_\omega \mathcal{F}^1(\mathbb{P}^n)(e)$$

$$\begin{array}{ccc} \omega & \overset{i^*}{\longleftarrow} & \omega_\varepsilon \in \Omega^1_{\mathbb{P}^n \times D|D}(e) \\ \mathbb{P}^n & \overset{i}{\longrightarrow} & \mathbb{P}^n \times D \\ \downarrow & & \downarrow \\ Spec(\mathbb{C}) & \longrightarrow & D \end{array}$$

Unfoldings

$$U(\omega)$$

$$\begin{array}{ccc} \omega & \overset{i^*}{\longleftarrow} & \widetilde{\omega}_\varepsilon \in \Omega^1_{\mathbb{P}^n \times D|\mathbb{C}}(e) \\ \mathbb{P}^n & \overset{i}{\longrightarrow} & \mathbb{P}^n \times D \\ \downarrow & & \downarrow \\ Spec(\mathbb{C}) & \longrightarrow & Spec(\mathbb{C}) \end{array}$$

where $D = Spec(\mathbb{C}[\varepsilon]/\varepsilon^2)$ and $\omega_\varepsilon \wedge d\omega_\varepsilon = 0$ and $\widetilde{\omega}_\varepsilon \wedge \widetilde{\omega}_\varepsilon = 0$.

The geometric idea is that a deformation defines a foliation for every fixed parameter ε and an unfolding is a foliation in $\mathbb{P}^n \times Spec(D)$.

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where $D = Spec(\mathbb{C}[\varepsilon]/\varepsilon^2)$ and $\omega_\varepsilon \wedge d\omega_\varepsilon = 0$ and $\widetilde{\omega}_\varepsilon \wedge \widetilde{\omega}_\varepsilon = 0$.

The geometric idea is that a deformation defines a foliation for every fixed parameter ε and an unfolding is a foliation in $\mathbb{P}^n \times Spec(D)$.

There are two ways of making a **first order** perturbation of $\omega \in \Omega^1_{\mathbb{P}^n|\mathbb{C}}(e)$ such that $\omega \wedge d\omega = 0$:

Deformations

$$D(\omega) = T_\omega \mathcal{F}^1(\mathbb{P}^n)(e)$$

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Both types of perturbations can be written as:

$$\omega_\varepsilon = \omega + \varepsilon\eta \qquad \text{(deformations)}$$
$$\widetilde{\omega}_\varepsilon = \omega + \varepsilon\eta + h d\varepsilon \qquad \text{(unfoldings)}$$

The **integrability condition** applied to ω_ε and $\widetilde{\omega}_\varepsilon$ allows to **parametrize** $D(\omega)$ and $U(\omega)$ as

$$D(\omega) = \left\{\eta \in H^0(\Omega^1_{\mathbb{P}^n}(e)) \ : \ \omega \wedge d\eta + d\omega \wedge \eta = 0\right\} / \mathbb{C}.\omega$$

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So, when a first order deformation comes from a first order unfolding?

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The theory of unfoldings of codimension one foliations was developed in the 80's by **Tatsuo Suwa** in a **local analytic setting**

For $\varpi \in \Omega^1_{\mathbb{C}^{n+1},p}$ a **germ of an integrable differential 1-form**, we have

$$U_h(\varpi) = \left\{(h,\eta) \in \mathcal{O}_{\mathbb{C}^{n+1},p} \times \Omega^1_{\mathbb{C}^{n+1},p} \;:\; h\, d\varpi = \varpi \wedge (\eta - dh)\right\} / \mathbb{C}.(0,\varpi).$$

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$$\begin{aligned} U_h(\varpi) &\xrightarrow{\pi_1} \mathcal{O}_{\mathbb{C}^{n+1},p} \\ (h,\eta) &\longmapsto h \end{aligned}$$

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This fact was used by T. Suwa to **classify unfoldings of rational and logarithmic foliations**.

More or less the same can be reproduced in the algebraic setting (A. M.). I mean, we can also define a graded ideal $I(\omega) \subset S = \mathbb{C}[x_0, \ldots, x_n]$ such that

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What about the **singular locus** of a codimension 1 foliation in $\mathbb{P}^n$?

As you may know, every global differential form has singular points in $\mathbb{P}^n$ (Jouanolou, Equations de Pfaff algebriques). The integrability condition $\omega \wedge d\omega = 0$ makes that set to have codimension ≥ 2. Why?

Because of the **Koszul complex** associated to ω:

$$K^{\bullet}(\omega): \qquad S \xrightarrow{\omega\wedge} \Omega^1_S \xrightarrow{\omega\wedge} \Omega^2_S \xrightarrow{\omega\wedge} \dots$$

We clearly have that $d\omega \in \mathcal{Z}^2(K^{\bullet}(\omega))$ and, by a matter of degrees, we also have that $[d\omega] \neq 0$ in $H^2(K^{\bullet}(\omega))$.

Our statement comes from:

The following are equivalent (Malgrange, Frobenius avec singularites, I):

- i) $codim(Sing(\omega)) \geq k$
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What can you have in the singular locus in codimension 2?

There is the **Kupka set**, defined as

$$K_{set}(\omega) = \{p \in Sing(\omega) \ : \ d\omega(p) \neq 0\}$$

That set has the following properties:

i) It is generically smooth of codimension 2
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iii) Locally, around $p \in K_{set}(\omega)$, ω has a normal form. It can be written as the pullback of a differential form in $\mathbb{C}^2$

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Let us denote as $C(\eta)$ the ideal defined by the polynomial coefficients of the given differential form. Then, $C(\omega)$ is the ideal of the singular locus of ω.

Writing ω as

$$\omega = \sum_{i=0}^{n} A_i \, dx_i \qquad \Rightarrow \qquad C(\omega) = (A_0, \dots, A_n) \; .$$

We defined the **Kupka scheme** as the projective variety defined by the following homogeneous ideal:

$$K(\omega) = (C(\omega) : C(d\omega)) \; .$$

And we proved:

- If ω is 'generic' then

$$\sqrt{I(\omega)} = \sqrt{K(\omega)}$$

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Toric varieties

X_T^q a simplicial complete toric variety of dimension q.

Definition

A toric variety is an algebraic variety X which contains a torus $T \simeq (\mathbb{C}^*)^q$ as a Zariski open set, in such way that the natural action of T on itself extends to an algebraic action of T on X.
Examples: $\mathbb{P}^n$, $\mathbb{P}^n(\bar{a})$, $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$, $\mathcal{H}_r$, ...

Guiding principle:

- X_T^q "geometric object" $\leftrightsquigarrow$ Σ fan "simplicial and combinatorial object" .
- X_T^q $\leftrightsquigarrow$ S ring of homogeneous coordinates

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Cox ring

Ingredients for X_T^q:

1. $\Sigma(1) = \{v_1, \ldots, v_m\} \subset \mathbb{Z}^q$, $m = |\Sigma(1)| \rightsquigarrow$ (skeleton).
2. $a = \{a^j = (a_1^j, \ldots, a_m^j)\}_{j=1}^{m-q}$ basis of relations among the rays: $\sum_{i=1}^m a_i^j v_i = 0$ $\rightsquigarrow$ (charge matrix).
3. $\Sigma(d)$... $d \geq 2 \rightsquigarrow$ (rest of the fan, exceptional set $\rightsquigarrow Z$).

There is $S = \mathbb{C}[z_1, \ldots, z_m]$ homogeneous coordinate ring such that:

a. $v_i \in \Sigma(1)$ we have D_i a T-invariant divisor $(z_i = 0)$.
b. $Cl(X) \simeq \mathbb{Z}^{m-q} \times H$, $deg(z_i) = [D_i] \mapsto (a_i = (a_i^1, \ldots, a_i^{m-q}), h_i) \in \mathbb{Z}^{m-q} \times H$.
c. $S = \bigoplus_{D \in Cl(X)} S_D$ is $Cl(X)$-graded (Cox ring).
d. A quasi coherent sheaf of $\mathcal{O}_{X_T^q}$ is given by a graded S-module M. A subvariety of X_T^q by an homogeneous ideal $I \subset S$.

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How to describe foliations on X_T^q?

Let X be normal variety. A singular foliation of dimension t (or codimension $k = q - t$) on X is a nonzero coherent subsheaf $\mathcal{F} \subset TX$ of generic rank t which is closed under [,] and saturated: $TX/\mathcal{F}$ torsion free.

Idea: "Dualizing we need a line bundle and a twisted differential form on the regular part of X_T^q"

- $j : X_r \hookrightarrow X$, $codim(X - X_r) \geq 2$ and has finite quotient singularities.
- $\hat{\Omega}^\bullet_X := (\Omega^\bullet_X)^{\vee\vee} = j_*(\Omega^\bullet_{X_r})$ (Zariski forms).

Toric Euler sequence:

$$0 \to \hat{\Omega}^1_X \to \oplus_{i=1}^m \mathcal{O}_X(-D_i) \to Cl(X) \otimes \mathcal{O}_X \to 0$$

Radial Euler fields

$R_j = \sum_{i=1}^m a_i^j z_i \frac{\partial}{\partial z_i}$ with $j = 1, \ldots, m - q$.

We consider: $\alpha \in H^0(X, \hat{\Omega}^k_X \times \mathcal{O}_X(D))$ **satisfying certain equations**
$\rightsquigarrow Ker(\alpha) = \mathcal{F} \subset \mathcal{T}_X$.

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Let X be normal variety. A singular foliation of dimension t (or codimension $k = q - t$) on X is a nonzero coherent subsheaf $\mathcal{F} \subset TX$ of generic rank t which is closed under [,] and saturated: $TX/\mathcal{F}$ torsion free.

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- $j : X_r \hookrightarrow X$, $codim(X - X_r) \geq 2$ and has finite quotient singularities.
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$$0 \to \hat{\Omega}_X^1 \to \oplus_{i=1}^m \mathcal{O}_X(-D_i) \to Cl(X) \otimes \mathcal{O}_X \to 0$$

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$R_j = \sum_{i=1}^m a_i^j z_i \frac{\partial}{\partial z_i}$ with $j = 1, \ldots, m - q$.

We consider: $\alpha \in H^0(X, \hat{\Omega}_X^k \times \mathcal{O}_X(D))$ **satisfying certain equations** $\rightsquigarrow Ker(\alpha) = \mathcal{F} \subset \mathcal{T}_X$.

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Parameter spaces of singular toric foliations

Conditions in Cox coordinates for $\alpha \in H^0(X, \hat{\Omega}^k_X(D))$ with $D = \sum d_i D_i$:

1. $\alpha = \sum A_I dz_{i_1} \wedge \cdots \wedge dz_{i_k} \in \Omega^k_S$ of degree $D \mapsto \sum d_i a_i$ **(Multi-homogeneity)**
2. $i_{R_j}(\alpha) = 0 \quad (\forall j = 1, \ldots, m-q)$ **(Descent conditions)**
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Rational maps. What about $F : \mathbb{P}^n \dashrightarrow X_T^q$?

Rational maps in Cox coordinates:

Let $e_1 v_1 + \cdots + e_m v_m = 0$ be a relation among the rays of $X = X_T^q$. Every $F = (F_1, \ldots, F_m) \in \mathbb{C}[x_0, \ldots, x_n]^m$ such that F_i is homogeneous of degree e_i, induces a rational map $\tilde{F} : \mathbb{P}^n \dashrightarrow X$ that fits in the diagram

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- **(Cox)** If X is smooth, every regular map $\tilde{F} : \mathbb{P}^n \to X$ arises from $F : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^m - Z$.
- **(Brown-Buczyński)** Every rational map $\phi : Y \dashrightarrow X$ between two toric varieties admits a complete description in Cox coordinates (formal roots).
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What does complete means?

It means that the lifting has the right base locus. That is:

$$Reg(\phi) = \mathbb{P}^n \backslash \pi(\{F^{-1}(Z)\})$$

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By considering the cone of maximal dimension we get that exists an open set $U_\sigma \simeq \mathbb{C}^q$. Then we just dehomogenize and homogenize there and we get the polynomial lifting $F : \mathbb{C}^n \backslash \{0\} \dashrightarrow \mathbb{C}^m \backslash Z$.

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suppose F' is a polynomial lifting wich is not defined along $V(f)$, where f is an irreducible polynomial.

Let $u_i = mult_f(F'_i)$ be the multiplicity of F'_i along f and $\tau = Cone(v_{i_1}, \dots, v_{i_k}) \in \Sigma_X$ be the cone of minimal dimension satisfying $\sum_{i=1}^m u_i v_i \in \tau$. Let $u' \in \mathbb{Q}_+^m$ satisfy $u'_k = 0$ for $k \notin \{i_1, \dots, i_k\}$ and $\sum_{i=1}^m u_i v_i = \sum_{j=1}^k u'_{i_j} v_{i_j}$. By construction,

$$F_1 = \left(f^{u'_1 - u_1}, \dots, f^{u'_m - u_m}\right) \cdot F'$$

is a multi-valued lifting.

Moreover, F_1 does not have a general point of $V(f)$ in its base locus. Since τ is a smooth cone, we can assume that $u' \in \mathbb{N}^m$ and therefore F_1 is polynomial. Applying this algorithm a finite number of times we get a complete polynomial lifting F as claimed.

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An example:

consider the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}(1,1,2)$ defined in homogeneous coordinates by $F = (z_0^2,\ z_0z_1,\ z_0z_2^3)$.

Then if we consider the polynomial $f = z_0$ we get that the multiplicities are given by the vector $u = (2,1,1)$.

Since we can generate the fan of $\mathbb{P}^2(1,1,2)$ with the rays: $v_0 = (-2,-1) \leftrightsquigarrow z_0$, $v_1 = (0,1) \leftrightsquigarrow z_1$ and $v_2 = (1,0) \leftrightsquigarrow z_2$. We get that the vector $2.v_0 + 1.v_1 + 1.v_2 + = (-3,-1) \in \tau$. Then we can write τ with v_0 and v_1 as $\tau = \frac{3}{2}v_0 + \frac{1}{2}v_1$.

With this, we have that $u' = (3/2, 1/2, 0)$ and $u' - u = (-1/2, -1/2, -1)$. Finally

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Let X be a simplicial complete toric variety and $\phi : \mathbb{P}^n \dashrightarrow X$ be a dominant rational map such that $\operatorname{codim}(\phi^{-1}(Sing(X))) \geq 2$. Then ϕ admits a complete polynomial lifting.

Foliations induced by fibers of rational maps

Canonical sheaf

For a toric variety X_T^q, the canonical sheaf is given by $\omega_{X_T} = \mathcal{O}_{X_T}(-\sum_{i=1}^m D_i)$ reflexive sheaf of rank 1 $\rightsquigarrow [-\sum D_i] = K_X \in Cl(X_T)$ canonical Weil divisor class.

Volume form

The volume form Ω_X in X_T^q can be described in homogeneous coordinates as:
$\Omega_X = i_{R_1} \dots i_{R_{m-q}} dz_1 \wedge \cdots \wedge dz_m = \sum_{|I|=q} b_I \hat{z}_I dz_I \in H^0(X_T^q, \hat{\Omega}^q_{X_T}(-K_X))$.

Definition

Let $F : \mathbb{P}^n \dashrightarrow X_T^q$ be a rational map with a complete lifting of degree $\bar{e}$. Write $\mathcal{F}_F$ for the foliation given by the fibers of F:

- $\mathcal{F}_F$ is a singular projective foliation of codimension q.
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Varieties of foliations given by fibers

We consider the following rational map:

$$\phi_{\bar{e},X} : \bigoplus_{i=1}^{m} \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(e_i)) \dashrightarrow \mathcal{F}_q(\mathbb{P}^n, \textstyle\sum e_i)$$
$$(F_1, \ldots, F_m) \mapsto \omega = F^*\Omega_X.$$

Define $\mathcal{R}_q(n, X, \bar{e}) \subset \mathcal{F}_q(\mathbb{P}^n, \sum e_i)$ as the Zariski closure of the image of $\phi_{\bar{e},X}$.

Weighted projective case

If $X = \mathbb{P}^q(\bar{e})$, then $\mathcal{R}_q(n, X, \bar{e})$ determines an irreducible and generically reduced component of $\mathcal{F}_q(\mathbb{P}^n, \sum e_i)$ (Cukierman-Pereira-Vainsencher).

Proposition (GMV.)

Let X_T^q be a complete simplicial toric variety. Then $\mathcal{R}_q(n, X_T^q, \bar{e})$ fills an irreducible component of $\mathcal{F}_q(\mathbb{P}^n, \sum e_i)$ if and only if X_T^q is a weighted projective space or a fake weighted projective space.

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Definitions

Focus on the situtation where $X = X_T^2$ **a complete simplicial toric surface:**

$\mathcal{F}_1(X, D) \underset{\text{open}}{\subset} \mathbb{P}H^0(X, \hat{\Omega}_X^1(D))$ because $\alpha \wedge d\alpha \in H^0(X, \hat{\Omega}_X^3(D^{\otimes 2})) = 0$.

- **Twisted 1-forms** $\hat{\Omega}_X^1(D)$: $\alpha = \sum A_i(z)dz_i$ with $i_{R_j}(\alpha) = \sum a_i^j z_i A_i = 0$.
- **(Homogeneous) Vector fields** $TX(D + K_X)$: $[Y] = \sum B_j \frac{\partial}{\partial z_j}$ (mod $\sum f_i R_i$). Assume $H^1(X, \mathcal{O}_X(D + K_X)) = 0$.

Rational pull-backs

For $F : \mathbb{P}^n \dashrightarrow X$ with a polynomial lifting of degree $\bar{e} = (e_i)$, then

$$\omega = F^*(\alpha) = \sum_i A_i(F)dF_i \in \mathcal{F}_1(\mathbb{P}^n, \bar{d} \cdot \bar{e}),$$

where $\alpha \in \mathcal{F}_1(X, D)$ ("generic") and $D = \sum d_i D_i \in \mathit{Eff}(X)$.

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Varieties of foliations given by pull-backs

Definition

$$\phi = \phi_{(\bar{e},D)} : \mathcal{F}_1(X,D) \times \left(\prod_{i=1}^{m} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e_i)) \backslash \tilde{Z} \right) \Big/ G \dashrightarrow \mathcal{F}_1(\mathbb{P}^n, \bar{d} \cdot \bar{e})$$

$$(\alpha, (F_1, \ldots, F_m)) \longmapsto \omega = F^*(\alpha).$$

and define $PB_1(n, X, D, \bar{e}) = \overline{Im(\phi_{(\bar{e},D)})}$ (Zariski closure)

- $PB_1(n, \mathbb{P}^2, d, e)$ irreducible component of $\mathcal{F}_1(\mathbb{P}^n, d \cdot e)$ (Cerveau-Lins Neto-Edixhoven).

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The variety $PB_1(n, X, -K_X, \bar{e})$ is contained in the variety of logarithmic foliations $\mathcal{L}_1(n, \bar{e})$. Moreover, $PB_1(n, X, -K_X, \bar{e})$ coincides with $\mathcal{L}_1(n, \bar{e})$ if and only if X is a weighted projective surface or a fake weighted projective surface.

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Genericity conditions

Notation: $\alpha = \sum_{i=1}^{m} A_i(z)dz_i \in H^0(X_T^2, \hat{\Omega}^1(D))$ and $F : \mathbb{P}^n \dashrightarrow X_T^2$.

Definition

The pair (F, α) is *generic* if the following holds:

1. The critical values of F, $C_V(F)$, are such that $C_V(F) \cap Sing(\alpha) = \emptyset$. Also, $Sing(\omega)$ is reduced along $C(F)$ (the critical points of F).
2. $C(\alpha)$ is radical ($\sqrt{C(\alpha)} = C(\alpha)$) and has codimension ≥ 2.
3. The affine variety associated to the ideal $C(d\alpha)$ has codimension ≥ 3, that is $K(\alpha) = C(\alpha)$.

Kupka set of foliations on toric surfaces

A generic foliation on $\mathbb{P}^2$ has all of its singular points of Kupka type.

When a foliation on $\mathbb{P}^2(a_i)$ has all of its singular points of Kupka type?

Theorem (GMV.)

A generic vector field $[Y] = [\sum_{j=0}^{2} B_j \frac{\partial}{\partial z_j}] \in H^0(\mathbb{P}^2(a_i), T\mathbb{P}^2(a_i) \otimes \mathcal{O}(\ell))$ induces a foliation with all its singular points of Kupka type if and only if $\ell + a_0 \equiv 0(a_i)$ or $\ell + a_1 \equiv 0(a_i)$ or $\ell + a_2 \equiv 0(a_i)$ $\forall i$. Moreover, in that case, $Sing(\alpha) = \mathcal{K}(\alpha)$.

Idea: In homogeneous coordinates, we can assume that $div(Y) = 0$. Then we use:
$d\left(i_Y \Omega_{\mathbb{P}^2(a_0,a_1,a_2)}\right) = div(Y)\Omega_{\mathbb{P}^2(a_0,a_1,a_2)} + \ell(\imath_Y dz_0 \wedge dz_1 \wedge dz_2) = \ell(\imath_Y dz_0 \wedge dz_1 \wedge dz_2)$.

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Theorem (GMV.)

Let X be a regular toric surface and $\mathcal{L} \in Pic(X)$ such that $TX(\mathcal{L})$ is generated on global sections. If $Y \in H^0(X, TX(\mathcal{L}))$ is generic, then

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What about the singular scheme of $\omega = F^*(\alpha)$?

Lemma

Let (F, α) be an generic pair in X_T^2. Then (if $m > 3$)

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Let (F, α) be an generic pair in X_T^2, with $F^* : S_X \to S_{\mathbb{P}^n}$ flat. Then $K(\omega) = F^*(K(\alpha))$.

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Let (F, α) be a generic pair in X_T^2 with F flat. Then the Kupka ideal of $\omega = F^*(\alpha)$ is $K(\omega) = \langle A_1(F), \ldots, A_m(F) \rangle$. In addition, $Sing(\omega) = \mathcal{K}(\omega) \cup C(F, \alpha)$.

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Consider a first order deformation of $\omega = F^*(\alpha)$ of the following form:
$(F + \varepsilon G)^*(\alpha + \varepsilon\beta) = F^*(\alpha) + \varepsilon\, \eta$:

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For (α, F) generic and $\omega = F^*(\alpha)$, when $D(\omega) = \mathrm{Im}(d\phi_{(\alpha,F)})$? Meaning that $\eta \in D(\omega)$ is of the form $\eta = \eta_1 + \eta_2$ as before?

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Deformations from unfoldings in the pull-back case

Recall that:

$$0 \longrightarrow IF(\omega) \longrightarrow U(\omega) \longrightarrow D(\omega)$$
$$(h,\eta) \longmapsto \eta$$

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Let $F : \mathbb{P}^n \dashrightarrow X_T^2$, $\alpha \in \mathcal{F}_1(X_T^2, D)$, and $\omega = F^*(\alpha)$.

Let (F,α) be a generic pair with F flat, then $\eta \in D(\omega)$ comes from a first order unfolding iff η is of type η_1. Also:

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