EQUIVARIANT CHERN CHARACTER

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A C* algebra (or a Banach algebra) with unit 1_A

$$\mathrm{K}_0(A) := \begin{array}{l}\text{Grothendieck group of finitely}\\ \text{generated (left) projective}\\ A\text{-modules}\end{array}$$

$n = 1, 2, 3, \ldots$

$$M_n(A) = \{n \times n \text{ matrices } [a_{ij}] : a_{ij} \in A\}$$

$M_n(A)$ is a C* algebra (or a Banach algebra) with unit

$$\begin{pmatrix} 1_A & 0 & \ldots & 0 \\ 0 & 1_A & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1_A \end{pmatrix}$$

$$\mathsf{GL}(1,A) \subset \mathsf{GL}(2,A) \subset \mathsf{GL}(3,A) \subset \dots$$

$$\mathsf{GL}(A) := \bigcup_{n=1}^{\infty} \mathsf{GL}(n,A)$$

$\mathsf{GL}(A)$ is topologized by the direct limit topology i.e. $U \subset \mathsf{GL}(A)$ is open iff $U \cap \mathsf{GL}(n,A)$ is open in $\mathsf{GL}(n,A)$ for all $n = 1,2,3,\dots$

$j = 1,2,3,\dots$

$$\mathsf{K}_j(A) := \pi_{j-1}(\mathsf{GL}(A))$$

Bott periodicity

$$\Omega^2\,\mathsf{GL}(A) \sim \mathsf{GL}(A)$$

$$\mathsf{K}_j(A) \simeq \mathsf{K}_{j+2}(A), \quad j = 1,2,3,\dots$$

$$\mathsf{K}_0(A), \quad \mathsf{K}_1(A)$$

X locally compact Hausdorff topological space

$X^+ = X \cup \{p_\infty\}$ one point compactification of X

$C_0(X) = \{\alpha\colon X^+ \to \mathbb{C} : \alpha$ is continuous $\alpha(p_\infty) = 0\}$

$C_0(X)$ is a C*-algebra

$x \in X^+$, $\alpha, \beta \in C_0(X)$, $\lambda \in \mathbb{C}$

$$\begin{aligned}(\alpha+\beta)x &= \alpha x + \beta x\\ (\alpha\beta)x &= (\alpha x)(\beta x)\\ (\lambda\alpha)x &= \lambda(\alpha x)\\ \|\alpha\| &= \sup_{x\in X^+} |\alpha(x)|\\ \alpha^* x &= \overline{\alpha x}\end{aligned}$$

X loacally compact Hausdorff topological space

$X^+ = X \cup \{p_\infty\}$ one point compactification of X

$$\mathrm{K}_0(C_0(X)) = \ker \begin{pmatrix} \mathrm{K}^0(X^+) \to \mathrm{K}^0(p_\infty) = \mathbb{Z} \\ E \mapsto \dim_{\mathbb{C}}(E_{p_\infty} \end{pmatrix}$$

E $\mathbb{C}$ vector bundle on X^+

X locally compact Hausdorff topological space

$\mathsf{K}_*(C_0(X))$ is Atiyah-Hirzerbruch K-theory

This is topological K-theory with compact supports

Atiyah-Hirzerbruch notation for this K-theory is $\mathsf{K}^*(X)$

$$\mathsf{K}_j(C_0(X)) = \mathsf{K}^j(X)$$

X compact Hausdorff $\Longrightarrow$

$\mathsf{K}_0(C_0(X)) = \mathsf{K}^0(X) =$ Grothendieck group of $\mathbb{C}$ vector bundles on X

Chern character

X locally compact Hausdorff topological space

$$\mathrm{ch}\colon \mathsf{K}_j(C_0(X)) \to \bigoplus_l \mathsf{H}_c^{j+2l}(X;\mathbb{Q}), \quad j = 0, 1$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathsf{K}_j(C_0(X)) \xrightarrow{\cong} \bigoplus_l \mathsf{H}_c^{j+2l}(X;\mathbb{Q})$$

- Čech cohomology

- Alexander Spanier cohomology

(with compact supports)

X locally compact Hausdorff topological space

Γ discrete (countable) group

$\Gamma \times X \to X$ continuous action of Γ on X

$C_r^*(\Gamma, X) = C_0(X) \rtimes_r \Gamma$ is the reduced crossed-product C*-algebra for the action of Γ on X

Definition of $C_r^*(\Gamma, X)$

Extend the given action $\Gamma \times X \to X$ to $\Gamma \times X^+ \to X^+$ by

$$\gamma p_\infty = p_\infty \quad \forall \gamma \in \Gamma$$

Γ then acts on $C_0(X)$ by C* algebra automorphisms $\Gamma \times C_0(X) \to C_0(X)$

$$(\gamma f)x = f(\gamma^{-1}x) \quad f \in C_0(X),\ \gamma \in \Gamma,\ x \in X$$

Form the purely algebraic crossed-product algebra $C_0(X) \rtimes_{\mathrm{alg}} \Gamma$

$$C_0(X) \rtimes_{\mathrm{alg}} \Gamma = \left\{ \begin{array}{l} \text{finite formal sums } \sum_{\gamma\in\Gamma} f_\gamma[\gamma] : \\ f_\gamma \in C_0(X) \end{array} \right\}$$

$$\left(\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right) + \left(\sum_{\gamma\in\Gamma} h_\gamma[\gamma]\right) = \sum_{\gamma\in\Gamma} (f_\gamma + h_\gamma)[\gamma]$$

$$\lambda \left(\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right) = \sum_{\gamma\in\Gamma} (\lambda f_\gamma)[\gamma] \quad \lambda \in \mathbb{C}$$

$$(f[\gamma])(h[g]) = (f)(\gamma h)[\gamma g] \quad \gamma, g \in \Gamma,\ f, h \in C_0(X)$$

Complete $C_0(X)\rtimes_{\text{alg}}$ to obtain $C^*_r(\Gamma, X)$

$$l^2(\Gamma) = \{u\colon \Gamma \to \mathbb{C} : \sum_{\gamma\in\Gamma} \overline{u(\gamma)}u(\gamma) < \infty\}$$

$l^2(\Gamma)$ is a Hilbert space

$$(u+v)\gamma = u\gamma + v\gamma$$

$$(\lambda u)\gamma = \lambda(u\gamma)$$

$$\langle u,\, v\rangle = \sum_{\gamma\in Ga} \overline{u(\gamma)}v(\gamma)$$

$\mathcal{L}^2(l^2(\Gamma))$ is the C*-algebra of all bounded operators $T\colon l^2(\Gamma) \to l^2(\Gamma)$ with operator norm

$$\|T\| = \sup_{\langle u,u\rangle=1} (\langle Tu,\, Tu\rangle^{\frac{1}{2}})$$

Each $x \in X$ determines a homomorphism of algebras

$$\tau_x\colon C_0(X) \rtimes_{\text{alg}} \Gamma \to \mathcal{L}(l^2\Gamma)$$

$$(\tau_x(f[\gamma])u)(g) = f(gx)u(\gamma^{-1}g)$$

$$u \in l^2(\Gamma),\ x \in X,\ \gamma, g \in \Gamma,\ f \in C_0(X)$$

$C^*_r(\Gamma, X)$ is the completion of $C_0(X) \rtimes_{\text{alg}} \Gamma$ in the norm

$$\left\|\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right\| = \sup_{x\in X}\left\|\tau_x\left(\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right)\right\|$$

$$\mathsf{ch}\colon \mathsf{K}_j(C_r^*(\Gamma, X)) \to ?$$

Γ finite $\implies$
$\mathsf{K}_*(C_r^*(\Gamma, X))$ is Atiyah-Segal equivariant K-theory, denoted $\mathsf{K}_\Gamma^*(X)$

Theorem 1 (Slominska)**.** Γ *finite*

$$\mathsf{ch}\colon \mathsf{K}_j^\Gamma(X) \to \bigoplus_l \mathsf{H}_c^{j+2l}(\widehat{X}/\Gamma; \mathbb{C})$$

$$\mathsf{K}_j^\Gamma(X) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \bigoplus_l \mathsf{H}_c^{j+2l}(\widehat{X}/\Gamma; \mathbb{C})$$

If Γ is not finite, then $\mathsf{K}_j(C_r^*(\Gamma, X))$ can be viewed as the natural extension of Atiyah-Segal equivariant K-theory to the case when Γ is not finite

$$\mathsf{ch}\colon \mathsf{K}_j(C_r^*(\Gamma, X)) \to ?$$

M C^∞ manifold, $\partial M = \emptyset$

Γ discrete (countable) group

$\Gamma \times M \to M$ smooth action of Γ on M

"smooth" = each $\gamma \in \Gamma$ acts by a diffeomorphism

$C^*_r(\Gamma, M) = C_0(M) \rtimes_r \Gamma$ is the reduced crossed-product C* algebra for the action of Γ on $C_0(M)$

$$\mathrm{ch}\colon\ \mathsf{K}_j(C^*_r(\Gamma, M)) \to ?$$

Theorem 2 (J. Raven)**.** *If BC for Γ with coefficient algebra $C_0(M)$ is true, then there is a Chern character*

$$\mathrm{ch}\colon\ \mathsf{K}_j(C^*_r(\Gamma, M)) \to \bigoplus_l \mathsf{H}_{j+2l}(\Gamma; \Omega^*_c(\widehat{M})),\ j = 0, 1$$

and

$$\mathsf{K}_j(C^*_r(\Gamma, M)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \bigoplus_l \mathsf{H}_{j+2l}(\Gamma; \Omega^*_c(\widehat{M}))$$

$\widehat{M}$

$\gamma \in \Gamma$, $M^\gamma := \{p \in M : \gamma p = p\}$

If $\text{order}(\gamma) = \infty$, then M^γ can be anything
e.g. M^γ can be a Cantor set

Lemma 3. *If* $\text{order}(\gamma) < \infty$, *then* M^γ *is a* C^∞ *submanifold of* M*.*

Proof. $\text{order}(\gamma) < \infty \implies$ The subgroup J of Γ generated by γ is a finite (cyclic) group $\implies$
Can choose on M a C^∞ Riemannian metric which is J-invariant. Then γ acts by an isometry and M^γ is a totally geodesic C^∞ submanifold of M. $\square$

$$\widehat{M} := \coprod_{\substack{\gamma \in \Gamma \\ \text{order}(\gamma) < \infty}} M^\gamma$$

$$= \{(\gamma, p) \in \Gamma \times M : \gamma p = p\}$$

$\widehat{M}$ is a C^∞ manifold

$\Omega_c^*(\widehat{M})$ = the de Rham complex of $\mathbb{C}$-valued compactly supported C^∞ differential forms on $\widehat{M}$

$$0 \to \Omega_c^0(\widehat{M}) \xrightarrow{d} \Omega_c^1(\widehat{M}) \xrightarrow{d} \dots \xrightarrow{d} \Omega_c^n(\widehat{M}) \to 0$$

$\Gamma \times \widehat{M} \to \widehat{M}$ smooth action of Γ on $\widehat{M}$

$$g(\gamma, p) = (g\gamma g^{-1}, gp),\ g \in \Gamma,\ (\gamma, p) \in \widehat{M}$$

$$\Gamma \times \Omega_c^*(\widehat{M}) \to \Omega_c^*(\widehat{M})$$

$\Omega_c^*(\widehat{M})$ is a complex of Γ-modules

$\mathrm{H}_l(\Gamma; \Omega_c^*(\widehat{M}))$ is the l-th cohomology of Γ with coefficients $\Omega_c^*(\widehat{M})$

$\mathbb{C}[\Gamma]$

$\mathbb{C}[\Gamma]$ is the purely algebraic group algebra

$$\mathbb{C}[\Gamma] := \left\{ \text{Finite formal sums } \sum_{\gamma \in \Gamma} \lambda_\gamma[\gamma] : \lambda_\gamma \in \mathbb{C} \right\}$$

Algebra homomorphism $\varepsilon \colon \mathbb{C}[\Gamma] \to \mathbb{C}$

$$\varepsilon\left(\sum_{\gamma \in \Gamma} \lambda_\gamma[\gamma]\right) = \sum_{\gamma \in \Gamma} \lambda_\gamma$$

A (left) Γ-module is a $\mathbb{C}$ vector space V with a given group homomorphism $\Gamma \to \mathrm{GL}_{\mathbb{C}}(V)$

Equivalently a (left) Γ-module is a unital (left) $\mathbb{C}[\Gamma]$-module

V Γ-module

$\mathsf{H}_l(\Gamma; V) = l$-th cohomology of Γ with coefficients V, $l = 0, 1, 2, 3, \ldots$

$$\mathsf{H}_l(\Gamma; V) := \mathsf{Tor}^l_{\mathbb{C}[\Gamma]}(\mathbb{C}, V)$$

$$0 \leftarrow V \xleftarrow{\partial} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} V \xleftarrow{\partial} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \xleftarrow{\partial}$$

$$\partial(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes v) =$$

$$\varepsilon(a_0) a_1 \otimes \cdots \otimes a_n \otimes v$$

$$+ \sum_{j=0}^{n-1} (-1)^{j+1} a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_n \otimes v$$

$$+ (-1)^{n+1} a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n v$$

$$a_0, a_1, \ldots, a_n \in \mathbb{C}[\Gamma],\ v \in V$$

$$0 \leftarrow \underbrace{V \xleftarrow{\partial} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} V}_{\varepsilon(a)v - av \leftarrow a \otimes v} \xleftarrow{\partial} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \xleftarrow{\partial}$$

$$a \in \mathbb{C}[\Gamma],\ v \in V,\ \gamma \in \Gamma$$

$$\begin{aligned} \mathsf{H}_0(\Gamma; V) &= V/(\varepsilon(a)v - av) \\ &= V/(v - \gamma v) \\ &= V_\Gamma \end{aligned}$$

V_Γ is the Γ-coinvariants

$$V^\Gamma := \{v \in V : \gamma v = v\ \forall\ \gamma \in V\}$$

V^Γ is the Γ-invariants

$\mathsf{H}_l(\Gamma; V)$ is the l-th derived functor of

$$V \mapsto V_\Gamma$$

Let

$$\Psi = \left\{ 0 \xrightarrow{d} V^0 \xrightarrow{d} V^1 \xrightarrow{d} \dots \right\}$$

be a complex of (left) Γ-modules

To define $\mathsf{H}_l(\Gamma;\Psi)$, $l \in \mathbb{Z}$ form (first quadrant bicomplex) $\{A^{i,j}\}$

$$A^{i,j} := \underbrace{\mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \otimes \dots \otimes \mathbb{C}[\Gamma]}_{i} \otimes V^j$$

$$\begin{array}{ccc} & & A^{i,j+1} \\ & & \uparrow \\ A^{i-1,j} & \xleftarrow{\partial} & A^{i,j} \end{array}$$

$$a_0 \otimes \cdots \otimes a_{i-1} \otimes dv$$

$$\begin{array}{c} \varepsilon(a_0) a_1 \otimes \cdots \otimes a_n \otimes v \\ + \sum_{j=0}^{n-1} (-1)^{j+1} a_0 \otimes \dots \\ a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_n \otimes v \\ + (-1)^{n+1} a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n v \end{array} \longleftarrow a_0 \otimes \cdots \otimes a_{i-1} \otimes v$$

Totalize this bicomplex $\{A^{i,j}\}$ by setting

$$D_l := \bigoplus_{i-j=l} A^{i,j} \quad l \in \mathbb{Z}$$

$$D_* = \{ \dots D_{-1} \leftarrow D_0 \leftarrow D_1 \leftarrow \dots \}$$

D_* is a complex of $\mathbb{C}$ vector spaces

$$\mathsf{H}_l(\Gamma;\Psi) := \mathsf{H}_l(D_*)$$

$$\mathsf{H}_l(\Gamma;\Omega_c^*(\widehat{M})), \quad l \in \mathbb{Z}$$

Two extreme cases

1. The acton of Γ on M is proper

2. $M = *$

Ad. 1.

"proper" = The map $\Gamma \times M \to M \times M$, $(\gamma, p) \mapsto (\gamma p, p)$ is proper (i.e. the preimage of any compact set in $M \times M$ is compact)

Equivalently, if $\Delta \subset M$ is any compact subset of M, then $\{\gamma \in \Gamma : \Delta \cap \gamma\Delta \neq \emptyset\}$ is finite

action of Γ on M is smooth and proper $\Longrightarrow$

M/Γ is an orbifold

action of Γ on M is smooth and proper $\Longrightarrow$

action of Γ on $\widehat{M}$ is smooth and proper $\Longrightarrow$

$\widehat{M}/\Gamma$ is an orbifold

When the action of Γ is smooth and proper, $\mathsf{H}_*(\Gamma; \Omega_c^*(\widehat{M})) = ?$

Answer: When the action of Γ on M is smooth and proper

$$\mathsf{H}_*(\Gamma; \Omega_c^*(\widehat{M})) = \mathsf{H}_c^*(\widehat{M}/\Gamma; \mathbb{C})$$

Why ? action of Γ on M is smooth and proper $\Longrightarrow$

$\mathsf{H}_j(\Gamma; \Omega_c^r(\widehat{M})) = 0$ for $j > 0$, $r = 0, 1, 2 \implies$

$\mathsf{H}_j(\Gamma; \Omega_c^r(\widehat{M}))$ is the homology of

$$0 \to (\Omega_c^0(\widehat{M}))_\Gamma \xrightarrow{d} (\Omega_c^1(\widehat{M}))_\Gamma \xrightarrow{d} (\Omega_c^2(\widehat{M}))_\Gamma \xrightarrow{d} \ldots$$

Moriyoshi's lemma

Γ (countable) discrete group

W C^∞ manifold

$\Gamma \times W \to W$ smooth proper action of Γ on W

$\omega \in \Omega_c^*(W)$ smooth compactly supported $\mathbb{C}$-valued differential form on W

Notation: For $\gamma \in \Gamma$, $\gamma_* \omega$ denotes ω translated by γ

Terminology: A closed set $\Delta \subset W$ is Γ-compact if $\gamma\Delta = \Delta$ for all $\gamma \in \Gamma$ and Δ/Γ is compact

Then $\sum_{\gamma\in\Gamma}\gamma_*\omega$ makes sense and $\sum_{\gamma\in\Gamma}\gamma_*\omega$ is a Γ-invariant differential form with Γ-compact support

$$\sum_{\gamma\in\Gamma}\gamma_*\omega \in \left[\Omega^*_{\Gamma-\text{compact}}(W)\right]^\Gamma$$

Consider the map

$$\eta\colon \Omega^*_c(W) \to \left[\Omega^*_{\Gamma-\text{compact}}(W)\right]^\Gamma$$

$$\eta(\omega) = \sum_{\gamma\in\Gamma}\gamma_*\omega$$

For all $g\in\Gamma$ $\eta(g_*\omega) = \eta(\omega)$, so η factors through $[\Omega^*_c(W)]^\Gamma$

$$\begin{tikzcd}
\Omega^*_c(W) \arrow[r] \arrow[d] & \left[\Omega^*_{\Gamma-\text{compact}}(W)\right]^\Gamma \\
{[\Omega^*_c(W)]_\Gamma} \arrow[ur] &
\end{tikzcd}$$

Lemma 4.

$$[\Omega_c^*(W)]_\Gamma \xrightarrow{\cong} \left[\Omega^*_{\Gamma-\text{compact}}(W)\right]^\Gamma$$

Lemma 5.

$$\mathsf{H}_*\left(\left[\Omega^*_{\Gamma-\text{compact}}(W)\right]^\Gamma\right) = \mathsf{H}_c^*(W/\Gamma;\mathbb{C})$$

Proof. this is a slight extension of the de Rham theorem i.e.

$$0 \to \left[\Omega^0_{\Gamma-\text{compact}}(W)\right]^\Gamma \to \left[\Omega^1_{\Gamma-\text{compact}}(W)\right]^\Gamma$$

$$\to \left[\Omega^2_{\Gamma-\text{compact}}(W)\right]^\Gamma \to \dots$$

is a resolution of the constant sheaf on W/Γ. $\square$

Corollary 6. *If the action of Γ on M is smooth and proper then*

$$\mathsf{H}_*(\Gamma;\Omega_c^*(\widehat{M})) = \mathsf{H}_c^*(\widehat{M}/\Gamma;\mathbb{C})$$

Corollary 7. *(Since BC for Γ with coefficient algebra $C_0(M)$ is true when the action of Γ is proper) The Chern character*

$$\mathrm{ch}\colon \mathsf{K}_j(C_r^*(\Gamma,M)) \to \bigoplus_l \mathsf{H}_c^{j+2l}(\widehat{M}/\Gamma;\mathbb{C})$$

exists.

$M = *$

Let $S\Gamma = \{\gamma \in \Gamma : \text{order}(\gamma) < \infty\}$

$$\widehat{*} = S\Gamma$$

$$\Omega_c^r(\widehat{*}) = 0 \text{ for } r > 0$$

$$\Omega_c^0(\widehat{*}) = F\Gamma$$

$$F\Gamma = \{\text{Finite formal sums} \sum_{\gamma \in S\Gamma} \lambda_\gamma[\gamma] : \lambda_\gamma \in \mathbb{C}\}$$

$F\Gamma$ is a Γ-module

$$\left(\sum_{\gamma \in S\Gamma} \lambda_\gamma[\gamma]\right) + \left(\sum_{\gamma \in S\Gamma} \mu_\gamma[\gamma]\right) = \sum_{\gamma \in S\Gamma} (\lambda_\gamma + \mu_\gamma)[\gamma]$$

$$\lambda \left(\sum_{\gamma \in S\Gamma} \lambda_\gamma[\gamma]\right) = \sum_{\gamma \in S\Gamma} \lambda\lambda_\gamma[\gamma], \ \lambda \in \mathbb{C}$$

$$g \left(\sum_{\gamma \in S\Gamma} \lambda_\gamma[\gamma]\right) = \sum_{\gamma \in S\Gamma} \lambda_\gamma[g\gamma g^{-1}], \ g \in \Gamma$$

Problem: Forget about BC and give a direct construction of the Chern character

$$\text{ch}: \mathsf{K}_j(C_r^*(\Gamma, M)) \to \bigoplus_l \mathsf{H}_c^{j+2l}(\widehat{M}/\Gamma; \mathbb{C})$$

(Assuming action of Γ on M is smooth and proper)

The direct construction is done as follows

For simplicity, shall assume $j = 0$ and M/Γ compact

Remark: M/Γ compact $\Longrightarrow$ $\widehat{M}/\Gamma$ compact

Theorem 8 (W. lück and R. Oliver). *Let W be a C^∞ manifold with a given smooth proper and co-compact action of Γ. Then*

$$\mathsf{K}_0(C_r^*(\Gamma, W)) = \begin{array}{l}\textit{Grothendieck group}\\ \textit{of } \Gamma\textit{-equivariant}\\ \mathbb{C} \textit{ vector bundles on } W\end{array}$$

Localized Chern character

W as in the Lüeck-Oliver theorem

$$\mathrm{ch}_{(\mathrm{local})} \colon \mathsf{K}_0(C_r^*(\Gamma, W)) \to \bigoplus \mathsf{H}^{2l}(W/\Gamma; \mathbb{C})$$

is constructed as follows

Let F be a Γ-equivariant C^∞ $\mathbb{C}$ vector bundle on W

Choose a Γ-equivariant connection D for F

Consider the Γ-equivariant differential form

$$\mathrm{ch}(K) = \mathrm{Tr}\left(\exp\left(\frac{K}{2\pi i}\right)\right)$$

$$K = \mathrm{curvature}(D)$$

$$\mathrm{ch}(K) \in [\Omega^*(W)]^\Gamma \implies$$

$\mathrm{ch}(K)$ determines an element in $\bigoplus \mathrm{H}^{2l}(W/\Gamma;\mathbb{C})$

This is $\mathrm{ch}_{(\mathrm{local})}(F)$

Local Chern character

Alternate (more topological) construction of

$$\mathrm{ch}_{(\mathrm{local})}\colon \mathrm{K}_0(C_r^*(\Gamma, W)) \to \bigoplus \mathrm{H}^{2l}(W/\Gamma;\mathbb{C})$$

W as in the Lüeck-Oliver theorem

Let F be a Γ-equivariant $\mathbb{C}$ vector bundle on W

$$\begin{array}{c} E\Gamma \times_\Gamma F \\ \downarrow \\ E\Gamma \times_\Gamma W \end{array}$$

φ = classifying map for $E\Gamma \times_\Gamma F$

$$\varphi\colon E\Gamma \times_\Gamma W \to BU(r)$$

r = fiber dimension of $E\Gamma \times_\Gamma F$

$$\mathrm{ch}_{(\text{universal})} \in \prod_{l=0}^{\infty} \mathrm{H}^{2l}(BU(r);\mathbb{Q})$$

$$\varphi^*(\mathrm{ch}_{(\text{universal})}) \in \prod_{l=0}^{\infty} \mathrm{H}^{2l}(E\Gamma \times_\Gamma W;\mathbb{Q})$$

$$\begin{tikzcd} & E\Gamma \times_\Gamma W \arrow[dl] \arrow[dr] & \\ B\Gamma & & W/\Gamma \end{tikzcd}$$

$$\mathrm{H}^*(W/\Gamma;\mathbb{Q}) \cong \mathrm{H}^*(E\Gamma \times_\Gamma W;\mathbb{Q})$$

$$\underbrace{\varphi^*(\mathrm{ch}_{(\text{universal})})}_{\text{this is } \mathrm{ch}_{(\text{local})}(F)} \in \prod_{l=0}^{\infty} \mathrm{H}^{2l}(E\Gamma \times_\Gamma W;\mathbb{Q})$$

$$\mathrm{ch}\colon \mathrm{K}_0(C_r^*(\Gamma, M)) \to \bigoplus_l \mathrm{H}^{2l}(\widehat{M}/\Gamma;\mathbb{C})$$

Action of Γ on M assumed to be smooth, proper, and co-compact

$$\widehat{M} = \{(\gamma, p) \in \Gamma \times M : \gamma p = p\}$$

$$\Gamma \times \widehat{M} \to \widehat{M}$$

$$g(\gamma, p) = (g\gamma g^{-1}, gp), \quad g \in \Gamma,\ (\gamma, p) \in \widehat{M}$$

Let F be a Γ-equivariant vector bundle on $\widehat{M}$

Define $\theta\colon F \to F$ for $v \in F_{(\gamma,p)}$

$$\theta(v) = \gamma v \in F_{(\gamma,p)}$$

F is any Γ-equivariant $\mathbb{C}$ vector bundle on $\widehat{M}$

θ is an automorphism of F

θ is of finite order, $\theta^m = Id$ for some positive integer m

$$F = \underbrace{F_1}_{\zeta_1} \oplus \underbrace{F_2}_{\zeta_2} \oplus \ldots \oplus \underbrace{F_t}_{\zeta_t}$$

Each $\zeta_j \in \mathbb{C}$ is a root of unity

$$\mathsf{K}_0(C_r^*(\Gamma, \widehat{M})) \to \bigoplus_l \mathsf{H}^{2l}(\widehat{M}/\Gamma; \mathbb{C})$$

$$F \mapsto \sum_{\nu=1}^{t} \zeta_\nu \, \mathsf{ch}_{(\mathsf{local})}(F_\nu)$$

$$\widehat{M} \xrightarrow{\rho} M$$

$$(\gamma, p) \mapsto p$$

$$\mathsf{K}_0(C_r^*(\Gamma, M)) \to \mathsf{K}_0(C_r^*(\Gamma, \widehat{M}))$$

$$E \mapsto \rho^* E$$

$$\mathsf{K}_0(C_r^*(\Gamma, \widehat{M})) \to \bigoplus_l \mathsf{H}^{2l}(\widehat{M}/\Gamma; \mathbb{C})$$

$$F \mapsto \sum_{\nu=1}^{t} \zeta_\nu \, \mathsf{ch}_{(\mathsf{local})}(F_\nu)$$

$$\mathsf{ch}\colon \mathsf{K}_0(C_r^*(\Gamma, M)) \to \bigoplus_l \mathsf{H}^{2l}(\widehat{M}/\Gamma; \mathbb{C})$$

$$\mathsf{K}_0(C_r^*(\Gamma, M)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \bigoplus_l \mathsf{H}^{2l}(\widehat{M}/\Gamma; \mathbb{C})$$

$$\mathsf{H}_l(\Gamma;\Omega_c^*(\widehat{*})) = \mathsf{H}_l(\Gamma;F\Gamma) \quad l = 0,1,2,\ldots$$

$$\begin{tikzcd} \mathsf{K}_j^\Gamma(\underline{E}\Gamma) \arrow[r] \arrow[d] & \mathsf{K}_j(C_r^*(\Gamma)) \\ \bigoplus_l \mathsf{H}_l(\Gamma;F\Gamma) & \end{tikzcd}$$

$$\mathsf{K}_j^\Gamma(\underline{E}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \bigoplus_l \mathsf{H}_{j+2l}(\Gamma;F\Gamma)$$

$\Gamma \times M \to M$ smooth action

$$\mathsf{ch}\colon\ \mathsf{K}_j(C_r^*(\Gamma,M)) \xrightarrow{?} \bigoplus_l \mathsf{H}_{j+2l}(\Gamma;\Omega_c^*(\widehat{M}))$$

X locally compact Hausdorff topological space

$\Gamma \times X \to X$ continuous action

$$\mathsf{ch}\colon\ \mathsf{K}_j(C_r^*(\Gamma,X)) \xrightarrow{?} \bigoplus_l \mathsf{H}_{j+2l}(\Gamma;C_c^*(\widehat{X}))$$

$C_c^*(\widehat{X}) =$ Alexander-Spanier cochains with compact supports on $\widehat{X}$

$$\widehat{X} = \{(\gamma,x) \in \Gamma \times X : \mathrm{order}(\gamma) < \infty,\ \gamma x = x\}$$