WHAT IS AN EQUIVARIANT INDEX?

Paul Baum

Warsaw, 8 May 2006

Γ discrete (countable) group

M C^∞-manifold, $\partial M = \emptyset$

$\Gamma \times M \to M$ smooth, proper, co-compact action of Γ on M

"**smooth**" = each $\gamma \in \Gamma$ acts on M by diffeomorphisms

"**proper**" = if Δ is any compact subset of M, then $\{\gamma \in \Gamma : \Delta \cap \gamma\Delta \neq \emptyset\}$ is finite

"**co-compact**" = the quotient space M/Γ is compact

REMARKS

1. If $p \in M$, then $\{\gamma \in \Gamma : \gamma p = p\}$ is a finite subgroup of Γ

2. M/Γ is a compact orbifold

3. M is compact $\Longleftrightarrow$ Γ is finite

$\Gamma \times M \to M$ smooth, proper, co-compact action of Γ on M

Let D be a Γ-equivariant elliptic differential operator (or ψDO) on M

What is the (equivariant) index of D ?

$$\text{Index}_{\Gamma}(D) = ?$$

EXAMPLE

$$M = \mathbb{R}$$
$$\Gamma = \mathbb{Z}$$
$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$$
$$(n, t) \mapsto n + t$$
$$D = -i\frac{d}{dx}$$

$$C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$$

$-i\frac{d}{dx}$ is an unbounded operator on the Hilbert space $L^2(\mathbb{R})$

$$-i\frac{d}{dx} \colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$$

$-i\frac{d}{dx}$ is **essentially self-adjoint** i.e.

$$-i\frac{d}{dx} \colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$$

has a unique self-adjoint extension

What is the (equivariant) index of $-i\frac{d}{dx}$?

$$\text{Index}_{\mathbb{Z}}\left(-i\frac{d}{dx}\right) = ?$$

H a Hilbert space

An **unbounded operator** on H is a pair $(\mathcal{D}, T)$ such that

1. $\mathcal{D}$ is a vector subspace of H

2. $\mathcal{D}$ is dense in H

3. $T\colon \mathcal{D} \to H$ is a $\mathbb{C}$-linear map

4. $(\mathcal{D}, T)$ is closable (i.e. the closure of graph T in $H \oplus H$ is the graph of a $\mathbb{C}$-linear map

$$P(\overline{\text{graph}\, T}) \to H$$

where $P\colon H \oplus H \to H$, $P(u, v) = u$)

$(\mathcal{D}, T)$ is **symmetric** $\iff$

$$\langle Tu,\, v\rangle = \langle u,\, Tv\rangle \quad \forall\, u, v \in \mathcal{D}$$

CAUTION

symmetric $\neq$ self-adjoint

symmetric $\Longleftarrow$ self-adjoint

If $(\mathcal{D}, T)$ is an unbounded operator on H, then

$$\mathcal{D}(T^*) := \left\{ u \in H : \begin{array}{l} v \mapsto \langle u, Tv \rangle \text{ extends} \\ \text{from } \mathcal{D} \text{ to } H \\ \text{to be bounded} \\ \text{linear functional} \end{array} \right\}$$

For $u \in \mathcal{D}(T^*)$ and $v \in H$, $\langle u, Tv \rangle = \langle T^* u, v \rangle$

$$T^* \colon \mathcal{D}(T^*) \to H$$

$(\mathcal{D}, T)$ is **selfadjoint** if $(\mathcal{D}, T) = (\mathcal{D}(T^*), T^*)$

$$C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$$

$(C_c^\infty(\mathbb{R}), -i\frac{d}{dx})$ has a unique self-adjoint extension $(\mathcal{D}, -i\frac{d}{dx})$

$$\mathcal{D} = \left\{ u \in L^2(\mathbb{R}) : \begin{array}{l} -i\frac{d}{dx} \text{ in the distribution} \\ \text{sense is in } L^2(\mathbb{R}) \end{array} \right\}$$

$$= \left\{ u \in L^2(\mathbb{R}) : x\widehat{u} \in L^2(\mathbb{R}) \right\}$$

$\widehat{u}$ = the Fourier transform of u

$$x \colon \mathbb{R} \to \mathbb{R}, \quad x(t) = t \quad \forall\, t \in \mathbb{R}$$

$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$$
$$(n, t) \mapsto n + t$$

fundamental domain is $[0, 1]$

$$C_c^\infty((0,1)) \subset L^2([0,1])$$

$(C_c^\infty((0,1)), -i\frac{d}{dx})$ has one self-adjoint extension for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$

Fix $\theta \in [0, 1]$

$C_\theta^\infty([0,1]) := \{u \in C^\infty([0,1]) : u(1) = e^{2\pi i\theta} u(0)\}$

$$L^2([0,1]) \supset C_\theta^\infty([0,1]) \supset C_\theta^\infty((0,1))$$

$$e^{2\pi i(n+\theta)x}, \quad n = 0, \pm 1, \pm 2, \ldots$$

is an orthonormal basis for $L^2([0,1])$

$$e^{2\pi i(n+\theta)x} \in C_\theta^\infty([0,1])$$

$$-i\frac{d}{dx} e^{2\pi i(n+\theta)x} = 2\pi(n+\theta) e^{2\pi i(n+\theta)x}$$

$$e^{2\pi i(n+\theta)x}, \quad n = 0, \pm 1, \pm 2, \ldots$$

is an orthonormal basis for $L^2([0,1])$ consisting of eigen-vectors of the operator $-i\frac{d}{dx}$

The eigen-values are $2\pi(n+\theta)$, $n = 0, \pm 1, \pm 2, \ldots$

Set

$$\mathcal{D}_\theta := \left\{ \sum_{n=-\infty}^{\infty} \lambda_n e^{2\pi i(n+\theta)x} \in L^2([0,1]) : \right.$$

$$\left. \sum_{n=-\infty}^{\infty} 2\pi(n+\theta)\lambda_n e^{2\pi i(n+\theta)x} \in L^2([0,1]) \right\}$$

$$L^2([0,1]) \supset \mathcal{D}_\theta \supset C_\theta^\infty([0,1]) \supset C^\infty((0,1))$$

$(\mathcal{D}_\theta, -i\frac{d}{dx})$ is an unbounded self-adjoint operator on $L^2([0,1])$

For $\theta = 0$ and $\theta = 1$ we have the same unbounded self-adjoint operator

Except for this, the unbounded self-adjoint operators $(\mathcal{D}_\theta, -i\frac{d}{dx})$ are all distinct

Spectrum of $(\mathcal{D}_\theta, -i\frac{d}{dx})$ is $\{2\pi(n+\theta) : n = 0, \pm 1, \pm 2, \ldots\}$.

We shall now convert $(\mathcal{D}_\theta, -i\frac{d}{dx})$ to a bounded operator on $L^2([0,1])$

Functional calculus. Apply the function $\frac{x}{\sqrt{1+x^2}}$ to $(\mathcal{D}_\theta, -i\frac{d}{dx})$ to obtain $T_\theta \colon L^2([0,1]) \to L^2([0,1])$.

Spectrum of T_θ is

$$\frac{x}{\sqrt{1+x^2}}\left(\mathrm{sp}\left(\mathcal{D}_\theta, -i\frac{d}{dx}\right)\right) \cup \{-1,1\}$$

$$= \frac{2\pi(n+\theta)}{\sqrt{1+(2\pi(n+\theta))^2}} \cup \{-1,1\}$$

$$T_\theta(e^{2\pi i(n+\theta)x}) = \frac{2\pi(n+\theta)}{\sqrt{1+(2\pi(n+\theta))^2}} e^{2\pi i(n+\theta)x}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

$\theta \mapsto T_\theta$ is a loop of bounded self-adjoint operators on $L^2([0,1])$.

This loop $\theta \mapsto T_\theta$ should be viewed as giving the index of

$$(\mathcal{D}, -i\frac{d}{dx}) \qquad -i\frac{d}{dx} \colon \mathcal{D} \to L^2(\mathbb{R})$$

WHY ?

H Hilbert space

$T\colon H \to H$ bounded operator on H

$$\|T\| = \sup_{\langle v, v\rangle = 1} \langle Tv, Tv\rangle^{\frac{1}{2}}$$

operator norm

A bounded operator $T\colon H \to H$ is **Fredholm** if

$$\dim_{\mathbb{C}}(\ker T) < \infty$$

and

$$\dim_{\mathbb{C}}(\operatorname{coker} T) < \infty$$

Let T be a Fredholm operator on H

$$\mathrm{Index}(T) := \dim_{\mathbb{C}}(\ker T) - \dim_{\mathbb{C}}(\operatorname{coker} T)$$

$$\mathcal{L}(H) := \{\text{bounded operators } T\colon H \to H\}$$
$$\mathcal{F}(H) := \{T \in \mathcal{L}(H) : T \text{ is Fredholm}\}$$

$\mathcal{F}(H)$ is topologized by the operator norm topology

$$\pi_0(\mathcal{F}(H)) = \mathbb{Z}$$
$$T \mapsto \mathrm{Index}(T)$$

$S, T \in \mathcal{F}(H)$ are in the same connected component of $\mathcal{F}(H) \iff \mathrm{Index}(T) = \mathrm{Index}\, S$

$\mathcal{F}(H)$ is a classifying space for K^0

K^0 = Atiyah-Hirzerbruch K-theory

Let X be any compact Hausdorff topological space

$$\mathsf{K}^0(X) := \begin{array}{l}\text{Grothendieck group of}\\ \mathbb{C}\text{-vector bundles on } X\end{array}$$

Notation: X, Y topological spaces

$$[X,Y] = \left\{ \begin{array}{l}\text{Homotopy classes of continuous}\\ \text{maps } f\colon X \to Y\end{array} \right\}$$

$$= \{\text{Continuous maps } f\colon X \to Y\} / \sim$$

$$\sim = \text{homotopy}$$

Theorem 1 (Atiyah, Janich)**.** *Let X be any compact Hausdorff topological space. Then*

$$\mathsf{K}^0(X) = [X, \mathcal{F}(H)]$$

$$\mathcal{F}_{\text{s.a.}}(H) := \left\{ T \in \mathcal{L}(H) : \begin{array}{l} T \text{ is Fredholm} \\ \text{and selfadjoint} \end{array} \right\}$$

$\mathcal{F}_{\text{s.a.}}(H)$ has three connected components

$$\mathcal{F}_{\text{s.a.}}(H) = \mathcal{F}^{-}_{\text{s.a.}}(H) \cup \mathcal{F}_{\text{s.a.}}(H)^{\#} \cup \mathcal{F}^{+}_{\text{s.a.}}(H)$$

Essential spectrum

$$\mathcal{L}^{\text{inv}}(H) := \left\{ T \in \mathcal{L}(H) : \begin{array}{l} \exists\, S \in \mathcal{L}(H) \text{with} \\ ST = TS = Id \end{array} \right\}$$

$$Id(v) = v \quad \forall\, v \in H$$

$$\mathcal{L}(H) \supset \mathcal{F}(H) \supset \mathcal{L}^{\text{inv}}(H)$$

$T \in \mathcal{L}(H)$

$$\text{Spectrum}(T) := \{\lambda \in \mathbb{C} : (\lambda Id - T) \notin \mathcal{L}^{inv}(H)\}$$

$$\text{Essential spectrum}(T) := \{\lambda \in \mathbb{C} : (\lambda Id - T) \notin \mathcal{F}(H)\}$$

$$\text{Essential spectrum}(T) \subset \text{Spectrum}(T)$$

$$T \in \mathcal{F}(H) \Longleftrightarrow 0 \notin \text{Essential spectrum}(T)$$

$$\mathcal{F}^{-}_{\text{s.a.}} = \left\{ T \in \mathcal{F}_{s.a.}(H) : \begin{array}{l} \text{Essential spectrum}(T) \\ \subset (-\infty, 0) \end{array} \right\}$$

$$\mathcal{F}^{+}_{\text{s.a.}} = \left\{ T \in \mathcal{F}_{s.a.}(H) : \begin{array}{l} \text{Essential spectrum}(T) \\ \subset (0, \infty) \end{array} \right\}$$

$$\mathcal{F}^{\#}_{\text{s.a.}} = \left\{ \begin{array}{l} T \in \mathcal{F}_{s.a.}(H): \\ \quad \text{Essential spectrum}(T) \cap (-\infty, 0) \neq \emptyset \\ \quad \text{Essential spectrum}(T) \cap (0, \infty) \neq \emptyset \end{array} \right\}$$

$\mathcal{F}^{\#}_{\text{s.a.}}(H)$ is a classifying space for K^1

K^1 = Atiyah-Hirzerbruch K-theory

Let X be any compact Hausdorff topological space

$$\mathsf{K}^1(X) := \lim_{n\to\infty} [X, \mathsf{GL}(n, \mathbb{C})]$$

Theorem 2 (Atiyah, Singer)**.** *Let X be any compact Hausdorff topological space. Then*

$$\mathsf{K}^1(X) = [X, \mathcal{F}^{\#}_{\text{s.a.}}(H)]$$

Bott periodicity

$$\Omega\mathcal{F}(H) \sim \mathcal{F}^{\#}_{\text{s.a.}}(H)$$

$$\Omega\mathcal{F}^{\#}_{\text{s.a.}}(H) \sim \mathcal{F}(H)$$

$$\pi_j(\mathcal{F}(H)) = \begin{cases} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$$

$$\pi_j(\mathcal{F}^{\#}_{\text{s.a.}}(H)) = \begin{cases} 0 & j \text{ even} \\ \mathbb{Z} & j \text{ odd} \end{cases}$$

EXAMPLE

$$-i\frac{d}{dx}\colon \mathcal{D} \to L^2(\mathbb{R})$$

$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$$

$$(n, t) \mapsto n + t$$

$$\mathbb{S}^1 \to \mathcal{F}^{\#}_{\text{s.a.}}(L^2([0,1]))$$

$$e^{2\pi i\theta} \mapsto T_\theta$$

This loop is the generator of

$$\pi_1(\mathcal{F}^{\#}_{\text{s.a.}}(L^2([0,1]))) = \mathbb{Z} = \mathsf{K}^1(\mathbb{S}^1)$$

$\mathbb{S}^1$ is the Pontrjagin dual of $\mathbb{Z}$

G abelian locally compact Hausdorff topological group

Pontrjagin dual $\widehat{G}$ is again an abelian locally compact Hausdorff topological group

$$\widehat{G} := \mathsf{Hom}(G, \mathbb{S}^1)$$

$\widehat{G}$ is compact $\Longleftrightarrow$ G is discrete

$\Gamma \times M \to M$ smooth proper co-compact action of Γ on M

D Γ-equivariant elliptic differential (or ψDO) operator on M

Assume:

1. Γ is abelian

2. D is essenitally self-adjoint

Let Δ be a fundamental domain for the action of Γ on M

Each $\varphi \in \widehat{\Gamma}$ determines a boundary condition for $D|_\Delta$

Using this boundary condition, construct a bounded self-adjoint operator T_φ

$$\widehat{\Gamma} \to \mathcal{F}^{\#}_{\text{s.a.}}(H)$$

$$\varphi \mapsto T_\varphi$$

$$\text{Index}_\Gamma(D) \in \text{K}^1(\widehat{\Gamma})$$

Remark: $\widehat{\Gamma}$ is viewed here as a compact Hausdorff topological space. The group structure of $\widehat{\Gamma}$ is not being used.

$\Gamma \times M \to M$

D

Assume:

1. Γ is abelian

2. D has no self-adjoint property

$$\widehat{\Gamma} \to \mathcal{F}(H)$$

$$\text{Index}_\Gamma(D) \in \mathsf{K}^0(\widehat{\Gamma})$$

EXAMPLE

$$\Gamma = \mathbb{Z} \oplus \mathbb{Z}, \quad M = \mathbb{R}^2$$

$$(\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}^2$$

$$((n_1, n_2), (t_1, t_2)) \mapsto (n_1 + t_1, n_2 + t_2)$$

$$D = \overline{\partial} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$$

$$\widehat{\mathbb{Z} \oplus \mathbb{Z}} = \mathbb{S}^1 \times \mathbb{S}^1$$

$$\begin{array}{rl} \mathsf{K}^0(\mathbb{S}^1 \times \mathbb{S}^1) = & \mathbb{Z} \oplus \mathbb{Z} \\ & \;\mathbf{1} \quad L \end{array}$$

$$\mathbf{1} = (\mathbb{S}^1 \times \mathbb{S}^1) \times \mathbb{C}$$

$$L = \text{ Hopf line bundle}$$

$$\text{Index}_{\mathbb{Z}\oplus\mathbb{Z}}(\overline{\partial}) \in \mathsf{K}^0(\widehat{\mathbb{Z} \oplus \mathbb{Z}}) = \mathsf{K}^0(\mathbb{S}^1 \times \mathbb{S}^1)$$

$$\text{Index}_{\mathbb{Z}\oplus\mathbb{Z}}(\overline{\partial}) = L - \mathbf{1}$$

Application: Dirac operato formulation of Baum-Connes conjecture

Γ is a (countable) discrete group

$C_r^*(\Gamma)$ denotes the reduced C*-algebra of Γ

$C_r^*(\Gamma)$ is the completion of the purely algebraic group algebra $\mathbb{C}[\Gamma]$ via the (left) regular representation of Γ

$\mathsf{K}_j(C_r^*(\Gamma))$ denotes the j-th K-theory group of $C_r^*(\Gamma)$, $j = 0, 1, 2, \ldots$

Bott periodicity: $\mathsf{K}_j(C_r^*(\Gamma)) \cong \mathsf{K}_{j+2}(C_r^*(\Gamma))$, $j = 0, 1, 2, \ldots$

If Γ is abelian, then $\mathsf{K}_j(C_r^*(\Gamma)) \cong \mathsf{K}^j(\widehat{\Gamma})$ where $\mathsf{K}^j(\widehat{\Gamma})$ is the Atiyah-Hirzerbruch K-theory of the Pontrjagin dual $\widehat{\Gamma}$

Moral: If Γ is not abelian, then $\mathsf{K}_j(C_r^*(\Gamma))$ replaces $\mathsf{K}^j(\widehat{\Gamma})$

We shall now define an abelian group $\mathsf{K}_j^{\mathrm{top}}(\Gamma)$, $j = 0, 1$

Definition of $\mathsf{K}_j^{\mathrm{top}}$, $j = 0, 1$

Consider pairs (M, E) such that

1. M is a C^∞-manifold, $\partial M = \emptyset$, with a given smooth, proper co-compact action of Γ

$$\Gamma \times M \to M$$

2. M has a given Γ-equivariant $\mathrm{Spin}^{\mathrm{c}}$-structure

3. E is a Γ-equivariant vector bundle on M

$$\mathsf{K}_0^{\mathsf{top}}(\Gamma) \oplus \mathsf{K}_1^{\mathsf{top}}(\Gamma) = \{(M, E)\}/\sim$$

Addition will be disjoint union

$$(M, E) + (M, E') = (M \cup M', E \cup E')$$

Each fiber of E is a finite dimensional vector space over $\mathbb{C}$

$$\dim_{\mathbb{C}}(E_p) < \infty \quad p \in M$$

The equivalence relation

Isomorphism (M, E) is isomorphic to (M', E') iff $\exists$ a Γ-equivariant diffeomorphism

$$\psi \colon M \to M'$$

preserving the Γ-equivariant Spin^c-structures on M, M' and with

$$\psi^*(E') \cong E$$

The equivalence relation $\sim$ will be generated by three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

Bordism (M_0, E_0) is **bordant** to (M_1, E_1) iff $\exists$ (W, E) such that:

1. W is a C^∞ manifold with boundary, with a given smooth proper co-compact action of Γ
$$\Gamma \times W \to W$$
2. W has a given equivariant Spin^c-structure
3. E is a Γ-equivariant vector bundle on W
4. $(\partial W, E|_{\partial W}) \cong (M_0, E_0) \cup (-M_1, E_1)$

Direct sum - disjoint union

Let E, E' be two Γ-equivariant vector bundles on M
$$(M, E) \cup (M, E') \sim (M, E \oplus E')$$

Vector bundle modification

(M, E)

Let F be Γ-equivariant Spin^c vector bundle on M

Assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M$$

for every fiber F_p of F

$$\mathbf{1} = M \times \mathbb{R} \quad \gamma(p,t) = (\gamma p, t)$$

$$\gamma \in \Gamma \quad (p,t) \in \mathbf{1}$$

$$S(F \oplus \mathbf{1}) := \text{unit sphere bundle of } F \oplus \mathbf{1}$$

$$(M, E) \sim (S(F \oplus \mathbf{1}), \beta_+ \otimes \pi^* E)$$

$$\begin{array}{c} S(F \oplus \mathbf{1}) \\ \big\downarrow \pi \\ M \end{array}$$

This is a fibration with even-dimensional spheres as fibers

$F \oplus \mathbf{1}$ is a Γ-eqivariant Spin^c vector bundle on M with odd dimensional fibers. Let Σ be the spinor bundle for $F \oplus \mathbf{1}$

$$\text{Cliff}_{\mathbb{C}}(F_p \oplus \mathbb{R}) \otimes \Sigma_p \to \Sigma_p$$

$$\pi^* \Sigma = \beta_+ \oplus \beta_-$$

$$(M, E) \sim (S(F \oplus \mathbf{1}), \beta_+ \otimes \pi^* E)$$

$$\{(M,E)\}/\sim = \mathsf{K}_0^{\mathsf{top}}(\Gamma) \oplus \mathsf{K}_1^{\mathsf{top}}(\Gamma)$$

$\mathsf{K}_j^{\mathsf{top}}(\Gamma) =$ subgroup of $\{(M,E)\}/\sim$ consisting of all (M,E) such that every connected component of M has dimension $\equiv j \mod 2$, $j = 0, 1$

Notation: for (M,E) D_E is the Dirac operator of M tensored with E

$F =$ spinor bundle of M

$$D_E\colon C_c^\infty(M, F\otimes E) \to C_c^\infty(M, F\otimes E)$$

$$\mathsf{K}_j^{\mathsf{top}}(\Gamma) \to \mathsf{K}_j(C_r^*(\Gamma)) \quad j = 0, 1$$

$$(M, E) \mapsto \mathsf{Index}(D_E)$$

Conjecture (BC). (P. Baum, A. Connes) For any (countable) discrete group

$$\mathsf{K}_j^{\mathsf{top}}(\Gamma) \to \mathsf{K}_j(C_r^*(\Gamma)) \quad j = 0, 1$$

is an isomorphism

Corollary. If BC conjecture is true for Γ, then

1. Every element of $\mathsf{K}_j(C_r^*(\Gamma))$ is of the form Index(D_E) for some (M, E) (surjectivity)

2. (M, E) and (M', E') have

$$\mathsf{Index}(D_E) = \mathsf{Index}(D_{E'})$$

   if and only if it is possible to pass from (M, E) to (M', E') by a finite sequence of the three elementary moves

   - Bordism
   - Direct sum - disjoint union
   - Vector bundle modification

   (injectivity)

Corollaries of BC

- Novikov conjecture
- Stable Gromov-Lawson-Rosenberg conjecture
- Idempotent conjecture
- Kadison-Kaplansky conjecture
- Mackey analogy
- Construction of the discrete series via Dirac induction (Parthasarathy, Atiyah-Schmid)
- Homotopy invariance of ρ-invariants (Keswani, Piazza-Schick)

Theorem.(N. Higson, G. Kasparov) If Γ is a discrete group which is amenable (or a-t-menable), then BC is true for Γ.

Theorem.(I. Mineyev, G. Yu, V. Lafforgue) If Γ is a discrete group which is hyperbolic (in Gromov's sense), then BC is true for Γ.