Generalized bialgebras and triples of operads

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Contents

Abstract. We introduce the notion of generalized bialgebra, which includes the classical notion of bialgebra (Hopf algebra) and many others. We prove that, under some mild conditions, a connected generalized bialgebra is completely determined by its primitive part. This structure theorem extends the classical Poincaré-Birkhoff-Witt theorem and the Cartier-Milnor-Moore theorem, valid for cocommutative bialgebras, to a large class of generalized bialgebras. Technically we work in the theory of operads which permits us to give a conceptual proof of our main theorem. It unifies several results, generalizing PBW and CMM, scattered in the literature. We treat many explicit examples and suggest a few conjectures.

Introduction

The aim of this paper is to prove that, under some simple conditions, there is a structure theorem for generalized bialgebras.

First we introduce the notion of "generalized bialgebras", which includes the classical notions of bialgebras, Lie bialgebras, infinitesimal bialgebras, dendriform bialgebras and many others. A type of generalized bialgebras is determined by the coalgebra structure $\mathcal{C}^c$, the algebra structure $\mathcal{A}$ and the compatibility relations between the operations and the cooperations. For $\mathcal{C}^c = As^c$ (coassociative coalgebra) and $\mathcal{A} = As$ (associative algebra) with Hopf compatibility relation, we get the classical notion of bialgebra (Hopf algebra). We make the following assumption:

`(H0)` *the compatibility relations are distributive.*

It means that any composition of an operation followed by a cooperation can be rewritten as cooperations first and then operations. Then, we make an assumption on the free $\mathcal{A}$-algebra with respect to the bialgebra structure:

`(H1)` *the free $\mathcal{A}$-algebra $\mathcal{A}(V)$ is naturally a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra.*

At this point we are able to determine a new algebra structure, denoted $\mathcal{P}$, such that the primitive part of any connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra is a $\mathcal{P}$-algebra. In other words we show that the $\mathcal{A}$-operations which are well-defined on the primitive part are stable by composition. Of course we get $\mathcal{P} = Lie$ in the classical case. One should observe that, even when the types $\mathcal{A}$ and $\mathcal{C}$ are described by explicit generators and relations, there is no obvious way to get such a presentation for the type $\mathcal{P}$. Therefore one needs to work with "abstract types of algebras", that is with operads.

The forgetful functor $\mathcal{A}$-alg $\to$ $\mathcal{P}$-alg from the category of $\mathcal{A}$-algebras to the category of $\mathcal{P}$-algebras admits a left adjoint which we denote by $U : \mathcal{P}\text{-alg} \to \mathcal{A}\text{-alg}$. The main result unravels the algebraic structure and the coalgebraic structure of any connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra:

Structure Theorem for generalized bialgebras. *Let $\mathcal{C}^c$-$\mathcal{A}$ be a bialgebra type which satisfies (H0) , (H1) and*
(H2epi) the coalgebra map $\varphi(V) : \mathcal{A}(V) \to \mathcal{C}^c(V)$ is split surjective.

Then, for any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ with primitive part $\mathrm{Prim}\,\mathcal{H}$, the following are equivalent:
(a) $\mathcal{H}$ is connected,
(b) $\mathcal{H}$ is isomorphic to $U(\mathrm{Prim}\,\mathcal{H})$,
(c) $\mathcal{H}$ is cofree over its primitive part, i.e. isomorphic to $\mathcal{C}^c(\mathrm{Prim}\,\mathcal{H})$.

In this statement the space $\mathrm{Prim}\,\mathcal{H}$ is defined as

$$\mathrm{Prim}\,\mathcal{H} = \{x \in \mathcal{H} \mid \delta(x) = 0 \text{for any cooperation } \delta \text{ of arity } \geq 2\}.$$

As said above, the tool to determine $\mathcal{P}$ and also to prove the structure theorem, is the *operad* theory. A triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ as above is said to be *good* if the structure theorem holds. There are many cases and many results in the literature which fall into this framework. They come from algebraic topology, noncommutative geometry, universal algebra, representation theory, algebraic combinatorics, computer sciences. Here are some of them.

The case of (classical) cocommutative bialgebras (i.e. Com^c-As-bialgebras, with Hopf compatibility relation) is well-known. Here $\mathcal{P}$ is Lie, that is, the primitive part of a classical bialgebra is a Lie algebra. The functor U is the universal enveloping functor

$$U : Lie\text{-alg} \to As\text{-alg}.$$

The isomorphism $\mathcal{H} \cong U(\mathrm{Prim}\,\mathcal{H})$ is the Cartier-Milnor-Moore theorem. The isomorphism $U(\mathfrak{g}) \cong S^c(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra, is essentially the Poincaré-Birkhoff-Witt theorem. It implies that, given a basis for $\mathfrak{g}$, we can make up a basis for $U(\mathfrak{g})$ from the commutative polynomials over a basis of $\mathfrak{g}$ (classical PBW theorem). In short, there is a good triple

$$(Com, As, Lie)\ .$$

When the bialgebra type satisfies the stronger hypothesis

`(H2iso)` *the $\mathcal{C}^c$-coalgebra map $\varphi(V) : \mathcal{A}(V) \to \mathcal{C}^c(V)$ is an isomorphism,*

then the operad $\mathcal{P}$ is trivial that is $\mathcal{P}$-alg is the category `Vect` of vector spaces. The triple is $(\mathcal{C}, \mathcal{A}, Vect)$ and the structure theorem becomes the

Rigidity Theorem for generalized bialgebras. *Let $\mathcal{C}^c$-$\mathcal{A}$ be a bialgebra type which satisfies `(H0)` , `(H1)` and `(H2iso)` . Then any connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is free and cofree:*

$$\mathcal{A}(\mathrm{Prim}\,\mathcal{H}) \cong \mathcal{H} \cong \mathcal{C}^c(\mathrm{Prim}\,\mathcal{H})\ .$$

There are many good triples of the form $(\mathcal{A}, \mathcal{A}, Vect)$. For instance, when $\mathcal{A} = Com$ and the compatibility relation is the Hopf relation, then this is the classical Hopf-Borel theorem (originally phrased in the framework of graded vector spaces). One has to keep in mind that, in the notion $\mathcal{A}^c$-$\mathcal{A}$-bialgebra, the compatibility relation(s) is an important feature. For instance the rigidity theorem for $\mathcal{A} = As$ (operad of associative algebras) does not hold for the Hopf compatibility relation, but does hold when the compatibility relation is the *unital infinitesimal relation*:

$$\delta(xy) = x \otimes y + x_{(1)} \otimes x_{(2)}y + xy_{(1)} \otimes y_{(2)}\ ,$$

where we have put $\delta(x) = x_{(1)} \otimes x_{(2)}$.

Similarly, there is a rigidity theorem for Lie^c-Lie-bialgebras, but the compatibility relation is not the one of classical Lie bialgebras (see Chapter 4 section 4.4). It gives a criterion to check if a given Lie algebra is free.

In most cases characteristic zero is necessary, but here is an example of a good triple of operads which is valid over any field $\mathbb{K}$ (characteristic free). The algebra type has two associative operations denoted $\prec$ and $\succ$ which satisfy moreover the relation

$$(x \succ y) \prec z = x \succ (y \prec z).$$

We call them OU-algebras because the free algebra admits an elegant description in terms of the Over and the Under operation on planar binary trees. The coalgebra type is determined by a coassociative cooperation δ and the compatibility relations is the unital infinitesimal relation for both pairs $(\delta, \prec)$ and $(\delta, \succ)$. We can show that there is a structure theorem in this case and that the primitive structure is simply a magmatic structure. The magmatic operation $x \cdot y$ is given by

$$x \cdot y := x \succ y - x \prec y.$$

In short, there is a good triple

$$(As, OU, Mag) \ .$$

This example, which is treated in details in Chapter 5, is remarkable, not only because all the operads are binary, quadratic and Koszul (like in the (Com, As, Lie) case), but because they are also set-theoretic and regular.

The proof of the main theorem relies on the construction of a certain "universal" idempotent which maps a given bialgebra onto its primitive part. This construction does not depend on the explicit presentation of $\mathcal{A}$ or $\mathcal{C}$. It is a conceptual construction. In the classical case the idempotent so obtained is precisely the *Eulerian idempotent*, which is an important object since it permitted us, for instance, to give an explicit description of the Baker-Campbell-Hausdorff formula [**42**], and to split the Hochschild chain complex in the commutative case [**41, 43**]. See also [**10**] for an application to the Kashiwara-Vergne conjecture. Our construction gives an analogue of the Eulerian idempotent for each triple of operads.

Here is the content of this paper. The first chapter contains elementary facts about "types of algebras and bialgebras" from the operadic point of view. The proofs of the theorems are performed in this framework, not only because of its efficiency, but also because some of the types of algebras that we encounter are not defined by generators and relations. We introduce the notion of "connected coalgebra" used in the hypotheses of the main theorem. The reader who is fluent in operad theory can easily bypass this chapter.

The second chapter contains the main results of this paper together with their proof. First, we study the algebraic structure of the primitive part of a generalized bialgebra of type $\mathcal{C}^c$-$\mathcal{A}$. In general a product of two primitive elements is not primitive. However the primitive part is stable under some operations. We determine all of them under the hypotheses (H0) and (H1) and we get the "maximal" algebraic structure for the primitive part. We call it the primitive operad and denote it by $\mathcal{P}$.

Then we study more particularly the generalized bialgebra types for which this primitive structure is trivial. Though it will become a particular case of the general theorem, we prefer to treat it independently, because of its importance, because the proofs are easier and because several cases reduce to this one. The result is the *rigidity theorem* for triples of the form $(\mathcal{C}, \mathcal{Z}, Vect)$. Then we move to the general theorem. We establish that the conditions (H0) , (H1) and (H2epi) ensure that the *structure theorem*, referred to above, is valid for the $\mathcal{C}^c$-$\mathcal{A}$-bialgebras.

Here we phrased these conditions in terms of bialgebras, but they, of course, can be phrased in terms of operads (more precisely in terms of props). In short we say that the triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is a *good triple of operads* when these hypotheses are fulfilled.

The previous case (rigidity theorem) is a particular case of this one since (H2iso) is a particular case of (H2epi) and since $\mathcal{P} = Vect$ in this case (so the universal algebra is the free algebra).

There are several consequences to the structure theorem. For instance any good triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ gives rise to a natural isomorphism

$$\mathcal{A}(V) \cong \mathcal{C}^c \circ \mathcal{P}(V) \ .$$

It generalizes the classical fact that the underlying vector space of the symmetric algebra over the free Lie algebra is isomorphic to the tensor module

$$T(V) \cong S(Lie(V)).$$

In the known cases the proof of the structure theorem uses an ad hoc construction of an idempotent, which depends very much on the type of bialgebras at hand. The key point of our proof is to construct an "abstract" idempotent which works universally.

Third Chapter. In the first three sections of this chapter we give recipes to construct good triples of operads. An important consequence of our formulation is that, starting with a good triple $(\mathcal{C}, \mathcal{A}, \mathcal{P})$, we can construct many others by moding out by some primitive operations. If J is a set of primitive operations and (J) (resp. $((J))$) is the operadic ideal it generates in $\mathcal{P}$ (resp. $\mathcal{A}$), then $(\mathcal{C}, \mathcal{A}/((J)), \mathcal{P}/(J))$ is also a good triple. In particular any good triple $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ determines a good triple of the form $(\mathcal{C}, \mathcal{Z}, Vect)$, where $\mathcal{Z} = \mathcal{A}/((\bar{\mathcal{P}}))$.

Assuming that the tensor product of two $\mathcal{A}$-algebras is still a $\mathcal{A}$-algebra (Hopf operad, multiplicative operad, for instance), there is a natural way of constructing a notion of As^c-$\mathcal{A}$-bialgebra which verifies (H0) and (H1) .

For quadratic operads there is a notion of Koszul dual operad. It should permit us to construct new triples of operads by taking the Koszul dual.

In order to keep the proofs into the most simple form we treated the case of algebraic operads over a characteristic zero field. But the structure theorem admits several generalizations. First, if we work with regular operads, then the characteristic zero hypothesis is not necessary anymore. In order to simplify the exposition we worked with the tensor category of vector spaces `Vect`, but everything is valid into any symmetric monoidal category, for example the category of sign-graded vector spaces (super vector spaces) or the category of **S**-modules . The formulas are the same provided that one applies the Koszul sign rule. In characteristic p it is expected that similar results hold.

The structure theorem can be "dualized" in the sense that the role of the algebra structure and the coalgebra structure are exchanged. The role of the primitive part is played by the indecomposable part.

In the last two sections we explain the relationship with the theory of "rewriting systems" and we give some application to the representation theory of the symmetric groups.

In the fourth chapter we study some explicit examples in details. We show how several results in the literature can be interpreted as giving rise to a good triple of operads. Since any good triple $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ gives rise to a (quotient) triple of the form $(\mathcal{C}, \mathcal{Z}, Vect)$, we put in the same section the triples which have the same quotient triple:

• $(Com, Com, Vect)$ and Hopf compatibility relation. This section deals with the classical case of Com^c-As-bialgebras (cocommutative bialgebras) and some of its variations: $(Com, Parastat, NLie)$, $(Com, Mag, Sabinin)$. Our structure theorem for the triple (Com, As, Lie) is equivalent to the Poincaré-Birkhoff-Witt (PBW) theorem plus the Cartier-Milnor-Moore (CMM) theorem. We show that our universal idempotent identifies to the Eulerian idempotent.

• $(As, As, Vect)$ and unital infinitesimal compatibility relation. It contains the case of 2-associative bialgebras, that is the triple $(As, 2as, Brace)$, and also $(As, Dipt, \mathbf{B}_\infty)$, $(As, Mag, MagFine)$. It is important because it permits us to handle the structure of classical cofree Hopf algebras [**51**].

• $(As, Zinb, Vect)$ and semi-Hopf compatibility relation. It contains the dendriform and dipterous bialgebras. It is interesting for its role in the study of the graph-complexes à la Kontsevich obtained by replacing the Lie homology by the Leibniz homology [**43, 11**].

• $(Lie, Lie, Vect)$. This is a completely new case. It gives a criterion to show that a Lie algebra is free.

• $(Nap, PreLie, Vect)$. It is due to M. Livernet [**40**]. A variation $(Nap, Mag, \mathrm{Prim}_{NAP}PreLie)$ needs more work to find a small presentation of the primitive operad.

• Then we survey some examples of the form $(\mathcal{A}, \mathcal{A}, Vect)$, we formulate a conjecture related to a question of M. Markl and we introduce an example coming from computer sciences (interchange bialgebra). Finally we present a triple involving k-ary operations and cooperations.

In the final Chapter we treat in details the case of the triple (As, OU, Mag) mentioned before. We prove that the triple is good and we make explicit the analogue of the PBW theorem. We prove that the operad OU is Koszul. We treat the case $(OU, OU, Vect)$ and we comment on further generalizations.

In the Appendix we provide a tableau of compatibility relations and a tableau of triples summarizing the examples treated in Chapter 4.

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Notation, convention. In this paper $\mathbb{K}$ is a field, which is, sometimes, supposed to be of characteristic zero. Its unit is denoted $1_{\mathbb{K}}$ or just 1. All vector spaces are over $\mathbb{K}$ and the category of vector spaces is denoted by $\mathtt{Vect}$. The vector space spanned by the elements of a set X is denoted $\mathbb{K}[X]$. The tensor product of vector spaces over $\mathbb{K}$ is denoted by $\otimes$. The tensor product of n copies of the space V is denoted by $V^{\otimes n}$. For $v_i \in V$ the element $v_1 \otimes \cdots \otimes v_n$ of $V^{\otimes n}$ is denoted by $(v_1, \ldots, v_n)$ or simply by $v_1 \ldots v_n$. For instance in the *tensor module*

$$T(V) := \mathbb{K} \oplus V \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

we denote by $v_1 \ldots v_n$ an element of $V^{\otimes n}$, but in $T(V)^{\otimes k}$ we denote by $v_1 \otimes \cdots \otimes v_k$ the element such that $v_i \in V \subset T(V)$ is in the ith factor. The *reduced tensor module* $\overline{T}(V) := V \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$ can be considered either as a subspace of $T(V)$ or as a quotient of it.

A linear map $V^{\otimes n} \to V$ is called an *n-ary operation* on V and a linear map $V \to V^{\otimes n}$ is called an *n-ary cooperation* on V. The symmetric group is the automorphism group of the finite set $\{1, \ldots, n\}$ and is denoted S_n. It acts on $V^{\otimes n}$ on the left by $\sigma \cdot (v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$. The action is extended to an action of $\mathbb{K}[S_n]$ by linearity. We denote by τ the *switching map* in the symmetric monoidal category $\mathtt{Vect}$, that is $\tau(u \otimes v) = v \otimes u$ (in the nongraded case).

A *magmatic algebra* is a vector space A equipped with a binary operation $A \otimes A \to A$, usually denoted $(a, b) \mapsto a \cdot b$. In the unital case it is assumed that there is an element 1, called the unit, which satisfies $a \cdot 1 = a = 1 \cdot a$. In the literature a magmatic algebra is sometimes referred to as a nonassociative algebra.

Quotienting by the associativity relation $(ab)c = a(bc)$ we get the notion of *associative algebra*. Quotienting further by the commutativity relation $ab = ba$ we get the notion of *commutative algebra*. So, in the terminology "commutative algebra", associativity is understood.

References. References inside the paper include the Chapter. So "see 2.3.4" means see paragraph 3.4 in Chapter 2.

CHAPTER 1

Algebraic operads

We briefly recall the definition, notation and terminology of the operad framework (see for instance [**54**]). The reader who is familiar with algebraic operads and props can skip this first chapter, except for the notion of connectedness of a coalgebra, which seems to be new.

1.1. S-module

1.1.1. S-module and Schur functor. An **S**-*module* $\mathcal{P}$ is a family of right S_n-modules $\mathcal{P}(n)$ for $n \geq 0$. Its associated *Schur functor* $\mathcal{P} : \mathtt{Vect} \to \mathtt{Vect}$ is defined as

$$\mathcal{P}(V) := \bigoplus_{n\geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$$

where S_n acts on the left on $V^{\otimes n}$ by permuting the factors. We also use the notation $\mathcal{P}(V)_n := \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$ so that $\mathcal{P}(V) := \bigoplus_{n\geq 0} \mathcal{P}(V)_n$.

In this paper we always assume that $\mathcal{P}(n)$ is finite dimensional. In general we assume that $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbb{K}\mathrm{id}$ (*connected operad*). In a few cases we assume instead that $\mathcal{P}(0) = \mathbb{K}$, so that the algebras can be equipped with a unit. The natural projection map which sends $\mathcal{P}(V)_n$ to 0 when $n > 1$ and $\mathcal{P}(V)_1$ to itself, that is $\mathbb{K}\,\mathrm{id} \otimes V = V$ is denoted

$$\mathrm{proj} : \mathcal{P}(V) \to V.$$

A morphism of **S**-modules $f : \mathcal{P} \to \mathcal{P}'$ is a family of S_n-morphisms $f(n) : \mathcal{P}(n) \to \mathcal{P}'(n)$. They induce a morphism of Schur functors: $f(V) : \mathcal{P}(V) \to \mathcal{P}'(V)$.

1.1.2. Composition of S-modules. Let $\mathcal{P}$ and $\mathcal{Q}$ be two **S**-modules . It can be shown that the composite $\mathcal{Q} \circ \mathcal{P}$ of the Schur functors (as endomorphisms of `Vect`) is again the Schur functor of an **S**-module , also denoted $\mathcal{Q} \circ \mathcal{P}$. The explicit value of $(\mathcal{Q} \circ \mathcal{P})(n)$ involves sums, tensor products and induced representations of the representations $\mathcal{Q}(i)$ and $\mathcal{P}(i)$ for all $i \leq n$. This composition makes the category of **S**-modules into a monoidal category (also called tensor category) whose neutral element is the identity functor. Observe that this is not a symmetric monoidal category since the composition of functors is far from being symmetric.

1.1.3. Generating series. The *generating series* of an $\mathbf{S}$-module $\mathcal{P}$ is defined as

$$f^{\mathcal{P}}(t) := \sum_{n\geq 1} \frac{\dim \mathcal{P}(n)}{n!} t^n \ .$$

It is immediate to check that the generating series of a composite is the composite of the generating series:

$$f^{\mathcal{Q}\circ\mathcal{P}}(t) = f^{\mathcal{Q}}(f^{\mathcal{P}}(t)).$$

1.2. Algebraic operad

1.2.1. Definition. By definition an *algebraic operad*, or *operad* for short, is a Schur functor $\mathcal{P}$ equipped with two transformations of functors $\iota : \mathrm{Id}_{\mathtt{Vect}} \to \mathcal{P}$ and $\gamma : \mathcal{P}\circ\mathcal{P} \to \mathcal{P}$ which makes it into a monoid. In other words we assume that γ is associative and that ι is a unit for γ. Such an object is also called a *monad* in $\mathtt{Vect}$.

The identity functor $\mathrm{Id}_{\mathtt{Vect}}$ is itself an operad that we denote by $Vect$ (instead of $\mathrm{Id}_{\mathtt{Vect}}$) when we consider it as an operad. We call it the *identity operad*.

We usually assume that the operad is *connected*, that is $\mathcal{P}(0) = 0$.

1.2.2. Algebra over an operad. By definition an *algebra over the operad* $\mathcal{P}$, or a $\mathcal{P}$-*algebra* for short, is a vector space A equipped with a linear map $\gamma_A : \mathcal{P}(A) \to A$ such that the following diagrams are commutative:

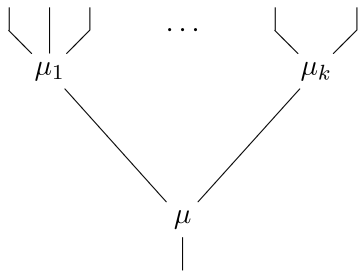
$$\begin{tikzcd}
\mathcal{P}\circ\mathcal{P}(A) \arrow[r, "\mathcal{P}(\gamma_A)"] \arrow[d, "\gamma(A)"'] & \mathcal{P}(A) \arrow[d, "\gamma_A"] \\
\mathcal{P}(A) \arrow[r, "\gamma_A"] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "\iota(A)"] \arrow[dr, "="'] & \mathcal{P}(A) \arrow[d, "\gamma_A"] \\
 & A
\end{tikzcd}$$

There is an obvious notion of morphism of $\mathcal{P}$-algebras. The category of $\mathcal{P}$-algebras is denoted $\mathcal{P}$-alg. Since we made the assumption $\mathcal{P}(0) = 0$, these algebras are nonunital.

If we want a unit for the $\mathcal{P}$-algebras, then we take $\mathcal{P}(0) = \mathbb{K}$ and the image of $1 \in \mathbb{K}$ by $\gamma_A : \mathcal{P}(0) \to A$ is the unit of A.

The operation $\mathrm{id} \in \mathcal{P}(1)$ is the identity operation: $\mathrm{id}(a) = a$ for any $a \in A$. For $\mu \in \mathcal{P}(k)$ and $\mu_1 \in \mathcal{P}(n_1), \ldots, \mu_k \in \mathcal{P}(n_k)$ the composite $\gamma(\mu; \mu_1, \cdots, \mu_k) \in \mathcal{P}(n_1+\cdots+n_k)$ is denoted $\mu\circ(\mu_1, \cdots, \mu_k)$ or $\mu(\mu_1, \cdots, \mu_k)$ if no confusion can arise.

$\mu(\mu_1, \cdots, \mu_k) :=$



So, an operad can also be described as a family of linear maps

$$\gamma : \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \to \mathcal{P}(i_1 + \cdots + i_k)$$

which assemble to give a monad $(\mathcal{P}, \gamma, \iota)$.

The restriction of γ_A to $\mathcal{P}(n) \otimes_{S_n} A^{\otimes n}$ is denoted $\gamma_n : \mathcal{P}(n) \otimes_{S_n} A^{\otimes n} \to A$ if no confusion can arise.

Given an element $\mu \in \mathcal{P}(n)$ and an n-tuple $(a_1, \ldots, a_n)$ of elements of A, we can construct

$$\mu(a_1, \ldots, a_n) = \gamma_n(\mu \otimes (a_1, \ldots, a_n)) \in A\ .$$

Hence $\mathcal{P}(n)$ is referred to as the "space of n-ary operations" for $\mathcal{P}$-algebras. The integer n is called the "arity" of the operation μ (or its degree if no confusion can arise).

The category of algebras over the operad $Vect$ is simply the category of vector spaces `Vect`. Hence we have $Vect(1) = \mathbb{K}$ and $Vect(n) = 0$ for $n \neq 1$.

1.2.3. Free $\mathcal{P}$-algebra. By definition a $\mathcal{P}$-algebra A_0 is *free* over the vector space V if it is equipped with a linear map $i : V \to A_0$ and if it satisfies the following universal property:

any map $f : V \to A$, where A is a $\mathcal{P}$-algebra, extends uniquely into a $\mathcal{P}$-algebra morphism $\tilde{f} : A_0 \to A$:

$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "f"] & \\ A_0 \arrow[rr, dashed, "\tilde{f}"] & & A \end{tikzcd}$$

Observe that the free algebra over V is well-defined up to a unique isomorphism.

For any vector space V one can equip $\mathcal{P}(V)$ with a structure of $\mathcal{P}$-algebra by setting $\gamma_{\mathcal{P}(V)} := \gamma(V) : \mathcal{P}(\mathcal{P}(V)) \to \mathcal{P}(V)$. The axioms defining the operad $\mathcal{P}$ show that $(\mathcal{P}(V), \gamma(V))$ is the free $\mathcal{P}$-algebra over V.

Categorically, $\mathcal{P}$ is left adjoint to the forgetful functor which assigns, to a $\mathcal{P}$-algebra A, its underlying vector space:

$$\mathrm{Hom}_{\mathcal{P}\text{-alg}}(\mathcal{P}(V), A) \cong \mathrm{Hom}_{\mathtt{Vect}}(V, A).$$

1.2.4. Operadic ideal. For a given operad $\mathcal{P}$ and a family of operations $\{\nu\}$ in $\mathcal{P}$ the ideal $\mathcal{I}$, generated by this family, is the sub-$\mathbf{S}$-module $\mathcal{I}$ linearly generated by all the compositions $\mu \circ (\mu_1, \cdots, \mu_k)$ where at least one of the operations is in the family. The quotient $\mathcal{P}/\mathcal{I}$, defined as $(\mathcal{P}/\mathcal{I})(n) = \mathcal{P}(n)/\mathcal{I}(n)$, is an operad.

If $\mathcal{Q}$ is a suboperad of $\mathcal{P}$, then we denote by $\overline{\mathcal{Q}}$ the sub $\mathbf{S}$-module of $\mathcal{Q}$ such that $\bar{\mathcal{Q}}(1) = 0$ and $\bar{\mathcal{Q}}(n) = \mathcal{Q}(n)$ for $n \geq 2$. We denote by $(\bar{\mathcal{Q}})$ the operadic ideal generated by $\bar{\mathcal{Q}}$ in $\mathcal{P}$. So the quotient $\mathcal{P}/(\bar{\mathcal{Q}})$ is an operad.

1.2.5. Type of algebras and presentation of an operad. For a given type of algebras defined by generators and relations (supposed to be multilinear), the associated operad is obtained as follows. Let $\mathcal{P}(V)$ be the free algebra of the given type over V. Let $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$ be a based n-dimensional vector space. The multilinear part of $\mathcal{P}(V)$ of degree n (i.e. linear in each variable) is a subspace of $\mathcal{P}(V)_n$ denoted $\mathcal{P}(n)$. It is clear that $\mathcal{P}(n)$ inherits an action of the symmetric group. The universal property of the free algebra $\mathcal{P}(V)$ permits us to give a structure of operad on the Schur functor $\mathcal{P}$. The category of $\mathcal{P}$-algebras is precisely the category of algebras we started with.

The operad $\mathcal{P}$ can also be constructed by taking the free operad over the generating operations and quotienting by the ideal (in the operadic sense) generated by the relators.

For instance the free operad on one binary operation μ (with no symmetry) is the magmatic operad Mag. In degree n we get $Mag(n) = \mathbb{K}[Y_{n-1}] \otimes \mathbb{K}[S_n]$ where Y_{n-1} is the set of planar binary rooted trees with $n-1$ internal vertices (and n leaves), cf. 5.1.2. The tree $| \in Y_0$ codes for id $\in Mag(1)$ and the tree  $\in Y_1$ codes for the generating operation $\mu \in Mag(2)$.

The operad As of (nonunital) associative algebras is the quotient of Mag by the ideal generated by the relator

$$\mu \circ (\mu \otimes \mathrm{id}) - \mu \circ (\mathrm{id} \otimes \mu) \in Mag(3).$$

Observe that a morphism of operads $\mathcal{P} \to \mathcal{Q}$ gives rise to a functor between the corresponding categories of algebras in the other direction:

$$\mathcal{Q}\text{-alg} \longrightarrow \mathcal{P}\text{-alg}.$$

1.2.6. Binary and quadratic operad. An element $\mu \in \mathcal{P}(2)$ defines a map

$$\mu : A^{\otimes 2} \to A, \quad a \otimes b \mapsto \mu(a, b) \ ,$$

called a *binary operation*. Sometimes such an operation is denoted by a symbol, for instance $*$, and we write $a * b$ instead of $\mu(a, b)$. We allow ourselves to talk about "the operation $a * b$ ".

An operad is said to be *binary*, resp. $\langle k \rangle$-ary, if it is generated by binary, resp. $\langle k \rangle$-ary, operations (elements in $\mathcal{P}(2)$, resp. $\mathcal{P}(k)$). An operad is said to be *quadratic* if the relations are made of monomials involving only the composition of two operations. In the binary case, it means that the relations are of the form

$$\sum_i \mu_i(\nu_i \otimes \mathrm{Id}) = \sum_j \mu_j(\mathrm{Id} \otimes \nu_j)$$

where the elements $\mu_i, \nu_i, \mu_j, \nu_j$ are binary operations (not necessarily the generating ones). Sometimes, in the literature, the adjective quadratic is used in place of binary and quadratic (see for instance [**26**]). Most classical types of algebras are defined by binary quadratic operads: associative, commutative, Lie, Poisson, pre-Lie, Leibniz, dendriform, 2-associative, alternative, magmatic, etc. Some are generated by n-ary operations, but are still quadratic: Lie triples, Jordan triples, $A_\infty, C_\infty, L_\infty, Brace, \mathbf{B}_\infty, \mathcal{M}ag^\infty$, etc.

1.2.7. Regular operad. Let $\mathcal{P}$ be an operad whose associated type of algebras has the following property. The generating operations do not satisfy any symmetry property and, in the relations, the variables stay in the same order. Then, it is easy to show that $\mathcal{P}(n) = \mathcal{P}_n \otimes \mathbb{K}[S_n]$ for some vector space $\mathcal{P}_n$. Here $\mathbb{K}[S_n]$ stands for the regular representation. Moreover the operadic structure is completely determined by composition maps

$$\gamma_{i_1 \cdots i_n} : \mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \cdots \otimes \mathcal{P}_{i_n} \longrightarrow \mathcal{P}_{i_1 + \cdots + i_n} \ .$$

Such operads are called *regular operads*. The object $(\mathcal{P}_n, \gamma_{i_1 \cdots i_n})_{n \geq 1}$ is called a *nonsymmetric operad* (though that, strictly speaking, it is not an operad). The operads

$$As, Dend, Dipt, 2as, Mag, A_\infty, Mag^\infty$$

are regular.

1.2.8. Set-theoretic operad. So far we have defined an operad in the monoidal category of vector spaces (and $\otimes$), but we could choose the category of sets (and $\times$). This would give us the notion of *set-operad*. Since any set X gives rise to a vector space $\mathbb{K}[X]$, any set-operad gives rise to an algebraic operad. Such an operad is said to be *set-theoretic*.

1.2.9. Classical examples: the three graces. The classical examples of algebraic operads are the operad As of associative algebras, the operad Com of commutative algebras (understood to be associative) and the operad Lie of Lie algebras. In each case the free algebra is well-known, so the operad is easy to describe: $As(V)$ is the (nonunital) algebra of noncommutative polynomials over V (reduced tensor algebra $\overline{T}(V)$), $Com(V)$ is the (nonunital) algebra of polynomials over V (reduced symmetric algebra $\overline{S}(V)$), $Lie(V)$ is the subspace of $As(V)$ generated by V under the bracket operation $[x, y] = xy - yx$. It follows that in the associative case we get $As(n) = \mathbb{K}[S_n]$ (regular representation). In the commutative case we get $Com(n) = \mathbb{K}$ (trivial representation). In the Lie case we get $Lie(n) = \mathrm{Ind}_{C_n}^{S_n}(\mathbb{K})$ (induced representation from the trivial representation over the cyclic group C_n).

Observe that *As* is regular, *As* and *Com* are set-theoretic, but *Lie* is not regular nor set-theoretic.

1.3. Coalgebra and cooperad

1.3.1. Coalgebra over an operad. By definition a *coalgebra over the operad* $\mathcal{C}$ is a vector space C equipped with S_n-equivariant maps

$$\mathcal{C}(n) \otimes C \to C^{\otimes n}, \quad \delta \otimes c \mapsto \delta(c) = c_1^\delta \otimes \cdots \otimes c_n^\delta$$

(we omit the summation symbol on the right hand side) which are compatible with the operad structure of $\mathcal{C}$. In particular, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}(k) \otimes \mathcal{C}(i_1) \otimes \cdots \otimes \mathcal{C}(i_k) \otimes C \arrow[r] \arrow[d] & \mathcal{C}(i_1) \otimes C \otimes \cdots \otimes \mathcal{C}(i_k) \otimes C \arrow[d] \\
\mathcal{C}(i_1 + \cdots + i_k) \otimes C \arrow[d, equal] & C^{\otimes i_1} \otimes \cdots \otimes C^{\otimes i_k} \arrow[d, equal] \\
\mathcal{C}(n) \otimes C \arrow[r] & C^{\otimes n}
\end{tikzcd}$$

In this framework the elements of $\mathcal{C}(n)$ are called *n-ary cooperations*.

1.3.2. Primitive part, connectedness (conilpotency). Let $\mathcal{C}$ be an operad such that $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = \mathbb{K}\,\mathrm{id}$. We suppose that there is only a finite number of generating cooperations in each degree (i.e. arity) so that $\mathcal{C}(n)$ is finite dimensional. The identity operation id is not considered as a generating operation. Let C be a $\mathcal{C}$-coalgebra. We define a filtration on C as follows:

$$F_1C = \mathrm{Prim}\, C := \{x \in C \mid \delta(x) = 0 \text{ for any generating cooperation } \delta\}.$$

The space $\mathrm{Prim}\, C$ is called the *primitive part* of C, and its elements are said to be *primitive*. Then we define the filtration by:

$$F_rC := \{x \in C \mid \delta(x) = 0 \text{ for any } \delta \in \mathcal{C}(n), n > r\}.$$

By definition C is said to be *connected*, or *conilpotent*, if $C = \bigcup_{r\geq 1} F_rC$.

If the operad $\mathcal{C}$ is binary, then this definition is equivalent to

$$F_rC := \{x \in C \mid \delta(x) \in (F_{r-1}C)^{\otimes|\delta|} \text{ for any generating cooperation } \delta\}.$$

1.3.3. Cofree coalgebra. By definition a $\mathcal{C}$-coalgebra C_0 is said to be *cofree* over the vector space V if it is connected, equipped with a map $s : C_0 \to V$ and if it satisfies the following universal property:

any map $p : C \to V$, where C is a connected $\mathcal{C}$-coalgebra, extends uniquely into a $\mathcal{C}$-coalgebra morphism $\tilde{p} : C \to C_0$:

$$\begin{tikzcd} C \arrow[rr, dashed, "\tilde{p}"] \arrow[dr, "p"'] & & C_0 \arrow[dl, "s"] \\ & V & \end{tikzcd}$$

The cofree coalgebra over V is well-defined up to a unique isomorphism. Observe that we are working within the category of "connected" coalgebras. If we were working in the whole category of coalgebras, the notion of cofree object would be different.

Let $\mathcal{C} = As$. The cofree coassociative coalgebra over V is the reduced tensor module $\overline{T}(V)$ equipped with the deconcatenation operation:

$$\delta(v_1 \dots v_n) = \sum_{1 \leq i \leq n-1} v_1 \dots v_i \otimes v_{i+1} \dots v_n \ .$$

1.3.4. Cooperad and coalgebra over a cooperad. Taking the linear dual of an operad $\mathcal{C}$ gives a *cooperad* denoted $\mathcal{C}^c$. Let us recall that a cooperad is a comonoid structure on a Schur functor. As a vector space $\mathcal{C}^c(n) = \mathcal{C}(n)^* = \mathrm{Hom}(\mathcal{C}(n), \mathbb{K})$. We equip this space with the following right S_n-module structure:

$$f^{\sigma}(\mu) := f(\mu^{\sigma^{-1}}),$$

for $f \in \mathcal{C}(n)^*, \mu \in \mathcal{C}(n)$ and $\sigma \in S_n$.

The cooperadic composition is denoted $\theta : \mathcal{C}^c \to \mathcal{C}^c \circ \mathcal{C}^c$. There is an obvious notion of *coalgebra* C over a cooperad $\mathcal{C}^c$ given by maps:

$$\theta(C) : C \longrightarrow \mathcal{C}^c(C)^{\wedge} := \prod_{n \geq 1} \mathcal{C}^c(n) \otimes_{S_n} C^{\otimes n}.$$

It coincides with the notion of coalgebra over an operad when the cooperad is the linear dual of the operad. Here we use the characteristic zero hypothesis to identify invariants and coinvariants under the symmetric group action. We always assume that $\mathcal{C}^c(0) = 0$, $\mathcal{C}^c(1) = \mathbb{K}\mathrm{id}$ and that $\mathcal{C}(n)$ is finite dimensional. The elements of $\mathcal{C}^c(1)$ are called the trivial cooperations and an element $f \in \mathcal{C}^c(n), n \geq 2$, is called a *nontrivial cooperation*.

The projection of $\theta(C)$ to the n-th component is denoted

$$\theta_n : C \to \mathcal{C}^c(n) \otimes_{S_n} C^{\otimes n}$$

if no confusion can arise. Let $\langle -, - \rangle : \mathcal{C}(n) \otimes \mathcal{C}^c(n) \to \mathbb{K}$ be the evaluation pairing. The relationship with the notation introduced in 1.3.1 is given by the commutative diagram

$$
\begin{tikzcd}
 & \mathcal{C}(n)\otimes\mathcal{C}^c(n)\otimes C^{\otimes n} \arrow[dr, "\langle -,-\rangle\otimes\mathrm{id}"] & \\
\mathcal{C}(n)\otimes C \arrow[ur, "\mathrm{id}\otimes\theta_n"] \arrow[rr] & & C^{\otimes n}
\end{tikzcd}
$$

$$\delta\otimes c \qquad\qquad \mapsto \qquad\qquad \delta(c)=c_1^\delta\otimes\cdots\otimes c_n^\delta$$

1.3.5. Lemma. *If the coalgebra C is connected (cf. **??**), then the map $\theta(C)$ factors through the direct sum:*

$$\theta(C): C \longrightarrow \mathcal{C}^c(C) := \oplus_{n\geq 1}\mathcal{C}^c(n)\otimes_{S_n} C^{\otimes n}.$$

Proof. Since for any $x\in C$ there is an integer r so that $\delta(x)=0$ for any cooperation δ such that $|\delta|>r$, it follows that there is only finitely many nonzero component in $\theta(C)(x)$. □

1.3.6. Cofree coalgebra and cooperad. From the axioms of a cooperad it follows that $\mathcal{C}^c(V)$ is the cofree $\mathcal{C}^c$-coalgebra over V.

Explicitly the universal lifting $\tilde{p}: C\to\mathcal{C}^c(V)$ induced by $p: C\to V$ is obtained as the composite

$$C \xrightarrow{\theta(C)} \mathcal{C}^c(C) \xrightarrow{\mathcal{C}^c(p)} \mathcal{C}^c(V).$$

Suppose that the operad $\mathcal{C}$ is binary quadratic, generated by operations $\mu_1,\mu_2,\dots$ and relations of the form

$$\sum_{i,j}\alpha_{ij}\mu_i(\mu_j\otimes\mathrm{Id}) = \sum\beta_{ij}\mu_i(\mathrm{Id}\otimes\mu_j),\quad \alpha_{ij},\beta_{ij}\in\mathbb{K}.$$

Then a coalgebra C over $\mathcal{C}$ is defined by the cooperations $\mu_i^*: C\to C\otimes C$ satisfying the relations:

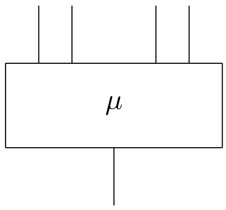
$$\sum_{i,j}\alpha_{ij}(\mu_j^*\otimes\mathrm{Id})\mu_i^* = \sum_{i,j}\beta_{ij}(\mathrm{Id}\otimes\mu_j^*)\mu_i^*.$$

1.3.7. Invariants versus coinvariants. Saying that a binary cooperation $\delta: C\to C\otimes C$ is symmetric means that its image lies in the invariant subspace $(C\otimes C)^{S_2}$. In characteristic zero the natural map from invariants to coinvariants is an isomorphism $(C\otimes C)^{S_2}\cong(C\otimes C)_{S_2}$. Therefore, if $\mathcal{C}(2)=\mathbb{K}$ (trivial representation), then δ defines a map $C\to\mathcal{C}(2)\otimes_{S_2}C^{\otimes 2}$.

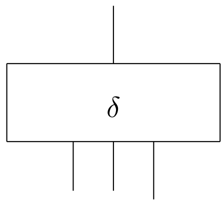
1.3.8. Regular cooperad. If the operad $\mathcal{C}$ is regular (cf. 1.2.7), then the equivalence between the two notions of coalgebra does not need the characteristic zero hypothesis. Indeed, since $\mathcal{C}(n)=\mathcal{C}_n\otimes\mathbb{K}[S_n]$, we simply take $\mathcal{C}^c(n):=\mathcal{C}_n^*\otimes\mathbb{K}[S_n]$.

1.4. Prop

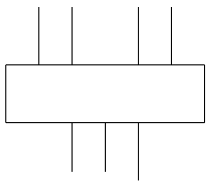
1.4.1. Definition. In the algebra framework (i.e. operad framework) an operation μ can be seen as a box with n inputs and one output:



In the coalgebra framework (i.e. cooperad framework) a cooperation δ can be seen as a box with 1 input and m outputs:



If we want to deal with bialgebras, then we need boxes with multiple inputs and multiple outputs:



called *multivalued operations* or properations. Hence we have to replace the **S**-modules by the $\mathbf{S}^{op}$-**S**-modules , i.e. families $\mathcal{P}(m,n)$ of $S_m^{op} \times S_n$-bimodules. There is a way of defining a monoidal product on $\mathbf{S}^{op}$-**S**-modules , denoted $\boxtimes$, which extends the monoidal product on **S**-modules . By definition a *prop* (also denoted PROP in the literature), is a monoid in the monoidal category of $\mathbf{S}^{op}$-**S**-modules (cf. for instance [**71**]).

Equivalently it can be defined as a $\mathbb{K}$-linear monoidal category $([\mathcal{P}], \otimes)$ whose objects are $[0], [1], \ldots, [n], \ldots$, and such that $[1]^{\otimes n} = [n]$. The relationship with the preceding definition is given by

$$\mathcal{P}(m,n) = \mathrm{Hom}_{[\mathcal{P}]}([m],[n]) \ .$$

Observe that an operad (resp. a cooperad) is a particular case of prop for which $\mathcal{P}(m,n) = 0$ when $m \geq 2$ (resp. $\mathcal{P}(m,n) = 0$ when $n \geq 2$).

There is a notion of *gebra* over a prop which generalizes the notion of algebra (resp. coalgebra) over an operad (resp. cooperad). If $\mathcal{H}$ is a gebra over the prop $\mathcal{P}$, then there exist maps

$$\mathcal{P}(m,n)\otimes\mathcal{H}^{\otimes n}\longrightarrow\mathcal{H}^{\otimes m}\ .$$

A type of gebra can be defined by generators and relations. When the generators are either operations (i.e. elements in $\mathcal{P}(1,n)$) and/or cooperations (i.e. elements in $\mathcal{P}(m,1)$), the gebras are called *generalized bialgebras*, or simply bialgebras if there is no ambiguity with the classical notion of bialgebras.

Since, in this paper, we are dealing only with generalized bialgebra types we will not use this general notion of prop, except for terminological use. However, in the expected generalizations of the results presented here, the notion of prop will prove central.

CHAPTER 2

Generalized bialgebra and triple of operads

We introduce the notion of *generalized bialgebra* and its primitive part. We prove that, under some hypotheses `(H0)` and `(H1)` , a generalized bialgebra type determines an operad called the *primitive operad*. Primitive elements in a generalized bialgebra do not, in general, give a primitive element under an operation. However they do when this operation is primitive. So, the primitive part of a generalized bialgebra has the property of being an algebra over the primitive operad.

Then we treat the case where the primitive operad is trivial (i.e. `Vect`). We prove that under the hypotheses `(H0)` (distributive compatibility condition), `(H1)` (the free algebra is a bialgebra), and `(H2iso)` (free isomorphic to cofree), any connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra is both free and cofree. This is the *thm:rigidity*. The key of the proof is the construction of a universal idempotent $e_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ whose image is the space of primitive elements $\mathrm{Prim}\,\mathcal{H}$.

For a given prop $\mathcal{C}^c$-$\mathcal{A}$ (satisfying `(H0)` and `(H1)`) whose primitive operad is $\mathcal{P}$, we call $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ a triple of operads. Our aim is to find simple conditions under which the "structure theorem" holds for $(\mathcal{C}, \mathcal{A}, \mathcal{P})$. This structure theorem says that any connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra is isomorphic to $U(\mathrm{Prim}\,\mathcal{H})$ as an algebra and is cofree over $\mathrm{Prim}\,\mathcal{H}$ as a coalgebra. These simple conditions are `(H0)` , `(H1)` and `(H2epi)` (the coalgebra map $\mathcal{A}(V) \to \mathcal{C}^c(V)$ is surjective and admits a splitting).

Then we give some immediate consequences of the main theorem.

In this chapter we suppose that the ground field is of characteristic zero. We indicate in the next chapter how to avoid this hypothesis in certain cases.

2.1. Generalized bialgebra

We consider a certain type of prop generated by operations and cooperations. A gebra over such prop is called a generalized bialgebra.

2.1.1. Compatibility relation and generalized bialgebra. Let $\mathcal{A}$ and $\mathcal{C}$ be two algebraic operads. We always assume that there is a finite number of generating operations in each arity. As a consequence $\mathcal{C}(n)$ and $\mathcal{A}(n)$ are finite dimensional vector spaces.

By definition a $(\mathcal{C}^c, \between, \mathcal{A})$*-bialgebra*, or $\mathcal{C}^c$-$\mathcal{A}$*-bialgebra* for short, also called *generalized bialgebra*, is a vector space $\mathcal{H}$ which is an $\mathcal{A}$-algebra, a $\mathcal{C}$-coalgebra, and such that the operations of $\mathcal{A}$ and the cooperations of $\mathcal{C}$

acting on $\mathcal{H}$ are related by some *compatibility relations*, denoted $\between$, read "between" (some equalities involving composition of operations and cooperations). This kind of structure is an example of an *algebraic prop* as mentioned in 1.4 (cf. for instance [**71**]). We will not use this general notion here, though, sometimes, we will say "the prop $\mathcal{C}^c$-$\mathcal{A}$" instead of "the type of $(\mathcal{C}^c, \between, \mathcal{A})$-bialgebras".

A compatibility relation between the operation μ and the cooperation δ is *distributive*, if it is of the form

$$\delta \circ \mu = \sum_i (\mu_1^i \otimes \cdots \otimes \mu_m^i) \circ \omega \circ (\delta_1^i \otimes \cdots \otimes \delta_n^i) \qquad (\between)$$

where

$$\begin{cases} \mu \in \mathcal{A}(n),\ \mu_1^i \in \mathcal{A}(k_1), \ldots, \mu_m^i \in \mathcal{A}(k_m), \\ \delta \in \mathcal{C}(m),\ \delta_1^i \in \mathcal{C}(l_1), \ldots, \delta_n^i \in \mathcal{C}(l_n), \\ k_1 + \cdots + k_m = l_1 + \cdots + l_n = r, \\ \omega \in \mathbb{K}[S_r]. \end{cases}$$

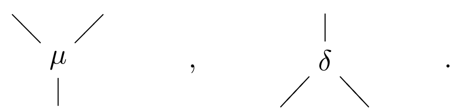
Hence distributivity means that the composite of an operation and a cooperation can be re-written as cooperations first and then operations. Observe that the identity is both an operation and a cooperation.

Hypothesis (H0) : There is a distributive compatibility relation for any pair (δ, μ) where μ is an operation and δ is a cooperation.

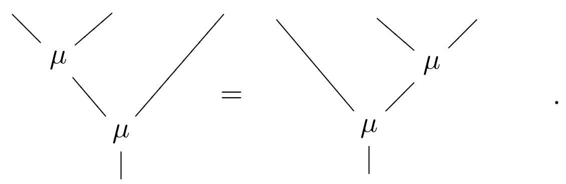
Of course, it suffices to check this hypothesis for μ a generating operation and δ is a generating cooperation.

For a given relation $\between$ we denote by Φ the right-hand side term.

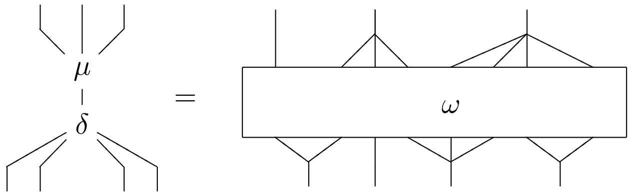
2.1.2. Diagrams. It will prove helpful to write the compatibility relations as diagrams instead of long algebraic expressions. For instance, for a binary operation μ and a binary cooperation δ we draw



For instance the associativity property of μ, which is written $\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$ algebraically, becomes



Example of a compatibility relation for the pair (δ, μ) with $n = 3, m = 4$ and $r = 8$:



Here we have $l_1 = 1, l_2 = 3, l_3 = 4; k_1 = 2, k_2 = 1, k_3 = 3, k_4 = 2$ and so $r = 1+3+4 = 2+1+3+2$. Observe that, in the general case, the right-hand side term Φ is a sum of such compositions. We split Φ into two summands

$$\Phi = \Phi_1 + \Phi_2,$$

as follows. The summand Φ_1 contains all the terms for which $r = n$ and Φ_2 contains all the terms for which $r > n$ see section 2.1 for the meaning of r and n. There is no term with $r < n$ since we assume that $\mathcal{C}(0) = 0$. The important point of this splitting is the following: for each summand

$$(\mu_1^i \otimes \cdots \otimes \mu_m^i) \circ \omega \circ (\delta_1^i \otimes \cdots \otimes \delta_n^i)$$

of Φ_2 at least one of the cooperations δ_k^i is nontrivial (i.e. of arity ≥ 2). In Φ_1 the only cooperation which pops up is the identity.

When both operads $\mathcal{A}$ and $\mathcal{C}$ are regular and, in the compatibility relations, there is no crossing (in particular the only permutations ω are the identity), then we say that this is a *regular case* and that $\mathcal{C}^c$-$\mathcal{A}$ is a *regular prop*.

2.1.3. Examples of distributive compatibility relations.

2.1.3.1. *Example 1.* Hopf algebra (classical bialgebra, Hopf relation):

A classical bialgebra is a unital associative algebra equipped with a counital coproduct Δ which satisfies the *Hopf compatibility relation*

$$\Delta(xy) = \Delta(x)\Delta(y).$$

Since, here, we want to work without unit nor co-unit, we work over the augmentation ideal and with the *reduced comultiplication* δ defined by

$$\delta(x) := \Delta(x) - x \otimes 1 - 1 \otimes x.$$

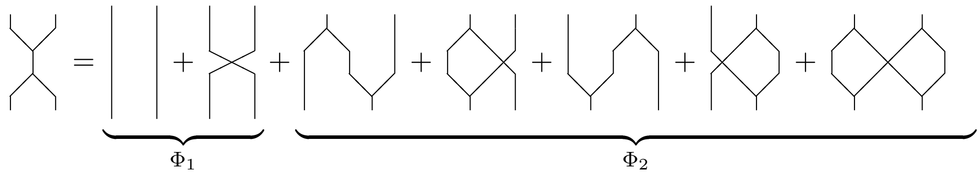
The classical Hopf compatibility relation becomes $\between_{Hopf}$:

$\delta(xy) =$

$$x\otimes y + y\otimes x + x_{(1)}\otimes x_{(2)}y + x_{(1)}y\otimes x_{(2)} + xy_{(1)}\otimes y_{(2)} + y_{(1)}\otimes xy_{(2)} + x_{(1)}y_{(1)}\otimes x_{(2)}y_{(2)}$$

under the notation $\delta(x) := x_{(1)} \otimes x_{(2)}$ (Sweedler notation with summation sign understood).

Pictorially the relation $\between_{Hopf}$ reads:



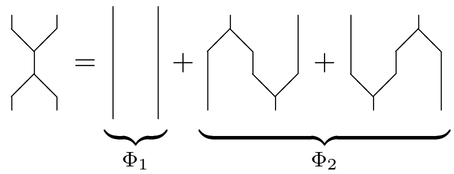
2.1.3.2. *Example 2.* Unital infinitesimal bialgebra (As^c-As-bialgebra, u.i. relation).

The motivation for this case is the tensor algebra that we equip with the deconcatenation coproduct (instead of the shuffle coproduct). In the nonunital framework the compatibility relation satisfied by the concatenation product and the (reduced) deconcatenation coproduct is

$$\delta(xy) = x \otimes y + x_{(1)}y \otimes x_{(2)} + xy_{(1)} \otimes y_{(2)}$$

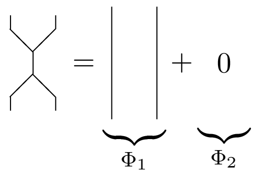
under the notation $\delta(x) := x_{(1)} \otimes x_{(2)}$.

Pictorially we get $\between_{ui}$:



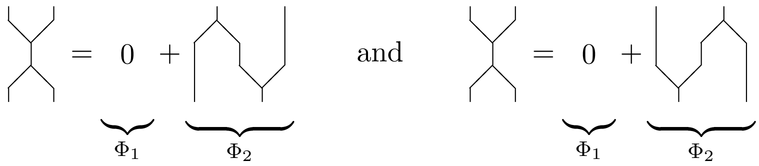
See [**51**] and 4.2 for more details. The prop defined by this type of generalized bialgebras is regular.

2.1.3.3. *Example 3.* Bimagmatic bialgebra (Mag^c-Mag-bialgebra, magmatic relation). The motivation for this case is the magmatic algebra $Mag(V)$. We equip it with the magmatic coproduct obtained by identifying the classical basis of $Mag(\mathbb{K})$ (planar binary trees) with its dual. In the nonunital framework the compatibility relation is $\between_{mag}$:



This is a regular prop.

2.1.3.4. *Example 4.* Frobenius algebra (As^c-As-bialgebra, Com^c-Com-bialgebra Frobenius relation):



This is a regular case if the algebra and the coalgebra are not supposed to be commutative.

These examples and many more will be treated in Chapter 4. See 6.1 for a list of some compatibility relations.

2.2. The primitive operad

The primitive part of a generalized bialgebra is, in general, not stable under the operations of the operad $\mathcal{A}$. However it may be stable under *some* operations. In this section we describe the maximal suboperad $\mathcal{P}$ of $\mathcal{A}$ such that the primitive part $\operatorname{Prim}\mathcal{H}$ of the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is a $\mathcal{P}$-algebra. Both operads $\mathcal{A}$ and $\mathcal{C}$ are supposed to be finitely generated and connected, that is $\mathcal{A}(0) = 0 = \mathcal{C}(0)$.

2.2.1. The primitive part of a bialgebra. Let $\mathcal{H}$ be a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra. By definition the *primitive part* of the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$, denoted $\operatorname{Prim}\mathcal{H}$, is

$$\operatorname{Prim}\mathcal{H} := \{x \in \mathcal{H} \mid \delta(x) = 0 \text{ for all } \delta \in \mathcal{C}^c(n), n \geq 2\}.$$

Hence, if $\mathcal{C}$ is generated by $\delta_1, \ldots, \delta_k, \ldots$, then we have

$$\operatorname{Prim}\mathcal{H} = \operatorname{Ker}\delta_1 \cap \ldots \cap \operatorname{Ker}\delta_k \cap \ldots \ .$$

By definition an element $\mu \in \mathcal{A}(n)$ is called a *primitive operation* if, for any independent variables $x_1, \ldots, x_n$, the element $\mu(x_1, \ldots, x_n) \in \mathcal{A}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$ is primitive. Let $(\operatorname{Prim}_{\mathcal{C}}\mathcal{A})(n) \subset \mathcal{A}(n)$ be the *space of primitive operations* for $n \geq 1$:

$$(\operatorname{Prim}_{\mathcal{C}}\mathcal{A})(n) := \{\mu \in \mathcal{A}(n) \mid \mu \text{ is primitive}\}.$$

By functoriality of the hypothesis, $(\operatorname{Prim}_{\mathcal{C}}\mathcal{A})(n)$ is a sub-S_n-module of $\mathcal{A}(n)$ and so we obtain an inclusion of Schur functors

$$\operatorname{Prim}_{\mathcal{C}}\mathcal{A} \rightarrowtail \mathcal{A}.$$

2.2.2. Theorem (The primitive operad). *Let $(\mathcal{C}^c, \between, \mathcal{A})$ be a type of generalized bialgebras over a characteristic zero field $\mathbb{K}$. We suppose that the following hypotheses are fulfilled:*

(H0) any pair (δ, μ) satisfies a distributive compatibility relation,
(H1) the free $\mathcal{A}$-algebra $\mathcal{A}(V)$ is equipped with a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure which is functorial in V.
Then the Schur functor $\mathcal{P}$, given by $\mathcal{P}(V) := (\operatorname{Prim}_{\mathcal{C}}\mathcal{A})(V)$, is a suboperad

of $\mathcal{A}$. For any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ the space $\operatorname{Prim}\mathcal{H}$ *is a $\mathcal{P}$-algebra and the inclusion* $\operatorname{Prim}\mathcal{H} \rightarrowtail \mathcal{H}$ *is a morphism of $\mathcal{P}$-algebras.*

Proof. First we remark that the elements of $V \subset \mathcal{A}(V)$ are primitive, hence $V \subset \mathcal{P}(V)$, and id is a primitive operation. Indeed, since the bialgebra structure of $\mathcal{A}(V)$ is functorial in V, any cooperation δ on $\mathcal{A}(V)$ respects the degree. For $n \geq 2$ the degree one part of $\mathcal{A}(V)^{\otimes n}$ is trivial. Since V is of degree one in $\mathcal{A}(V)$, we get $\delta(V) = 0$. Hence any element of V is primitive and the functor $\iota : \mathrm{Id} \to \mathcal{A}$ factors through $\mathcal{P}$.

To prove that the Schur functor $\mathcal{P} : \mathtt{Vect} \to \mathtt{Vect}$ is an operad, it suffices to show that it inherits a monoid structure $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ from the monoid structure of $\mathcal{A}$. In other words it suffices to show that composition of primitive operations, under the composition in $\mathcal{A}$, provides a primitive operation:

$$\begin{tikzcd} \mathcal{P} \circ \mathcal{P} \arrow[r, dotted] \arrow[d, rightarrowtail] & \mathcal{P} \arrow[d, rightarrowtail] \\ \mathcal{A} \circ \mathcal{A} \arrow[r, "\gamma"] & \mathcal{A} \end{tikzcd}$$

We use the hypothesis of distributivity of the compatibility relation between operations and cooperations, cf. 2.1.

Let $\mu, \mu_1, \ldots, \mu_n$ be operations, where $\mu \in \mathcal{P}(n)$. We want to prove that the composite $\mu \circ (\mu_1, \ldots, \mu_n)$ is primitive when all the operations are primitive. It suffices to show that $\delta \circ \mu \circ (\mu_1, \ldots, \mu_n)$ applied to the generic element $(x_1, \ldots, x_s)$ is 0 for any nontrivial cooperation δ.

By 2.1 we know that $\delta \circ \mu = \Phi_1 + \Phi_2$, where Φ_1 involves only operations, and Φ_2 is of the form

$$\Phi_2 = \sum_i (\mu_1^i \otimes \cdots \otimes \mu_m^i) \circ \omega \circ (\delta_1^i \otimes \cdots \otimes \delta_n^i)$$

where, for any i, at least one of the cooperations $\delta_k^i,\ k = 1, \ldots, n$, is nontrivial. We evaluate this expression on a generic element $(x_1, \ldots, x_n)$. On the left-hand side $\mu(x_1, \ldots, x_n)$ is primitive by hypothesis, so $(\delta \circ \mu)(x_1, \ldots, x_n) = 0$. On the right-hand side $\Phi_2(x_1, \ldots, x_n) = 0$ because the evaluation of a nontrivial cooperation on a generic element (which is primitive) is 0. Hence we deduce that $\Phi_1(x_1, \ldots, x_n) = 0$. Therefore the operation Φ_1 is 0.

Let us now suppose that, not only μ is primitive, but $(\mu_1, \ldots, \mu_n)$ are also primitive operations. By the preceding argument we get

$$\Phi_2 \circ (\mu_1, \ldots, \mu_n) = \sum_i \Psi \circ (\delta_1^i \mu_1 \otimes \cdots \otimes \delta_n^i \mu_n)$$

where, for any i, at least one of the cooperations $\delta_k^i,\ k = 1, \ldots, n$ is nontrivial. Hence, summarizing our arguments, the evaluation on $(x_1, \ldots, x_s)$ give:

$$\begin{array}{rcl}(\delta \circ \mu \circ (\mu_1, \ldots, \mu_k))(x_1, \ldots, x_s) & = & \Phi_2 \circ (\mu_1, \ldots, \mu_n)(x_1, \ldots, x_s) \\ & = & \Psi \circ (\delta_1^i \mu_1^i (x_1 \ldots), \ldots, \delta_n^i \mu_n^i (\ldots x_s)) \\ & = & 0,\end{array}$$

because a nontrivial cooperation applied to a primitive element gives 0.

In conclusion we have shown that, when $\mu, \mu_1, \ldots, \mu_n$ are primitive, then

$$\delta \circ \mu \circ (\mu_1, \ldots, \mu_n)(x_1, \ldots, x_s) = 0.$$

Hence the operation $\mu \circ (\mu_1, \ldots, \mu_n)$ is primitive. As a consequence the image of the composite

$$\mathcal{P} \circ \mathcal{P} \rightarrowtail \mathcal{A} \circ \mathcal{A} \xrightarrow{\gamma} \mathcal{A}$$

lies in $\mathcal{P}$ as expected, and so $\mathcal{P}$ is a suboperad of $\mathcal{A}$.

From the definition of the primitive part of the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ it follows that $\mathrm{Prim}\, \mathcal{H}$ is a $\mathcal{P}$-algebra. Since $\mathcal{P}$ is a suboperad of $\mathcal{A}$, $\mathcal{H}$ is also a $\mathcal{P}$-algebra and the inclusion $\mathrm{Prim}\, \mathcal{H} \to \mathcal{H}$ is a $\mathcal{P}$-algebra morphism. □

2.2.3. Examples. Theorem 2.2.2 proves the existence of an operad structure on $\mathcal{P} = \mathrm{Prim}_{\mathcal{C}} \mathcal{A}$, however, even when $\mathcal{A}$ and $\mathcal{C}$ are described by generators and relations, it is often a challenge to find a small presentation of $\mathcal{P}$ and then to find explicit formulas for the functor $F : \mathcal{A}\text{-alg} \to \mathcal{P}\text{-alg}$.

In the case of the classical bialgebras, the primitive operad is *Lie* and the functor $F : As\text{-alg} \to Lie\text{-alg}$ is the classical Liezation functor: $F(A)$ is A as a vector space and the bracket operation is given by $[x, y] = xy - yx$, cf. 4.1.2.

In the case of u.i. bialgebras the primitive operad is $Vect$ and the functor F is simply the forgetful functor, cf. 4.2.2.

In the magmatic bialgebras case the primitive operad is $Vect$ and the functor F is simply the forgetful functor, cf. **??**.

In the case of Frobenius bialgebras the primitive operad is As (that is the whole operad) and the functor F is the identity.

We end this section with a result which will prove helpful in the sequel.

2.2.4. Lemma. *Let $\mathcal{C}^c$-$\mathcal{A}$ be a generalized bialgebra type verifying the hypotheses* (H0) *and* (H1) *of Theorem 2.2.2. Let $\varphi(V) : \mathcal{A}(V) \to \mathcal{C}^c(V)$ be the unique coalgebra map induced by the projection map $proj : \mathcal{A}(V) \to V$. Denote by*

$$\langle -, - \rangle : \mathcal{C}(n) \times \mathcal{C}^c(n) \to \mathbb{K}$$

the pairing between the operad and the cooperad.

For any cooperation $\delta \in \mathcal{C}(n)$ the image of $(\mu; x_1 \cdots x_n) \in \mathcal{A}(V)$ is

$$\delta(\mu; x_1 \cdots x_n) = \langle \delta, \varphi_n(\mu) \rangle x_1 \otimes \cdots \otimes x_n \in V^{\otimes n} \subset \mathcal{A}(V)^{\otimes n}.$$

Proof. Let us recall from 1.3.4 that the map $\varphi_n : \mathcal{A}(n) \to \mathcal{C}^c(n)$ is given by the composite

$$\mathcal{A}(V) \xrightarrow{\theta_n} \mathcal{C}^c(n) \otimes_{S_n} (\mathcal{A}(V)^{\otimes n}) \xrightarrow{\mathrm{Id}\otimes \mathrm{proj}^{\otimes n}} \mathcal{C}^c(n) \otimes_{S_n} V^{\otimes n}.$$

By assumption the bialgebra structure of $\mathcal{A}(V)$ is functorial in V. Therefore $\theta_n(\mu \otimes (x_1, \ldots, x_n))$ is linear in each variable x_i. Hence it lies in $\mathcal{C}^c(n) \otimes_{S_n} \mathcal{A}(V)_1{}^{\otimes n} = \mathcal{C}^c(n) \otimes_{S_n} V^{\otimes n}$. So we have proved that $\theta_n(\mu \otimes (x_1, \ldots, x_n)) = \varphi_n(\mu) \otimes (x_1, \ldots, x_n)$.

By definition, the coalgebra structure of $\mathcal{A}(V)$

$$\mathcal{C}(n) \otimes \mathcal{A}(V) \to \mathcal{A}(V)^{\otimes n}$$

is dual to (cf. 1.3.4)

$$\theta_n : \mathcal{A}(V) \to \mathcal{C}^c(n) \otimes \mathcal{A}(V)^{\otimes n}$$

via the pairing $\langle -, - \rangle$. Hence we get

$$\delta(\mu \otimes (x_1, \ldots, x_n)) = \langle \delta, \varphi_n(\mu) \rangle x_1 \otimes \cdots \otimes x_n.$$

$\square$

2.3. Rigidity theorem

We first study the generalized bialgebra types for which the primitive operad is trivial. The paradigm is the case of cocommutative commutative bialgebras (over a characteristic zero field). The classical theorem of Hopf and Borel [**6**], can be phrased as follows:

Theorem (Hopf-Borel). *In characteristic zero any connected cocommutative commutative bialgebra is both free and cofree over its primitive part.*

In other words such a bialgebra $\mathcal{H}$ is isomorphic to $S(\mathrm{Prim}\,\mathcal{H})$ (symmetric algebra over the primitive part), see 4.1.8 for more details. Recall that, here, we are working in the monoidal category of vector spaces. The classical Hopf-Borel theorem was originally phrased in the monoidal category of sign-graded vector spaces cf. 3.5.

Our aim is to generalize this theorem to the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra types for which $\mathcal{P} = Vect$.

2.3.1. Hypotheses. In this section we make the following assumptions on the given $\mathcal{C}^c$-$\mathcal{A}$-bialgebra type:

`(H0)` for any pair (δ, μ) of generating operation μ and generating cooperation δ there is a distributive compatibility relation,

`(H1)` the free $\mathcal{A}$-algebra $\mathcal{A}(V)$ is naturally equipped with a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure.

`(H2iso)` the natural coalgebra map $\varphi(V) : \mathcal{A}(V) \to \mathcal{C}^c(V)$ induced by the projection $\mathrm{proj} : \mathcal{A}(V) \twoheadrightarrow V$ is an isomorphism of $\mathbf{S}$-modules $\varphi : \mathcal{A} \cong \mathcal{C}^c$.

2.3.2. Proposition. *Let $\mathcal{C}^c$-$\mathcal{A}$ be a type of bialgebras which verifies hypotheses* `(H0)` *,* `(H1)` *and* `(H2iso)`*. Then the primitive operad is the identity operad: $\mathcal{P} = Vect$.*

Proof. It follows from `(H0)` , `(H1)` and Theorem 2.2.2 that there exists a primitive operad $\mathcal{P}$. Let $\mu \in \mathcal{P}(n)$ be a nonzero n-ary operation for $n \geq 2$. Since $\varphi : \mathcal{P}(n) \cong \mathcal{C}^c(n)$ is an isomorphism by `(H2iso)`, there exists a cooperation $\delta \in \mathcal{C}(n)$ such that $\langle \delta, \varphi\mu \rangle = 1$. Let $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$. It follows from Lemma 2.2.4 that

$$\delta \circ \mu(x_1, \ldots, x_n) = x_1 \otimes \cdots \otimes x_n \in V^{\otimes n} \subset \mathcal{A}(V)^{\otimes n}.$$

Therefore $\delta \circ \mu \neq 0$ and there is a contradiction. Hence we have $\mathcal{P}(n) = 0$ for any $n \geq 2$. $\square$

2.3.3. Proposition. *Let $\mathcal{C}^c$-$\mathcal{A}$ be a type of bialgebras which verifies hypotheses* `(H0)` *,* `(H1)` *and* `(H2iso)`*. Let $\mathcal{H}$ be a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra and let $V \to \operatorname{Prim}\mathcal{H}$ be a linear map. Then the unique algebra lifting $\tilde{\alpha} : \mathcal{A}(V) \to \mathcal{H}$ of the composite $\alpha : V \to \operatorname{Prim}\mathcal{H} \rightarrowtail \mathcal{H}$ is a bialgebra map.*

Proof. First let us observe that, by Proposition 2.3.2, we have $\operatorname{Prim}\mathcal{A}(V) = V$. Since $\tilde{\alpha}$ is an algebra map by construction, we need only to prove that it is a coalgebra map. We work by induction on the filtration of $\mathcal{A}(V)$. When x is primitive, that is x lies in $F_1\mathcal{A}(V) = V$, then $\tilde{\alpha}(x) = \alpha(x)$ is primitive by hypothesis. Let $x \in \mathcal{A}(V)$ be an obstruction of minimal filtration degree m. From the definition of the filtration by the cooperations there exists some cooperation which provides an obstruction of minimal filtration degree $m-1$. But, since for $m = 1$ there is no obstruction, we get a contradiction and $\tilde{\alpha}$ is a coalgebra morphism. $\square$

2.3.4. The universal idempotent e. Let $\mathcal{H}$ be a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra . We define a linear map $\omega^{[n]} : \mathcal{H} \to \mathcal{H}$ for each $n \geq 2$ as the following composite

$$\omega^{[n]} : \mathcal{H} \xrightarrow{\theta_n} \mathcal{C}^c(n) \otimes_{S_n} \mathcal{H}^{\otimes n} \overset{\varphi^{-1}\otimes \mathrm{Id}}{\cong} \mathcal{A}(n) \otimes_{S_n} \mathcal{H}^{\otimes n} \xrightarrow{\gamma_n} \mathcal{H}.$$

We define a linear map $e : \mathcal{H} \to \mathcal{H}$ by the formula:

$$e = e_{\mathcal{H}} := (\mathrm{Id} - \omega^{[2]})(\mathrm{Id} - \omega^{[3]}) \cdots (\mathrm{Id} - \omega^{[n]}) \cdots$$

(infinite product). We also denote by $e_{\mathcal{H}}$, or simply e, the surjective map $\mathcal{H} \twoheadrightarrow \mathrm{Im}(e)$.

Before stating and proving the main theorem of this section, we prove some technical results on the universal idempotent e.

We denote by $\tilde{\iota}_{\mathcal{H}} : \mathcal{A}(\operatorname{Prim}\mathcal{H}) \to \mathcal{H}$ the unique algebra lifting induced by the inclusion map $\iota_{\mathcal{H}} : \operatorname{Prim}\mathcal{H} \rightarrowtail \mathcal{H}$.

2.3.5. Proposition. *If the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is connected, then the map $e = e_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is well-defined and satisfies the following properties:*

a) $e_{\mathcal{A}(V)} = proj_V : \mathcal{A}(V) \to V$,
b) the image of $e_{\mathcal{H}}$ is $\operatorname{Prim}\mathcal{H}$,
c) e is an idempotent: $e^2 = e$.

Proof. Let $(\delta^n_{i_1}, \ldots, \delta^n_{i_k})$ be a linear basis of $\mathcal{C}(n)$ and let $(\bar\delta^n_{i_1}, \ldots, \bar\delta^n_{i_k})$ be the dual basis (of $\mathcal{C}^c(n)$). We know by Lemma 2.2.4 that for any $x \in C$ we have

$$\theta_n(x) = \sum_j \bar\delta^n_{i_j} \otimes \delta^n_{i_j}(x) \ .$$

From the connectedness assumption on C (cf. 1.3.2) it follows that, for any $x \in C$, there exists an integer r such that $x \in F_r\mathcal{H}$. Hence we have $\theta_n(x) = 0$ whenever $n > r$ and therefore $\omega^{[n]} = 0$ on $F_r\mathcal{H}$ whenever $n > r$. As a consequence

$$e(x) = \big((\mathrm{Id} - \omega^{[2]})(\mathrm{Id} - \omega^{[3]}) \cdots (\mathrm{Id} - \omega^{[r]})\big)(x),$$

and so $e(x)$ is well-defined.

Proof of (a). We consider the following diagram (where $\underline{\otimes}$ means $\otimes_{S_n}$):

$$\begin{tikzcd}
\mathcal{A}(n)\underline{\otimes}V^{\otimes n} \arrow[r, "\varphi\otimes\mathrm{Id}"] \arrow[d, tail] & \mathcal{C}^c(n)\underline{\otimes}V^{\otimes n} \arrow[r, "\varphi^{-1}\otimes\mathrm{Id}"] \arrow[d, tail] & \mathcal{A}(n)\underline{\otimes}V^{\otimes n} \arrow[r, "="] \arrow[d, tail] & \mathcal{A}(n)\underline{\otimes}V^{\otimes n} \arrow[d, tail] \\
\mathcal{A}(V) \arrow[r, "\theta_n"] & \mathcal{C}^c(n)\underline{\otimes}\mathcal{A}(V)^{\otimes n} \arrow[r, "\varphi^{-1}\otimes\mathrm{Id}"] & \mathcal{A}(n)\underline{\otimes}\mathcal{A}(V)^{\otimes n} \arrow[r, "\gamma_n"] & \mathcal{A}(V)
\end{tikzcd}$$

where the composition in the last line is $\omega^{[n]}$. The left hand side square is commutative by definition of φ, cf. 1.3.4. The middle square is commutative by construction. The right hand side square is commutative by definition of the $\mathcal{A}$-structure of $\mathcal{A}(V)$, cf. 1.2.3. As a consequence the whole diagram is commutative. Since, in the diagram, the lower composite is $\omega^{[n]}$ and the upper composite is the identity, we deduce that the restriction of $\omega^{[n]}$ on the n-th component $\mathcal{A}(V)_n$ is the inclusion into $\mathcal{A}(V)$. As a consequence $\mathrm{Id} - \omega^{[n]}$ is 0 on the n-th component for any $n \geq 2$. So e is the projection on $V = \mathcal{A}(V)_1$ parallel to the higher components, since $e(x) = x$ for any primitive element.

Proof of (b). First we remark that the statement is true for $\mathcal{H} = \mathcal{A}(V)$ by virtue of (a). Since α is a bialgebra morphism by Proposition 2.3.3, there is a commutative diagram:

$$\begin{tikzcd}
\mathcal{A}(V) \arrow[r, "\tilde\alpha"] \arrow[d, "e_{\mathcal{A}(V)}"'] & \mathcal{H} \arrow[d, "e_{\mathcal{H}}"] \\
V \arrow[r, "\alpha"] & \mathrm{Prim}\,\mathcal{H}
\end{tikzcd}$$

where $V = \mathrm{Prim}\,\mathcal{H}$. Statement (a) implies that $e_{\mathcal{H}}$ is surjective.

Proof of (c). From the definition of e we observe that $e(x) = x$ for any $x \in \mathrm{Prim}\,\mathcal{H}$ because $\omega^{[n]}(x) = 0$ for any $n \geq 2$. Since $e(x)$ is primitive by (b) we get $e^2 = e$.

$\square$

2.3.6. Corollary. *Let $\mathcal{C}^c$-$\mathcal{A}$ be a type of generalized bialgebras which verifies hypotheses* `(H0)` *and* `(H1)` *. For any connected bialgebra $\mathcal{H}$ the natural algebra map $\tilde{\iota} : \mathcal{A}(\operatorname{Prim}\mathcal{H}) \to \mathcal{H}$ induced by the inclusion $\iota : \operatorname{Prim}\mathcal{H} \to \mathcal{H}$ is surjective.*

Proof. If $x \in \operatorname{Prim}\mathcal{H} = \operatorname{Im}\iota$, then clearly $x \in \operatorname{Im}\iota$. Let us now work by induction on the filtration of $\mathcal{H}$. Assume that $F_{m-1}\mathcal{H} \subset \operatorname{Im}\tilde{\iota}$ and let $x \in F_m\mathcal{H}$. In the formula

$$x = e(x) + \big(\sum \omega^{[i]}(x) - \sum \omega^{[i]} \circ \omega^{[j]}(x) + \cdots\big)$$

the first summand $e(x)$ is in $\operatorname{Prim}\mathcal{H} \subset \operatorname{Im}\tilde{\iota}$ by Proposition 2.3.5. The second summand is also in $\operatorname{Im}\tilde{\iota}$ because it is the sum of elements which are products of elements in $\operatorname{Prim}\mathcal{H}$ by induction. Therefore we proved $x \in \operatorname{Im}\tilde{\iota}$ for any $x \in \mathcal{H}$, so $\tilde{\iota}$ is surjective. □

2.3.7. Theorem (Rigidity theorem). *Let $\mathcal{C}^c$-$\mathcal{A}$ be a type of generalized bialgebras (over a characteristic zero field) verifying the following hypotheses:*
`(H0)` *the operad $\mathcal{C}$ is finitely generated and for any pair (δ, μ) of generating operation μ and generating cooperation δ there is a distributive compatibility relation,*
`(H1)` *the free $\mathcal{A}$-algebra $\mathcal{A}(V)$ is naturally equipped with a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure,*
`(H2iso)` *the natural coalgebra map $\varphi(V) : \mathcal{A}(V) \to \mathcal{C}^c(V)$ is an isomorphism.*

Then any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is free and cofree over its primitive part:

$$\mathcal{A}(\operatorname{Prim}\mathcal{H}) \cong \mathcal{H} \cong \mathcal{C}^c(\operatorname{Prim}\mathcal{H}).$$

Proof. By Proposition 2.3.2 the map $\tilde{\iota} : \mathcal{A}(\operatorname{Prim}\mathcal{H}) \to \mathcal{H}$ is a bialgebra morphism. On the other hand the projection $e : \mathcal{H} \to \operatorname{Prim}\mathcal{H}$ induces a coalgebra map $\tilde{e} : \mathcal{H} \to \mathcal{C}^c(\operatorname{Prim}\mathcal{H})$ by universality (cf. 1.3.3). We will prove that both morphisms are isomorphisms and that the composite

$$\mathcal{A}(\operatorname{Prim}\mathcal{H}) \xrightarrow{\tilde{\iota}} \mathcal{H} \xrightarrow{\tilde{e}} \mathcal{C}^c(\operatorname{Prim}\mathcal{H})$$

is φ.

By Proposition 2.3.3 and the fact that the idempotent is functorial in the bialgebra, there is a commutative diagram

$$\begin{tikzcd} & \mathcal{H} \arrow[dr, "e"] & \\ \mathcal{A}(\operatorname{Prim}\mathcal{H}) \arrow[ur, "\tilde{\iota}"] \arrow[rr, "\tilde{\alpha}"] & & \operatorname{Prim}\mathcal{H} \end{tikzcd}$$

which induces, by universality of the cofree coalgebra, the commutative diagram:

$$\begin{tikzcd} & \mathcal{H} \arrow[dr, "\tilde{e}"] & \\ \mathcal{A}(\mathrm{Prim}\,\mathcal{H}) \arrow[ur, "\tilde{\imath}"] \arrow[rr, "\varphi"] & & \mathcal{C}^c(\mathrm{Prim}\,\mathcal{H}) \end{tikzcd}$$

Since φ is an isomorphism by `(H2iso)`, it follows that $\tilde{\imath}$ is injective.

In Proposition 2.3.6 we proved that $\tilde{\imath}$ is surjective, therefore $e = \tilde{\imath} : \mathcal{A}(\mathrm{Prim}\,\mathcal{H}) \longrightarrow \mathcal{H}$ is a bialgebra isomorphism and, as a consequence, $\tilde{e}$ is also an isomorphism. □

2.3.8. COROLLARY. *Let $\mathcal{H}$ be a connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra and let $\mathcal{H}^2$ denote the image in $\mathcal{H}$ of $\bigoplus_{n\geq 2} \mathcal{A}(n)_{S_n}\mathcal{H}^{\otimes n}$. Then one has $e_{\mathcal{H}}(\mathcal{H}^2) = 0$.*

Proof. By the rigidity theorem it suffices to show that this assertion is valid when $\mathcal{H}$ if free. By definition of e we have $e_{\mathcal{A}(V)} = \mathrm{proj}_V$, whose kernel is precisely $\mathcal{A}(V)^2$. □

2.3.9. Explicit universal idempotent. Let us suppose that $\mathcal{A}$ and $\mathcal{C}$ are given by generators and relations, and that one knows how to describe $\mathcal{A}(n)$ and $\mathcal{C}(n)$ explicitly in terms of these generators. Then it makes sense to look for an explicit description of e in terms of the elements of $\mathcal{A}(n)$ and $\mathcal{C}(n)$. In the cases already treated in the literature (cf. for instance [**65, 51, 40, 32, 33, 9, 18**], the first step of the proof of the rigidity theorem consists always in writing down such an explicit idempotent. In the case at hand (i.e. under `(H2iso)`) the universal idempotent and the explicit idempotent coincide because, on $\mathcal{A}(V)$, it is the projection onto V parallel to the other components $\oplus_{n>1}\mathcal{A}(n) \otimes_{S_n} V^{\otimes n}$.

The exact form of the compatibility relation(s) depends on the choice of the presentation of $\mathcal{A}$ and of $\mathcal{C}$. Let us suppose that hypotheses `(H0)` , `(H1)` and `(H2iso)` hold. Once a linear basis $(\mu_1^{[n]}, \ldots, \mu_k^{[n]})$ of $\mathcal{A}(n)$ is chosen, then we can choose, for basis of $\mathcal{C}(n)$, its dual $(\delta_1^{[n]}, \ldots, \delta_k^{[n]})$ under the isomorphism φ: $\langle \varphi(\mu_i^{[n]}), \delta_j^{[n]} \rangle = 1$ if $i = j$ and 0 otherwise. Then the compatibility relation of the pair $(\delta_j^{[n]}, \mu_i^{[n]})$ is such that $\Phi_1 = \mathrm{id}_n$ if $i = j$ and 0 otherwise.

2.4. Triple of operads

We introduce the notion of *triple of operads*

$$(\mathcal{C}, \mathcal{A}, \mathcal{P}) = (\mathcal{C}, \between, \mathcal{A}, F, \mathcal{P})$$

deduced from the prop $\mathcal{C}^c$-$\mathcal{A}$, that is from a notion of $\mathcal{C}^c$-$\mathcal{A}$-bialgebra. We construct and study the universal enveloping functor

$$U : \mathcal{P}\text{-alg} \to \mathcal{A}\text{-alg}\ .$$

2.4.1. Triple of operads. Let $(\mathcal{C}, \between, \mathcal{A})$ be a type of generalized bialgebras. Suppose that the hypotheses `(H0)` and `(H1)` are fulfilled, cf. 2.3.1. Then it determines an operad $\mathcal{P} := \mathrm{Prim}_{\mathcal{C}}\mathcal{A}$ and a functor $F : \mathcal{A}\text{-}alg \to \mathcal{P}\text{-}alg$. Observe that the operad $\mathcal{P}$ is the largest suboperad of $\mathcal{A}$ such that any $\mathcal{P}$-operation applied on primitive elements give a primitive element. For any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ the inclusion $\mathrm{Prim}\,\mathcal{H} \rightarrowtail \mathcal{H}$ becomes a morphism of $\mathcal{P}$-algebras. We call this whole structure a *triple of operads* and we denote it by

$(\mathcal{C}, \between, \mathcal{A}, F, \mathcal{P})$, or $(\mathcal{C}, \between, \mathcal{A}, \mathcal{P})$, or more simply $(\mathcal{C}, \mathcal{A}, \mathcal{P})$.

2.4.2. The map φ and the hypothesis `(H2epi)`. Since, by hypothesis `(H1)` , $\mathcal{A}(V)$ is a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra , the projection map $\mathrm{proj}_V : \mathcal{A}(V) \twoheadrightarrow V$ determines a unique coalgebra map (cf. 2.2.4):

$$\varphi(V) : \mathcal{A}(V) \to \mathcal{C}^c(V).$$

We recall from 1.3.6 that $\varphi(V)$ is the composite

$$\mathcal{A}(V) \xrightarrow{\theta(\mathcal{A}(V))} \mathcal{C}^c(\mathcal{A}(V)) \xrightarrow{\mathcal{C}^c(\mathrm{proj})} \mathcal{C}^c(V).$$

We denote by $\varphi : \mathcal{A} \to \mathcal{C}^c$ the underlying functor of $\mathbf{S}$-modules and by $\varphi_n : \mathcal{A}(n) \to \mathcal{C}^c(n)$ its degree n component.

We make the following assumption:

`(H2epi)` the natural coalgebra map $\varphi(V)$ is surjective and admits a natural coalgebra map splitting $s(V) : \mathcal{C}^c(V) \to \mathcal{A}(V)$, i.e. $\varphi(V) \circ s(V) = \mathrm{Id}_{\mathcal{C}^c(V)}$.

2.4.3. Universal enveloping functor. The functor

$$F : \mathcal{A}\text{-alg} \longrightarrow \mathcal{P}\text{-alg}$$

is a *forgetful functor* in the sense that the composition

$$\mathcal{A}\text{-alg} \xrightarrow{F} \mathcal{P}\text{-alg} \longrightarrow \mathtt{Vect}$$

is the forgetful functor $\mathcal{A}\text{-alg} \to \mathtt{Vect}$. In other words, in passing from an $\mathcal{A}$-algebra to a $\mathcal{P}$-algebra we keep the same underlying vector space. Hence this forgetful functor has a left adjoint denoted by

$$U : \mathcal{P}\text{-alg} \longrightarrow \mathcal{A}\text{-alg}$$

and called the *universal enveloping algebra functor* (by analogy with the classical case $U : Lie\text{-alg} \to As\text{-alg}$). Let us recall that adjointness means the following: for any $\mathcal{P}$-algebra L and any $\mathcal{A}$-algebra A there is a binatural isomorphism

$$\mathrm{Hom}_{\mathcal{A}\text{-alg}}(U(L), A) = \mathrm{Hom}_{\mathcal{P}\text{-alg}}(L, F(A)).$$

2.4.4. Proposition. *Let L be a $\mathcal{P}$-algebra. The universal enveloping algebra of L is given by*

$$U(L) = \mathcal{A}(L)/\sim$$

where the equivalence relation $\sim$ is generated, for any $x_1, \ldots, x_n$ in $L \subset \mathcal{A}(L)$, by

$$\mu^{\mathcal{P}}(x_1, \ldots, x_n) \sim (\mu^{\mathcal{A}}; x_1, \ldots, x_n), \quad \mu^{\mathcal{P}} \in \mathcal{P}(n),$$

where $\mu^{\mathcal{P}} \mapsto \mu^{\mathcal{A}}$ under the inclusion $\mathcal{P}(n) \subset \mathcal{A}(n)$.

Proof. We have $\mu^{\mathcal{P}}(x_1, \ldots, x_n) \in L = \mathcal{A}(1) \otimes L$ and $(\mu^{\mathcal{A}}; x_1, \ldots, x_n) \in \mathcal{A}(n) \otimes_{S_n} L^{\otimes n}$. So the equivalence relation does not respect the graduation. However, it respects the filtration given by

$$F_n\mathcal{A}(V) := \bigoplus_{j \leq n} \mathcal{A}(V)_j.$$

Let us show that the functor $L \mapsto U(L) := \mathcal{A}(L)/\sim$ is left adjoint to the forgetful functor F. Let A be an $\mathcal{A}$-algebra and let $f : L \to F(A)$ be a $\mathcal{P}$-morphism. There is a unique $\mathcal{A}$-algebra extension of f to $\mathcal{A}(L)$ since $\mathcal{A}(L)$ is free. It is clear that this map passes to the quotient by the equivalence relation and so defines an $\mathcal{A}$-morphism $U(L) \to A$.

In the other direction, let $g : U(L) \to A$ be a $\mathcal{A}$-morphism. Then its restriction to L is a $\mathcal{P}$-morphism $L \to F(A)$ by Theorem 2.2.2. It is immediate to verify that these two constructions are inverse to each other. Therefore we have an isomorphism

$$\mathrm{Hom}_{\mathcal{A}\text{-alg}}(U(L), A) \cong \mathrm{Hom}_{\mathcal{P}\text{-alg}}(L, F(A)),$$

which proves that U is left adjoint to F. □

2.4.5. Proposition. *Under the hypotheses* (H0) , (H1) *and* (H2epi) *the universal enveloping algebra $U(L)$ of the $\mathcal{P}$-algebra L is a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra .*

Proof. Since we mod out by an ideal, the quotient is an $\mathcal{A}$-algebra. By hypothesis the free algebra $\mathcal{A}(L)$ is a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra . The coalgebra structure of $U(L)$ is induced by the coalgebra structure of $\mathcal{A}(L)$. For any nontrivial cooperation δ we have

$$\delta(\mu^{\mathcal{P}}(x_1, \ldots, x_n)) = 0$$

since $\mu^{\mathcal{P}}(x_1, \ldots, x_n)$ lies in L, and we have

$$\delta(\mu^{\mathcal{A}}(x_1, \ldots, x_n)) = 0$$

because $\delta(\mu^{\mathcal{A}})$ is a primitive operation. Hence for any cooperation δ we have $\delta(\text{relator}) = 0$. Then δ is also 0 on $\mathrm{Ker}(U(L) \to \mathcal{A}(L))$ by the distributivity property of the compatibility relation $\between$. □

2.5. Structure Theorem for generalized bialgebras

In this section we show that any triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$, which satisfies (H2epi) , gives rise to a structure theorem analogous to the classical CMM+PBW theorem valid for the triple (Com, As, Lie) (cf. 4.1.3). It says that any connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra is cofree over its primitive part as a coalgebra and that, as an algebra, it is the universal enveloping algebra over its primitive part.

2.5.1. THEOREM (Structure Theorem for generalized bialgebras). *Let $\mathcal{C}^c$-$\mathcal{A}$ be a type of generalized bialgebras over a field of characteristic zero. Suppose that the following hypotheses are fulfilled:*
`(H0)` for any pair (δ, μ) of generating operation μ and generating cooperation δ there is a distributive compatibility relation,
`(H1)` the free $\mathcal{A}$-algebra $\mathcal{A}(V)$ is naturally equipped with a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure,
`(H2epi)` the natural coalgebra map $\varphi(V) : \mathcal{A}(V) \to \mathcal{C}^c(V)$ is surjective and admits a natural coalgebra map splitting $s(V) : \mathcal{C}^c(V) \to \mathcal{A}(V)$.

Then for any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ the following are equivalent:
a) the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is connected,
b) there is an isomorphism of bialgebras $\mathcal{H} \cong U(\operatorname{Prim} \mathcal{H})$,
c) there is an isomorphism of connected coalgebras $\mathcal{H} \cong \mathcal{C}^c(\operatorname{Prim} \mathcal{H})$.

We need a construction and two Lemmas before entering the proof of the structure Theorem. We first introduce a useful terminology.

2.5.2. The versal idempotent e. The choice of a coalgebra splitting s permits us to construct a functorial idempotent $e = e_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ as follows. First we define $\omega^{[n]} : \mathcal{H} \to \mathcal{H}$ as the composite

$$\omega^{[n]} : \mathcal{H} \xrightarrow{\theta_n} \mathcal{C}^c(n) \otimes_{S_n} \mathcal{H}^{\otimes n} \xrightarrow{s(n)\otimes \mathrm{Id}} \mathcal{A}(n) \otimes_{S_n} \mathcal{H}^{\otimes n} \xrightarrow{\gamma_n} \mathcal{H}.$$

We define a linear map $e : \mathcal{H} \to \mathcal{H}$ by the formula:

$$e = (\mathrm{Id} - \omega^{[2]})(\mathrm{Id} - \omega^{[3]}) \cdots (\mathrm{Id} - \omega^{[n]}) \cdots \ .$$

By the very same argument as in the proof of Proposition 2.3.5 we show that e is well-defined though it is given by an infinite product.

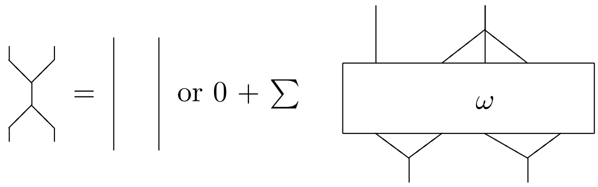
We will show below that e is an idempotent ($e^2 = e$). We call it the *versal idempotent* (and not universal) since it depends on the choice of a splitting. Different choices lead to different idempotents.

2.5.3. LEMMA. *We assume hypotheses `(H0)` , `(H1)` and `(H2epi)`. Let $\delta_1^c, \ldots, \delta_k^c$ be a basis of $\mathcal{C}^c(n)$. Let $\mu_i := s(V)(\delta_i^c) \in \mathcal{A}(n)$ and complete it into a basis $\mu_1 \ldots, \mu_k, \mu_{k+1}, \ldots, \mu_l$ of $\mathcal{A}(n)$.*
Then one has:

$$\begin{cases} \delta_i \circ \mu_i & = & \mathrm{id} + \textit{higher terms}, \\ \delta_j \circ \mu_i & = & 0 + \ \textit{higher terms, when } j \neq i, \end{cases}$$

where "higher terms" means a sum of some multivalued operations which begin with at least one nontrivial cooperation (Φ_2-type multivalued operations).

Proof. Graphically, for $n = 2$ the statement that we want to prove is



Since we are interested only in the Φ_1-type part, it is sufficient to compute the value of $\delta_j \circ \mu_i$ on the generic element $x_1 \cdots x_n$ of $\mathcal{A}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$. By Proposition 2.2.4 we get

$$\begin{aligned} \delta_j \circ \mu_i(x_1 \cdots x_n) &= \langle \delta_j, \varphi_n(\mu_i) \rangle x_1 \otimes \cdots \otimes x_n, \\ &= \langle \delta_j, \delta_i^c \rangle x_1 \otimes \cdots \otimes x_n, \\ &= (\text{id or } 0) x_1 \otimes \cdots \otimes x_n, \end{aligned}$$

depending on $j = i$ or $j \neq i$. □

2.5.4. Lemma. *If the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is connected, then the map $e = e_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is well-defined and satisfies the following properties:*

a) the image of e is $\operatorname{Prim} \mathcal{H}$,
b) e is an idempotent.

Proof. First we observe that, if x is primitive, then $e(x) = x$. Indeed, it is clear that $\omega^{[n]}(x) = 0$ for any $n \geq 2$ since $\omega^{[n]}$ begins with a nontrivial cooperation. Hence we have $e(x) = \operatorname{Id}(x) = x$.

Proof of a). We will prove by induction on n that the image of $F_n\mathcal{H}$ by e lies in $\operatorname{Prim} \mathcal{H}$. It is true for $n = 1$, since $F_1\mathcal{H} = \operatorname{Prim} \mathcal{H}$. We use the notation of the previous Lemma.

For $n = 2$ we have

$$\theta_2(x) = \sum_{i=1}^{i=k} \mu_i \circ \delta_i(x).$$

On $F_2\mathcal{H}$ we have $e = \operatorname{Id} - \omega^{[2]}$. We want to prove that for any $x \in F_2\mathcal{H}$ we have $(\operatorname{Id} - \omega^{[2]})(x) \in F_1\mathcal{H} = \operatorname{Prim} \mathcal{H}$, that is, for any $\delta_j \in \mathcal{C}(2)$, $\delta_j(x) = \delta_j \omega^{[2]}(x)$.

We have

$$\begin{cases} \delta_j \omega^{[2]}(x) &= \sum_{i=1}^{k} \delta_j \mu_i \delta_i(x), \\ &= \delta_i(x) + \text{ higher terms}, \end{cases}$$

by Lemma 2.5.3. So we have

$$\delta_j(\operatorname{Id} - \omega^{[2]})(x) = \delta_j(x) - \delta_j(x) + \sum \delta_i \circ \text{ higher terms}(x).$$

Since $x \in F_2\mathcal{H}$, we have $\delta_i \circ \text{higher terms}(x) = 0$ and therefore $\delta_j(\operatorname{Id} - \omega^{[2]})(x) = 0$ as expected.

A similar proof shows that $x \in F_n\mathcal{H}$ implies $(\operatorname{Id} - \omega^{[n]})(x) \in F_{n-1}\mathcal{H}$. Hence, putting all pieces together, we have shown that $x \in F_n\mathcal{H}$ implies $e(x) \in F_1\mathcal{H} = \operatorname{Prim} \mathcal{H}$. The expected assertion follows from the connectedness of $\mathcal{H}$.

Proof of b) Since $e(x) = x$ when x is primitive and since $e(x) \in \mathrm{Prim}\,(x)$ for any $x \in \mathcal{H}$, it is clear that $e \circ e = e$. $\square$

2.5.5. Proof of the structure Theorem. $(a) \Rightarrow (b)$. Since the functor $U : \mathcal{P}\text{-alg} \to \mathcal{A}\text{-alg}$ is left adjoint to the forgetful functor $F : \mathcal{A}\text{-alg} \to \mathcal{P}\text{-alg}$, the adjoint to the inclusion map $\iota : \mathrm{Prim}\,\mathcal{H} \to \mathcal{H}$ is an algebra map $\alpha : U(\mathrm{Prim}\,\mathcal{H}) \to \mathcal{H}$.

Let us show that α is surjective. If $x \in \mathcal{H}$ is in $\mathrm{Prim}\,\mathcal{H}$, then obviously it belongs to the image of α. For any $x \in \mathcal{H}$ there is an integer m such that $x \in F_m\mathcal{H}$ by the connectedness hypothesis. We now work by induction and suppose that α is surjective on $F_{m-1}\mathcal{H}$. From the formula

$$x = e(x) + \sum_i \omega^{[i]}(x) - \sum_{i,j} \omega^{[i]}\omega^{[j]}(x) + \cdots$$

and the fact that $\omega^{[n]}$ consists in applying a cooperation first and then an operation, it follows that $x - e(x)$ is the sum of products of elements in $F_{m-1}\mathcal{H}$. From the inductive hypothesis we deduce that $x - e(x)$ is in the image of α. Since $e(x) \in \mathrm{Prim}\,\mathcal{H}$ by Proposition 2.3.4 we have proved that any $x \in \mathcal{H}$ belongs to the image of α and so α is surjective.

Let us show that α is injective. The inductive argument as in the proof of Theorem 2.3.7 shows the injectivity of α.

In conclusion the algebra map $\alpha : U(\mathrm{Prim}\,\mathcal{H}) \to \mathcal{H}$ is surjective and injective, so it is an isomorphism. It is also a coalgebra map by 2.3.3 and 2.4.5, so it is a bialgebra isomorphism.

$(b) \Rightarrow (c)$. Let L be a $\mathcal{P}$-algebra. Since U is left adjoint to the functor $F : \mathcal{A}\text{-alg} \to \mathcal{P}\text{-alg}$ the map $\varphi(L) : \mathcal{A}(L) \to \mathcal{C}^c(L)$ factors through $U(L)$:

$$\begin{tikzcd} \mathcal{A}(L) \arrow[rr, "\varphi(L))"] \arrow[dr, two heads] & & \mathcal{C}^c(L) \\ & U(L) \arrow[ur] & \end{tikzcd}$$

We first show that the map $U(L) \to \mathcal{C}^c(L)$ is an isomorphism when L is a free $\mathcal{P}$-algebra. In this case we know that $\mathrm{Prim}\,U(L) = L$ since $U(\mathcal{P}(V)) = \mathcal{A}(V)$ (left-adjointness), and $\mathrm{Prim}\,\mathcal{A}(V) = \mathcal{P}(V)$ by definition of $\mathcal{P}$. Since $U(L)$ is a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra and $\mathrm{Prim}\,U(L) = L$ by Proposition 2.4.5, there is a surjection $e_{U(L)} : U(L) \to L$. By the universality of the cofree coalgebra there is a lifting $\tilde{e}_{U(L)}$ and so a commutative diagram:

$$\begin{tikzcd} U(L) \arrow[rr, "\tilde{e}_{U(L)}"] \arrow[dr, two heads] & & \mathcal{C}^c(L) \arrow[dl] \\ & L & \end{tikzcd}$$

Observe that the composite $\mathcal{A}(L) \twoheadrightarrow U(L) \to \mathcal{C}^c(L)$ is the map $\varphi(L)$, hence $\tilde{e}_{U(L)}$ is compatible with the filtration.

We claim that the associated morphism $gr(U(L)) \to gr\mathcal{C}^c(L) = \mathcal{C}^c(L)$ on the graded objects is an isomorphism. Indeed, the quotient $F_nU(L)/F_{n-1}U(L)$ consists in moding out $\mathcal{A}(L)_1 \oplus \cdots \oplus \mathcal{A}(L)_n$ by the subspace J_n generated by the elements $\mu^{\mathcal{P}}(x_1 \ldots x_n) - \mu^{\mathcal{A}}(x_1 \ldots x_n)$ and by $\mathcal{A}(L)_1 \oplus \cdots \oplus \mathcal{A}(L)_{n-1}$. On the other hand, the quotient $F_n\mathcal{C}^c(L)/F_{n-1}\mathcal{C}^c(L)$ is simply $\mathcal{C}^c(L)_n$, which is the quotient of $\mathcal{A}(L)_n$ by the homogeneous degree n part of J_n. These two quotients are the same because the relations

– $u + v \sim 0$ and $v \sim 0$,

and

– $u \sim 0$ and $v \sim 0$,

are equivalent (recall that $\mu^{\mathcal{P}}(x_1 \ldots x_n)$ is in degree 1).

Let us now prove that $U(L) \to \mathcal{C}^c(L)$ is an isomorphim for any Lie algebra L. Let

$$L_1 \to L_0 \twoheadrightarrow L \to 0$$

be a free resolution of L. Since the morphism $U(V) \to \mathcal{C}^c(V)$ is natural in V, the isomorphisms for L_0 and L_1 imply the isomorphism for L.

The implication $(c) \Rightarrow (a)$ is a tautology. □

2.5.6. Good triple of operads. If a triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ satisfies the structure theorem, then we call it a *good triple of operads*. So Theorem 2.5.1 shows that, if the triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ satisfies the hypothesis `(H2epi)` , then it is a good triple. Conversely, if the triple is good, then the coalgebra isomorphism $\mathcal{A}(V) \cong \mathcal{C}^c(\mathcal{P}(V))$ composed with the projection induced by proj : $\mathcal{P}(V) \to V$ defines $\mathcal{A}(V) \to \mathcal{C}^c(V)$, which is a splitting of $\varphi(V)$. Hence the hypothesis `(H2epi)` is fulfilled. Therefore the triple $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is good if and only if the hypothesis `(H2epi)` is fulfilled.

2.5.7. About the verification of the hypotheses.

2.5.7.1. *(H0).* The hypothesis `(H0)` (distributivity of the compatibility relation) is, in general, immediate to check by direct inspection. Observe that, when the operads $\mathcal{C}$ and $\mathcal{A}$ are given by generators and relations, it suffices to check the compatibility relations on the pairs (δ, μ) when they are both generators.

2.5.7.2. *(H1).* In order to verify hypothesis `(H1)` there are, essentially, three strategies.

(1). When the free algebra $\mathcal{A}(V)$ is known explicitly (for instance a basis of $\mathcal{A}(n)$ is identified with some explicit combinatorial objects), then one can usually construct explicitly the generating cooperations and check that they satisfy the relations of $\mathcal{C}^c$ and the compatibility relations.

(2). Another strategy consists in taking advantage of the distributivity of the compatibility relations. One constructs inductively the cooperations on $\mathcal{A}(V) = \oplus_{n\geq 1}\mathcal{A}(V)_n$ by sending V to 0 and then by using the compatibility relations. Then one checks, again inductively, that they satisfy the relations

of $\mathcal{C}^c$ and the compatibility relations. This is very close to the techniques used in "rewriting systems", cf. citeLafont97 and 4.8.
(3). The third strategy consists in viewing a given triple as a quotient of a good triple. It is given in Proposition 3.1.1 below.

2.5.7.3. *(H2epi)*. The map φ is in fact a map of **S**-modules . Therefore it sends the degree n part of $\mathcal{A}$ to the degree n part of $\mathcal{C}^c$. So in order to check surjectivity, it is sufficient to compute the composite $\delta \circ \mu$ for any pair (δ, μ) where μ is a linear generator of $\mathcal{A}(n)$ and δ is a linear generator of $\mathcal{C}^c(n)$. From this functorial property of φ we deduce that the element $\delta \circ \mu(x_1, \ldots, x_n)$ is of the form $\sum_\sigma a_\sigma(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Let $a(\mu, \delta)$ be the coefficient a_{id} of this sum. The map φ is

$$\varphi_n(\mu) = \sum_\delta a(\mu, \delta)\delta$$

where the sum is over a basis of $\mathcal{C}^c(n)$.

2.6. A few consequences of the structure theorem

We derive a few consequences of the structure theorem, namely by applying it to the free algebra $\mathcal{A}(V)$. It gives some criterion to check if a given triple of operads has some chances to be good.

2.6.1. From the structure theorem to the rigidity theorem. The rigidity theorem is a Corollary of the structure theorem. Indeed, if the hypothesis `(H2iso)` holds, then `(H2epi)` holds (unique choice for the splitting) and the primitive operad is $Vect$ by Proposition 2.3.2. Hence the triple $(\mathcal{C}, \mathcal{A}, Vect)$ is a good triple and the functor F is simply the forgetful functor to `Vect`. The left adjoint functor of F is the free $\mathcal{A}$-algebra functor, so item (b) in the structure theorem becomes $\mathcal{H} \cong \mathcal{A}(\mathrm{Prim}\, \mathcal{H})$. So $\mathcal{H}$ is free and cofree over its primitive part, as claimed in Theorem 2.3.7.

2.6.2. Dualization. Observe that if $(\mathcal{C}, \mathcal{A}, Vect)$ is a good triple of operads, then so is the triple $(\mathcal{A}, \mathcal{C}, Vect)$. The compatibility relation(s) is obtained by dualization, i.e. reading $\between$ upsidedown. The new map φ is simply the linear dual of the former one.

2.6.3. Theorem. *If $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is a good triple of operads over the field $\mathbb{K}$, then there is an equivalence of categories between the category of connected (i.e. conilpotent) $\mathcal{C}^c$-$\mathcal{A}$-bialgebras and the category of $\mathcal{P}$-algebras:*

$$\{con.\ \mathcal{C}^c\text{-}\mathcal{A}\text{-}bialg\} \underset{\mathrm{Prim}}{\overset{U}{\leftrightarrows}} \{\mathcal{P}\text{-}alg\}\ .$$

Proof. We already know that if L is a free $\mathcal{P}$-algebra, i.e. $L = \mathcal{P}(V)$, then $\mathrm{Prim}\, U(L) = \mathrm{Prim}\, U(\mathcal{P}(V)) = \mathrm{Prim}\, \mathcal{A}(V) = \mathcal{P}(V) = L$. By the same argument as in the proof of $(b) \Rightarrow (c)$ it is true for any $\mathcal{P}$-algebra L.

In the other direction, let $\mathcal{H}$ be a connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra . By item (b) in Theorem 2.5.1 we have an isomorphism $\mathcal{H} \cong U(\mathrm{Prim}\,\mathcal{H})$. $\square$

2.6.4. Proposition. *If $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is a good triple of operads, then there is an isomorphism of Schur functors:*

$$\mathcal{A} \cong \mathcal{C}^c \circ \mathcal{P}\ .$$

Proof. It suffices to apply the structure theorem to the free algebra $\mathcal{A}(V)$, which is a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra by hypothesis. It is connected because $\mathcal{A}(1) = \mathbb{K}$.

Since the composite of left adjoint functors is still left adjoint, we have $U(\mathcal{P}(V)) = \mathcal{A}(V)$. Hence, by the structure Theorem 2.5.1, we get the expected isomorphism. $\square$

2.6.5. Corollary. *If $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is a good triple of operads, then there is an identity of formal power series:*

$$f^{\mathcal{A}}(t) = f^{\mathcal{C}}(f^{\mathcal{P}}(t)) = t.$$

Proof. Since $\mathcal{C}^c(n)$ is finite dimensional, the two Schur functors $\mathcal{C}$ and $\mathcal{C}^c$ have well-defined generating series which are equal. The formula follows from Proposition 2.6.4 and the computation of the generating series of a composite of Schur functors, cf. 1.1.3. $\square$

2.6.6. Searching for good triples. Observe that this relationship entertwining the generating series gives a criterion to the possible existence of a good triple. Indeed, let us suppose that we start with a forgetful functor $\mathcal{A}\text{-alg} \xrightarrow{F} \mathcal{P}\text{-alg}$ and we would like to know if it can be part of a good triple $(\mathcal{C}, \between, \mathcal{A}, F, \mathcal{P})$. Then, there should exist a power series $c(t) = \sum_{n\geq 1} \frac{c(n)}{n!} t^n$ where the coefficients $c(n)$ are integers (and $c(1) = 1$), such that $f^{\mathcal{A}}(t) = c(f^{\mathcal{P}}(t))$.

For instance, if $(Com, \mathcal{A}, \mathcal{P})$ is a good triple of operads and if $\mathcal{B}\text{-alg} \to \mathcal{A}\text{-alg}$ is a forgetful functor, then there is no good triple for the composite $\mathcal{B}\text{-alg} \to \mathcal{P}\text{-alg}$ (unless $\mathcal{B} = \mathcal{A}$).

2.6.7. Frobenius character. There is an invariant which is finer than the generating series. It consists in taking the Frobenius character series of the Schur functor. Indeed the isomorphism

$$\mathcal{A}(n) \cong \sum_{i_1+\cdots+i_k=n} \mathcal{C}^c(k) \otimes_{S_k} \mathrm{Ind}_{S_{i_1}+\cdots+S_{i_k}}^{S_n} \big(\mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k)\big)$$

implies that the composite of the Frobenius characters of $\mathcal{C}^c$ and $\mathcal{P}$ is the Frobenius character of $\mathcal{A}$.

CHAPTER 3

Applications and variations

In this chapter we give a few applications of the structure theorem, some generalizations and we give some general constructions to obtain a good triple of operads. Concrete examples will be given in the next chapter.

One of the most easy ways of constructing a good triple from an existing triple is to mod out by primitive relators. It gives rise to many examples.

There are some techniques to obtain triples of the form $(As, \mathcal{A}, \mathrm{Prim}_{As}\mathcal{A})$. For instance one can assume that $\mathcal{A}$ is an Hopf operad, that is, the Schur functor is a functor to coalgebras. Another assumption is to suppose that there exists an associative operation verifying some good properties (multiplicative operad).

In the regular case, we do not need the characteristic zero hypothesis to get a good triple, so, under this hypothesis, the structure theorem is valid in a characteristic free context.

We show how Koszul duality should help to construct good triples out of existing ones.

Our basic category is the category `Vect` of vector spaces. It is a symmetric monoidal category and this is exactly the structure that we used. So there is an immediate extension of our main theorem to any symmetric monoidal category, for instance the category of sign-graded vector spaces and the category of **S**-modules .

We can reverse the roles of algebraic and coalgebraic structures. Then the primitives are replaced by the indecomposables and we obtain a "dual" result.

The classical result (PBW+CMM) admits a characteristic p variant. We expect similar generalizations in characteristic p and we explain how to modify the operad framework to do so.

We mention briefly the relationship with rewriting systems in computer sciences.

Finally, we give an application to a natural problem in representation theory of the symmetric groups.

3.1. Quotient triple

3.1.1. PROPOSITION. *Let $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ be a good triple of operads and let $r \in \mathcal{P}(n), (n \geq 2)$, be a nontrivial primitive operation. Then the triple of operads $(\mathcal{C}, \mathcal{A}/((r)), \mathcal{P}/(r))$, where (r), resp.$((r))$, is the operadic ideal in $\mathcal{P}$, resp. $\mathcal{A}$, generated by r, is a good triple. As a consequence, there is a good* *triple*

$$(\mathcal{C}, \mathcal{A}/(\!(\bar{\mathcal{P}})\!), Vect).$$

Proof. First we check immediately that the type of algebras $\mathcal{C}$-$\mathcal{A}/(\!(r)\!)$ satisfies the hypotheses `(H0)` , `(H1)` and `(H2epi)`. Indeed, the operad $\mathcal{A}/(\!(\bar{\mathcal{P}})\!)$ is a quotient of $\mathcal{A}$, so the compatibility relations are the same. The space $\mathcal{A}(V)$ is a $\mathcal{C}^c$-coalgebra and we mod out by elements which give 0 under any nontrivial cooperation since they are primitive. So $(\mathcal{A}/(\!(r)\!))(V)$ is a $\mathcal{C}^c$-coalgebra. As for hypothesis `(H2epi)` , it suffices to take the composite $\mathcal{C}^c \to \mathcal{A}(V) \twoheadrightarrow (\mathcal{A}/(\!(r)\!))(V)$ to get a splitting. This argument proves the Proposition by applying Theorem 2.5.1.

When moding out by all the generating operations of $\overline{\mathcal{P}}$ the map $\mathcal{A}/(\!(\overline{\mathcal{P}})\!) \to \mathcal{C}^c$ becomes an isomorphism and so, by Theorem 2.3.7, we get the last statement. □

3.1.2. Quotient triple of a triple. The triple $(\mathcal{C}, \mathcal{A}/(\!(\bar{\mathcal{P}})\!), Vect)$ associated to the triple $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is called its *quotient triple*.

3.1.3. Remark. Observe that in many cases, included the classical case, the **S**-module isomorphism $\mathcal{C} \cong \mathcal{A}/(\!(\bar{\mathcal{P}})\!)$ is in fact an operad isomorphism. But this not always true, see for instance examples 4.3.1 and 4.5.2.

3.1.4. THEOREM (Analogue of the classical PBW Theorem). *Let* $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ *be a good triple of operads, and let* $\mathcal{Z} := \mathcal{A}/(\!(\bar{\mathcal{P}})\!)$ *be the quotient operad of* $\mathcal{A}$ *by the ideal generated by the primitive operations. Then, for any* $\mathcal{P}$*-algebra* L *there is an isomorphism of* $\mathcal{Z}$*-algebras:*

$$\mathcal{Z}(L) \to gr\ U(L)\ .$$

Proof. First we observe that $gr\ U(L)$ is a $\mathcal{Z}$-algebra by direct inspection of the structure of $U(L)$, cf. 2.4.4. The composite map

$$L \to \mathcal{A}(L) \to U(L) \to gr_1 U(L) \subset U(L)$$

induces a $\mathcal{Z}$-algebra map $\mathcal{Z}(L) \to gr\ U(L)$. The commutativity of the diagram

$$\begin{tikzcd}
\mathcal{Z}(L) \arrow[r] \arrow[dr] & gr\ U(L) \arrow[r] \arrow[d, two heads] & \mathcal{C}^c(L) \arrow[dl] \\
& L &
\end{tikzcd}$$

shows that the composite of the horizontal arrows is the isomorphism $\mathcal{Z}(L) \to \mathcal{C}^c(L)$ coming from the good triple $(\mathcal{C}, \mathcal{Z}, Vect)$. Since $gr\ U(L) \to \mathcal{C}^c(L)$ is an isomorphism (cf. the proof of $(b) \Rightarrow (c)$ in Theorem 2.5.1), we are done. □

3.1.5. Remark on PBW. In the classical case (Com, As, Lie) (see 4.1 for details) the isomorphism of commutative algebras

$$S(\mathfrak{g}) \to gr\ U(\mathfrak{g})$$

is often called the Poincaré-Birkhoff-Witt theorem (cf. [**29**]).

3.1.6. Split triple of operads. Let $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ be a triple of operads and let $\mathcal{Z} := \mathcal{A}/((\bar{\mathcal{P}}))$ be the quotient operad. We say that $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is a *split triple* if there is a morphism of operads $s : \mathcal{Z} \rightarrowtail \mathcal{A}$ such that the composite $\mathcal{Z} \rightarrowtail \mathcal{A} \twoheadrightarrow \mathcal{Z}$ is the identity and the map $s(V) : \mathcal{Z}(V) \to \mathcal{A}(V)$ is a $\mathcal{C}^c$-$\mathcal{Z}$-bialgebra morphism. For instance the triple (Com, As, Lie) is not split, and the triple (As, OU, Mag) (cf. 5) admits two different splittings.

3.1.7. Proposition. *Let $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ be a split triple of operads. Then any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is also a $\mathcal{C}^c$-$\mathcal{Z}$-bialgebra and the idempotent $e_{\mathcal{H}}$ is the same in both cases.*

Proof. Since, by hypothesis, the splitting s induces a morphism of bialgebras on the free algebras, it induces a morphism of props $\mathcal{C}^c$-$\mathcal{Z} \to \mathcal{C}^c$-$\mathcal{A}$, or, equivalently, a functor between the category of bialgebras $\mathcal{C}^c$-$\mathcal{A}$-*bialg* and $\mathcal{C}^c$-$\mathcal{Z}$-*bialg*. In the construction of the idempotent $e_{\mathcal{H}}$ for $\mathcal{C}^c$-$\mathcal{A}$-bialgebras we need a coalgebra splitting $\mathcal{C}^c \to \mathcal{A}$. We can take the composite $\mathcal{C}^c \cong \mathcal{Z} \xrightarrow{s} \mathcal{A}$. From the construction of $e_{\mathcal{H}}$ (cf. 2.3.4) it follows that we get precisely the universal idempotent for $\mathcal{C}^c$-$\mathcal{Z}$-bialgebras. □

3.2. Hopf operad, multiplicative operad

Under some reasonable assumption on an operad $\mathcal{P}$ we can show that the tensor product of two $\mathcal{P}$-algebras is still a $\mathcal{P}$-algebra. As a consequence one can equip the free $\mathcal{P}$-algebra with a coassociative cooperation. It gives rise to a notion of As^c-$\mathcal{P}$-bialgebras. Quite often the assumption are easy to verify and show immediately that hypothesis `(H1)` is fulfilled. We present two cases: Hopf operad and multiplicative operad.

3.2.1. Hopf operad. By definition a *Hopf operad* is an operad $\mathcal{P}$ in the category of coalgebras (cf. [**58**] for instance). Moreover we assume that $\mathcal{P}(0) = \mathbb{K}$, so $\mathcal{P}$-algebras have a unit (the image of $1 \in \mathcal{P}(0)$). Explicitly the spaces $\mathcal{P}(n)$ are coalgebras, i.e. they are equipped with a coassociative map $\Delta : \mathcal{P}(n) \to \mathcal{P}(n) \otimes \mathcal{P}(n)$, compatible with the operad structure. For instance a set-theoretic operad gives rise to a Hopf operad by using the diagonal on sets. As a consequence the tensor product of two $\mathcal{P}$-algebras A and B is a $\mathcal{P} \otimes \mathcal{P}$-algebra (where $\mathcal{P} \otimes \mathcal{P}$ is the Hadamard product) and the map Δ makes it into a $\mathcal{P}$-algebra.

An algebra over $\mathcal{P}$ in this framework is a coalgebra equipped with a coalgebra map $\mathcal{P}(A) \to A$. Hence we get a notion of As^c-$\mathcal{P}$-bialgebra. In particular the free $\mathcal{P}$-algebra is a As^c-$\mathcal{P}$-bialgebra since there is a unique $\mathcal{P}$-algebra morphism

$$\mathcal{P}(V) \to \mathcal{P}(V) \otimes \mathcal{P}(V)$$

which extends $v \mapsto v \otimes 1 + 1 \otimes v$. Several examples of triples of operads $(As, \mathcal{P}, \mathrm{Prim}_{As}\mathcal{P})$ are of this type (see Chapter 4).

3.2.2. Multiplicative operad [**45**]**.** Let $\mathcal{P}$ be a binary quadratic regular operad which contains an associative operation, denoted $*$ (this hypothesis is sometimes called "split associativity" as $*$ comes, often, as the sum of the generating operations). We call it a *multiplicative operad*. We suppose that there is a partial unit 1 in the following sense. We give ourselves two maps

$$\alpha : \mathcal{P}(2) \to \mathcal{P}(1) = \mathbb{K} \quad \text{and} \quad \beta : \mathcal{P}(2) \to \mathcal{P}(1) = \mathbb{K}$$

which give a meaning to $x \circ 1$ and $1 \circ x$ for any $\circ \in \mathcal{P}(2)$ and any $x \in A = \mathcal{P}$-algebra:

$$x \circ 1 = \alpha(\circ)x \qquad 1 \circ x = \beta(\circ)x.$$

We always assume that 1 is a two sided unit for $*$ (i.e. $\alpha(*) = 1 = \beta(*)$). Observe that we do not require $1 \circ 1$ to be defined. Let $A_+ = A \oplus \mathbb{K}1_A$ be the augmented algebra. For two $\mathcal{P}$-algebras A and B the augmentation ideal of $A_+ \otimes B_+$ is

$$A \otimes \mathbb{K}1_B \ \oplus \ \mathbb{K}1_A \otimes B \ \oplus \ A \otimes B$$

The *Ronco's trick* consists in constructing an operation $\circ$ on the augmentation ideal as follows:

$$(a \otimes b) \circ (a' \otimes b') = (a * a') \otimes (b \circ b')$$

whenever all the terms are defined, and (when $b = 1_B = b'$)

$$(a \otimes 1_B) \circ (a' \otimes 1_B) = (a \circ a') \otimes (1_{A \otimes B}).$$

Observe that the relations of $\mathcal{P}$ are verified for any $a, a' \in A$ and $b, b' \in B$. If they are also verified in all the other cases, then the choice of α and β is said to be *coherent* with $\mathcal{P}$.

It was proved in [**45**] that, under this coherence assumption, the free $\mathcal{P}$-algebra $\mathcal{P}(V)$ is equipped with a coassociative coproduct δ. It is constructed as follows. By hypothesis there is a $\mathcal{P}$-algebra structure on

$$\mathcal{P}(V) \otimes \mathbb{K}1 \ \oplus \ \mathbb{K}1 \otimes \mathcal{P}(V) \ \oplus \ \mathcal{P}(V) \otimes \mathcal{P}(V)$$

We define Δ from $\mathcal{P}(V)$ to the $\mathcal{P}$-algebra above as the unique $\mathcal{P}$-algebra map which sends $v \in V$ to $v \otimes 1 + 1 \otimes v$. The projection to $\mathcal{P}(V) \otimes \mathcal{P}(V)$ gives the expected map δ.

From this construction we get a well-defined notion of As^c-$\mathcal{P}$-bialgebra for which the hypotheses `(H0)` and `(H1)` are fulfilled. Of course, this construction can be re-written in the nonunital context. But, then, the formulas are more complicated to handle (see 4.1.1).

In many cases we get a good triple of operads $(As, \mathcal{P}, \mathrm{Prim}_{As}\mathcal{P})$, see [**45**]. Several cases will be described in Chapter 4. The interesting point point about these examples is that $\mathcal{P}(V)$ is a Hopf algebra in the classical sense. In fact many *combinatorial Hopf algebras* can be constructed this way.

A refinement of this method gives triples of the form $(\mathcal{A}, \mathcal{A}, Vect)$, cf. 4.6.3.

3.3. The regular case

In the preceding chapter we always made the hypothesis: $\mathbb{K}$ is a characteristic zero field. The reason was the following. In the interplay between operad and cooperad we had to identify invariants and coinvariants, cf. 1.3.4. There is an environment for which this hypothesis is not necessary, it is the regular case (for the characteristic p case, see 3.7).

We suppose that $\mathcal{C}$ and $\mathcal{A}$ are regular operads (cf. 1.2.7) and that, in the compatibility relations, the only permutation which is involved is the identity (cf 2.1.2). Such a type is called a *regular type* of generalized bialgebras (regular prop). In hypothesis `(H2epi)` we suppose that there is a cooperad splitting of the form $\mathcal{C}^c_n \to \mathcal{A}_n$, i.e. not involving the symmetric group. This is called the *regular version* of `(H2epi)` . Then the very same proof as in the structure Theorem can be performed and we get the following result.

3.3.1. Theorem (Structure Theorem for regular generalized bialgebras). *Let $\mathcal{C}^c$-$\mathcal{A}$ be a regular type of generalized bialgebras over a field $\mathbb{K}$. Suppose that the hypotheses `(H1)` and regular `(H2epi)` are fulfilled.*

Then the good triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ has the following property.

For any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ the following are equivalent:
a) the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is connected,
b) there is an isomorphism of bialgebras $\mathcal{H} \cong U(\mathrm{Prim}\,\mathcal{H})$,
c) there is an isomorphism of connected coalgebras $\mathcal{H} \cong \mathcal{C}^c(\mathrm{Prim}\,\mathcal{H})$.

The following rigidity theorem is a Corollary of the Structure Theorem in the regular case.

3.3.2. Theorem (Rigidity Theorem, regular case). *Let $\mathcal{C}^c$-$\mathcal{A}$ be a regular type of generalized bialgebras over a field $\mathbb{K}$. Suppose that the hypotheses `(H1)` and `(H2iso)` are fulfilled.*

Then any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is free and cofree over its primitive part:

$$\mathcal{A}(\mathrm{Prim}\,\mathcal{H}) \cong \mathcal{H} \cong \mathcal{C}^c(\mathrm{Prim}\,\mathcal{H}).$$

Explicit examples will be given in the next section.

3.4. Koszul duality and triples

We provide a method to construct good triples of operads by using the Koszul duality for operads.

3.4.1. Koszul duality of quadratic operads. Let us recall briefly from [**26**] and [**22**] that any quadratic operad $\mathcal{P}$ gives rise to a dual operad $\mathcal{P}^!$. It is also quadratic and $(\mathcal{P}^!)^! = \mathcal{P}$. For instance $As^! = As, Com^! = Lie, Mag^! = Nil$. When the operad $\mathcal{P}$ is binary, then the generating series of $\mathcal{P}$ and of $\mathcal{P}^!$ are related by the formula:

$$f^{\mathcal{P}^!}(-f^{\mathcal{P}}(-t)) = t,$$

(see [**26**], or the Appendix of [**44**] for a brief account of Koszul duality of operads). When the operad $\mathcal{P}$ is k-ary, one needs to introduce the skew-generating series

$$g^{\mathcal{P}}(t) := \sum_{n\geq 1}(-1)^k \frac{\dim \mathcal{P}((k-1)n+1)}{n!}\ t^{((k-1)n+1)}.$$

If the operad $\mathcal{P}$ is Koszul, then Vallette proved in [**72**]) the formula:

$$f^{\mathcal{P}^!}(-g^{\mathcal{P}}(-t)) = t.$$

In the most general case (generating operations of any arity), it is best to work with a series in two variables, cf. [**72**] for details.

3.4.2. Extension of operads. Let us say that the sequence of operads

$$\mathcal{P} \rightarrowtail \mathcal{A} \twoheadrightarrow \mathcal{Z}$$

is an *extension of operads* if $\rightarrowtail$ is a monomorphism, $\twoheadrightarrow$ is an epimorphism, and if there is an isomorphism of **S**-modules (which is part of the structure) $\mathcal{A} \cong \mathcal{Z} \circ \mathcal{P}$ such that $\mathrm{Id}_{\mathtt{Vect}} \rightarrowtail \mathcal{Z}$ induces $\mathcal{P} \rightarrowtail \mathcal{A}$, and $\mathcal{P} \twoheadrightarrow \mathrm{Id}_{\mathtt{Vect}}$ induces $\mathcal{A} \twoheadrightarrow \mathcal{P}$. Under this hypothesis we say that $\mathcal{A}$ is an *extension* of $\mathcal{Z}$ by $\mathcal{P}$. For instance As and $Pois$ are both extensions of Com by Lie.

In many cases where all the operads are quadratic we can check that

$$\mathcal{Z}^! \rightarrowtail \mathcal{A}^! \twoheadrightarrow \mathcal{P}^!$$

is also an extension of operads (Exercise: show that it works at the level of generating functions). For instance the classical extension

$$Lie \rightarrowtail As \twoheadrightarrow Com$$

is self-dual.

Suppose that $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is a good triple and that $\mathcal{P}^!$ is part of a good triple $(\mathcal{Q}^!, \mathcal{P}^!, Vect)$. We recall that we have also a good triple $(\mathcal{C}, \mathcal{Z}, Vect)$. Then comparing the generating functions we can expect the existence of a good triple of the form:

$$(\mathcal{Q}^!, \mathcal{A}^!, \mathcal{Z}^!).$$

In the case where $\mathcal{Q} = \mathcal{P}$ and $\mathcal{C} = \mathcal{Z}$, we would get the triple

$$(\mathcal{P}^!, \mathcal{A}^!, \mathcal{C}^!).$$

Similar structures have been studied in [**19**].

3.5. Graded version

We briefly explore the change of symmetric monoidal category.

3.5.1. Graded vector space. Until now our ground category was the category of vector spaces over a field. The only property of `Vect` that we used is that it is a symmetric monoidal category. Hence we can replace it by any other symmetric monoidal category, like, for instance, the category of *graded vector spaces* (more accurately we should say the *sign-graded vector spaces*). Recall that the objects are the graded vector spaces $\{V_n\}_{n\geq 0}$ and the symmetric isomorphism (twisting map) is given by

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x \ ,$$

where x and y are homogeneous elements of degree $|x|$ and $|y|$ respectively.

3.5.2. Structure theorem in the graded case. The main result (cf. 2.5.1) holds in this more general setting because, as already said, our proofs use only the symmetric monoidal properties of `Vect`. The result is not significantly different when the operads are regular (since the symmetric group does not play any role). However it is different for general operads since, for instance, the free commutative algebra over an odd-degree vector space is the exterior algebra instead of the symmetric algebra. In particular when the vector space is finite dimensional, the exterior algebra is finite dimensional, while the symmetric algebra is not.

In algebraic topology it is the sign-graded framework which is relevant.

3.5.3. Structure theorem for S-modules. The category of **S**-modules can be equipped with a symmetric product as follows. Let M and N be two **S**-modules . We define their tensor product $M \otimes N$ by

$$(M \otimes N)(V) := M(V) \otimes N(V) \ .$$

Here we use the interpretation of **S**-modules in terms of endofunctors of $Vect$ (Schur functors, see 1.1). We let the reader write the theorem explicitly.

3.5.4. Generalization to colored operads. A category is a generalization of a monoid in the sense that the composition of two elements is defined only if certain conditions are fulfilled (source of one = target of the other). Similarly there is a generalization of the notion of operads in which the composition of operations is defined only if some conditions are fulfilled among the operations. This is called a *colored operad* or a *multicategory.* One should be able to write a structure theorem in this framework. See [**69**] for an example.

3.5.5. Generalization of the regular case. In the regular case we don't even use the symmetry properties of the monoidal category `Vect`. Hence we can extend our theorem to other monoidal categories. For instance we can replace $(\texttt{Vect}, \otimes)$ by the category of **S**-modules equipped with the composition product $\circ$, cf. 1.1.2. Then, there are notions of (generalized) algebras, coalgebras and bialgebras in this context and we can extend our main theorem. For instance the analogue of associative algebra (resp. associative coalgebra) is the notion of operad (resp. cooperad). So a *unital* *infinitesimal bioperad* $\mathcal{B}$ is an $\mathbf{S}$-module $\mathcal{B}$ equipped with an operad structure $\gamma : \mathcal{B} \circ \mathcal{B} \to \mathcal{B}$ and a cooperad structure $\theta : \mathcal{B} \to \mathcal{B} \circ \mathcal{B}$ satisfying the following compatibility relation

$$\theta\gamma = -\mathrm{id}_{\mathcal{B}\circ\mathcal{B}} + (\mathrm{id}_{\mathcal{B}} \circ \theta)(\gamma \circ \mathrm{id}_{\mathcal{B}}) + (\theta \circ \mathrm{id}_{\mathcal{B}})(\mathrm{id}_{\mathcal{B}} \circ \gamma)$$

which is nothing but the unital infinitesimal relation 2.1.3.2. Observe that we denoted the composition of functors by concatenation (for instance $\theta\gamma$) to avoid confusion with the composition of $\mathbf{S}$-modules . The generalization of our theorem says that the only example of unital infinitesimal bioperad is the free operad.

Observe that there is no such object as Hopf bioperad since the monoidal category $(\mathbf{S} - modules\mathbf{S})$-**modules**$, \circ)$ is not symmetric.

We plan to come back to this notion of generalized bioperads in a subsequent paper.

3.6. Coalgebraic version

Let $\mathcal{C}^c$-$\mathcal{A}$ be a bialgebra type. The notion of "indecomposable" is dual to the notion of "primitive". By definition, the *indecomposable space* of a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ is the quotient

$$\mathrm{Indec}\,\mathcal{H} := \mathcal{H}/\mathcal{H}^2$$

where $\mathcal{H}^2$ is the image of $\bigoplus_{n\geq 2} \mathcal{A}(n) \otimes_{S_n} \mathcal{H}^{\otimes n}$ in $\mathcal{H}$ under γ. Observe that it depends only on the $\mathcal{A}$-algebra structure of $\mathcal{H}$. In general $\mathrm{Indec}\,\mathcal{H}$ is not a $\mathcal{C}^c$-coalgebra, but we will construct a quotient cooperad of $\mathcal{C}^c$ on which $\mathrm{Indec}\,\mathcal{H}$ is a coalgebra.

3.6.1. PROPOSITION. *Suppose that the bialgebra type $\mathcal{C}^c$-$\mathcal{A}$ satisfies* (H0) *and*

(H1^c) *the cofree $\mathcal{C}^c$-coalgebra $\mathcal{C}^c(V)$ is equipped with a natural $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure.*

Then the $\mathbf{S}$-module $\mathcal{Q}^c(V) = \mathrm{Indec}_{\mathcal{A}}\mathcal{C}^c(V) := \mathcal{C}^c(V)/\mathcal{C}^c(V)^2$ inherits a cooperad structure from $\mathcal{C}^c$.

Moreover for any $\mathcal{C}^c$-$\mathcal{A}$-bialgebra $\mathcal{H}$ the indecomposable space $\mathrm{Indec}\,\mathcal{H}$ *is a $\mathcal{Q}^c$-coalgebra, and the surjection $\mathcal{H} \to \mathrm{Indec}\,\mathcal{H}$ is a $\mathcal{Q}^c$-coalgebra morphism.*

Proof. It suffices to dualize the proof of 2.5.1. □

Example. Let As^c-Com be the type "commutative (classical) bialgebras". Then $\mathcal{Q} = Lie$ and the surjection $As^c \to Lie^c$ is simply the dual of the inclusion $Lie \to As$. Explicitly the coLie structure of the coassociative coalgebra (C, δ) is given by $(\mathrm{id} - \tau)\delta$ where τ is the twisting map.

Since $\mathcal{Q}^c$ is a quotient of $\mathcal{C}^c$, there is a forgetful functor

$$F^c : \mathcal{C}^c\text{-}coalg \to \mathcal{Q}^c\text{-}coalg$$

which admits a *right* adjoint, that we denote by

$$F^c : \mathcal{Q}^c\text{-}coalg \to \mathcal{C}^c\text{-}coalg.$$

So now we have all the ingredients to write a structure theorem in the dual case .

Observe that $U^c(C)$ acquires a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure from the $\mathcal{C}^c$-$\mathcal{A}$-bialgebra structure of $\mathcal{C}^c(V)$.

The dual PBW has been proved in [**56**]. See also [**23**] §4.2 for the dual PBW and dual CMM theorems. The Eulerian idempotent has been worked out in this context by M. Hoffman [**31**]

3.7. Generalized bialgebras in characteristic p

First, observe that in the regular framework (cf. 3.3), there is no characteristic assumption, therefore the structure theorem holds in characteristic p. In [**12**] and [**57**] the authors give a characterisic p version of the PBW theorem and of the CMM theorem. So there is a characteristic p version of the structure theorem for cocommutative (classical) bialgebras. But the notion of Lie algebra has to be replace by the notion of p-restricted Lie algebras.

3.7.1. p-restricted Lie algebras. By definition a *p-restricted Lie algebra* is a Lie algebra over a characteristic p field which is equipped with a unary operation $x \mapsto x^{[p]}$ called the Frobenius operation. It is supposed to satisfy all the formal properties of the iterated bracket

$$\underbrace{[x, [x, [\cdots, [x}_{p \text{ times}}, -]\cdots]]]$$

in an associative algebra (cf. loc.cit.).

In the PBW and CMM theorems the forgetful functor is replaced by the functor As-alg $\to$ p-Lie-alg where the bracket is as usual and the Frobenius is the iterated bracket as above. Since this functor admits a left adjoint U all the ingredients are in place for a structure theorem in that case (cf. loc.cit.).

3.7.2. Operads in characteristic p. Since the Frobenius is not a linear operation (it is polynomial of degree p), a p-restricted Lie algebra is not an algebra over some operad in the sense of 1.2.2. Note that, in our definition of operad, we defined the Schur functor $\mathcal{P}(V) := \bigoplus_n \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$ by using the coinvariants. If, instead, we had taken the invariants, then there would be no difference in characteristic 0, but it would be different in characteristic p. In short, there is a way to handle p-restricted Lie algebras in the operad framework by playing with the two different kinds of Schur functors. In fact B. Fresse showed in [**21**] how to work with any operad in characteristic p along this line. It gives rise to the notion of $\mathcal{P}$-algebra with *divided symmetries*. For instance, in the commutative case, it gives rise to the divided power operation. If the operad is regular the notions of $\mathcal{P}$-algebra and $\mathcal{P}$-algebra with divided symmetries are equivalent.

3.7.3. Structure theorem in characteristic p. Now we have all the ingredients to write down a structure theorem for generalized bialgebras in characteristic p, including a toy-model. I conjecture that such a theorem exists. In fact some cases, with $\mathcal{C} = As$ or Com, have already been proved, see [**60**] and the references in this paper.

Observe that there are two levels of difficulty. First write the general theorem and its proof, second handle explicit cases. Recall for instance that, for Poisson algebras, we have to work with two divided operations: the Frobenius operation and the divided power operation. The relationship between these two are quite complicated formulas (cf. [**23**]).

3.8. Relationship with rewriting systems

The rewriting theory aims at computing a monoid (or a group) starting from a presentation. The idea is to write any relation under the form $u_1 \dots u_k = v_1 \dots v_l$ and to think of it as a "rewriting procedure" $u_1 \dots u_k \to v_1 \dots v_l$. In this setting one can define the notions of convergence, confluence, noetherianity, critical peak. There is a way to extend the rewriting theory to operads and even props, see Y. Lafont [**36**]. For instance a distributive compatibility relation like $\delta \circ \mu = \Phi$ (cf. 2.1.1 can be thought of as a rewriting procedure $\delta \circ \mu \to \Phi$. The aim is to find a *reduced form* for the multivalued operations. In this setting Hypothesis `(H1)` can be proved by verifying that the rewriting systems is convergent (cf. loc.cit.). See 4.8 for an example taken out of [**35**].

3.9. Application to representation theory

Given an $\mathbf{S}$-module $\mathcal{A}$ and a sub $\mathbf{S}$-module $\mathcal{P}$ it is usually difficult to decide whether there exists an $\mathbf{S}$-module $\mathcal{Z}$ such that $\mathcal{A} = \mathcal{Z} \circ \mathcal{P}$ (recall that in this framework the composition $\circ$ is called the *plethysm*). We will show that, in certain cases, we can give a positive answer to this question.

3.9.1. Proposition. *Let $\mathcal{A}$ be an operad and let $\mathcal{P}$ be a suboperad of $\mathcal{A}$. The following condition is sufficient to ensure that there is an isomorphism of $\mathbf{S}$-modules $\mathcal{A} \cong \mathcal{Z} \circ \mathcal{P}$, where $\mathcal{Z} := \mathcal{A}/(\overline{\mathcal{P}})$.*

Condition: there exists an operad $\mathcal{C}$ and a good triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ giving rise to the inclusion $\mathcal{P} \subset \mathcal{A}$.

Proof. If $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is good, then so is $(\mathcal{C}, \mathcal{A}/(\overline{\mathcal{P}}), Vect) = (\mathcal{C}, \mathcal{Z}, Vect)$ by Proposition 3.1. So we get isomorphisms of $\mathbf{S}$-modules $\mathcal{C}^c \cong \mathcal{Z}$ as $\mathbf{S}$-modules and $\mathcal{A} \cong \mathcal{C}^c \circ \mathcal{P}$ (Proposition 2.6.4), which imply $\mathcal{A} \cong \mathcal{Z} \circ \mathcal{P}$. □

CHAPTER 4

Examples

The problem of determining if a triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$, or more accurately $(\mathcal{C}, \between, \mathcal{A}, F, \mathcal{P})$, is good may crop up in different guises. Most of the time the starting data is the prop $(\mathcal{C}, \between, \mathcal{A})$, that is the type of bialgebras. Verifying (H0) is, most of the time, immediate by direct inspection. The first problem is to verify the hypotheses (H1) and (H2epi) . The second problem (and often the most difficult) is to find a small presentation of the operad $\mathcal{P} = \mathrm{Prim}_{\mathcal{C}}\mathcal{A}$ and make explicit the functor $F : \mathcal{A}$-alg $\to \mathcal{P}$-alg.

Another kind of problem is to start with a forgetful functor $F : \mathcal{A}$-alg $\to$ $\mathcal{P}$-alg (i.e. $\mathcal{P}$ is a suboperad of $\mathcal{A}$) and to try to find $\mathcal{C}$ and $\between$ so that $(\mathcal{C}, \between, \mathcal{A}, F, \mathcal{P})$ is a good triple.

In both problems Corollary 2.6.5 relating the generating series of $\mathcal{C}, \mathcal{A}$ and $\mathcal{P}$ is a good criterion since the knowledge of two of the operads determines uniquely the generating series of the third.

As said in the introduction the (uni)versal idempotent e is a very powerful tool. In Chapter 3 it is constructed abstractly. To get it explicitly in a given example is often a challenge.

In this chapter we present several concrete cases. For many of them, existing results in the literature permits us to prove the hypotheses and to find a small presentation of the primitive operad. In some cases the technique is very close to the *rewriting techniques* in computer sciences. Proposition 3.1.1 is quite helpful in proving that $(\mathcal{C}, \mathcal{A}, \mathrm{Prim}_{\mathcal{C}}\mathcal{A})$ is a good triple since it reduces several cases to one case. For instance when $\mathcal{A}$ is generated by one operation, it suffices to prove the hypotheses for the prop $\mathcal{C}^c$-Mag.

We have seen that any good triple $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ gives rise to a triple of the form $(\mathcal{C}, \mathcal{Z}, Vect)$ (the quotient triple) by moding out by the primitive operad $\mathcal{P}$. We put in the same section the triples which have the same quotient triple (with a few exceptions).

We give only the proofs of the statements which are not already in the literature.

In section 1 we treat $(Com, Com, Vect)$ with Hopf compatibility relation. It includes the classical case (Com, As, Lie) as well as $(Com, Parastat, Nil)$ and $(Com, Mag, Sabinin)$. The rigidity theorem for $(Com, Com, Vect)$ is the Hopf-Borel theorem and the structure theorem for (Com, As, Lie) is the union of the CMM theorem and the PBW theorem. We prove that, in this classical case, the universal idempotent in precisely the well-known Eulerian idempotent.

In section 2 we treat $(As, As, Vect)$ with unital infinitesimal compatibility relation. It includes the case $(As, Mag, MagFine)$ and the case $(As, 2as, \mathbf{B}_\infty)$ which is important because the category of cofree Hopf algebras is equivalent to the category of $\mathbf{B}_\infty$-algebras. The triple (As, OU, Mag) where a OU-algebra is a space equipped with two associative operations satisfying further the relation

$$(x \succ y) \prec z = x \succ (y \prec z),$$

should be in this section. It will be treated in full detail in the next Chapter.

In section 3 we treat $(As, Zinb, Vect)$ with semi-Hopf compatibility relation. It includes the case $(As, Dipt, \mathbf{B}_\infty)$ and the case $(As, Dend, brace)$ (due to María Ronco [**68**]) which is important since it permits us to unravel the structure of a free brace algebra.

In section 4 we treat $(Lie, Lie, Vect)$. It should be noted that the notion of Lie^c-Lie-bialgebra is NOT what is commonly called Lie bialgebras because the compatibility relation is different. In particular there is a nontrivial Φ_1 term in our case (cf. 2.1.2).

In section 5 we treat $(NAP, PreLie, Vect)$ due to M. Livernet [**40**] and the triple $(NAP, Mag, \mathrm{Prim}_{NAP}Mag)$.

In section 6 we describe several cases of the form $(\mathcal{A}, \mathcal{A}, Vect)$.

In section 7 we describe the interchange bialgebra case. Here the operads are no more quadratic but cubic.

In section 8 we treat a case where the generating operations and cooperations are of arity k.

When there is no Φ_1 term in the compatibility relation(s) (see 2.1.2), every operation is a primitive operation and there is nothing to prove. This is why we do not treat the Frobenius case.

4.1. Hopf algebras: the classical case

In this section we treat several triples admitting the triple $(Com, Com, Vect)$ (Hopf-Borel) as a quotient triple. It includes the classical case (Com, As, Lie). The compatibility relation in these cases is the Hopf relation.

4.1.1. The Hopf compatibility relation. First, let us recall some elementary facts about unital associative algebras. The tensor product $A \otimes B$ of the two unital associative algebras A and B is itself a unital associative algebra with product given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ and with unit $1_A \otimes 1_B$. The free unital associative algebra over V is the tensor algebra

$$T(V) = \mathbb{K} \oplus V \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

whose product is the concatenation:

$$(v_1 \cdots v_p)(v_{p+1} \cdots v_n) = v_1 \cdots v_n.$$

Let $V \to T(V) \otimes T(V)$ be the map given by $v \mapsto v \otimes 1 + 1 \otimes v$. Since $T(V)$ is free, there is a unique extension as algebra homomorphism denoted

$$\Delta : T(V) \longrightarrow T(V) \otimes T(V) \ .$$

It is easy to show, from the universal property of the free algebra, that Δ is coassociative and cocommutative. The fact that Δ is an algebra morphism reads

$$\Delta(xy) = \Delta(x)\Delta(y) \ ,$$

which is the *classical Hopf compatibility relation*. Hence the tensor algebra, equipped with this comultiplication, is a classical cocommutative bialgebra. The map Δ can be made explicit in terms of shuffles, cf. [**42**].

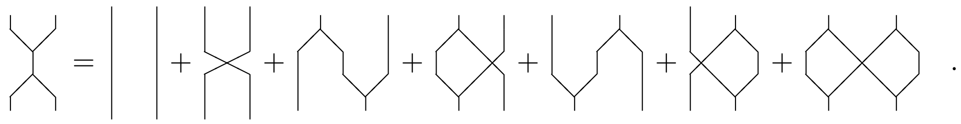
In order to work in a non-unital framework, we need to restrict ourself to the augmentation ideal of the bialgebra and to introduce the *reduced coproduct* δ

$$\delta(x) := \Delta(x) - x \otimes 1 - 1 \otimes x \ .$$

As already mentioned (cf. 2.1.3.1) the compatibility relation between the product μ and the coproduct δ becomes $\between_{Hopf}$:

$$\delta(xy) = x \otimes y + y \otimes x + \delta(x)(y \otimes 1 + 1 \otimes y) + (x \otimes 1 + 1 \otimes x)\delta(y) + \delta(x)\delta(y),$$

where $xy = \mu(x, y)$. Diagrammatically it reads:



Observe that this is a distributive compatibility relation.

4.1.2. The triple (Com, As, Lie)**.** By definition a Com^c-As-bialgebra is (in the non-unital framework) a vector space $\mathcal{H}$ equipped with a (nonunital) associative operation μ, a commutative associative comultiplication δ, satisfying the Hopf compatibility relation $\between_{Hopf}$. Obviously hypothesis `(H0)` is fulfilled. As we already mentioned in 2.1.3.1 a Com^c-As-bialgebra is equivalent to a classical bialgebra[1] by the map $\mathcal{H} \mapsto \mathcal{H}_+ := \mathbb{K}1 \oplus \mathcal{H}$.

The free As-algebra over V is the reduced tensor algebra

$$As(V) = \overline{T}(V) = V \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

equipped with the concatenation product. From the property of the tensor algebra recalled above, it follows that $\overline{T}(V)$ is a cocommutative bialgebra, in other words it satisfies the hypothesis `(H1)`. We claim that the operad $\mathcal{P}$ deduced from Theorem 2.5.1 is the operad *Lie* of Lie algebras. Indeed it is well-known that $\mathrm{Prim}\,\overline{T}(V) = Lie(V)$, cf. for instance [**75**] for a short proof.

The map $\varphi : As \to Com^c$ is given by $x_1 \cdots x_n \mapsto x_1 \cdots x_n$ in degree n, where on the left side we have a noncommutative polynomial, and on the

[1]Here we only deal with connected bialgebras for which an antipode always exists. So there is an equivalence between connected bialgebras and Hopf algebras

right side we have a commutative polynomial. In other words $\varphi_n : As(n) \to Com^c(n)$ is the map $\mathbb{K}[S_n] \to \mathbb{K}, \sigma \mapsto 1_{\mathbb{K}}$. This map has a splitting in characteristic zero, given by $x_1 \cdots x_n \mapsto \frac{1}{n!}\sum_{\sigma\in S_n}\sigma(x_1\cdots x_n)$. It is a coalgebra morphism for the coalgebra structure of $\overline{T}(V) = As(V)$ constructed above. Hence the hypothesis `(H2epi)` is fulfilled. So the triple (Com, As, Lie) is a good triple of operads and the structure Theorem holds for this triple. Translated in terms of unital-counital cocommutative bialgebras it gives:

4.1.3. Theorem (CMM+PBW). *Let $\mathbb{K}$ be a field of characteristic zero. For any cocommutative bialgebra $\mathcal{H}$ the following are equivalent:*
a) $\mathcal{H}$ is connected,
b) there is an isomorphism of bialgebras $\mathcal{H} \cong U(\mathrm{Prim}\,\mathcal{H})$,
c) there is an isomorphism of connected coalgebras $\mathcal{H} \cong S^c(\mathrm{Prim}\,\mathcal{H})$.

Here the functor U is the classical universal enveloping algebra functor from the category of Lie algebras to the category of unital associative algebras (or more accurately classical cocommutative bialgebras).

Of course, this is a classical result. In fact $(a) \Rightarrow (b)$ is the Cartier-Milnor-Moore Theorem which first appeared in Pierre Cartier's seminar lectures [**12**][2] and was later popularized by Milnor and Moore in [**57**].

Then $(b) \Rightarrow (c)$ is, essentially, the Poincaré-Birkhoff-Witt Theorem. In fact it is slightly stronger since it not only gives a basis of $U(\mathfrak{g})$ from a basis of the Lie algebra $\mathfrak{g}$ but it also provides an isomorphism of *coalgebras* $U(\mathfrak{g}) \to S^c(\mathfrak{g})$. This is Quillen's version of the PBW theorem, cf. [**65**] Appendix B. In this appendix Dan Quillen gives a concise proof of the PBW Theorem and of the CMM Theorem. The idempotent that he uses in his proof is the *Dynkin idempotent*

$$x_1 \dots x_n \mapsto \frac{1}{n}[\dots[[x_1, x_2], x_3], \dots, x_n].$$

We will show that the idempotent given by our proof (cf. 2.3.5) is the Eulerian idempotent (cf.[**66, 42**]).

Theorem 3.1.4 applied to the triple (Com, As, Lie) gives the most common version of the PBW Theorem:

$$gr\ U(\mathfrak{g}) \cong S(\mathfrak{g}).$$

Observe that the implication $(a) \Rightarrow (c)$ had been proved earlier by Jean Leray (cf. [**38**]), who had shown that the associativity hypothesis of the product was not necessary for this implication (see 4.1.12 for an explanation in terms of triples of operads).

For historical notes on the PBW theorem one may consult the paper by P.-P. Grivel [**29**].

[2] In this paper a bialgebra is called a "hyperalgebra".

4.1.4. Eulerian idempotent. [**42**] Let $\mathcal{H}$ be a connected cocommutative bialgebra (nonunital framework). The convolution of two linear maps $f, g : \mathcal{H} \to \mathcal{H}$ is defined as

$$f \star g := \mu \circ (f \otimes g) \circ \delta \ .$$

It is known that δ can be made explicit in terms of shuffles. By definition the (first) *Eulerian idempotent* $e^{(1)} : \mathcal{H} \to \mathcal{H}$ is defined as

$$e^{(1)} := \log^\star(uc + J) = J - \frac{J^{\star 2}}{2} + \frac{J^{\star 3}}{3} - \dots$$

where $J = \mathrm{Id}_{\mathcal{H}}$. For $\mathcal{H} = \overline{T}(V)$, the nonunital tensor algebra, $e^{(1)}$ sends $V^{\otimes n}$ to itself and we denote by $e_n^{(1)} : V^{\otimes n} \to V^{\otimes n}$ the restriction to $V^{\otimes n}$. Explicitly, it is completely determined by an element $e_n^{(1)} = \sum_\sigma a_\sigma \sigma \in \mathbb{Q}[S_n]$ since, by the Schur Lemma,

$$e_n^{(1)}(x_1, \dots, x_n) = \sum_\sigma a_\sigma (x_{\sigma(1)} \cdots x_{\sigma(n)}),$$

for some coefficients $a(\sigma)$ (here we let σ act on the right).

The higher Eulerian idempotents are defined as

$$e^{(i)} := \frac{(e^{(1)})^{\star i}}{i!} \ .$$

From the relationship between the exponential series and the logarithm series, it comes:

$$\mathrm{Id}_n = e_n^{(1)} + \cdots + e_n^{(n)} \ .$$

4.1.5. Proposition. *For any connected cocommutative bialgebra $\mathcal{H}$ the* versal idempotent e *is equal to the Eulerian idempotent:*

$$e := \Pi_{n\geq 2}(\mathrm{Id} - \omega^{[n]}) = \sum_{n\geq 1} (-1)^n \frac{J^{\star n}}{n} =: e^{(1)} \ .$$

Proof. It suffices to prove this equality for $\mathcal{H} = \overline{T}(V)$. From the definition of $\omega^{[n]}$ we get its expression in terms of shuffles. We get

$$\omega^{[n]} = e^{(n)} + e^{(n+1)} + \cdots \ .$$

Hence we deduce

$$\mathrm{Id} - \omega^{[n]} = e^{(1)} + \cdots + e^{(n-1)} \ .$$

Since the idempotents $e^{(i)}$ are orthogonal to each other (cf. [**42**]) we get

$$e = \Pi_{n\geq 2}(\mathrm{Id} - \omega^{[n]}) = e^{(1)}(e^{(1)} + e^{(2)})(e^{(1)} + e^{(2)} + e^{(3)})(\cdots)\cdots = e^{(1)}.$$

□

4.1.6. Explicit formula for the PBW isomorphism. Since the Eulerian idempotent can be computed explicitly in the symmetric group algebra, one can give explicit formulas for the isomorphism

$$T(V) \cong S^c(Lie(V)) \ .$$

In low dimension we get:

$$x = x,$$

$$xy = \frac{1}{2}[x,y] + \frac{1}{2}(xy + yx),$$

$$\begin{aligned} xyz &= \frac{1}{6}([[x,y],z] + [x,[y,z]]) \\ &+ \frac{1}{4}\big(x[y,z] + [y,z]x + y[x,z] + [x,z]y + z[x,y] + [x,y]z\big) \\ &+ \frac{1}{6}\sum_{\sigma \in S_3} \sigma(xyz) \ . \end{aligned}$$

4.1.7. Remark. In the case of classical bialgebras, not necessarily cocommutative, i.e. As^c-As-bialgebra with $\between = \between_{Hopf}$, the hypotheses (H0) and (H1) are also fulfilled. However the condition (H2epi) is not fulfilled, since the map $\varphi : \overline{T}(V) \to \overline{T}^c(V)$ factors through $\overline{S}^c(V)$. This is due to the cocommutativity of the coproduct on the free associative algebra.

4.1.8. The triple $(Com, Com, Vect)$**.** As mentioned in 3.1 if we mod out by relators in Lie, then we get a new triple of operads. For instance if we mod out by $\overline{Lie}$ (cf. 1.2.4 for the notation), then we get a good triple of operad:

$$(Com, Com, Vect).$$

In the unital framework the free commutative algebra $Com(V)$ is the symmetric algebra $S(V)$, which is the polynomial algebra $\mathbb{K}[x_1, \ldots, x_r]$ when $V = \mathbb{K}x_1 \oplus \ldots \oplus \mathbb{K}x_r$. Similarly the cofree coalgebra $Com^c(V)$ can be identified with $\mathbb{K}[x_1, \ldots, x_r]$ and the coproduct is given by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$. Under these identifications the map $\varphi(V) : Com(V) \to Com^c(V)$ is not the identity, but is given by $x_1^{i_1} \cdots x_r^{i_r} \mapsto \frac{x_1^{i_1}}{i_1!} \cdots \frac{x_r^{i_r}}{i_r!}$. This phenomenon can be phrased differently as follows. On the vector space of polynomials in one variable one can put two different commutative algebra structures:

$$(I) \qquad x^p x^q := x^{p+q},$$

$$(II) \qquad x^p x^q := \binom{p+q}{p} x^{p+q},$$

where $\binom{p+q}{p}$ is the binomial coefficient $\frac{(p+q)!}{p!q!}$. Of course, over a characteristic zero field, they are isomorphic ($x^n \mapsto \frac{x^n}{n!}$). By dualization we obtain two coalgebra structures (I^c) and (II^c). In order to make $\mathbb{K}[x]$ into a Hopf algebra we have to combine either (I) and (II^c) or (II) and (I^c).

The rigidity theorem for the cocommutative commutative connected bialgebras is the classical Hopf-Borel theorem recalled in the introduction of 2.3. Let us recall that the classical version (the one which is used in algebraic topology) is phrased in the graded framework (cf. 3.5). Here we gave the claim in the nongraded framework.

4.1.9. The triple $(Com, Parastat, NLie)$**.** Let us start with the triple (Com, As, Lie) and quotient by the relation

$$[[x, y], z] = 0.$$

Since the elements $[[x, y], z]$ are primitive we can apply Proposition 3.1. The new algebra type, that is associative algebras for which this relation holds, is called *parastatistics algebras*. The primitive type is simply nilpotent algebras whose product is antisymmetric. The structure theorem was proved in [**47**] (it follows easily from the classical one). This triple is interesting on two grounds. First, the parastatistics algebras (and their sign-graded version) appear naturally in theoretical physics. Second, the parastatistics operad is interesting from a representation theory point of view because $Parastat(n)$ is the sum of *one* copy of each irreducible type of S_n-representations.

4.1.10. The triple $(Com, Mag, Sabinin)$**.** Let Com^c-Mag be the magmatic cocommutative bialgebra type. The compatibility relation is the Hopf compatibility relation. Let us recall that a *magmatic algebra* is a vector space equipped with a binary operation, without further hypothesis. This is the nonunital case. In the unital case we assume further that there is an element 1 which is a unit on both sides. It is easy to show that the free magmatic algebra can be equipped with a cocommutative cooperation as follows. Working in the unital framework we put on the tensor product of two unital magmatic algebras a unital magmatic structure by

$$(a \otimes b) \cdot (a' \otimes b') = (a \cdot a') \otimes (b \cdot b') \text{ and } 1_{A\otimes B} = 1_A \otimes 1_B.$$

So the free unital magmatic algebra $Mag_+(V)$ tensored with itself is still a unital magmatic algebra. There is a unique morphism $Mag_+(V) \to Mag_+(V) \otimes Mag_+(V)$ which extends the map

$$V \to Mag_+(V) \otimes Mag_+(V), v \mapsto v \otimes 1 + 1 \otimes v.$$

This cooperation is immediately seen to be coassociative and cocommutative. Restricting the whole structure to the augmentation ideal gives a Com^c-Mag structure on the free magmatic algebra $Mag(V)$. Hence hypothesis `(H1)` holds. Explicitly the cooperation δ is given by

$$\delta(t; v_1, \ldots, v_n) = \sum_{i=1}^{n-1} \sum_{\sigma} (t^{\sigma}_{(1)}; v_{\sigma(1)}, \ldots, v_{\sigma(i)}) \otimes (t^{\sigma}_{(2)}; v_{\sigma(i+1)}, \ldots, v_{\sigma(n)})$$

where σ is an $(i, n-i)$-shuffle and the trees $t^{\sigma}_{(1)}$ and $t^{\sigma}_{(2)}$ are subtrees of t corresponding to the shuffle decomposition (cf. [**33**]).

A coalgebra splitting s to $\varphi(V)$ is obtained by

$$s(v_1 \dots v_n) = \sum_{\sigma \in S_n} \frac{1}{n!}(\mathrm{comb}^l_n; v_{\sigma(1)} \dots v_{\sigma(n)})$$

where comb^l_n is the left comb (cf. 5.1.2). So Hypothesis `(H2epi)` is fulfilled. Hence the structure theorem holds for Com^c-Mag-bialgebras. It was first proved by R. Holtkamp in [**33**]. Earlier studies on this case can be found in the pioneering work of M. Lazard [**37**] in terms of "analyseurs" and also in [**24**].

4.1.11. Sabinin algebras. The problem is to determine explicitly the primitive operad $\mathrm{Prim}_{Com}Mag$. Results of Shestakov and Urmibaev [**70**] and of Pérez-Izquierdo [**62**] show that it is the *Sabinin* operad. A *Sabinin algebra* can be defined as follows (there are other presentations). The generating operations are:

$$\langle x_1, \dots, x_m; y, z\rangle, \quad m \geq 0,$$

$$\Phi(x_1, \dots, x_m; y_1, \dots, y_n), \quad m \geq 1, n \geq 2,$$

with symmetry relations

$$\langle x_1, \dots, x_m; y, z\rangle = -\langle x_1, \dots, x_m; z, y\rangle,$$

$$\Phi(x_1, \dots, x_m; y_1, \dots, y_n) = \Phi(\omega(x_1, \dots, x_m); \theta(y_1, \dots, y_n)), \quad \omega \in S_m, \theta \in S_n,$$

and the relations are

$$\langle x_1, \dots, x_r, u, v, x_{r+1}, \dots, x_m; y, z\rangle - \langle x_1, \dots, x_r, v, u, x_{r+1}, \dots, x_m; y, z\rangle$$
$$+ \sum_{k=0}^{r} \sum_{\sigma} \langle x_{\sigma(1)}, \dots, x_{\sigma(k)}; \langle x_{\sigma(k+1)}, \dots, x_{\sigma(r)}; u, v\rangle, x_{r+1}, \dots, x_m; y, z\rangle$$

where σ is a $(k, r-k)$-shuffle,

$$K_{x,y,z}\big(\langle x_1, \dots, x_r, x; y, z\rangle +$$
$$\sum_{k=0}^{r} \sum_{\sigma} \langle x_{\sigma(1)}, \dots, x_{\sigma(k)}; \langle x_{\sigma(k+1)}, \dots, x_{\sigma(r)}; y, z\rangle, x\rangle\big) = 0$$

where $K_{x,y,z}$ is the sum over all cyclic permutations.

Observe that there is no relation between the generators $\langle -; -\rangle$ and the generators Φ. The functor $F : Mag\text{-alg} \to Sabinin\text{-alg}$ is constructed explicitly in [**70**] (also recalled in [**62**]). For instance $\langle y, z\rangle = -y \cdot z + z \cdot y$ and $(x \cdot y) \cdot z - (x \cdot y) \cdot z = -\frac{1}{2}\langle x; y, z\rangle + \Phi(x; y, z)$. It is easy to check that in $Mag(V)$ the two operations "bracket" and "associator" are not independent but related by the *nonassociative Jacobi identity*(cf. [**33**]):

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = \sum_{\mathrm{sgn}(\sigma)\sigma \in S_3} \sigma \; as(x, y, z) \; .$$

In the preceding presentation it corresponds to the cyclic relation with $r = 0$.

So, there is a good triple of operads

$$(Com, Mag, Sabinin)\ .$$

As a consequence, the generating series is $f^{Sab}(t) = \log(1 + (1/2)(1 - \sqrt{1-4t}))$ and the dimension of the space of n-ary operations is

$$\dim Sab(n) = 1, 1, 8, 78, 1104, \dots\ .$$

4.1.12. Remarks. Since the Com^c-Mag-bialgebra type satisfies `(H0)` , `(H1)` and `(H2epi)` , any connected bialgebra is cofree. This result has been proved earlier by Jean Leray in [**38**]. See also [**59, 20**] for a different generalization.

4.1.13. Quotients of As^c-Mag. The classical type (Com, As, Lie) is a quotient of the triple $(Com, Mag, Sabinin)$ (quotient by the associator, which is a primitive element and apply Proposition 3.1). In fact we have the following commutative diagram of operads:

$$\begin{tikzcd}
Sabinin \arrow[r, two heads] \arrow[d] & Lie \arrow[r, two heads] \arrow[d] & NLie \arrow[r, two heads] \arrow[d] & Vect \arrow[d] \\
Mag \arrow[r, two heads] & As \arrow[r, two heads] & Parastat \arrow[r, two heads] & Com
\end{tikzcd}$$

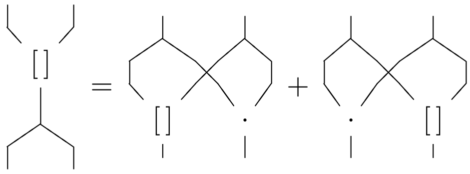
It may be worth to study other quotients of Mag by an ideal J (for instance $PreLie$ since $\langle x; y, z\rangle$ is the pre-Lie relator) and find a small presentation of the quotient $Sabinin/(J)$. Some results in this direction have been done for $Malcev$-algebras in [**63**]. It fits into this framework, since the Malcev relators:

$$x \cdot y - y \cdot x$$

$$((x \cdot y) \cdot z) \cdot t + (x \cdot (y \cdot z)) \cdot t - (x \cdot y) \cdot (z \cdot t) + x \cdot ((y \cdot z) \cdot t) + x \cdot (y \cdot (z \cdot t))$$

are primitive in $Mag(V)$.

4.1.14. Poisson bialgebras. Let us mention that there is a notion of Hopf-Poisson algebras (see for instance [**19**]), that is As^c-$Pois$-bialgebra and also cocommutative Hopf-Poisson algebras, that is Com^c-$Pois$-bialgebra. The compatibility relation for the pair $(\delta, \cdot)$ (where $a \cdot b$ is the commutative operation) is Hopf and the compatibility relation for the pair $(\delta, [\,])$ (where $[\,]$ is the Lie bracket) is given by



.

As in the classical case there is a good triple $(Com, Pois, Lie)$ since it is well-known that the free Poisson algebra $Pois(V)$ is precisely $Com(Lie(V))$.

4.1.15. A conjectural triple $(Com, ??, preLie)$**.** In [**14**] F. Chapoton and M. Livernet showed that the symmetric algebra over the free pre-Lie algebra in one generator can be identified with the dual of the Connes-Kreimer Hopf algebra [**16**]. The study of this case, as done in [**30**] for instance, suggests the existence of a triple of the form $(Com, \mathcal{A}, preLie)$ where the operad $\mathcal{A}$ is the unknown. It is expected that $\mathcal{A}(\mathbb{K}) \cong \mathcal{H}_{CK}^{*}$ and that this triple admits (Com, As, Lie) as a quotient. It is also interesting to remark that $\dim \mathcal{A}(n)$ should be equal to $(n+1)^{n-1}$ which is also $\dim Park(n)$ (parking functions).

4.2. Unital infinitesimal bialgebras

In this section we study some triples $(As, \mathcal{A}, \mathcal{P})$ which are over $(As, As, Vect)$ with compatibility relation the unital infinitesimal relation.

4.2.1. The unital infinitesimal compatibility relation. On the tensor algebra $T(V)$ the product is the concatenation product μ. Let us equip it with the deconcatenation coproduct given by

$$\Delta(v_1 \cdots v_n) := \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n.$$

The pair (Δ, μ) does not satisfy the Hopf compatibility relation, but does satisfy another relation:

$$\Delta(xy) = -x \otimes y + x_{(1)} \otimes x_{(2)} y + x y_{(1)} \otimes y_{(2)},$$

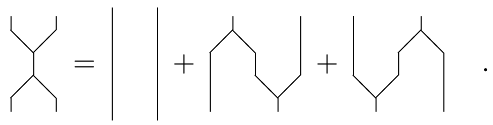
where $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

Since we want to work in the nonunital framework, we need to introduce the reduced coproduct δ defined by the equality $\Delta(x) = x \otimes 1 + 1 \otimes x + \delta(x)$.

The compatibility relation for the pair (δ, μ) is the *unital infinitesimal* (u.i.) compatibility relation $\between_{ui}$:

$$\delta(xy) = x \otimes y + \delta(x)(1 \otimes y) + (x \otimes 1)\delta(y),$$

Diagrammatically it reads (cf. 2.1 example 2):



4.2.2. The triple $(As, As, Vect)$**.** By definition a *unital infinitesimal bialgebra* ($(As^c$-$As)$-bialgebra) is a vector space equipped with an associative operation and a coassociative cooperation satisfying the u.i. compatibility relation. Hypothesis `(H0)` is obviously fulfilled. From the above discussion it follows that hypothesis `(H1)` is also fulfilled.

The next Proposition shows that hypothesis `(H2iso)` is fulfilled. So we get the rigidity theorem for the triple $(As, As, Vect)$. It was first announced in [**49**] and proved in [**51**] where details can be found. Let us just recall that the universal idempotent in this case is given by the geometric series:

$$e = \sum_{n\geq 1} (-1)^{n-1}\mathrm{id}^{\star n}$$

where $\star$ is the convolution product $f \star g := \mu \circ (f \otimes g) \circ \delta$.

4.2.3. PROPOSITION. *The map $\varphi(V) : \overline{T}(V) \to \overline{T}(V)^c$ is induced by the identification of the generator of As_n with its dual.*

Proof. The prop As^c-As is regular and As_n is one-dimensional. Let μ_n denote the generator of As_n, so $\mu_n(x_1, \ldots, x_n) = x_1 \ldots x_n$. In order to compute its image by φ it suffices to compute $\delta_n \circ \mu_n(x_1, \ldots, x_n)$ where δ_n is the dual of μ_n, that is the generator of As^c_n (cf. 2.5.7). From the compatibility relation we get

$$\delta_n \circ \mu_n(x_1, \ldots, x_n) = x_1 \otimes \ldots \otimes x_n \in V^{\otimes n} \subset As(V)^{\otimes n}$$

where $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$. Hence $\varphi(V)(\mu_n) = \delta_n$ and φ is an isomorphism. $\square$

4.2.4. The triple $(As, Mag, MagFine)$**.** This triple has been studied and proved to be good in [**34**]. The compatibility relation is $\between_{ui}$. The coproduct δ on $Mag(V) = \oplus_{n\geq 1}\mathbb{K}[Y_{n-1}] \otimes V^{\otimes n}$, where Y_n is the set of planar binary trees with n internal vertices (cf. 5.1.2, can be constructed as follows. Let t be p.b. trees whose leaves are numbered from left to right beginning at 0. We cut the tree along the path going from the ith vertex (standing between the leaves $i-1$ and i) to the root. It gives two trees denoted $t^i_{(1)}$ and $t^i_{(2)}$.We have

$$\delta(t; v_0 \cdots v_n) = \sum_{i=1}^{n} (t^i_{(1)}; v_0 \cdots v_{i-1}) \otimes (t^i_{(2)}; v_i \cdots v_n) \ .$$

Hypothesis `(H1)` can be proved either by using the explicit form of δ or by an inductive argument as explained in 2.5.7. The map $\varphi(V) : Mag(V) \to As^c(V)$ sends t to the generator 1_n of As^c_n. We choose the splitting $s_n : As_n \to Mag_n$ given by $s(1_n) = comb^l_n$ (left comb). So Hypothesis `(H2epi)` is fulfilled and we have a good triple of operads:

$$(As, Mag, \mathrm{Prim}\,_{As}Mag) \ .$$

It is proved in [**34**] that the primitive operad $\mathrm{Prim}\,_{As}Mag$ is generated by $n-2$ operations in arity n and that they satisfy no relations. So this operad is a magmatic operad (free operad). Since the dimension of $(\mathrm{Prim}\,_{As}Mag)_n$ is the Fine number, it is called the *magmatic Fine operad*, denoted $MagFine$. So there is a good triple of operads

$$(As, Mag, MagFine) \ .$$

4.2.4.1. *Relationship with previous work.* As a byproduct of the structure theorem for As^c-Mag-bialgebras we have that a connected coassociative algebra equipped with a magmatic operation satisfying the u.i. relation is cofree. Dually we have the following: an associative algebra equipped with a comagmatic operation, which is connected and satisfies the u.i. relation is free. A very similar result has been shown by I. Berstein in [**5**], who proved that a cogroup (in fact comonoid) in the category of associative algebras in free. M. Oudom remarked in [**59**] that coassociativity of the cooperation is not even necessary to prove the freeness. The difference with our case is in the compatibility relation. See also [**4**] for similar results in this direction.

4.2.4.2. *Quotient triples.* Of course the quotient triple of $(As, Mag, MagFine)$ is $(As, As, Vect)$. It would be interesting to find a small presentation of the intermediate quotient by the pre-Lie relator

$$\langle x; y, z\rangle := (x \cdot y) \cdot z - x \cdot (y \cdot z) - (x \cdot z) \cdot y + x \cdot (z \cdot y) \ ,$$

which gives the good triple

$$\big(As, PreLie, MagFine/(\langle x; y, z\rangle)\big) \ .$$

4.2.5. The triple $(As, 2as, \mathbf{B}_\infty)$**.** (cf. [**51**]). By definition a *2-associative algebra* or $2as$-algebra for short, is a vector space A equipped with two associative operations denoted $a \cdot b$ and $a * b$. In the unital case we assume that 1 is a unit for both operations. By definition a As^c-$2as$*-bialgebra* is a $2as$-algebra equipped with a coassociative cooperation δ, whose compatibility relations are as follows:

a) $\cdot$ and δ satisfy the u.i. compatibility relation (cf. 4.2.2),

b) $*$ and δ satisfy the Hopf compatibility relation (cf. 4.1).

The free $2as$-algebra can be explicitly described in terms on planar trees and on can show that it has a natural As^c-$2as$-bialgebra structure. Hypotheses `(H0)` , `(H1)` and `(H2epi)` are also easy to check (cf. loc.cit.) and so there is a good triple of operads $(As, 2as, \mathrm{Prim}_{As} 2as)$. It has been first proved in [**51**] where the primitive operad $\mathrm{Prim}_{As} 2as$ has be shown to be the operad of $\mathbf{B}_\infty$-algebras. This is a very important operad since there is an equivalence between the category of $\mathbf{B}_\infty$-algebras and the category of cofree Hopf algebras.

Let us give some details on this equivalence. By definition a $\mathbf{B}_\infty$*-algebra* (cf. [**25, 51**]) is a vector space A equipped with $(p+q)$-ary operations M_{pq} satisfying some relations. Let $T^c(A)$ be the cofree counital coalgebra on A. Let

$$* : T^c(A) \otimes T^c(A) \to T^c(A)$$

be the unique coalgebra morphism which extends the operations $M_{pq} : A^{\otimes p} \otimes A^{\otimes q} \to A$. Then the relations satisfied by the operations M_{pq} imply that $(T^c(A), *, \mathit{deconcatenation})$ is a cofree Hopf algebra. In the other direction, any cofree Hopf algebra determines a $\mathbf{B}_\infty$-structure on its primitive part. The details are to be found in [**51**].

Hence we deduce that

$$(As, 2as, \mathbf{B}_\infty)$$

is a good triple of operads (first proved in loc.cit.). One of the outcome of this result was to give an explicit description of the free $\mathbf{B}_\infty$-algebra. Indeed the operad $2as$ can be explicitly described in terms of planar rooted trees. Thanks to this description and the structure theorem, one can describe the operad $\mathbf{B}_\infty$ in terms of trees (cf. loc.cit.). Observe that moding out by the primitives gives the triple $(As, As, Vect)$.

Since the functor As-alg $\to 2as$-alg admits an obvious splitting (forgetful map), we can use it to construct the splitting of φ. Hence the idempotent e is the same as in the case of the triple $(As, As, Vect)$. It was shown in [**51**] that the universal idempotent is given by the geometric series:

$$e = \mathrm{Id} - \mathrm{Id} \star \mathrm{Id} + \cdots + (-1)^{n-1}\mathrm{Id}^{\star n} + \cdots \ .$$

Here $\star$ stands for the convolution product.

4.2.6. The triple (As, As^2, As). An As^2-*algebra* is, by definition, a vector space A equipped with two operations denoted $a \cdot b$ and $a * b$ such that the associativity relation

$$(a \circ_1 b) \circ_2 c = a \circ_1 (b \circ_2 c)$$

holds for any value of $\circ_i$ (i.e. either equal to $\cdot$ or to $*$). In a As^c-As^2-bialgebra the compatibility relations are the u.i. relations for both pairs $(\delta, \cdot)$ and $(\delta, *)$. It is immediate to verify that

$$\mathrm{Prim}_{\,As} As^2 = As \ .$$

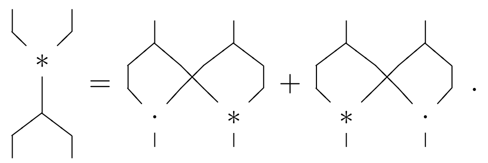
The associative product of the primitive operad is given by

$$x \circ y := x \cdot y - x * y \ .$$

Finally, we get a good triple of operads

$$(As, As^2, As) \ .$$

The operad As^2 appears in many places in the literature. For instance in [**28**] there is a notion of Com^c-As^2-bialgebra where the compatibility relation for the pair $(\delta, \cdot)$ is Hopf (unital version) and for the pair $(\delta, *)$ it is



4.2.7. The triple $(As, 2as, Mag^\infty)$. Consider As^c-$2as$-bialgebras with compatibility relations being both the u.i. compatibility relation. So this type of bialgebras is different from the one described in 4.2.5. It is immediate to check `(H0)` , `(H1)` and `(H2epi)` are fulfilled. So there is a good triple of operads $(As, 2as, \mathrm{Prim}\,_{As}2as)$. With some more work one can show that the primitive operad is the operad Mag^∞ which has one generating operation in each arity and no relation. So there is a good triple of operads

$$(As, 2as, Mag^\infty).$$

Observe that the triple (As, As^2, As) is a quotient of it.

4.3. Dendriform, dipterous and Zinbiel bialgebras

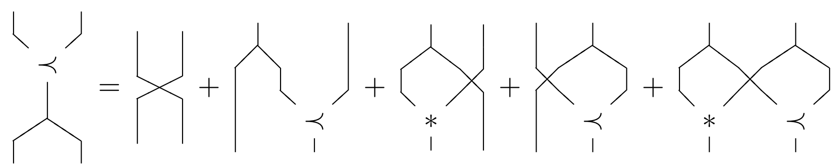
In this section we study some triples $(As, \mathcal{A}, \mathcal{A})$ which are over $(As, Zinb, Vect)$ with compatibility relation the semi-Hopf relation. Here $Zinb$ is the operad of Zinbiel algebras.

4.3.1. Zinbiel algebra and semi-Hopf compatibility relation. By definition a *Zinbiel algebra* is a vector space A equipped with an operation denoted $a \prec b$ satisfying the Zinbiel relation

$$(a \prec b) \prec c = a \prec (b \prec c + c \prec b).$$

We note immediately that the operation $a * b := a \prec b + b \prec a$ is associative (and commutative of course). The terminology comes from the fact that the Koszul dual operad is the operad of Leibniz algebras (cf. [**44**]).

By definition a As^c-*Zinb-bialgebra* is a Zinbiel algebra equipped with a coassociative cooperation δ, whose compatibility relation is *semi-Hopf*, denoted $\between^{l}_{semiHopf}$:



Observe that, as a consequence, the compatibility relation for the pair $(\delta, *)$ is Hopf (nonunital setting). It was obtained as a consequence of the semi-Hopf relation in the unital framework, given by

$$\Delta(x \prec y) = \Delta(x) \prec \Delta(y),$$

where the tensor product of the bialgebra with itself has been equipped with following Zinbiel structure:

$$(a \otimes b) \prec (a' \otimes b') = a * a' \otimes b \prec b',$$

whenever it is defined and

$$(a \otimes 1) \prec (a' \otimes 1) = a \prec a' \otimes 1,$$

otherwise (cf. M. Ronco [**67**] and 3.2.2). The behavior of $\prec$ with respect to the unit is given by

$$1 \prec x = 0, \qquad x \prec 1 = x \ .$$

The free Zinbiel algebra over V is the reduced tensor module $\overline{T}(V)$ and the relationship between the tensors and the Zinbiel algebra is given by

$$v_1 \cdots v_n = v_1 \prec (v_2 \prec (\cdots (v_{n-1} \prec v_n) \cdots)).$$

Explicitly, the Zinbiel product is given by the *half-shuffle*:

$$v_1 \dots v_p \prec v_{p+1} \cdots v_n = v_1 \sum_{\sigma \in SH(p-1,n-p)} \sigma(v_2 \cdots v_n)$$

where $SH(p-1, n-p)$ is the set of $(p-1, n-p)$-shuffles. As a consequence $(\overline{T}(V), *)$ is the (nonunital) shuffle algebra.

It can be shown that the free Zinbiel algebra satisfies both hypotheses `(H1)` and `(H2iso)` . It is a consequence of the work of M. Ronco [**67, 68**], see E. Burgunder [**8**] for a direct proof. Hence

$$(As, Zinb, Vect)$$

is a good triple of operads. This example is interesting because it shows that, for a certain algebraic structure $\mathcal{A}$, ($\mathcal{A} = As$ here), there can be several different coalgebraic structures $\mathcal{C}$ for which $(\mathcal{C}, \mathcal{A}, Vect)$ is a good triple. Here $\mathcal{C} = As$ or $Zinb$.

We can revert the roles of As and $Zinb$ in this example (cf. 3.6) and so there is a notion of $Zinb^c$-As-bialgebra. As a consequence

$$(Zinb, As, Vect)$$

is also a good triple (cf. 3.6).

4.3.2. Dipterous algebra. By definition a *dipterous algebra* is a vector space A equipped with two binary operations denoted $a * b$ and $a \prec b$ verifying the relations:

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y * z) \ , \\ (x * y) * z &= x * (y * z) \ . \end{aligned}$$

By definition a *As^c-Dipt-bialgebra* is a dipterous algebra equipped with a coassociative cooperation δ, whose compatibility relation is Hopf with $*$ and semi-Hopf with $\prec$. In fact one can put a unit on a dipterous algebra by requiring that 1 is a unit for $*$ and that

$$1 \prec a = 0, \quad a \prec 1 = a,$$

($1 \prec 1$ is not defined). This is a particular case of a multiplicative operad, see 3.2 and [**45**].

The free dipterous algebra can be described in terms of planar trees. It can be shown that the free dipterous algebra satisfies both hypotheses `(H1)` and (H2iso) (cf. [**49**]). The primitive operad was proven to be in loc.cit. the $\mathbf{B}_\infty$ operad (cf. 4.2.5). Hence

$$(As, Dipt, \mathbf{B}_\infty)$$

is a good triple.

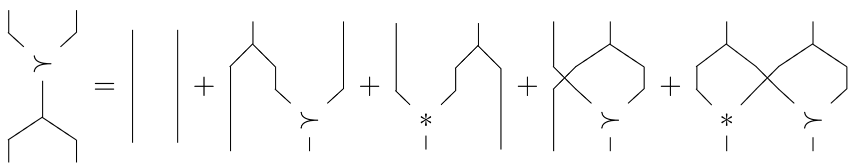
4.3.3. Dendriform algebra. A *dendriform algebra* A is determined by two binary operations $A \otimes A \to A$ called *left* $(a,b) \mapsto a \prec b$ and *right* $(a,b) \mapsto a \succ b$, satisfying the following three relations

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y * z)\ , \\ (x \succ y) \prec z &= x \succ (y \prec z)\ , \\ (x * y) \succ z &= x \succ (y \succ z)\ , \end{aligned}$$

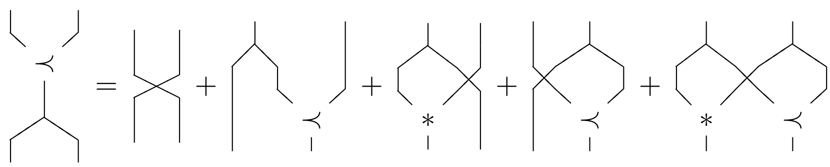
where $x * y := x \prec y + x \succ y$. From these axioms it follows that the operation $*$ is associative, hence a dendriform algebra is an example of dipterous algebra. The operad *Dend* is obviously regular. It has been shown in [**44**] that the associated operad *Dend* is such that $Dend_n = \mathbb{K}[Y_n]$ where Y_n is the set of planar binary rooted trees with $n+1$ leaves ($\#Y_n = \frac{(2n)!}{n!(n+1)!}$ is the Catalan number).

By definition a *As^c-Dend-bialgebra* is a dendriform algebra equipped with a coassociative cooperation δ, whose compatibility relations are as follows.

For the pair $(\delta, \succ)$ it is given by $\between^r_{semiHopf}$:



and for the pair $(\delta, \prec)$ it is given by $\between^l_{semiHopf}$:



Though *As* and *Dend* are regular, this prop is not regular because of the form of the compatibility relation. Observe that the sum of these two relations gives the Hopf compatibility relation for the pair $(\delta, *)$. Since *Dend* is a quotient of *Dipt* by the relation $(x \succ y) \prec z = x \succ (y \prec z)$ and since the operation $(x \succ y) \prec z - x \succ (y \prec z)$ in *Dipt* is primitive, it follows from Proposition 3.1 that $(As, Dend, \mathbf{B}_\infty /\sim)$ is a good triple. The quotient $\mathbf{B}_\infty /\sim$ is easy to compute, and it turns out to be the operad *Brace* of *brace algebras*, that we know recall.

The *Brace* operad admits one n-ary operation $\{-;-,\cdots,-\}$, for all $n\geq 2$, as generators and the relations are:

$$Br_{n,m}: \quad \{\{x;y_1,\ldots,y_n\};z_1,\ldots,z_m\} = \sum\{x;\ldots,\{y_1;\ldots\},\ldots,\{y_n;\ldots,\},\ldots\}.$$

On the right-hand side the dots are filled with the variables z_i's (in order) with the convention $\{y_k;\emptyset\}=y_k$. In a As^c-*Dend*-bialgebra the binary operation is given by

$$\{x;y\} := x \prec y - y \succ x$$

and the ternary operation is given by

$$\{x;y,z\} = x \prec (y \succ z) - y \succ x \prec z + (y \prec z) \succ x\ .$$

The first nontrivial relation, which relates the 2-ary operation and the 3-ary operation reads

$$Br_{1,1}: \qquad \{\{x;y\};z\} - \{x;\{y;z\}\} = \{x;y,z\} + \{x;z,y\}\ .$$

As a consequence we deduce that the associator of the 2-ary operation is right-symmetric:

$$\{\{x,y\},z\} - \{x,\{y,z\}\} = \{\{x,z\},y\} - \{x,\{z,y\}\}\ .$$

So the primitive part of a As^c-*Dend*-bialgebra is, in particular, a pre-Lie algebra.

It is easy to check that the quotient $\mathbf{B}_\infty/\sim$ is *Dend*, hence it follows that $(As, Dend, Brace)$ is a good triple of operads.

The Hopf structure of the free dendriform algebra was first constructed in [**48**]. See [**2**] for an alternative basis with nice behavior with respect to the coproduct. The primitive operad $\mathrm{Prim}_{As}Dend$ was first identified to be the brace operad by María Ronco in [**67**]. The structure theorem was proved in [**68**] and in [**13**]. It was the first example outside the classical framework and the one which motivated this theory.

If we mod out by the primitive operation $x \prec y - y \succ x$, then we get the good triple $(As, Zinb, Vect)$.

4.3.4. Tridendriform algebra. The notion of dendriform algebra admits several generalizations. One of them is the notion of *tridendriform algebra* (originally called dendriform trialgebra in [**50**]). It has three generating operations denoted $\prec$ (left), $\succ$ (right), and $\cdot$ (dot or middle). They satisfy the following 11 relations (one for each cell of the pentagon):

$$
\begin{aligned}
(x \prec y) \prec z &= x \prec (y * z) \ , \\
(x \succ y) \prec z &= x \succ (y \prec z) \ , \\
(x * y) \succ z &= x \succ (y \succ z) \ , \\
(x \succ y) \cdot z &= x \succ (y \cdot z) \ , \\
(x \prec y) \cdot z &= x \cdot (y \succ z) \ , \\
(x \cdot y) \prec z &= x \cdot (y \prec z) \ , \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z) \ ,
\end{aligned}
$$

where $x * y := x \prec y + x \succ y + x \cdot y$.

The operad *Tridend* is obviously binary, quadratic and regular. The free tridendriform algebra on one generator has been shown to be linearly generated by the set of all planar rooted trees in [**50**].

Using the existence of a partial unit one can put a structure of As^c-*Tridend*-bialgebra structure on $Tridend(V)$ as in [**45**]. The coefficients α and β (cf. 3.2.2) are given by:

$$x \prec 1 = x = 1 \succ x, \text{ and } 1 \prec x = x \succ 1 = 1 \cdot x = x \cdot 1 = 0 \ .$$

These choices are coherent with the operad structure of *Dend* and therefore, by [**45**] (see also 3.2.2), there is a well-defined notion As^c-*Dend*-bialgebra for which the hypotheses `(H0)` and `(H1)` are fulfilled.

Hypothesis `(H2epi)` is easy to check, and therefore the triple

$$(As, Tridend, \mathrm{Prim}_{\,As}Tridend)$$

is good. The operad $\mathrm{Prim}_{\,As}Tridend$ can be described explicitly as a mixture of the brace structure and the associative structure, cf. [**61**].

One of the interesting points about the good triple $(As, Tridend, Brace+As)$ is its quotient (As, CTD, Com), where CTD stands for the Commutative TriDendriform algebra operad. The commutativity property is

$$x \prec y = y \succ x, \text{ and } x \cdot y = y \cdot x \ .$$

Hence a CTD-*algebra* can be described by two generating operations $x \prec y$ and $x \cdot y$ (the second one being symmetric), satisfying the relations:

$$
\begin{aligned}
(x \prec y) \prec z &= x \prec (y * z) \ , \\
(x \cdot y) \prec z &= x \cdot (y \prec z) \ , \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z) \ .
\end{aligned}
$$

The good triple

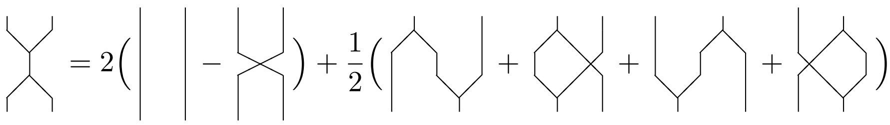
$$(As, CTD, Com)$$

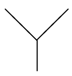
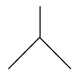
has been studied in [**46**] and shown to be strongly related with the quasi-shuffle algebras.

4.4. Lie^c-Lie-bialgebras

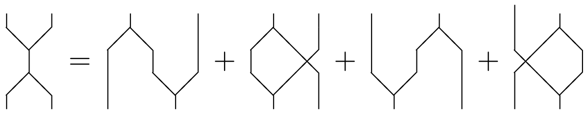
In this section we work over a characteristic zero field. We introduce the notion of Lie^c-Lie-bialgebra, different from the classical notion of Lie bialgebra, and we prove a rigidity theorem for Lie^c-Lie-bialgebras.

4.4.1. Definition. A *Lie^c-Lie-bialgebra* is a vector space A which is a Lie algebra for the bracket $[x, y]$, a Lie coalgebra for $\delta_{[]}$ and whose compatibility relation is $\between_{Lily}$:



Here  stands for the bracket $[-,-]$ and  stands for the cobracket $\delta_{[]}$.

Observe that the notion of Lie^c-Lie bialgebra is completely different from the notion of Lie bialgebras, since, in this latter case, the compatibility relation is the cocycle condition $\between_{biLie}$:



In particular for $\between_{Lily}$ there is a Φ_1-term, so there is chance for a rigidity theorem.

In order to show that the free Lie algebra $Lie(V)$ is equipped with a structure of Lie^c-Lie-bialgebra, we are going to use the tensor algebra $\overline{T}(V)$ for $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$. Hence $\overline{T}(V)$ is the space of noncommutative polynomials without constant term in the variables x_i's. The coproduct δ on $\overline{T}(V)$ is the deconcatenation coproduct (cf. 4.2.2). Recall that $Lie(V)$ is made of the Lie polynomials, that is the polynomials generated by the x_i's under the bracket operation. The degree of a homogeneous polynomial X is denoted $|X|$. We use the involution $X \mapsto \overline{X}$ on $\overline{T}(V)$ which is the identity on the x_i's and satisfies $\overline{(XY)} = \overline{Y}\ \overline{X}$.

4.4.2. LEMMA. *If $X \in Lie(V)$, then $\overline{X} = -(-1)^{|X|}X$ and*

$$2\delta(X) = X_1 \otimes X_2 - (-1)^{p+q}\overline{X}_2 \otimes \overline{X}_1,$$

where $\delta(X) =: X_1 \otimes X_2$ and $p = |X_1|, q = |X_2|$.

Proof. The proof is by induction on the degree n of X. We assume that these formulas are true for X, and then we prove them for $[X, z]$ where z is of degree 1.

For the first formula we get

$$\overline{[X,z]} = \overline{(Xz - zX)} = z\overline{X} - \overline{X}z = -(-1)^n([z,X]) = -(-1)^{n+1}[X,z],$$

as expected.

For the second formula, the u.i. compatibility relation and the induction hypothesis give:

$$\begin{aligned} \delta([X,z]) &= \delta(Xz - zX) \\ &= X \otimes z + X_1 \otimes X_2 z - z \otimes X - zX_1 \otimes X_2 \\ &= X \otimes z - z \otimes X + \frac{1}{2}\Big(X_1 \otimes X_2 z - zX_1 \otimes X_2 \\ &\qquad -(-1)^{p+q}\overline{X}_2 \otimes \overline{X}_1 z + (-1)^{p+q} z\overline{X}_2 \otimes \overline{X}_1\Big). \end{aligned}$$

On the other hand we have

$$\begin{aligned} &[X,z]_1 \otimes [X,z]_2 - (-1)^{p+q}\overline{[X,z]_1} \otimes \overline{[X,z]_2} \\ &= \tfrac{1}{2}\big(X \otimes z - z \otimes X + X_1 \otimes X_2 z - zX_1 \otimes X_2 - (-1)^{n+1} z \otimes \overline{X} \\ &\quad +(-1)^{n+1}\overline{X} \otimes z - (-1)^{p+q+1} z\overline{X}_2 \otimes \overline{X}_1 + (-1)^{p+1+q}\overline{X}_2 \otimes \overline{X}_1 z\big). \end{aligned}$$

The two expresssions are equal, because, since X is a Lie polynomial, we have $X = -(-1)^n\overline{X}$. □

4.4.3. Proposition. *Let $\delta_{[,]} := \delta - \tau\delta$. The image of $Lie(V)$ by $\delta_{[,]}$ is in $Lie(V) \otimes Lie(V)$.*

Proof. The proof is by induction on the degree n of $X \in Lie(V)$. It is immediate for $n = 1$. Suppose that $X \in Lie(V), \delta(X) = X_1 \otimes X_2$ and $X_1, X_2 \in Lie(V)$. We observe that, by Lemma 4.4.2 we have

$$\delta(X) =: X_1 \otimes X_2 = \frac{1}{2}\big(X_1 \otimes X_2 - X_2 \otimes X_1\big).$$

We are going to show that, for any element z of degree 1, we have $\delta_{[,]}([X,z]) \in Lie(V) \otimes Lie(V)$. We compute:

$$\begin{aligned} \delta_{[,]}([X,z]) &= (\delta - \tau\delta)(Xz - zX) \\ &= X \otimes z - z \otimes X + X_1 \otimes X_2 z - zX_1 \otimes X_2 \\ &\quad -z \otimes X + X \otimes z - X_2 z \otimes X_1 + X_2 \otimes zX_1 \\ &= 2\big(X \otimes z - z \otimes X\big) \\ &\quad +\frac{1}{2}\big(X_1 \otimes X_2 z - zX_1 \otimes X_2 - X_2 z \otimes X_1 + X_2 \otimes zX_1 \\ &\quad -X_2 \otimes X_1 z + zX_2 \otimes X_1 + X_1 z \otimes X_2 - X_1 \otimes zX_2\big) \\ &= 2\big(X \otimes z - z \otimes X\big) \\ &\quad +\frac{1}{2}\big(X_1 \otimes [X_2,z] - [z,X_1] \otimes X_2 - [X_2,z] \otimes X_1 + X_2 \otimes [z,X_1]\big). \end{aligned}$$

So we have proved that $\delta_{[,]}([X,z]) \in Lie(V) \otimes Lie(V)$. □

4.4.4. Proposition. *On $Lie(V)$ the bracket operation $[x,y]$ and the bracket cooperation $\delta_{[,]}$ satisfy the compatibility relation $\between_{Lily}$ (cf. 4.4.1).*

Proof. Let $X, Y \in Lie(V)$. We compute $\delta_{[,]}([X,Y])$:

$$\begin{aligned}
\delta_{[,]}([X,Y]) &= (\delta - \tau\delta)(XY - YX) \\
&= \delta(XY) - \delta(YX) - \tau\delta(XY) + \tau\delta(YX) \\
&= +X \otimes Y + X_1 Y \otimes X_2 + XY_1 \otimes Y_2 \\
&\quad -Y \otimes X - Y_1 X \otimes Y_2 - YX_1 \otimes X_2 \\
&\quad -Y \otimes X - X_2 \otimes X_1 Y - Y_2 \otimes XY_1 \\
&\quad +X \otimes Y + Y_2 \otimes Y_1 X + X_2 \otimes YX_1 \\
&= 2\big(X \otimes Y - Y \otimes X\big) \\
&\quad +[X_1, Y] \otimes X_2 + [X, Y_1] \otimes Y_2 + X_1 \otimes [X_2, Y] + Y_1 \otimes [X, Y_2] \\
&= 2\big(X \otimes Y - Y \otimes X\big) + \frac{1}{2}\Big([X_{[1]}, Y] \otimes X_{[2]} \\
&\quad +[X, Y_{[1]}] \otimes Y_{[2]} + X_{[1]} \otimes [X_{[2]}, Y] + Y_{[1]} \otimes [X, Y_{[2]}]\Big).
\end{aligned}$$

Observe that, in this computation, we have used the fact that, for any element $Z \in Lie(V)$, the element $\delta(Z) = Z_1 \otimes Z_2$ is antisymmetric, that is $Z_1 \otimes Z_2 = -Z_2 \otimes Z_1$ (cf. 4.4.2). As a consequence we have $\frac{1}{2}\delta_{[,]}(Z) = \delta(Z)$. □

4.4.5. THEOREM. *In characteristic zero, the prop Lie^c-Lie satisfies the hypotheses* `(H0) (H1) (H2iso)` *, therefore the triple* $(Lie, Lie, Vect)$ *(with* $\between = \between_{Lily}$*) is a good triple. Hence any connected Lie^c-Lie-bialgebras is both free and cofree.*

Proof. It is clear that the compatibility relation $\between_{Lily}$ is distributive. By Propositions 4.4.3 and 4.4.4 the hypothesis `(H1)` is fulfilled. Let us prove `(H2iso)` .

We have seen in the last proof that $\delta_{[,]}(Z) = 2\delta(Z)$ when $Z \in Lie(V)$. The cooperation δ induces the isomorphism map $\varphi_{As}(V) : As(V) \to As^c(V)$ which identifies the generator of As_n with its dual. Hence, restricted to $Lie(V)$ it is injective. So the map $\varphi_{Lie}(V) : Lie(V) \to Lie^c(V)$, induced by $\delta_{[,]}$ is injective. Since $Lie(n)$ and $Lie^c(n)$ have the same dimension, it is an isomorphism. □

4.4.6. The conjectural triple $(Lie, PostLie, \mathrm{Prim}\,{}_{As}PostLie)$**.** By definition, cf. [**73**] a *PostLie algebra* is a vector space A equipped with two operations $x \circ y$ and $[x,y]$ which satisfy the relations

$$[x,y] = -[y,x]$$
$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$
$$(x \circ y) \circ z - x \circ (y \circ z) - (x \circ z) \circ y + x \circ (z \circ y) = x \circ [y,z]$$
$$[x,y] \circ z = [x \circ z, y] + [x, y \circ z]$$

In particular a PostLie algebra is a Lie algebra for the bracket $[x,y]$. But it is also a Lie algebra for the operation $\{x,y\} := x \circ y - y \circ x + [x,y]$ (cf. loc.cit).

We conjecture that there exists a notion of Lie^c-$PostLie$ bialgebra such that the free PostLie algebra is such a bialgebra. Hopefully there is a good triple of operads $(Lie, PostLie, \mathrm{Prim}\,_{As}PostLie)$. The isomorphism of $PostLie = Lie \circ PBT$, where $PBT_n = \mathbb{K}[Y_{n-1}]$ proved in [**72**] is an evidence in favor of this conjecture. It is not clear what is the algebraic structure one should put on PBT to make it work (Mag is one option out of many).

4.5. NAP^c-$\mathcal{A}$-bialgebras

Triples of the form $(NAP^c, \mathcal{A}, \mathrm{Prim}\,_{NAP}\mathcal{A})$ come from the work of Muriel Livernet [**40**].

4.5.1. Pre-Lie algebras. By definition a *pre-Lie algebra* is a vector space A equipped with a binary operation $a \cdot b$ which satisfies the following relation

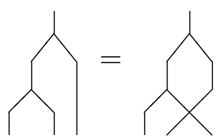
$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x \cdot z) \cdot y - x \cdot (z \cdot y)$$

(right-symmetry of the associator). The free pre-Lie algebra has been described in terms of abstract trees in [**14**].

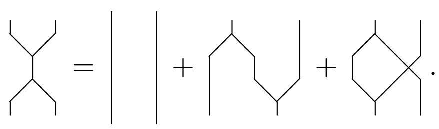
4.5.2. NAP-algebra. By definition a *non-associative permutative algebra*, or NAP algebra for short, is a vector space A equipped with a binary operation denoted ab which satisfies the following relation

$$(xy)z = (xz)y\ .$$

In fact we are going to use the notion of NAP-coalgebra, whose relation is pictorially as follows:



4.5.3. NAP^c-$PreLie$-bialgebra. By definition a NAP^c-$PreLie$-*bialgebra* is a pre-Lie algebra equipped with a NAP cooperation δ, whose compatibility relation $\between_{Liv}$ is as follows:



It has been shown by M. Livernet in [**40**], Proposition 3.2, that the free pre-Lie algebra is naturally a NAP^c-$PreLie$-bialgebra. She also proved the rigidity theorem for NAP^c-$PreLie$-bialgebras by providing an explicit idempotent. This result follows also from our general result, since the coalgebra map $PreLie(V) \to NAP^c(V)$ is an isomorphism (cf. loc.cit.). The explicit description of the universal idempotent in terms of generating operations and cooperations (as described in 2.3.9) is to be found in loc.cit.

4.5.4. NAP^c-Mag-bialgebras. The compatibility relation for NAP^c-Mag-bialgebras is $\between_{Liv}$. Hypothesis `(H0)` is clearly fulfilled. Here is a proof of Hypothesis `(H1)` .

4.5.5. PROPOSITION. *On the free magmatic algebra $Mag(V)$ there is a well-defined cooperation δ which satisfies the NAP^c relation, that is $(\delta \otimes \mathrm{Id})\delta = (\mathrm{Id} \otimes \tau)(\delta \otimes \mathrm{Id})\delta$, and the Livernet compatibility relation $\between_{Liv}$.*

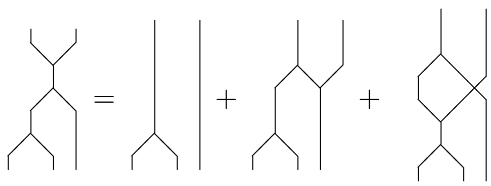
Proof. We use the inductive method described in 3.2. We let

$$\delta : Mag(V) \to Mag(V) \otimes Mag(V)$$

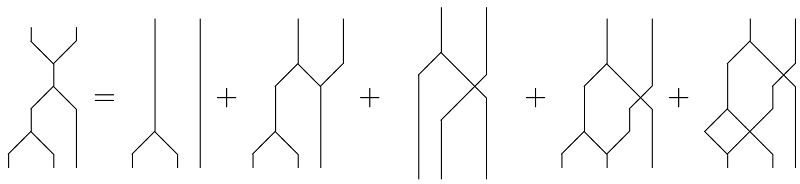
be the unique linear map which sends V to 0 and which satisfies the compatibility relation $\between_{Liv}$. Here the tensor product $Mag(V) \otimes Mag(V)$ is equipped with its standard magmatic operation. In low dimension we get

$$\begin{aligned} \delta(x \cdot y) &= x \otimes y, \\ \delta((x \cdot y) \cdot z) &= x \cdot y \otimes z + x \otimes y \cdot z + x \cdot z \otimes y, \\ \delta(x \cdot (y \cdot z)) &= x \otimes y \cdot z. \end{aligned}$$

Remark that the pre-Lie relator is primitive. We show that δ satisfies the NAP^c-relation (see the diagram above) inductively, by using the natural filtration of $Mag(V) = \oplus_{n \geq 1} \mathbb{K}[Y_{n-1}] \otimes V^{\otimes n}$. Applying $\between_{Liv}$ twice we get



and then



Let us denote by $\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5$ the five terms on the right-hand side of the last line. We check that $(\mathrm{Id} \otimes \tau)\omega_1 = \omega_3$, and that, under the NAP^c relation, we have $(\mathrm{Id} \otimes \tau)\omega_2 = \omega_4$ and $(\mathrm{Id} \otimes \tau)\omega_5 = \omega_5$. Hence we have proved that the NAP^c-relation holds. By Theorem 2.2.2 there is a triple of operads $(NAP, Mag, \mathrm{Prim}_{NAP} Mag)$. $\square$

4.5.6. Conjecture on NAP^c-Mag-bialgebras. We mentioned in the proof of Proposition 4.5.5 that the pre-Lie relator is primitive in the bialgebra $Mag(V)$. Moding out by the ideal generated by this pre-Lie relator gives the NAP^c-$PreLie$-bialgebra $PreLie(V)$. It follows that the map $\varphi(V)$ described in 2.4.2 is the composite

$$\varphi(V) : Mag(V) \twoheadrightarrow PreLie(V) \cong NAP^c(V) \ .$$

Conjecture. The coalgebra map $\varphi(V) : Mag(V) \twoheadrightarrow NAP^c(V)$ admits a coalgebra splitting.

It would follow that there is a good triple of operads

$$(NAP, Mag, \mathrm{Prim}_{\,NAP}Mag)$$

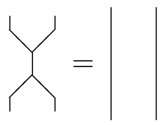
with quotient triple $(NAP, PreLie, Vect)$. The operad $\mathrm{Prim}_{\,NAP}Mag$ has no generating operation in arity 2, but has a generating operation in arity 3, namely the *pre-Lie relator*

$$\langle x; y, z\rangle := (x \cdot y) \cdot z - x \cdot (y \cdot z) - (x \cdot z) \cdot y + x \cdot (z \cdot y) \ .$$

4.6. Some examples of the form $(\mathcal{A}, \mathcal{A}, Vect)$

We have already discussed the cases $(Com, Com, Vect)$ and $(As, As, Vect)$. We give here some more examples, some of them being already in the literature. The main point is to unravel the compatibility relation. The triple $(OU, OU, Vect)$ will be treated in Chapter 5.

4.6.1. The triple $(Mag, Mag, Vect)$**.** The free magmatic algebra $Mag(V) = \oplus_{n\geq 1}\mathbb{K}[Y_{n-1}] \otimes V^{\otimes n}$ inherits a comagmatic coalgebra structure under the identification of the basis of Y_{n-1} of Mag_n with its dual. An immediate inspection shows that the magmatic operation and the comagmatic cooperation are related by the magmatic compatibility relation $\between_{mag}$:



Hence Hypotheses (H0) and (H1) are fulfilled for Mag^c-Mag-bialgebras. Since the map $\varphi(V) : Mag(V) \to Mag^c(V)$ is easily seen to be the identification of the basis of Mag_n with its dual, it is an isomorphism and Hypothesis (H2iso) is fulfilled. By Theorem 2.3.7

$$(Mag, Mag, Vect)$$

is a good triple of operads.

Exercise. Describe the idempotent e explicitly in terms of the generating operation and the generating cooperation. The answer is to be found in [**8**].

4.6.2. The triple $(2as, 2as, Vect)$**.** The operad $2as$ admits a basis made of planar trees. In fact, for $n \geq 2$, the space $2as_n$ is spanned by two copies of the set of planar trees with n leaves. So it is immediate to describe the $2as$-coalgebra structure on the same space. We put the compatibility relations given by the following tableau:

	$*$	$\cdot$
$\delta.$	$Hopf$	$u.i.$
δ_*	$u.i.$	$Hopf$

It was shown in [**51**] (see also 4.2.5) that $2as(V)$ equipped with $\delta.$ satisfies the first row of the compatibility relations. Inverting the role of $\cdot$ and $*$ it is clear that there is an associative cooperation δ_* which satisfies the second row of compatibility relations. Therefore the free $2as$-algebra is a $2as^c$-$2as$-bialgebra. It is interesting to note that, in this case, the isomorphism between the free $2as$-algebra and the free $2as$-coalgebra is not given by identifying the basis with its dual.

4.6.3. The triple $(\mathcal{A}, \mathcal{A}, Vect)$ for a multiplicative operad $\mathcal{A}$. Let $\mathcal{A}$ be a binary operad with split associativity (cf. 3.2.2 and [**45**]). We denote by $x * y$ the associative operation and by $\alpha(\circ), \beta(\circ)$ the coefficients such that

$$x \circ 1 = \alpha(\circ)x \quad \text{and} \quad 1 \circ x = \beta(\circ)x \ .$$

We always assume $\alpha(*) = 1_{\mathbb{K}} = \beta(*)$. Let us suppose moreover that all the relations in $\mathcal{A}$ are *generalized associativity*, that is

$$(x \circ_1 y) \circ_2 z = x \circ_3 (y \circ_4 z) \ ,$$

where $\circ_i \in \mathcal{A}(2)$. For each generating operation $\circ$ we denote by $\Delta_\circ$ the associated cooperation in the unital framework. By definition a $\mathcal{A}^c$-$\mathcal{A}$-bialgebra, called bi$\mathcal{A}$-bialgebra, has the following compatibility relations:

$$\Delta_\circ(x \bullet y) = \Delta_*(x) \bullet \Delta_\circ(y)$$

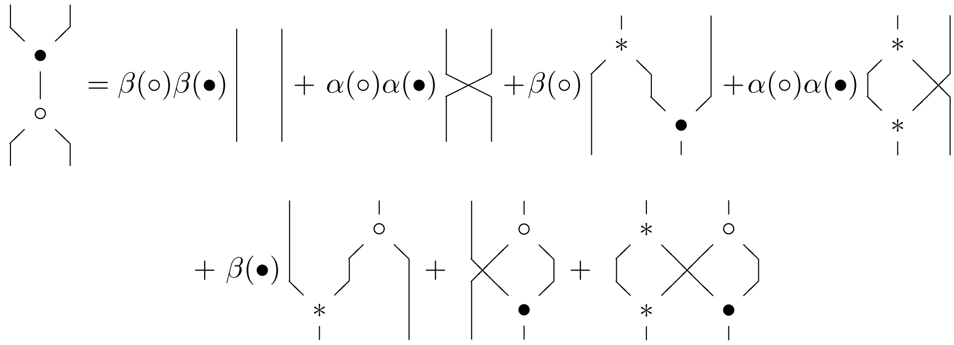
where, by definition,

$$(x \otimes y) \bullet (x' \otimes y') = (x * x') \otimes (y \bullet y')$$

(Ronco's trick, see 3.2.2 for the convention when $y = 1 = y'$). In order to formulate this compatibility relation in the nonunital framework we need to introduce the reduced cooperations $\delta_\circ$, defined by the equality

$$\Delta_\circ = \alpha(\circ)x \otimes 1 + \beta(\circ)1 \otimes x + \delta_\circ \ .$$

The compatibility relations become:



4.6.4. PROPOSITION. *If $\mathcal{A}$ is a multiplicative operad with generalized associativity relations, then the free $\mathcal{A}$-algebra has a natural structure of bi$\mathcal{A}$-bialgebra.*

Proof. Let us work in the unital framework (cf. 3.2.2). First, we construct a cooperation $\Delta_\circ$ on $\mathcal{A}(V)_+$ for each generating binary operation $\circ$ by induction as follows. First, $\Delta_\circ(v) = \alpha(\circ)v \otimes 1 + \beta(\circ)1 \otimes v$. Second, we use the compatibility relations to define $\Delta_\circ$ on $\mathcal{A}(V)_2$, then on $\mathcal{A}(V)_3$ and so forth. So the maps $\Delta_\circ$ are uniquely defined and satisfy the compatibility relations. Let us use the inductive argument to prove the generalized associativity relations.

One one hand we have:

$$\begin{aligned}\Delta_\circ((x \circ_1 y) \circ_2 z) &= \Delta_*(x \circ_1 y) \circ_2 \Delta_\circ(z)\\ &= (\Delta_*(x) \circ_1 \Delta_*(y)) \circ_2 \Delta_\circ(z)\end{aligned}$$

On the other hand we have:

$$\begin{aligned}\Delta_\circ(x \circ_3 (y \circ_4 z)) &= \Delta_*(x) \circ_3 \Delta_\circ(y \circ_4 z)\\ &= \Delta_*(x) \circ_3 (\Delta_*(y) \circ_4 \Delta_\circ(z))\end{aligned}$$

Assuming that the generalized associativity relations hold in some dimension (including the associativity of $*$), we prove from this computation that they hold one step further.

Again by induction we can show, by a straightforward verification, that these cooperations do satisfy the $\mathcal{A}^c$ relations. So $\mathcal{A}(V)_+$ is a unital-counital $\mathcal{A}^c$-$\mathcal{A}$-bialgebra and, by restriction, $\mathcal{A}(V)$ is a $\mathcal{A}^c$-$\mathcal{A}$-bialgebra. □

4.6.5. Examples. The operads

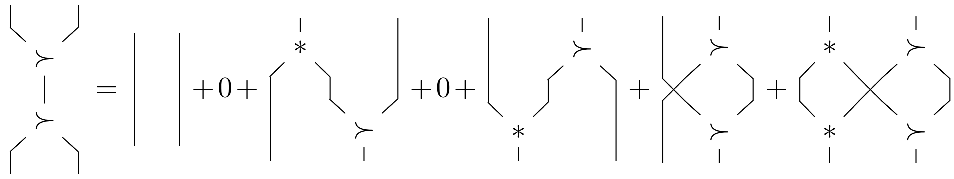
$$As, Dend, Dipt, 2as, OU, OU^!, Tridend, Dias, Trias, Quad, Ennea$$

(cf. [**44, 49, 51, 50, 3, 39, 17**]) are examples of multiplicative operads with generalized associative relations. In some of these examples the map φ is an isomorphism. However it is not always true: *As* is a counter-example. In the following section we study in more details the case $\mathcal{A} = Dend$. Observe that in the case of $2as$ (resp. OU) we get a type of bialgebras which is different from the type studied in 4.6.2 (resp. 5.7) since the compatibility relations are different.

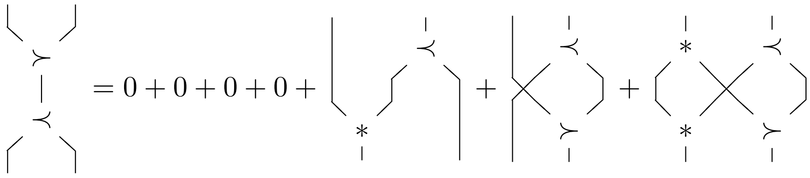
4.6.6. The triple $(Dend, Dend, Vect)$**.** Let us make explicit the particular case $\mathcal{A} = Dend$ which has been treated in details in [**18**]. Recall that the coefficients α and β are given by the relations

$$1 \prec x = 0 = x \succ 1 \quad \text{and} \quad x \prec 1 = x = 1 \succ x.$$

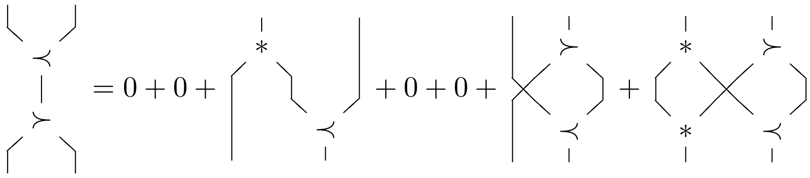
The compatibility relations for the reduced cooperations read as follows. For the pair $(\delta_\succ, \succ)$:



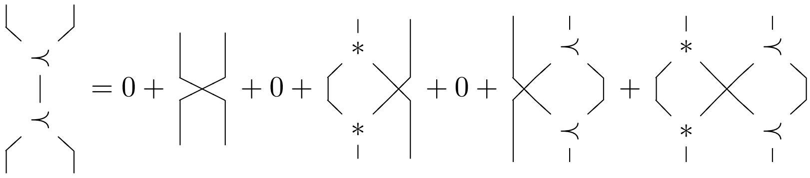
for the pair $(\delta_{\prec}, \succ)$:



for the pair $(\delta_{\succ}, \prec)$:



for the pair $(\delta_{\prec}, \prec)$:



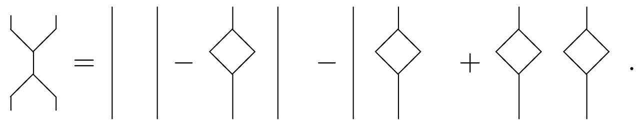
It has been shown by L. Foissy in [**18**] that the triple $(Dend, Dend, Vect)$ is good by using the explicit description of the free dendriform algebra [**44**] (compare with 3.2.2). So there is a rigidity theorem in this case.

4.6.7. The triple $(Nil, Nil, Vect)$**.** By definition a *Nil-algebra* is a vector space A equipped with a binary operation $a \cdot b$ such that any triple product is 0:

$$(x \cdot y) \cdot z = 0 = x \cdot (y \cdot z) \ .$$

Hence the operad Nil is binary, quadratic and regular. We have $Nil_1 = \mathbb{K}, Nil_2 = \mathbb{K}$ and $Nil_n = 0$ when $n \geq 3$.

By definition a Nil^c-Nil-bialgebra is determined by the following compatibility relation $\between_{nil}$:



On $Nil(V) = V \oplus V^{\otimes 2}$ the cooperation δ is given by $\delta(x) = 0$ and $\delta(x \cdot y) = x \otimes y$. We obviously have $(\delta \otimes \mathrm{id})\delta = 0 = (\mathrm{id} \otimes \delta)\delta$ as expected.

From the explicit formula of δ it follows that $\varphi(V) : Nil(V) \to Nil^c(V)$ is an isomorphism. Therefore $(Nil, Nil, Vect)$ is a good triple of operads.

Question. Is there a compatibility relation which makes $(Nil^3, Nil^3, Vect)$ into a good triple of operads ? Here Nil^3 is the operad of algebras equipped with a binary operation for which any quadruple product is 0.

4.6.8. The triple $(Mag_+^\infty, Mag_+^\infty, Vect)$. By definition a Mag_+^∞-algebra is a vector space equipped with an n-ary operation μ_n for any integer $n \geq 2$ and also equipped with a unit 1. Moreover we suppose that

$$\mu_n(a_1 \cdots a_{i-1} 1 a_{i+1} \cdots a_n) = \mu_{n-1}(a_1 \cdots a_{i-1} a_{i+1} \cdots a_n).$$

This triple has been treated in [**8**] along the same lines as the general case.

4.7. Pre-Lie algebras and a conjectural triple

4.7.1. From Pre-Lie algebras to Lie algebras. Let A be a pre-Lie algebra, cf. 4.5.1. It is immediate to check the the antisymmetrization $[x, y] := xy - yx$ of this operation is a Lie bracket. Therefore there is defined a forgetful functor $F : PreLie\text{-alg} \to Lie\text{-alg}$ which associates to a pre-Lie algebra A the Lie algebra $(A, [-, -])$.

4.7.2. The conjectural triple $(??, PreLie, Lie)$. In [**53**] Markl studied this functor. He mentioned the possible existence of connection with some triple of operads. Indeed it is very likely that there exists a notion of generalized bialgebras $\mathcal{C}^c$-$PreLie$ giving rise to a good triple of operads $(\mathcal{C}, PreLie, Lie)$. Not only we have to find the operad $\mathcal{C}$ but also the compatibility relations. The operad $\mathcal{C}$ would have at least one binary generating operation verifying the symmetry $xy = yx$ and one ternary operation verifying the symmetry $(x, y, z) = (y, z, x)$ (and probably more generators in higher arity). The compatibility relation between the binary coproduct and the pre-Lie product is probably of Hopf type. In low degrees the dimension of $\mathcal{C}(n)$ are $(1, 1, 4 = 3 + 1, 23, 181)$.

4.8. Interchange bialgebra

We introduce the notion of interchange algebra and interchange bialgebra and we prove that hypothesis `(H1)` holds. This example is extracted from the paper [**35**] by Yves Lafont.

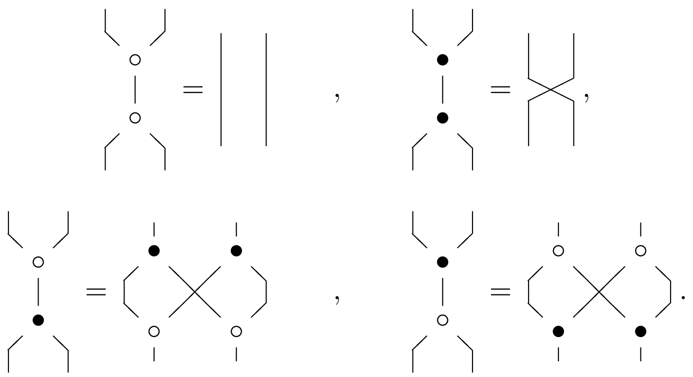
4.8.1. Interchange algebra and bialgebra. By definition an *interchange algebra* is a vector space A equipped with two operations $\circ$ and $\bullet$ satisfying the *interchange law*:

$$(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d) \ .$$

This law is quite common in category theory and algebraic topology since it is part of the axioms for a bicategory ($\circ$ = horizontal composition, $\bullet$ = vertical composition). It can be used to prove the commutativity of the higher homotopy groups.

Observe that this relation is not quadratic.

By definition an *interchange bialgebra* (IC^c-IC-bialgebra) $\mathcal{H}$ is both an interchange algebra and an interchange coalgebra with compatibility relations as follows:



4.8.2. PROPOSITION. *The free interchange algebra $IC(V)$ has a natural IC^c-IC-bialgebra structure.*

Proof. Since the operad IC is set-theoretic, $IC(V) \otimes IC(V)$ is still an IC-algebra and one can define maps $\delta_\circ$ and $\delta_\bullet : IC(V) \to IC(V) \otimes IC(V)$ sending V to 0 and satisfying the compatibility relations, cf. 3.2. By induction they can be shown to satisfy the interchange law (cf. strategy number (2) in 2.5.7). This proof is explained in terms of rewriting systems in [**35**]. $\square$

4.9. The $\langle k \rangle$-ary case

In the preceding examples the generating operations and cooperations were all binary (except sometimes for the primitive operad). In this section we give some examples with $k+1$-ary operations and cooperations for $k \geq 1$. There are many more, not yet explored.

4.9.1. Associative $k+1$-ary algebras. Let k be an integer greater than or equal to 1. Let $\mathcal{C}$ and $\mathcal{A}$ be two operads generated by $k+1$-ary operations. Here are two important examples taken from [**27**].

A *totally associative* $k+1$*-ary algebra* is a vector space A equipped with a $k+1$-ary operation $(a_0 \cdots a_k)$ satisfying the relations

$$\big((a_0 \cdots a_k)a_{k+1} \cdots a_{2k}\big) = \big(a_0 \cdots (a_i \cdots a_{i+k}) \cdots a_{2k}\big)$$

for any $i = 0, \ldots, k$. The operad is denoted $tAs^{\langle k\rangle}$.

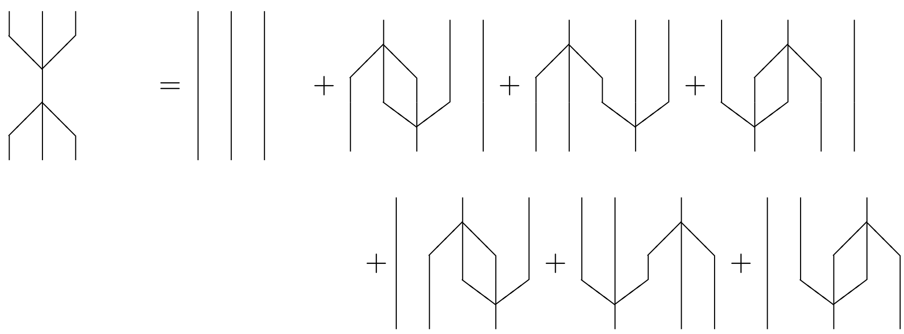
A *partially associative* k*-ary algebra* is a vector space A equipped with a k-ary operation $(a_0 \cdots a_k)$ satisfying the relations

$$\sum_{i=0}^{k} (-1)^{ki}\big(a_0 \cdots (a_i \cdots a_{i+k}) \cdots a_{2k}\big) = 0$$

for any $i = 0, \ldots, k$. These two operads were shown to be Koszul dual to each other by V Gnedbaye in [**27**].

In this context, a $\mathcal{C}^c$-$\mathcal{A}$-bialgebra (or generalized bialgebra) is a vector space $\mathcal{H}$ equipped with a $\mathcal{C}$-coalgebra structure, a $\mathcal{A}$-algebra structure, and each pair (δ, μ) of a generating operation and a generating cooperation is supposed to satisfy a distributive compatibility relation. Observe that in this case the Φ_1-term is an element of the group algebra $\mathbb{K}[S_{k+1}]$.

Here is an example for $k = 2$, denoted $\between_{tAs^{\langle 2\rangle}}$:



We let the reader figure out the similar relation for higher k's.

4.9.2. The triple $(tAs^{\langle k\rangle}, tAs^{\langle k\rangle}, Vect)$. By definition a $tAs^{\langle k\rangle}$-bialgebra is a vector space $\mathcal{H}$ equipped with a structure of $tAs^{\langle k\rangle}$-algebra, a structure of $tAs^{\langle k\rangle}$-coalgebra, related by the compatibility relation $\between_{tAs^{\langle k\rangle}}$ described above. For $k = 1$ this is the unital infinitesimal compatibility relation.

The free totally associative $(k+1)$-ary algebra over V is $tAs^{\langle k\rangle}(V) = \bigoplus_{n\geq 0} V^{\otimes 1+kn}$. We put a structure of $tAs^{\langle k\rangle}$-coalgebra on it by dualizing the natural basis. Then it is easy to prove by induction that the compatibility relation is precisely $\between_{tAs^{\langle k\rangle}}$. The map $\varphi(V)$ is the isomorphism of basis, hence the triple

$$(tAs^{\langle k\rangle}, tAs^{\langle k\rangle}, Vect)$$

is good and the rigidity theorem holds.

4.9.3. The triple $(tCom^{\langle k\rangle}, tCom^{\langle k\rangle}, Vect)$. By definition a $tCom^{\langle k\rangle}$-algebra is a totally associative $(k+1)$-ary algebra which is *commutative* in the sense

$$(a_0 \cdots a_k) = (a_{\sigma(0)} \cdots a_{\sigma(k)})$$

for any permutation $\sigma \in S_{k+1}$.

Exercise. Find the compatibility relation which gives a good triple of operads $(tCom^{\langle k\rangle}, tCom^{\langle k\rangle}, Vect)$.

It would be also interesting to work out the cases $pAs^{\langle k\rangle}$ and $pCom^{\langle k\rangle}$, and also the triple

$$(tCom^{\langle k\rangle}, tAs^{\langle k\rangle}, tLie^{\langle k\rangle}),$$

which is (Com, As, Lie) for $k = 1$.

CHAPTER 5

OU-bialgebras

In this chapter we study in details the operad of OU-algebras. They are defined by two associative operations verifying one more relation. We show that there exists a good triple of operad

$$(As, OU, Mag) \ ,$$

which is quite peculiar since the three operads are binary, quadratic, regular, set-theoretic and Koszul.

In order to prove that the operad OU is Koszul, we compute its dual and construct the chain complex giving rise to the homology of OU-algebras. It turns out that it is the total complex of a certain chain complex whose horizontal (resp. vertical) components are of Hochschild type.

5.1. OverUnder algebra

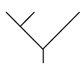
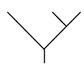
5.1.1. Definition. A *OverUnder algebra* (or OU-algebra for short) A is determined by two binary operations $A \otimes A \to A$ called *left* $(x, y) \mapsto x \prec y$ and *right* $(x, y) \mapsto x \succ y$ respectively, satisfying the following three relations

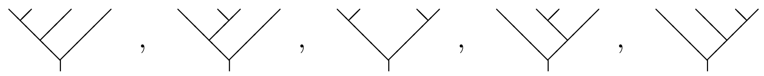
$$\begin{array}{rcl} (x \prec y) \prec z & = & x \prec (y \prec z) \ , \\ (x \succ y) \prec z & = & x \succ (y \prec z) \ , \\ (x \succ y) \succ z & = & x \succ (y \succ z) \ . \end{array}$$

So the two operations left and right are associative. From this definition it is clear that the operad OU is binary, quadratic, regular and set-theoretic.

In order to describe the free OU-algebra (or, equivalently, the operad), we need to introduce the planar binary rooted trees.

5.1.2. Planar binary trees. By definition a *planar binary rooted tree* (we simply say planar binary tree, or p.b. tree for short) is a finite planar graph with vertices which are either trivalent or univalent, with a pointed univalent vertex called the *root*. The other univalent vertices are called the *leaves*. The trivalent vertices are called the *internal vertices*. The set of planar binary rooted trees with $n + 1$ leaves is denoted Y_n:

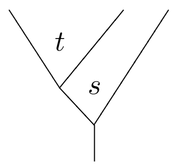
$Y_0 = \{ \, | \, \}$, $Y_1 = \{$  $\}$, $Y_2 = \{$  ,  $\}$

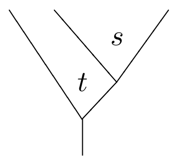
$Y_3 = \Big\{$  $\Big\}$

Recall that the number of elements in Y_n is the *Catalan number* $c_n = \frac{(2n)!}{n!(n+1)!}$ and the generating series is

$$c(t) := \sum_{n\geq 1} c_{n-1} t^n = \frac{1}{2}(1 - \sqrt{1-4t}) \ .$$

On these trees one can perform the following two kinds of grafting: the operation Over and the operation Under. By definition the operation *Over*, denoted t/s consists in grafting the tree t on the first leaf of s. Similarly, the operation *Under*, denoted $t\backslash s$ consists in grafting the tree s on the last leaf of t. Observe that for $t \in Y_p$ and $s \in Y_q$ we have $t/s \in Y_{p+q}, t\backslash s \in Y_{p+q}$.

$t/s =$ 

$t\backslash s =$ 

The Over and Under operations, together with their properties appear in the work of C. Brouder and A. Frabetti [**7**]. We will also use another type of grafting, denoted $t \vee s$ which consists in creating a new root and grafting the two trees to this root. Hence we have $t \vee s = Y_{p+q+1}$. Observe that any tree t (except $|$) is uniquely determined by its left part t^l and its right part t^r so that $t = t^l \vee t^r$. The *left comb* is the tree comb^l_n defined inductively as $\mathrm{comb}^l_1 =$  and $\mathrm{comb}^l_n =$  $/\mathrm{comb}^l_{n-1}$.

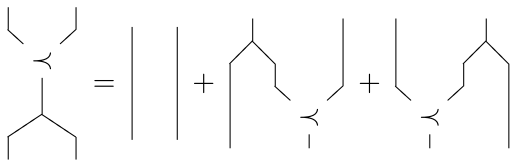
5.1.3. Proposition. *The free OU-algebra on one generator is spanned by the set of planar binary trees and the left (resp. right) operation is induced by the Over operation t/s (resp. Under operation $t\backslash s$). Hence the space of n-ary operations is $OU_n = \mathbb{K}[Y_n]$.*

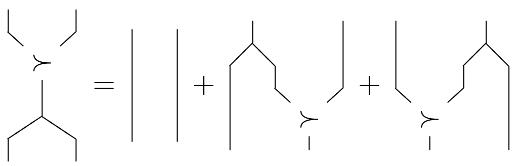
Proof. First, we verify immediately that $\bigoplus_{n\geq 1}\mathbb{K}[Y_n]$ equipped with the Over and Under operations is a OU-algebra generated by the unique element  of Y_1. Moreover we see that, for any p.b. tree $t = t^l \vee t^r$ we have $t = t^l/$  $\backslash t^r$.

Let us show that $\bigoplus_{n\geq 1}\mathbb{K}[Y_n]$ satisfies the universal condition. Let A be a OU-algebra and let $a \in A$. We define a map $\phi : \bigoplus_{n\geq 1}\mathbb{K}[Y_n] \to A$ inductively by $\phi($  $) = a$ and $\phi(t) = \phi(t^l \vee t^r) = \phi(t^l) \succ a \prec \phi(t^r)$. It is straightforward to check that this map is a OU-morphism (same proof as the dendriform case, see [**44**] Proposition 5.7 and [**64**]). Since we have no other choice for its value, it is the expected universal extension map. □

5.2. *OU*-bialgebras

5.2.1. Definition. By definition a As^c-OU-bialgebra, also called *Over-Under-bialgebra*, is a vector space $\mathcal{H}$ equipped with a OU-algebra structure, a coassociative coalgebra structure, and the compatibility relations are of unital infinitesimal type for both pairs $(\delta, \prec)$ and $(\delta, \succ)$:





Observe that this is a regular bialgebra type (hence a regular prop).

5.2.2. Proposition. *The free* OU*-algebra* $OU(V)$ *is a* As^c-OU*-bialgebra.*

Proof. Let us first define the cooperation

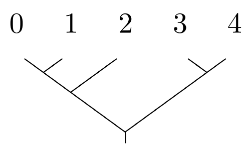
$$\delta : OU(\mathbb{K}) = \bigoplus_n \mathbb{K}[Y_n] \to \bigoplus_n \mathbb{K}[Y_n] \otimes \bigoplus_n \mathbb{K}[Y_n].$$

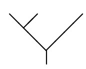
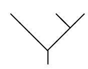
For any $t \in Y_n$ we define

$$\delta(t) = \sum_{1 \leq i \leq n-1} \delta_i(t) = \sum_{1 \leq i \leq n-1} r_i \otimes s_i$$

as follows. Let us number the leaves of t from left to right by the integers $0, 1, \ldots, n$. For any $i = 1, \ldots, n-1$ we consider the path going from the leaf number i to the root. The left part of t (including the dividing path) determines the tree r_i and the right part of t (including the dividing path) determines the tree s_i. In particular $\delta($  $) = 0$.

Example for $i = 2$:

$t =$ 

, $r_2 =$  , $s_2 =$  .

It is immediate to verify, by direct inspection, that δ is coassociative. Let us prove that

$$\delta(x/y) = x \otimes y + x_{(1)} \otimes x_{(2)} \prec y + x \prec y_{(1)} \otimes y_{(2)} \ ,$$

under the notation $\delta(x) = x_{(1)} \otimes x_{(2)}$. Let $x \in Y_p, y \in Y_q$, so that $x/y \in Y_{p+q}$. The element $\delta(x/y)$ is the sum of three different kinds of elements: either the dividing path starts from a leaf of x not being the last one, or starts from the last leaf of x, or starts from a leaf of y. In the first case we get $x \prec y_{(1)} \otimes y_{(2)}$, in the second case we get $x \otimes y$, in the third case we get $x_{(1)} \otimes x_{(2)} \prec y$. The proof for $\succ$ is similar.

To prove that $OU(V)$ is a As^c-OU-bialgebra for any V it suffices to extend δ to $\bigoplus_n \mathbb{K}[Y_n] \otimes V^{\otimes n}$ by

$$\delta(t; v_1 \cdots v_n) = \sum_{1 \leq i \leq n-1} (r_i; v_1 \cdots v_p) \otimes (s_i; v_{p+1} \cdots v_{p+q})$$

and use the property that $\overline{T}(V)$ is a u.i. bialgebra (cf. [**51**] of 4.2.2). $\square$

5.2.3. Remark. We could also prove this Proposition by using the inductive method described in 2.5.7 (it is a good exercise !).

5.2.4. Proposition. *The prop As^c-OU satisfies the hypothesis* `(H2epi)`.

Proof. Since we are dealing with a regular bialgebra type, it suffices to look at $OU(\mathbb{K}) = \bigoplus_n \mathbb{K}[Y_n]$. The map $\varphi : OU_n = \mathbb{K}[Y_n] \to \mathbb{K} = As^c_n$ is given by $\varphi(t) = \alpha$ where α is a scalar determined by the equation

$$\delta^{n-1}(t) = \alpha \ \curlyvee \ \otimes \cdots \otimes \ \curlyvee \ .$$

Here δ^{n-1} stands for the iterated comultiplication. From the explicit description of δ it comes immediately: $\delta^{n-1} = \ \curlyvee \ \otimes \cdots \otimes \ \curlyvee$. Hence $\alpha = 1$ and the map φ is given by $\varphi(t) = 1$.

Define a map $s_n : As_n = \mathbb{K} \to \mathbb{K}[Y_n]$ by $s_n(1) = comb^l_n$, where $comb^l_n$ is the left comb. It is immediate to check that $s_.$ induces a coalgebra map $s(V) : As(V) \to OU(V)$ which is a splitting to $\varphi(V)$. Hence hypothesis `(H2epi)` is fulfilled. $\square$

As a consequence the triple $(As, OU, \mathrm{Prim}_{As}OU)$ is a good triple and it satisfies the structure Theorem over any field $\mathbb{K}$ by 3.3.1. Let us now identify the operad $\mathrm{Prim}_{As}OU$.

5.2.5. Theorem. *The primitive operad* $\mathrm{Prim}_{As}OU$ *of the bialgebra type As^c-OU is the magmatic operad Mag and the functor*

$$F : OU\text{-}alg \to Mag\text{-}alg, \quad F(A, \prec, \succ) = (A, \cdot)$$

is determined by

$$x \cdot y := x \prec y - x \succ y \ .$$

5.2.6. Corollary. *There is a good triple of operads*

$$(As, OU, Mag)\ .$$

Before entering the proof of the Theorem and its Corollary we prove some useful technical Proposition.

5.2.7. Proposition. *Let* $(R, \cdot)$ *be a magmatic algebra. On* $As(R) = \overline{T}(R)$ *we define the operation* $a \succ b$ *as being the concatenation (i.e.* $\succ = \otimes$*) and we define the operation* $a \prec b$ *by* $a \prec b = a \cdot b + a \succ b$ *where the operation* $a \cdot b$ *is defined inductively as follows:*

$$\begin{aligned} (r \otimes a) \cdot b &= r \otimes (a \cdot b) \\ r \cdot (s \otimes b) &= (r \cdot s) \cdot b - r \cdot (s \cdot b) + (r \cdot s) \otimes b. \end{aligned}$$

Then $(As(R), \prec, \succ)$ *is a* OU*-bialgebra with the deconcatenation as coproduct.*

Proof. The last relation of OU-algebra (associativity of $\succ$) is immediate. The other two are proved by a straightforward induction argument on the degree. The compatibility relation for the pair $(\delta, \succ)$ is well-known (cf. 4.2.2). The compatibility relation for the pair $(\delta, \prec)$ is proved by induction. □

5.2.8. Proof of the Theorem 5.2.5 and the Corollary 5.2.6. Applying Proposition 5.2.7 to the free magmatic algebra $R = Mag(V)$, we get a OU-algebra $As(Mag(V))$. The inclusion map

$$V = As(Mag(V))_1 \to As(Mag(V))$$

induces a OU-map $OU(V) \to As(Mag(V))$. From the construction of $As(Mag(V))$ it follows that this map is surjective.

From 5.1.3 it follows that $\dim OU_n = c_n$. It is also known that $\dim(As \circ Mag)_n = c_n$ because

$$f^{As} f^{Mag}(t) = \frac{c(t)}{1 - c(t)} = \frac{c(t) - t}{t} = \sum_{n \geq 1} c_n t^n\ .$$

(Use the identity $c(t)^2 - c(t) + t = 0$). Therefore the surjective map $OU_n \to (As \circ Mag)_n$ is an isomorphism. Hence $OU \to As \circ Mag$ is an isomorphism. Since the comultiplication in $As(Mag(V))$ is the deconcatenation, its primitive part is $Mag(V)$. It follows that $\mathrm{Prim}_{As} OU(V) = Mag(V)$ as expected. □

5.2.9. Corollary. *As an associative algebra for the product* $\succ = /$ *the space* $OU(V)$ *is free over* $Mag(V)$.

Proof. By the structure theorem for As^c-OU-bialgebras we know that there is an isomorphism $OU(V) \cong \overline{T}^c(\mathrm{Prim}_{As} OU(V))$. Because of our choice of s, it turns out that the As-structure of the As^c-As-bialgebra $\overline{T}^c(\mathrm{Prim}_{As} OU(V))$ corresponds to $/ = \succ$, cf. 4.2.2. Hence $OU(V)$ is free for the operation $\succ$. □

We have an extension of operads

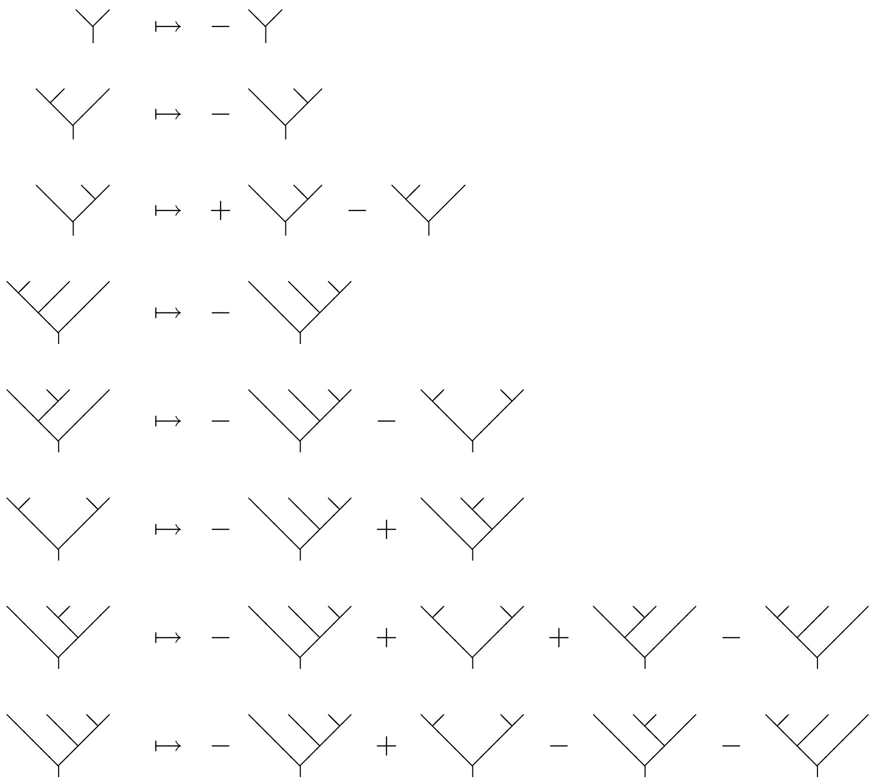
$$As \rightarrowtail OU \twoheadrightarrow Mag$$

in the sense of 3.4.2. It is even a split extension.

5.2.10. Remark on the map $\varphi : Mag \to OU$. Let us write Y_n as a union of two disjoint subsets Y_n^a and Y_n^b, where Y_n^a is made of the trees of the form $| \vee t$ for $t \in Y_{n-1}$. From the definition of $\varphi_n : Mag_n \to OU_n = \mathbb{K}[Y_n]$ and Theorem 5.2.5 it follows that the composition of maps

$$\mathbb{K}[Y_{n-1}] = Mag_n \to OU_n = \mathbb{K}[Y_n] \twoheadrightarrow \mathbb{K}[Y_n/Y_n^b] = \mathbb{K}[Y_n^a] \cong \mathbb{K}[Y_{n-1}]$$

is an isomorphism. It is a nontrivial isomorphism, given in low dimension by:



5.3. Explicit PBW-analogue isomorphism for OU

When $\mathcal{H} = OU(V)$ the isomorphism $\mathcal{H} \cong As^c(\mathrm{Prim}\,\mathcal{H})$ becomes $OU(V) \cong \overline{T}^c(Mag(V))$. Therefore we should be able to write any linear generator of OU_n as a tensor of elements in $Mag_k, k \leq n$. Since we choose the operation $\succ$ to split the map φ, we can replace the tensor by $\succ$ and write an equality in $OU(V)$ (analogous to what we did in the classical case, see 4.1.6). In low dimension it gives the following equalities:

OU		T^1Mag		T^2Mag		T^3Mag
x	$=$	x				
$x \succ y$	$=$	0	$+$	$x \succ y$		
$x \prec y$	$=$	$x \cdot y$	$+$	$x \succ y$		
$x \succ y \succ z$	$=$	0	$+$	0	$+$	$x \succ y \succ z$
$(x \prec y) \succ z$	$=$	0	$+$	$(x \cdot y) \succ z$	$+$	$x \succ y \succ z$
$x \succ y \prec z$	$=$	0	$+$	$x \succ (y \cdot z)$	$+$	$x \succ y \succ z$
$x \prec (y \succ z)$	$=$	$(x \cdot y) \cdot z$ $-x \cdot (y \cdot z)$	$+$	$(x \cdot y) \succ z$	$+$	$x \succ y \succ z$
$x \prec y \prec z$	$=$	$(x \cdot y) \cdot z$	$+$	$(x \cdot y) \succ z$ $+x \succ (y \cdot z)$	$+$	$x \succ y \succ z$

These formulas are consequences of Proposition 5.2.7.

5.4. Koszulity of the operad *OU*

5.4.1. Dual operad. Since the operad OU is quadratic, it admits a dual operad, denoted $OU^!$, cf. [**26**]. The $OU^!$-algebras are OU-algebras which satisfy the following additional relations:

$$(x \prec y) \succ z = 0 \qquad \text{and} \qquad 0 = x \prec (y \succ z)\ .$$

This is easy to check from the conditions given in [**44**] Appendix for a regular operad to be Koszul.

The free $OU^!$-algebra is easy to describe (analogous to the free diassociative algebra, see [**44**]). We have $OU^!_n = \mathbb{K}^n$, where the ith linear generator corresponds to

$$\underbrace{x \succ x \succ \cdots x}_{i-1} \succ x \prec \underbrace{x \cdots \prec x \prec x}_{n-i}\ .$$

5.4.2. The total bicomplex C^{OU}_{**}. Let A be a OU-algebra. We define a chain bicomplex $C^{OU}_{**}(A)$ as follows: $C^{OU}_{pq}(A) = A^{\otimes p+q+1}$ and

$$d^h(a_0 \cdots a_{p+q}) = \sum_{i=0}^{p-1} (-1)^i a_0 \cdots (a_i \succ a_{i+1}) \cdots a_{p+q}\ ,$$

$$d^v(a_0 \cdots a_{p+q}) = \sum_{j=p}^{p+q-1} (-1)^j a_0 \cdots (a_j \prec a_{j+1}) \cdots a_{p+q}\ .$$

The relation $d^h d^h = 0$ follows from the associativity of the operation $\succ$. The relation $d^v d^v = 0$ follows from the associativity of the operation $\prec$. The relation $d^h d^v + d^v d^h = 0$ follows from the relation entwining $\prec$ and $\succ$.

$$
\begin{tikzcd}
C^{OU}_{**}(A) & \cdot \arrow[d] & & & \\
& A^{\otimes 3} \arrow[d, "\mathrm{Id}\otimes\prec"'] & \cdot \arrow[l] \arrow[d] & & \\
& A^{\otimes 2} \arrow[d, "\prec"'] & A^{\otimes 3} \arrow[l, "\succ\otimes\mathrm{Id}"'] \arrow[d, "-\mathrm{Id}\otimes\prec"'] & \cdot \arrow[l] \arrow[d] & \\
& A & A^{\otimes 2} \arrow[l, "\succ"'] & A^{\otimes 3} \arrow[l, "\succ\otimes\mathrm{Id}"'] & \cdot \arrow[l]
\end{tikzcd}
$$

5.4.3. PROPOSITION. *The (operadic) homology of a OU-algebra A is the homology of the total complex of the bicomplex $C^{OU}_{**}(A)$ up to a shift.*

Proof. The operadic chain complex of a OU-algebra is given by

$$C^{OU}_n(A) = (OU^!_n)^*(A)$$

and the differential d is the unique coderivation which extends the OU-products.

From the description of the operad $OU^!$, cf. 5.4.1, we check immediately that $Tot\ C^{OU}_{**}(A) = C^{OU}_*(A)$. The fact that $d^h + d^v$ identifies to the operadic differential is also immediate. $\square$

5.4.4. THEOREM. *The operad OU is a Koszul operad.*

Proof. Let us recall some facts about Hochschild homology of non-unital algebras. Let R be a non-unital algebra and let M be a right R-module. The Hochschild complex of R with coefficients in M is:

$$C_*(R, M): \qquad \to \cdots M \otimes R^{\otimes n} \xrightarrow{b'} M \otimes R^{\otimes n-1} \to \cdots \to M$$

where $b'(a_0, \ldots, a_n) = \sum_{i=0}^{i=n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n)$ and $a_0 \in M, a_i \in R$. The homology groups are denoted by $H_*(R, M)$. If R is free over W, i.e. $R = \overline{T}(W)$, then one can prove the following (cf. for instance [**43**]):

$$
\begin{aligned}
H_0(R, M) &= M/MR, \\
H_n(R, M) &= 0 \quad \text{otherwise.}
\end{aligned}
$$

In order to prove the theorem it suffices to show that the Koszul complex is acyclic, or equivalently that the OU homology of the free OU-algebra $OU(V)$ is

$$H^{OU}_1(OU(V)) = V, \quad \text{and} \quad H^{OU}_n(OU(V)) = 0$$

for $n \geq 2$.

Since by Proposition 5.4.3 the chain complex of the OU-algebra A is the total complex of a bicomplex, we can use the spectral sequence associated to this bicomplex to compute it:

$$E^2_{pq} = H^v_q H^h_p(C^{OU}_{**}(A)) \Rightarrow H^{OU}_{p+q+1}(A)\ .$$

Since $A := OU(V)$ is free over $Mag(V)$ as an associative algebra for $\succ$ (cf. 5.2.9) and since the horizontal complex is the Hochschild complex (for $\succ$) with coefficients in $Mag(V)$, we get

$$H^v_q(C^{OU}_{p*}(A)) = 0, \text{ for } q \geq 1 \text{ and } H^v_0(C^{OU}_{p*}(A)) = A^{\otimes p} \otimes Mag(V) \ .$$

Hence the complex (E^1_{0*}, d^1) is the Hochschild complex (for $\prec$) of A with coefficients in $Mag(V)$. Its homology is

$$\begin{aligned} E^2_{00} &= Mag(V)/Mag(V)A = V, \\ E^2_{0q} &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence the spectral sequence tells us that $H^{OU}_n(A) = 0$ for $n > 1$ and that $H^{OU}_1(A) = V$. So we can deduce that OU is a Koszul operad. $\square$

5.4.5. Alternative proof. (Bruno Vallette, private communication) Since the operad OU is set-theoretic, one can apply the poset method of Vallette [**72, 15**] to prove its Koszulity. Here the poset is as follows. Let us fix an integer n. The poset $\Pi_{OU}(n)$ is made of ordered sequences $(t_1, \ldots, t_k)$ of p.b. trees such that $\sum_{i=1}^k |t_i| = n$ and $|t_i| \geq 1$. The covering relations defining the poset structure are

$$(t_1, \ldots, t_{k+1}) \to (t'_1, \ldots, t'_k)$$

if and only if the second sequence is obtained from the first by replacing two consecutive trees t_i, t_{i+1} either by t_i/t_{i+1} or by $t_i \backslash t_{i+1}$. One can show that the poset is "Cohen-Macaulay" by methods of [**15**], and so, by [**72**], that the associated chain complex is acyclic (except in top dimension). In fact the top dimension homology group is $OU^!_n$. This computation proves the Koszulity of the operad OU.

5.4.6. Question. Since $Mag^! = Nil$ and $As^! = As$ the construction proposed in 3.4 suggests the existence of a good triple of operads $(Nil, OU^!, As)$. Does it exists ?

5.5. On a quotient of OU

Let $OUpreLie$ be the operad which is a quotient of OU by the relation

$$(x \prec y) \succ z - x \prec (y \succ z) = (x \prec z) \succ y - x \prec (z \succ y) \ .$$

This operad is still binary and quadratic, but is not regular anymore since the added relation does not keep the variables in the same order.

5.5.1. Lemma. *In any OU-algebra the following equality holds:*

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x \prec y) \succ z - x \prec (y \succ z) \ .$$

Proof. Recall that $x \cdot y := x \prec y - x \succ y$. It is an immediate consequence of the relations. $\square$

It follows from this Lemma that the above relator defines a primitive operation. It follows from Proposition 3.1 that there is a good triple

$$(As, OUpreLie, PreLie)$$

since the quotient of the primitive operad by this relator is precisely the pre-Lie operad, cf. 4.5.1.

5.5.2. Proposition. *The dimension of* $OUpreLie(n)$ *is* $n^n \times n!$.

Proof. By Corollary 2.6.5 we have the following relation between the generating series:

$$f^{OUpreLie}(t) = f^{As}(f^{preLie}(t)) \ .$$

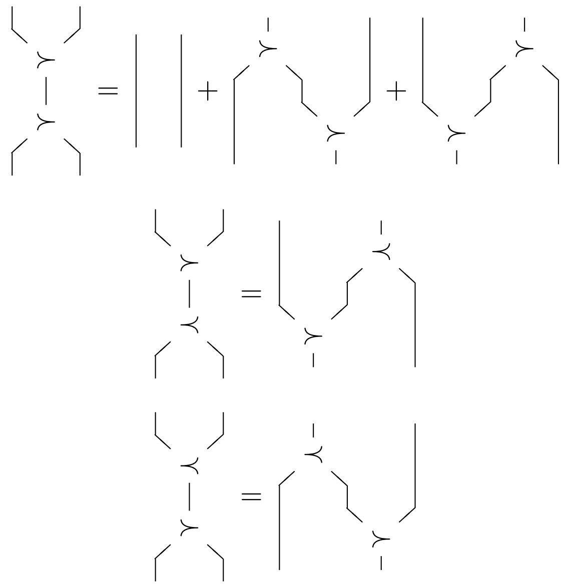
Since $f^{As}(t) = \frac{t}{1-t}$ and since $y := f^{preLie}(t)$ satisfies $y = t\exp(y)$, it follows that $\dim OUpreLie(n) = n^n \times n!$. $\square$

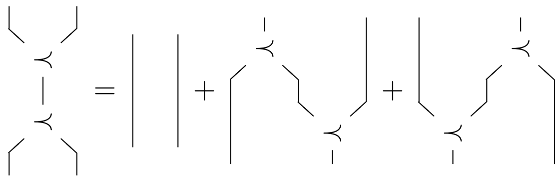
5.6. Grafting algebras

For OU-algebras we used the grafting on the first leaf and the grafting on the last leaf. However there is a more subtle structure which consists in using the grafting operations on any leaf. Strictly speaking it does not give an operad because, for a given integer i, the operation "grafting on the ith leaf" exists only when the elements have high enough degree. This "grafting algebra" structure has been studied in details by María Ronco in [**69**], where she proves the analogue of a structure theorem in this setting.

5.7. The triple $(OU, OU, Vect)$

5.7.1. OU**-bialgebra.** By definition a OU^c-OU*-bialgebra* is determined by the following compatibility relations:





5.7.2. Proposition. *Equipped with the dual basis coalgebra structure, the free OU-algebra $OU(V)$ is a natural OU^c-OU-bialgebra. So Hypothesis* `(H1)` *is fulfilled.*

Proof. We use the explicit description of $OU(V)$ in terms of planar binary trees given in 5.1.3. We check the case

$$\delta_{\prec}(x \prec y) = x \otimes y + x^{\prec}_{(1)} \otimes x^{\prec}_{(2)} \prec y + x \prec y^{\prec}_{(1)} \otimes y^{\prec}_{(2)},$$

where $\delta_{\prec}(x) = x^{\prec}_{(1)} \otimes x^{\prec}_{(2)}$.

First we describe $\delta_{\prec}(x)$ explicitly when x is a p.b. tree. Along the right edge of x we can cut between legs to obtain two trees denoted $x^{\prec}_{(1)}$ and $x^{\prec}_{(2)}$ such that $x = x^{\prec}_{(1)} \backslash x^{\prec}_{(2)}$. Then $\delta_{\prec}(x)$ is the sum of these $x^{\prec}_{(1)} \otimes x^{\prec}_{(2)}$ for all possible cuts.

Let x and y be two p.b. trees. Since $x \prec y = x \backslash y$ in the free algebra, the cuts on the right edge of $x \succ y$ are of three different types:

– either a cut in x,

– or a cut separating x from y,

– or a cut in y.

The first type of cuts gives the summands of $x^{\prec}_{(1)} \otimes (x_{(2)} \prec y)$; the second type of cuts gives $x \otimes y$; the third type of cuts gives the summands of $(x \prec y^{\prec}_{(1)}) \otimes y^{\prec}_{(2)}$.

Since in $x \succ y = x/y$ the right edge is the same as the right edge of y the cuts to obtain $\delta_{\prec}(y)$ are exactly the cuts of y. Therefore we get $\delta_{\prec}(x \succ y) = x \succ y^{\prec}_{(1)} \otimes y_{(2)} \prec$ as expected. The proof of the other two cases are analogous. $\square$

5.7.3. Proposition. *The map $\varphi(V) : OU(V) \to OU^c(V)$ identifies the basis of OU_n with its dual, which is a basis of OU^c. So Hypothesis* `(H2iso)` *is fulfilled.*

Proof. Let t and s be p.b. trees. We need to compute $\delta_{t(s)}$, which is of the form $\lambda\, x \otimes \cdots \otimes x$, where λ is a coefficient and $x =$  is the generator of $OU(\mathbb{K})$. From the compatibility relations it is immediately seen that $\lambda = 1$ if $t = s$ and that $\lambda = 0$ is $t \neq s$. $\square$

5.7.4. Corollary. *The triple $(OU, OU, Vect)$ is a good triple of operads.*

5.8. Towards NonCommutative Quantization

There is another possible choice of compatibility relations for which the free OU-algebra would still be a bialgebra. It consists in taking $\mathcal{C} = OU = \mathcal{A}$ and the u.i. compatibility relation for the four cases

$$(\delta_{\prec}, \prec), (\delta_{\succ}, \prec), (\delta_{\prec}, \succ), (\delta_{\succ}, \succ).$$

In this case the map $\varphi(V)$ is not surjective anymore because, for $OU(V)$, we have $\delta_{\prec} = \delta_{\succ} = \delta$ as described in 5.2. Hence $\varphi(V)$ factors through $As^c(V)$ (cf. 4.2.6. This phenomenon is similar to $\varphi(V)$ factorizing through $Com^c(V)$ in the As^c-As case with $\between = \between_{Hopf}$.

Analogously, for $\mathcal{C} = As^2 = \mathcal{A}$ and compatibility relations as above, the free As^2-algebra is a bialgebra, but the map φ is not surjective since it factors through As^c. The notion of infinitesimal associative bialgebra (with infinitesimal compatibility relation, cf. [**1**]) is going to play a role in the analysis of these bialgebras.

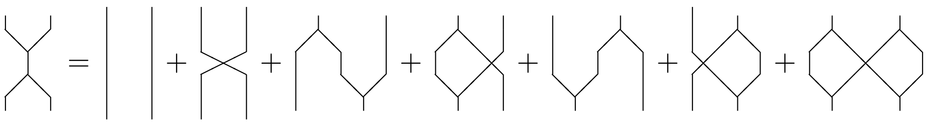
We intend to address these cases in a future paper.

CHAPTER 6

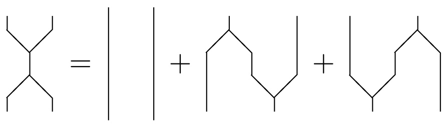
Appendix

6.1. Compatibility relations $\between$ mentioned in this paper

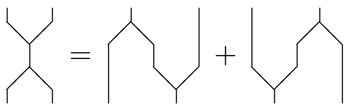
Hopf:



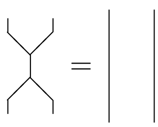
Unital infinitesimal:



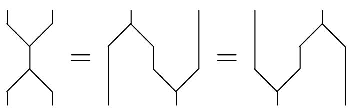
Infinitesimal:



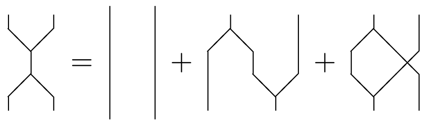
Magmatic:



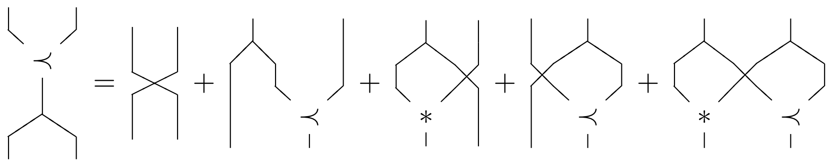
Frobenius:



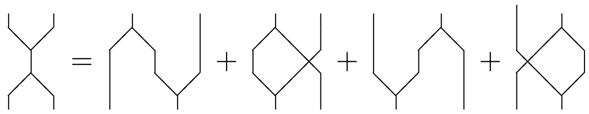
Livernet:



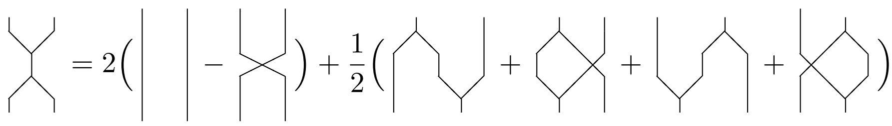
Semi-Hopf:



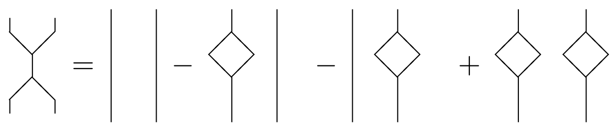
BiLie:



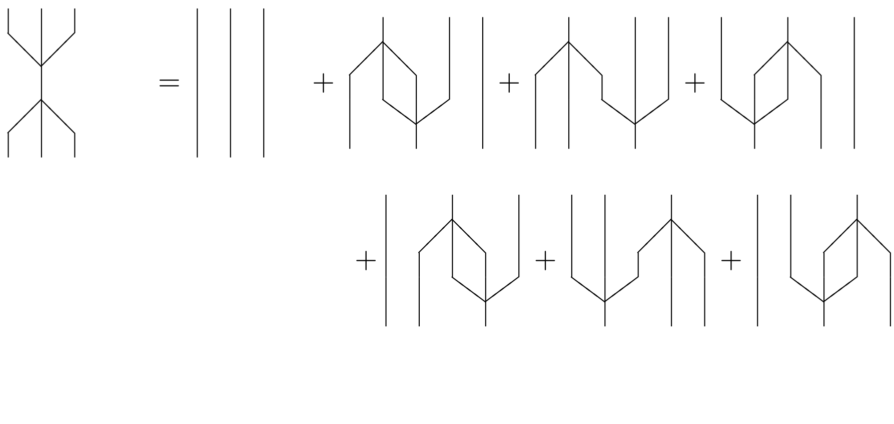
Lily:



Nilpotent:



Unital infinitesimal 3-ary:



6.2. Tableau of some good triples of operads

$\mathcal{C}$	$\mathcal{A}$	$\mathcal{P}$	$\between$	reference
Com	Com	$Vect$	Hopf	[**6**]
Com	$Parastat$	$NLie$	Hopf	[**47**]
Com	As	Lie	Hopf	[**12, 57**]
Com	??	As	Hopf+?	conjectural
Com	$Park$	$PreLie$	Hopf+?	conjectural
Com	Mag	$Sabinin$	Hopf	[**?, 33**]
Com	$Dipt$	$\mathrm{Prim}_{Com}Dipt$	Hopf	**??**
As	As	$Vect$	u.i.	[**51**]
As	$PreLie$	$\mathrm{Prim}_{As}PreLie$	u.i.	
As	Mag	$MagFine$	u.i.	[**34**]
As	$2as$	$\mathbf{B}_\infty$	u.i.+Hopf	[**51**]
As	As^2	As	u.i.+u.i.	4.2.6
As	$OUpreLie$	$PreLie$	u.i.+u.i.	see 5.5
As	OU	Mag	u.i.+u.i.	see 5.2.6
As	$2as$	Mag^∞	u.i.+u.i.	see 4.2.7
As	$Zinb$	$Vect$	semi-Hopf	[**8**]
As	$Dend$	$Brace$	semi-Hopf	[**68**]
As	$Dipt$	$\mathbf{B}_\infty$	semi-Hopf	[**49**]
As	CTD	Com	semi-Hopf	[**46**]
As	$Tridend$	$Brace + As$	semi-Hopf	[**61**]
$Zinb$	As	$Vect$	semi-Hopf	[**8**]
Mag	Mag	$Vect$	magmatic	[**9**]
NAP	$PreLie$	$Vect$	ad hoc	[**40**]
NAP	Mag	$\mathrm{Prim}_{NAP}Mag$		see 4.5
$2as$	$2as$	$Vect$	u.i.^{2}+Hopf^2	[**51**]
$2as$	$2mag$	$\mathrm{Prim}_{2as}2mag$	u.i.^{2}+Hopf^2	
$Dend$	$Dend$	$Vect$	hemisemi-Hopf	[**18**]
$Dend$	$2mag$	$\mathrm{Prim}_{Dend}2mag$	hemisemi-Hopf	
Lie	Lie	$Vect$	Lily (not biLie)	see 4.4
Lie	$PostLie$	$\mathrm{Prim}_{Lie}PostLie$	Lily+?	[**72**]
Nil	Nil	$Vect$	Nil	4.6.7
Nil	$Dias$	$Leib$	?	conjectural
Nil	$OU^!$	As	?	conjectural
IC	IC	$Vect$?	Hopf+ad hoc	4.8, [**35**]
$tCom^{\langle k\rangle}$	$tAs^{\langle k\rangle}$	$tLie^{\langle k\rangle}$	Hopf style	see 4.9.2
$tAs^{\langle k\rangle}$	$tAs^{\langle k\rangle}$	$Vect$	u.i.	see 4.9.2
???	$PreLie$	Lie	?	conjectural

The notation $\mathrm{Prim}_{\mathcal{C}}\mathcal{A}$ in the column "$\mathcal{P}$" means that we do not know yet about a small presentation of this operad.

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