DUALISING COMPLEXES AND TWISTED HOCHSCHILD (CO)HOMOLOGY FOR NOETHERIAN HOPF ALGEBRAS

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Abstract. We show that many noetherian Hopf algebras A have a rigid dualising complex R with $R \cong {}^{\nu}A^1[d]$. Here, d is the injective dimension of the algebra and ν is a certain k-algebra automorphism of A, unique up to an inner automorphism. In honour of the finite dimensional theory which is hereby generalised we call ν the *Nakayama automorphism* of A. We prove that $\nu = S^2\xi$, where S is the antipode of A and ξ is the left winding automorphism of A determined by the left integral of A. The Hochschild homology and cohomology groups with coefficients in a suitably twisted free bimodule are shown to be non-zero in the top dimension d, when A is an Artin-Schelter regular noetherian Hopf algebra of global dimension d. (Twisted) Poincaré duality holds in this setting, as is deduced from a theorem of Van den Bergh. Calculating ν for A using also the opposite coalgebra structure, we determine a formula for S^4 generalising a 1976 formula of Radford for A finite dimensional. Applications of the results to the cases where A is PI, an enveloping algebra, a quantum group, a quantised function algebra and a group algebra are outlined.

0. Introduction

0.1. The starting point for the work described in this paper was the observation by Hadfield and Krähmer [HK1] that, when calculating the Hochschild cohomology of the quantised function algebra $\mathcal{O}_q(SL(2))$, twisting the coefficient bimodule $\mathcal{O}_q(SL(2))$ by a suitable algebra automorphism ν, and so calculating $H^*(\mathcal{O}_q(SL(2)), {}^{\nu}\mathcal{O}_q(SL(2))^1)$, avoids the "dimension drop" which occurs if one works with untwisted coefficients. Our initial aim was to understand why this happens, and to extend their result to other quantised function algebras and then to other classes of Hopf algebra A.

0.2. **Rigid dualising complexes.** It transpires that twisting enters the theory at the level of the *rigid dualising complex* R of A, defined in paragraph (4.2). Let A be a noetherian Hopf algebra over the base field k. (We will assume except where stated otherwise that k is algebraically closed.) Conjecturally, A is *Artin-Schelter* (AS-)*Gorenstein* (Definition 1.2): this is true for numerous classes of Hopf algebras, as we review in §6; see also (0.5). We prove, as Proposition 4.5:

Theorem. *Let A be a noetherian AS-Gorenstein Hopf k-algebra with bijective antipode. Let d be the injective dimension of A. Then A has a rigid dualising complex*

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R,

$$R \cong {}^{\nu}A^1[d].$$

Here, ${}^{\nu}A^1$ denotes the A-bimodule A with left action twisted by a certain k-algebra automorphism ν of A, which, following the classical theory of Frobenius algebras which we are here generalising, we call the *Nakayama automorphism* of A. The automorphism ν is uniquely determined up to an inner automorphism.

0.3. **Nakayama automorphisms.** To describe the Nakayama automorphism of A we need to recall from [LWZ] the definition of the *left integral* of A. When A is AS-Gorenstein of dimension d, $\mathrm{Ext}^i_A({}_Ak, {}_AA)$ is 0 except when $i = d$, and $\mathrm{Ext}^d_A({}_Ak, {}_AA)$, the left integral $\int^l_A$ of A, has k-dimension 1. Thus $\mathrm{Ext}^d_A({}_Ak, {}_AA) \cong {}^1k^{\xi}$, where the right $A-$action on the trivial module k is twisted by a certain left winding automorphism ξ of A. (See §§1.3 and 2.5.) Let S be the antipode of A. We prove, also in Proposition 4.5:

Theorem. *Let A be as in Theorem 0.2. Then the Nakayama automorphism ν of A is $S^2\xi$.*

0.4. **Twisted Hochschild groups.** Hochschild (co)homology with coefficients in a twisted bimodule arises naturally when one studies the twisted (co)homology introduced by Kustermans, Murphy and Tuset [KMT]. Thus, let B be an algebra of finite global homological dimension, let σ be an algebra automorphism of B and let $HH^{\sigma}_i(B)$ be the ith σ-twisted Hochschild homology of B (see [KMT] or [HK1] for details). By [HK1, Proposition 2.1], when σ is diagonalisable, one has

$$HH^{\sigma}_i(B) \;\cong\; H_i(B, {}^{\sigma}B),$$

where the right hand side denotes Hochschild homology with coefficients in ${}^{\sigma}B^1$. We define the *twisted Hochschild dimension* of B to be

$$\mathrm{tHdim}\, B \;=\; \max\{i \mid H_i(B, {}^{\sigma}B) \neq 0 \quad \text{for some automorphism } \sigma \text{ of } B\}.$$

Similarly twisted Hochschild codimension, denoted by tHcdim, is defined by using twisted Hochschild cohomology; the definition is in (5.1). The phenomenon described in (0.1) is a special case of the following theorem, a central result of this paper.

Theorem (Theorems 3.4 and 5.3)**.** *Let A be a noetherian AS-Gorenstein Hopf algebra of finite global dimension d, with bijective antipode, and let ν be its Nakayama automorphism. Then*

(a) $H_d(A, A^{\nu}) \cong Z(A) \neq 0$. As a consequence $\mathrm{tHdim}\, A = d$.
(b) $H^d(A, {}^{\nu}A) \cong A/[A, A] \neq 0$. As a consequence $\mathrm{tHcdim}\, A = d$.

In fact, there is a Poincaré duality connecting the homology and cohomology of the algebras A of the above theorem. This is a consequence of a result of Van den Bergh [VdB1], which applies in the present context because of Theorem 0.2. The following result is obtained as Corollary 5.2.

Corollary. *Let A be as in Theorem 0.4. For every A-bimodule M and for all i,*

$$H^i(A, M) = H_{d-i}(A, {}^{\nu}M).$$

0.5. **Classes of examples.** Our results apply to all currently known classes of noetherian Hopf algebras (with bijective antipode S). In §6 we examine the details of the application, and in particular the nature of the Nakayama automorphism, for

- affine noetherian Hopf algebras satisfying a polynomial identity, (6.2);
- enveloping algebras of finite dimensional Lie algebras, (6.3);
- quantised enveloping algebras, (6.4);
- quantised function algebras of semisimple groups, (6.5),(6.6);
- noetherian group algebras, (6.7).

For all these classes we deduce that the twisted homological and cohomological dimensions equal the global homological dimension of the algebra.

As we explain in detail in §6, for some of the above classes - notably the second and third - the existence and form of the rigid dualising complex was previously known; and with it, therefore, the nature of what we are now calling the Nakayama automorphism. In other cases - notably the first and fourth - more work will be needed to describe ν. In the "classical" cases of enveloping algebras and group algebras Poincaré duality at the level of Lie algebra and group (co)homology dates back respectively to work of Koszul [Ko] and Hazewinkel [Ha], and to Bieri [Bi1], and can be retrieved from the present work.

0.6. **The antipode.** If A is a noetherian AS-Gorenstein Hopf algebra we can apply Theorem 0.2 also to the Hopf algebra $(A, \Delta^{\mathsf{op}}, S^{-1}, \epsilon)$. The two answers thus obtained for ν are necessarily equal to within an inner automorphism of A. Consequently we deduce

Theorem (Corollary 4.6)**.** *Let A be as in Theorem 0.2 with antipode S. Then $S^4 = \gamma \circ \phi \circ \xi^{-1}$, where γ is an inner automorphism and ϕ and ξ are the right and left winding automorphisms arising from the left integral of A.*

This result for finite dimensional Hopf algebras was proved by Radford [Ra] in 1976 - in that case, γ can be explicitly described and S has finite order. While the latter corollary is no longer valid when the Hopf algebra is infinite-dimensional, we can deduce the

Corollary (Propositions 4.6 and 6.2)**.** *Let A be as in Theorem 0.4 with antipode S. Let $io(A)$ be the integral order defined in* [LWZ, Definition 2.2] *and let $o(A)$ be the Nakayama order of A (see Definition 4.4(c)).*

(a) The Nakayama order $o(A)$ is equal to either $io(A)$ or $2\, io(A)$.
(b) Suppose $io(A)$ is finite. Then $S^{4\, io(A)}$ is an inner automorphism.
(c) Suppose A is a finite module over its centre. Then the antipode S and the Nakayama automorphism ν are of finite order up to inner automorphisms.

0.7. Some homological and Hopf algebra background is given in §§1 and 2 respectively. Twisted Hochschild homology is recalled and studied in §3, and Theorem 0.4(a) is proved. Rigid dualising complexes are defined in §4, and Theorems 0.2, 0.3 and 0.6 are proved. In §5 we study twisted Hochschild cohomology, and prove Theorem 0.4(b) and Corollary 0.4.

1. General preparations

1.1. **Standard notation.** Let k be a commutative base field, arbitrary for the moment. When we are working with quantum groups it will sometimes be necessary to assume that k contains $\mathbb{C}(q)$. Unless stated otherwise all vector spaces are over k, and an unadorned $\otimes$ will mean $\otimes_k$. An algebra or a ring always means a k-algebra with associative multiplication m_A and with unit 1. Every algebra homomorphism is a unitary k-algebra homomorphism. All A-modules will be by default left modules. Let A^{op} denote the opposite algebra of A and let A^e denote the enveloping algebra $A \otimes A^{\mathsf{op}}$. The category of left [resp. right] A-modules is denoted by A-Mod [resp. A^{op}-Mod]. So an A^e-module - that is, an object of A^e-Mod - is the same as an A-bimodule central over k.

When A is a Hopf algebra we shall use the symbols Δ, ϵ and S respectively for its coproduct, counit and antipode. The coproduct of $a \in A$ will be denoted by $\Delta(a) = \sum a_1 \otimes a_2$. For details concerning the above terminology, see for example [Mo, Chapter 1]. In many places we assume that

(1.1.1) *the antipode S of the Hopf algebra A is bijective.*

Some of the results in this paper are valid more generally, but the bijectivity of S is satisfied by all the examples which will concern us. Recently Skryabin proved the following.

Proposition. [Sk] *Let A be a noetherian Hopf algebra. If one of the following hold, then S is bijective.*

(a) A is semiprime.
(b) A is affine PI.

Thus (1.1.1) is a reasonable condition and may be satisfied by all noetherian Hopf algebras.

1.2. **Artin-Schelter Gorenstein algebras.** While our main interest is in Hopf algebras of various sorts, some of our results apply in fact to any algebra possessing a natural augmentation ϵ to the base field k, such as, for example, the connected $\mathbb{N}$-graded k-algebras, where ϵ is the graded map $A \to A/A_{\geq 1}(= k)$. Accordingly, we give the following definitions in this more general context.

Definition. Let A be a noetherian algebra.

(a) We shall say A has *finite injective dimension* if the injective dimensions of ${}_AA$ and A_A, $\operatorname{injdim}_A A$ and $\operatorname{injdim} A_A$, are both finite. In this case these integers are equal by [Za], and we write d for the common value. We say A is *regular* if it has finite global dimension, $\operatorname{gldim}_A A < \infty$. Right global dimension always equals left global dimension [We, Exercise 4.1.1]; and, when finite, the global dimension equals the injective dimension.
(b) Suppose now that A has a fixed augmentation $\epsilon : A \to k$. Let k also denote the trivial A-module $A/\ker\epsilon$. Then A is *Artin-Schelter Gorenstein*, which we usually abbreviate to *AS-Gorenstein*, if
  (AS1) $\operatorname{injdim}{}_AA = d < \infty$,
  (AS2) $\dim_k \operatorname{Ext}^d_A({}_Ak, {}_AA) = 1$ and $\operatorname{Ext}^i_A({}_Ak, {}_AA) = 0$ for all $i \neq d$,
  (AS3) the right A-module versions of (AS1,AS2) hold.
(c) If, further, $\operatorname{gldim} A = d$, then A is called *Artin-Schelter regular*, usually shortened to *AS-regular*.

We recall that the following question, first posed in [BG1, 1.15], and repeated in [Br1], remains open:

Question. Is every noetherian affine Hopf algebra AS-Gorenstein?

We shall note in Section 6 many of the classes of algebra for which a positive answer to this question is known.

Remark. The Artin-Schelter Gorenstein condition (AS2) given in the above definition is slightly different from the definition given in [BG1, 1.14]. When A is a Hopf algebra, the two definitions are equivalent, as we record in Lemma 3.2.

1.3. **Homological integrals.** Here is the natural extension to augmented algebras of a definition recently given in [LWZ] for Hopf algebras. This definition generalises a familiar concept from the case of a finite dimensional Hopf algebra [Mo, Definition 2.1.1]). The motive for the use of the term "integral" in the latter setting arises from the relation to the Haar integral.

Definition. Let A be a noetherian algebra with a fixed augmentation $\epsilon : A \to k$. Suppose A is AS-Gorenstein of injective dimension d. Any nonzero element in $\mathrm{Ext}^d_A({}_Ak, {}_AA)$ is called a *left homological integral* of A. We denote $\mathrm{Ext}^d_A({}_Ak, {}_AA)$ by $\int^l_A$. Any nonzero element in $\mathrm{Ext}^d_{A^{\mathsf{op}}}(k_A, A_A)$ is called a *right homological integral* of A. We write $\int^r_A = \mathrm{Ext}^d_{A^{\mathsf{op}}}(k_A, A_A)$. Abusing language slightly, we shall also call $\int^l_A$ and $\int^r_A$ the left and the right homological integrals of A respectively. When no confusion as to the algebra in question seems likely, we'll simply write $\int^l$ and $\int^r$ respectively.

2. Hopf algebra Preparations

2.1. **Restriction via Δ.** Let A be a Hopf algebra. Let $\Delta : A \to A \otimes A$ be the coproduct. Then we may view A as a subalgebra of the algebra $A \otimes A$ via Δ. Define a functor

$$\mathrm{Res}_A \ : \ (A \otimes A) - \mathsf{Mod} \ \to \ A - \mathsf{Mod}$$

by restriction of the scalars via Δ, so that Res_A is equivalent to the functor $\mathrm{Hom}_{A\otimes A}({}_{A\otimes A}(A \otimes A)_A, -)$.

Lemma. *Let A be a Hopf algebra and B be any algebra.*

(a) Res_A *is an exact functor.*
(b) Let N be an $A \otimes B^{\mathsf{op}}$-module. Then $\mathrm{Res}_A(A \otimes N)$ with restriction of B^{op}-module from N is isomorphic to ${}_AA \otimes {}_kN_B$ as $A \otimes B^{\mathsf{op}}$-modules where the left A-module on ${}_kN_B$ is trivial. The isomorphism from $\mathrm{Res}_A(A \otimes N)$ to ${}_AA \otimes {}_kN_B$ is given by

$$a \otimes n \mapsto \sum a_1 \otimes S(a_2)n,$$

where $\Delta(a) = \sum a_1 \otimes a_2$, with inverse $a' \otimes n' \mapsto \sum a'_1 \otimes a'_2 n'$.
(c) Res_A *preserves freeness. Similarly, $A \otimes A$ is a free right A-module via the coproduct Δ.*
(d) Res_A *preserves projectivity.*
(e) Res_A *preserves injectivity.*

Proof. (a) This is clear.

(b) By the left-module version of the fundamental theorem of Hopf modules [Mo, Theorem 1.9.4] $A \otimes N$ is free over A with basis given by a k-basis of N, because N is the space of coinvariants of the left A-comodule $A \otimes N$. It's easy to verify that the isomorphism given in the proof of [Mo, Theorem 1.9.4] has the desired form, and that it preserves the B^{op}-module action.

(c) This follows from (b) and its right-hand analogue, with $N = A$.

(d) This is true because $\mathrm{Res}_A(A \otimes A)$ is free by (c).

(e) Let M be an injective $A \otimes A$-module. Then

$$\mathrm{Hom}_A(-, \mathrm{Res}_A(M)) \;\cong\; \mathrm{Hom}_A(-, \mathrm{Hom}_{A\otimes A}({}_{A\otimes A}(A \otimes A)_A, M)) \;=:\; (*)$$

By the Hom $-\otimes$ adjunction

$$(*) \;\cong\; \mathrm{Hom}_{A\otimes A}({}_{A\otimes A}(A \otimes A)_A \otimes_A -, M) \;=:\; (**).$$

By (c) $(A \otimes A)_A$ is free, hence flat. Since M is $A \otimes A$-injective, the functor $(**)$ is exact. Hence $\mathrm{Res}_A(M)$ is A-injective. □

2.2. **The left adjoint action.** The algebra homomorphism $\mathrm{id}_A \otimes S : A \otimes A \to A \otimes A^{\mathsf{op}} = A^e$ induces a functor

$$F_{\mathrm{id}_A \otimes S} \;:\; \mathsf{Mod}(A^e) \;\to\; \mathsf{Mod}(A \otimes A).$$

If S is bijective, then $F_{\mathrm{id}_A \otimes S}$ is an invertible functor. The *left adjoint functor* L [Mo, Definition 3.4.1(1)] is defined to be

$$L = \mathrm{Res}_A \circ F_{\mathrm{id}_A \otimes S} \;:\; A^e - \mathsf{Mod} \;\to\; A - \mathsf{Mod}\,.$$

Let M be an A-bimodule. Then $L(M)$ is a left A-module defined by the action

$$a \cdot m \;=\; \sum a_1 m S(a_2).$$

In a similar way to Lemma 2.1, one can prove:

Lemma. *Let A be a Hopf algebra and L be the left adjoint functor defined above.*

(a) L is an exact functor.
(b) L preserves projectives.
(c) $L(A^e)$ is a free A-module.
(d) If S is bijective, then L preserves injectives.

Remark. If S is not bijective, it is unclear to us whether L preserves injectives. That's why we need bijectivity of S for Lemma 2.4(d).

2.3. **Twisted (bi)modules.** We extend slightly a standard notation for twisted one-sided modules, as follows. Let A be an algebra and let M be an A-bimodule. For every pair of algebra automorphisms σ, τ of A, we write ${}^{\sigma}M^{\tau}$ for the A-bimodule defined by

$$a \cdot m \cdot b \;=\; \sigma(a) m \tau(b)$$

for all $a, b \in A$ and all $m \in M$. When one or the other of σ, τ is the identity map we shall simply omit it, writing for example ${}^{\sigma}M$ for ${}^{\sigma}M^{1}$. If ϕ is another automorphism of A, then the map $x \mapsto \phi(x)$ for all $x \in A$ defines an isomorphism of A-bimodules ${}^{\sigma}A^{\tau} \to {}^{\phi\sigma}A^{\phi\tau}$. In particular,

$$(2.3.1) \qquad {}^{\sigma}A^{\tau} \;\cong\; A^{\sigma^{-1}\tau} \;\cong\; {}^{\tau^{-1}\sigma}A$$

as A-bimodules. If τ is an inner automorphism of A, given by conjugation $x \mapsto uxu^{-1}$ by the unit u of A, then the map on A given by left multiplication by u shows that

$$ {}^{\sigma}A^{\beta} \;\cong\; {}^{\tau\sigma}A^{\beta} \;\cong\; {}^{\sigma}A^{\beta\tau} \tag{2.3.2} $$

for all automorphisms σ and β of A.

Let M be an A-bimodule. Define

$$ Z_M(A) \;:=\; \{a \in A : am = ma \quad \forall\, m \in M\}. $$

Then $Z_M(A)$ is a subalgebra of A. For any algebra automorphism σ, write

$$ Z_A^{\sigma}(M) \;=\; \{m \in M \mid am = m\sigma(a) \quad \forall\, a \in A\}, $$

so $Z^{\sigma}(M)$ is a $Z_M(A)$-submodule of M. When $M = A$, $Z^{\sigma}(M)$ is the space of σ^{-1}-*normal* elements of A. We write $Z(M)$ for $Z^{\mathrm{id}_A}(M)$; of course $Z(A) = Z_A(A)$ is just the centre of A, and $Z^{\sigma}(A)$ is a $Z(A)$-module. We shall denote by $N(A)$ the multiplicative subsemigroup of $\mathrm{Aut}_{k\text{-alg}}(A)$ consisting of those automorphisms σ such that A contains a σ-normal element which is not a zero divisor.

Lemma. *Let A be an algebra and let M be an A-bimodule.*

(a) Under the canonical identification of M with M^{σ} as k-vector spaces,

$$ Z_A^{\sigma}(M) \;=\; Z(M^{\sigma}). $$

(b) Let σ and τ be algebra automorphisms of A. Then τ carries $Z(A^{\sigma\tau})$ to $Z(A^{\tau\sigma})$.

(c) Suppose that A is a Hopf algebra. If σ is a Hopf algebra automorphism of A, then ${}^{\sigma}(L(M)) \cong L({}^{\sigma}M^{\sigma})$. In particular,

$$ {}^{\sigma}(L({}^{\tau}A)) \cong L({}^{\sigma\tau}A^{\sigma}) \cong L({}^{\tau}A). $$

Proof. (a) is simply the definition.

(b) By (a) we may regard $Z(A^{\sigma\tau})$ and $Z(A^{\tau\sigma})$ as subsets of A. For every $x \in Z(A^{\sigma\tau})$, $ax = x\sigma\tau(a)$ for all $a \in A$. Then, for all $a \in A$,

$$ a\tau(x) \;=\; \tau(\tau^{-1}(a)x) \;=\; \tau(x\sigma\tau\tau^{-1}(a)) \;=\; \tau(x)\tau\sigma(a). $$

This means that $\tau : Z(A^{\sigma\tau}) \longrightarrow Z(A^{\tau\sigma})$ is an isomorphism.

(c) The second assertion is a special case of the first, so we consider the first one only. Let $*$, $\cdot$ and $\bullet$ be the left action of A on ${}^{\sigma}(L(M))$, $L(M)$ and $L({}^{\sigma}M^{\sigma})$ respectively. For every $a \in A$ and $m \in M$,

$$ a * m = \sigma(a) \cdot m = \sum \sigma(a)_1 m S(\sigma(a)_2) =: (*). $$

Let $\cdot_{\sigma}$ be the action of A on ${}^{\sigma}M^{\sigma}$. Since σ commutes with product, coproduct and antipode, then we continue

$$ (*) = \sum \sigma(a_1) m \sigma(S(a_2)) = \sum a_1 \cdot_{\sigma} m \cdot_{\sigma} S(a_2) = a \bullet m. $$

□

2.4. **Homological properties of the adjoint action.** For us, the usefulness of L rests in its role linking homological algebra over A with that over A^e, thus permitting the calculation of Hochschild cohomology. The key lemma is the following result of Ginzburg and Kumar [GK, Proposition p.197]:

Lemma. [GK] *Let A be a Hopf algebra and let M be an A-bimodule.*

(a) $\mathrm{Hom}_{A^e}(A, M) = \mathrm{Hom}_A(k, L(M)) = Z(M)$.

(b) If S is bijective, $\mathrm{Ext}^i_{A^e}(A, M) = \mathrm{Ext}^i_A(k, L(M))$ *for all i.*

Proof. (a) This was proved in [GK, Proposition p.197]. We offer a proof following the line of the proof of [Mo, Lemma 5.7.2(1)]. By definition,

$$\mathrm{Hom}_{A^e}(A, M) \;=\; \{m \in M \mid am = ma, \forall\, a \in A\} \;=\; Z(M)$$

and

$$\mathrm{Hom}_A(k, L(M)) \;=\; \{m \in M \mid \sum a_1 m S(a_2) = \epsilon(a)m, \forall\, a \in A\}.$$

If $\sum a_1 m S(a_2) = \epsilon(a)m$ for all $a \in A$, then

$$am \;=\; \sum a_1 m \epsilon(a_2) \;=\; \sum a_1 m S(a_2) a_3 \;=\; \sum \epsilon(a_1) m a_2 \;=\; ma.$$

Conversely, if $am = ma$ for all $a \in A$, then

$$\sum a_1 m S(a_2) \;=\; \sum a_1 S(a_2) m \;=\; \epsilon(a)m.$$

Thus (a) follows.

(b) Let I be an injective A^e-resolution of M. Since L is exact and preserves injective modules by Lemma 2.2(d), $L(I)$ is an injective resolution of $L(M)$. Therefore (b) follows from (a) by replacing M by its injective resolution. □

Remark. In [GK] part (b) was stated for any Hopf algebra. But it seems to us that the bijectivity of S is needed. This should also be compared to Proposition 3.3 in which case the bijectivity is not needed.

2.5. **Representations and winding automorphisms.** Let $G(A^\circ)$ be the group of group-like elements of the Hopf dual A° of the Hopf algebra A; that is, $G(A^\circ)$ is the set of algebra homomorphisms from A to k, which is a group under multiplication in A°, namely, convolution of maps, $\pi * \pi' := m_A(\pi \otimes \pi')\Delta$, with identity element the counit ϵ. When A is noetherian and is known to be AS-Gorenstein, so that the left integral $\int^l$ of A exists as explained in (1.3), we denote by $\pi_0 \in G(A^\circ)$ the canonical map

$$\pi_0 \;:\; A \to A/\mathrm{r.ann}(\textstyle\int^l).$$

Given $\pi \in G(A^\circ)$, let $\Xi^\ell[\pi]$ denote the *left winding automorphism* of A defined by

$$\Xi^\ell[\pi](a) \;=\; \sum \pi(a_1) a_2$$

for $a \in A$. Similarly, let $\Xi^r[\pi]$ denote the *right winding automorphism* of A defined by

$$\Xi^r[\pi](a) \;=\; \sum a_1 \pi(a_2).$$

Since the right winding automorphism will not be used often we simplify $\Xi^\ell[\pi]$ to $\Xi[\pi]$ and call $\Xi[\pi]$ the winding automorphism associated to π. It is easy to see that the inverse of $\Xi[\pi]$ is

$$(\Xi[\pi])^{-1} \;=\; \Xi[\pi S];$$

moreover, $\Xi[\epsilon] = \mathrm{id}_A$, and the map $\Xi : \pi \longrightarrow \Xi[\pi]$ is an antihomomorphism of groups $G(A^\circ) \longrightarrow \mathrm{Aut}_{k\text{-alg}}(A)$. When π_0 exists, we shall write

$$\xi \ := \ \Xi[\pi_0]. \tag{2.5.1}$$

Given an algebra automorphism (or even an algebra endomorphism) σ of A, we define $\Pi[\sigma]$ to be the map $\epsilon\sigma : A \to k$. Since $\epsilon S = \epsilon$,

$$\Pi[S^2] \ = \ \Pi[\mathrm{id}_A] \ = \ \epsilon.$$

Since, in general, $S^2 \neq \mathrm{id}_A$, it follows that in general $\Xi[\Pi[\sigma]] \neq \sigma$.

Here is a collection of elementary facts about the notations introduced above.

Lemma. *Let A be a Hopf algebra, let $\pi, \phi \in G(A^\circ)$ and let $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$.*

(a) $\Pi[\Xi[\pi]] = \pi$. *In particular,* Π *is a surjective map from* $\mathrm{Aut}_{k\text{-alg}}(A)$ *to* $G(A^\circ)$.
(b) $\Xi[\Pi[\Xi[\pi]]] = \Xi[\pi]$.
(c) $\pi S^2 = \pi$ *and so* $\Xi[\pi S^2] = \Xi[\pi]$.
(d) $\Xi[\pi]S^2 = S^2\Xi[\pi]$. *In particular, if the left integral of A exists then* $\xi S^2 = S^2\xi$.

Proof. These are straightforward calculations, so the proofs are omitted. □

2.6. **The right adjoint action.** Let M be an A-bimodule. The *right adjoint* A-module $R(M)$ [Mo, Definition 3.4.1(2)], which we shall often denote by M' (see also [FT, HK1]), is M as a k-vector space, with right A-action given by

$$m \cdot a \ = \ \sum S(a_2)ma_1$$

for all $a \in A$ and $m \in M$. Below we shall want to combine twisting with the right adjoint action, producing right modules of the form $({}^\sigma A^\tau)'$ which will be crucial in our calculation of Hochschild homology. Parallel results to those stated in (2.2), (2.3) and (2.4) can be derived for $R(-)$. We shall omit most of the statements. The following lemma will be used later. For $\pi \in G(A^\circ)$, we write k_π for the corresponding right A-module $A/\ker\pi$.

Lemma. *Let A be a Hopf algebra, let $\pi, \phi \in G(A^\circ)$ and let $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$. Let M be an A-bimodule. From (b) to (e) we assume S is bijective. In (d) and (e) we assume that the left integral of A exists. In (b) to (e) we identify the spaces of homomorphisms with vector subspaces of M or A as appropriate.*

(a) $k_\pi^\sigma \cong k_{\pi\sigma}$.
(b) $\mathrm{Hom}_{A^{\mathrm{op}}}(k, ({}^{S^{-2}}M^\sigma)') \ = \ Z_A^\sigma(M)$.
(c) $\mathrm{Hom}_{A^{\mathrm{op}}}(k_{\phi*\pi}, (M^{\Xi[\phi]})') \ = \ \mathrm{Hom}_{A^{\mathrm{op}}}(k_\pi, (M)')$. *As a consequence,*

$$\mathrm{Hom}_{A^{\mathrm{op}}}(k_{\phi*\pi}, ({}^{S^{-2}}A^{\sigma\Xi[\phi]})') \ = \ \mathrm{Hom}_{A^{\mathrm{op}}}(k_\pi, ({}^{S^{-2}}A^\sigma)').$$

(d) $\mathrm{Hom}_{A^{\mathrm{op}}}(\int^l, ({}^{S^{-2}}A^\sigma)') \ = \ Z(A^{\sigma\xi^{-1}})$. *That is,*

$$\mathrm{Hom}_{A^{\mathrm{op}}}(\textstyle\int^l, (A^\sigma)') \ = \ Z(A^{S^{-2}\sigma\,\xi^{-1}}) \ \cong \ Z(A^{S^{-2}\xi^{-1}\sigma}).$$

(e) $\mathrm{Hom}_{A^{\mathrm{op}}}(\int^l, ({}^{S^{-2}}A^\xi)') \ = \ Z(A) \ = \ \mathrm{Hom}_{A^{\mathrm{op}}}(\int^l, ({}^{\xi^{-1}S^{-2}}A)')$.

Proof. (a) follows from the definitions.

(b) By Lemma 2.3(a) we may assume $\sigma = \mathrm{id}_A$. The right A-action on $({}^{S^{-2}}M)'$ is then given by

$$m \cdot a = \sum S^{-1}(a_2)ma_1$$

for all $m \in M$ and $a \in A$, and so we need to prove that

$$\mathrm{Hom}_{A^{\mathrm{op}}}(k, ({}^{S^{-2}}M)') = Z(M).$$

The proof of Lemma 2.4(a) can be modified and we leave the details to the reader.

(c) We may, in a similar way to the proof of (b), identify both the spaces of maps in question with subspaces of the k-vector space M. The rest of the proof is straightforward.

The second assertion follows by taking $M = {}^{S^{-2}}A^{\sigma}$.

(d) As a right module, $\int^l$ is by definition isomorphic to k_{π_0}. So from (c) with $\pi = \pi_0$ and $\phi = \pi_0^{-1}$, together with (b) with A for M, we see that

$$\mathrm{Hom}_{A^{\mathrm{op}}}(\textstyle\int^l, ({}^{S^{-2}}A^{\sigma})') \quad = \quad Z_A^{\sigma}(A).$$

The first isomorphism in (d) now follows from Lemma 2.3(a). The second isomorphism in (d) follows from the first after we observe that $(A^{\sigma})' \cong ({}^{S^{-2}}A^{S^{-2}\sigma})'$, which is clear from (2.3.1).

(e) In view of (2.3.2), this is a special case of (d). □

3. Twisted Hochschild homology

3.1. **Definition.** Let A be an algebra and let σ be an algebra automorphism. A σ-twisted version of the standard cochain complex used to define cyclic cohomology is defined in [KMT, Section 2], and used to define *σ-twisted cyclic and Hochschild (co)homology*; the notation is $HC_*^{\sigma}(A)$ and $HH_*^{\sigma}(A)$ in the case of the homology groups, and analogously for the cohomology groups. As shown in [HK1, Proposition 2.1], when σ is diagonalisable, as will be the case in the most of the examples of particular interest to us, one has

$$HH_n^{\sigma}(A) \;\cong\; H_n(A, {}^{\sigma}A), \tag{3.1.1}$$

where the right hand side denotes Hochschild homology with coefficients in ${}^{\sigma}A$.

Definition. The *Hochschild dimension* of A is defined to be

$$\mathrm{Hdim}\, A = \max\{i \mid HH_i(A) = H_i(A, A) \neq 0\}.$$

The *twisted Hochschild dimension of* A is defined to be

$$\mathrm{tHdim}\, A \;=\; \max\{i \mid H_i(A, {}^{\sigma}A) \neq 0 \quad \text{for some automorphism } \sigma \text{ of } A\}.$$

Remark. Ideally the twisted Hochschild dimension should be defined in terms of twisted Hochschild homology $HH_n^{\sigma}(A)$. Unfortunately our computation in later sections uses knowledge of $H_n(A, {}^{\sigma}A)$; and we don't know in general when σ will be diagonalisable. Further, it's not immediately obvious whether $H_n(A, {}^{\sigma}A) \neq 0$ is equivalent to $HH_n^{\sigma}(A) \neq 0$ when σ is not diagonalisable. The above definition is therefore chosen so as to finesse this uncertainty.

In Section 6 we will comment on the diagonalisability of the relevant automorphisms for particular classes of examples.

Of course, in some cases diagonalisability is automatic. The following lemma is a standard result of linear algebra.

Lemma. *If σ has finite order and the characteristic of k is zero, then σ is diagonalisable.*

3.2. **Ischebeck's spectral sequence.** We need the following special case of Ischebeck's spectral sequence [Is, 1.8]. Let A be an algebra, let M be a left A-module and N a right A-module. Suppose that ${}_AM$ has a resolution by finitely generated projectives and that N_A has finite injective dimension. Then Ischebeck's spectral sequence is a convergent spectral sequence of vector spaces

$$E_2^{p,q} \;:=\; \operatorname{Ext}^p_{A^{\mathsf{op}}}(\operatorname{Ext}^{-q}_A(M,A),N) \implies \operatorname{Tor}^A_{-p-q}(N,M). \tag{3.2.1}$$

If N is an A-bimodule, then this is a sequence of left A-modules. In particular, if A has finite injective dimension, then we obtain the double Ext-spectral sequence of left A-modules

$$E_2^{p,q} \;:=\; \operatorname{Ext}^p_{A^{\mathsf{op}}}(\operatorname{Ext}^{-q}_A(M,A),A) \implies \mathbb{H}^{p+q}(M) \tag{3.2.2}$$

where $\mathbb{H}^n(M) = \begin{cases} 0 & n \neq 0 \\ M & n = 0 \end{cases}$.

As an immediate application we record here the fact that, when A is a noetherian Hopf algebra the (AS2) condition of Definition (1.2)(b) can be replaced by a weaker form, as follows:

Lemma. *Let A be a noetherian Hopf algebra of injective dimension d. If $\operatorname{Ext}^i_A(k,A)$ and $\operatorname{Ext}^i_{A^{\mathsf{op}}}(k,A)$ are finite dimensional over k for all i and if $\operatorname{Ext}^d_A(k,A) \neq 0$, then A is AS-Gorenstein.*

Proof. Suppose that the stated hypotheses on the Ext-groups hold. Using the double Ext-spectral sequence, or following the proof of [BG1, Theorem 1.13], one shows that $\operatorname{Ext}^i_A(M,A) = 0$ and $\operatorname{Ext}^i_{A^{\mathsf{op}}}(N,A) = 0$ for all $i \neq d$ and all finite dimensional left A-modules M and finite dimensional right A-modules N. By the proof of [WZ2, Lemma 4.8] $\dim \operatorname{Ext}^d_A(M,A) = \dim M \cdot \dim \operatorname{Ext}^d_A(k,A)$ for any finite dimension left A-module M. Thus

$$\dim \operatorname{Ext}^d_A(\operatorname{Ext}^d_{A^{\mathsf{op}}}(k,A),A) \;=\; \dim \operatorname{Ext}^d_{A^{\mathsf{op}}}(k,A) \cdot \dim \operatorname{Ext}^d_A(k,A).$$

By the double-Ext spectral sequence (3.2.2), $\operatorname{Ext}^d_A(\operatorname{Ext}^d_{A^{\mathsf{op}}}(k,A),A) \cong k$. Hence

$$\dim \operatorname{Ext}^d_{A^{\mathsf{op}}}(k,A) \;=\; \dim \operatorname{Ext}^d_A(k,A) \;=\; 1.$$

Thus we have proved (AS2). □

3.3. **The link with integrals.** We need two fundamental observations. The first is due to Feng and Tsygan [FT, Corollary (2.5)] (see also [HK1, Proposition 2.3]):

Proposition. [FT] *Let A be a Hopf algebra and let M be an A-bimodule. Then, for all $i \geq 0$, there are vector space isomorphisms*

$$H_i(A,M) \cong \operatorname{Tor}_i^{A^e}(A,M) \cong \operatorname{Tor}_i^A(M',k).$$

Consequently, $\operatorname{tHdim} A \leq \operatorname{gldim} A$.

Proof. By [We, Lemma 9.1.3], $H_i(A,M) \cong \operatorname{Tor}_i^{A^e}(A,M)$. To see the second isomorphism we use the fact that $A \otimes_{A^e} M \cong M' \otimes_A k$ and that the functor $(-)' : A^e\text{-}\mathsf{Mod} \to A^{\mathsf{op}}\text{-}\mathsf{Mod}$ preserves projectives and is exact. Finally, $\operatorname{tHdim} A \leq \operatorname{gldim} A$ follows from the isomorphisms. □

The second observation is a version of Poincaré duality for AS-regular Hopf algebras, which, as we shall note in Section 6, is sufficient to retrieve some classical results:

Lemma. *Let A be an augmented AS-regular algebra of global dimension d. Then for any right A-module N,*

$$\mathrm{Tor}^A_{d-i}(N,k) \quad \cong \quad \mathrm{Ext}^i_{A^{\mathrm{op}}}(\textstyle\int^l, N)$$

for all i.

Proof. The isomorphism follows from Ischebeck's spectral sequence (3.2.1), taking $M = k$, together with the Artin-Schelter condition (AS2). □

Noting (3.2.1), and combining the above proposition and lemma, we immediately deduce the following partial Poincaré duality for the Hochschild homology of AS-regular Hopf algebras.

Corollary. *Let A be an AS-regular Hopf algebra of global dimension d. Let M be an A-bimodule. Then, for every integer i,*

$$H_{d-i}(A,M) \;\cong\; \mathrm{Ext}^i_{A^{\mathrm{op}}}(\textstyle\int^l, (M)').$$

As a consequence, if σ is an algebra automorphism of A, then, for every i,

$$H_{d-i}(A,{}^\sigma A) \;\cong\; \mathrm{Ext}^i_{A^{\mathrm{op}}}(\textstyle\int^l, ({}^\sigma A)').$$

Remark. When S is bijective, this corollary can be deduced from Van den Bergh's general Poincaré duality result Proposition 5.1 – see Corollary 5.2. However it is unknown at present whether S is bijective in general, so it seems worth recording this corollary.

3.4. **Proof of Theorem 0.4(a).** In this subsection we assume that S is bijective. Recall from (2.5.1) that $\xi = \Xi[\pi_0]$, the left winding automorphism obtained from $\pi_0 : A \to A/(\mathrm{r.ann}(\int^l))$.

Theorem. *Let A be a noetherian AS-regular Hopf algebra of global dimension d.*

(a) Let σ be an algebra automorphism of A. Then $H_d(A,{}^\sigma A) \cong Z({}^{S^2\xi\,\sigma}A)$. Consequently, $H_d(A,{}^\sigma A) \neq 0$ if and only if $S^2\xi\,\sigma \in N(A)$.
(b) $H_d(A,{}^{\xi^{-1}S^{-2}}A) \cong Z(A) \neq 0$. Consequently, $\mathrm{tHdim}\,A = d$.
(c) $\mathrm{Hdim}\,A = d$ *if and only if $S^2\xi \in N(A)$.*

Proof. (a) By Corollary 3.3

$$H_d(A,{}^\sigma A) \;\cong\; \mathrm{Hom}_{A^{\mathrm{op}}}(\textstyle\int^l, ({}^\sigma A)') \;=\; \mathrm{Hom}_{A^{\mathrm{op}}}(\textstyle\int^l, (A^{\sigma^{-1}})').$$

By Lemmas 2.6(d) and 2.3(b),

$$\mathrm{Hom}_{A^{\mathrm{op}}}(\textstyle\int^l, (A^{\sigma^{-1}})') \;\cong\; Z(A^{S^{-2}\sigma^{-1}\xi^{-1}}) \;\cong\; Z(A^{\sigma^{-1}\,\xi^{-1}S^{-2}}) \;\cong\; Z({}^{S^2\xi\,\sigma}A).$$

The second assertion follows from the definition $N(A)$

(b,c) Let M be an A-bimodule. By Proposition 3.3, $H_i(A,M) = \mathrm{Tor}^A_i(M',k) = 0$ for all integers $i > d$. Hence $\mathrm{Hdim}\,A \le d$ and $\mathrm{tHdim}\,A \le d$. Therefore the assertions follow from (a). □

3.5. **Remarks.** (a) When A is an affine commutative Hopf algebra, and k has characteristic 0, the global dimension hypothesis always holds [Mo, 9.2.11 and 9.3.2], so that A is commutative Gorenstein and therefore AS-regular [Ba]. Moreover the homological integral is clearly in this case a trivial module, and S^2 is the identity map by [Mo, Corollary 1.5.12]. So Theorem 3.4(a) reduces in this case to the statement that

$$HH_d(A) \cong Z(A) = A,$$

which is a consequence of the Hochschild-Kostant-Rosenberg theorem [We, Theorem 9.4.7].

(b) If A has non-central regular normal elements, then there are algebra automorphisms σ other than $\xi^{-1}S^{-2}$ such that $H_d(A, {}^\sigma A) \neq 0$.

(c) The determination of the automorphism ξ in particular classes of examples, and its relation to other "canonical" automorphisms of the algebra, forms part of the content of later sections.

(d) To understand $H_{d-i}(A, {}^\sigma A)$ for all i, one will need to understand the minimal injective resolution of $({}^{S^2\xi}\,{}^\sigma A)'$. This seems to be quite hard in general.

4. AS-Gorenstein algebras and dualising complexes

4.1. **Dualising Complexes.** The noncommutative version of the dualising complex was introduced by Yekutieli in [Ye1], and has now become one of the standard homological tools of noncommutative ring theory. We review in the next four paragraphs the small amount of this theory which we shall need to draw on. Most of the details, including some of the necessary facts about derived categories, can be found in [Ye1, VdB2, YZ]. Let A be an algebra and let $\mathsf{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})$ denote the bounded derived category of left A-modules. A complex X of left A-modules is called *homologically finite* if $\bigoplus_i H^i(X)$ is a finitely generated A-module.

Definition. Let A be a noetherian algebra. A complex $R \in \mathsf{D}^{\mathrm{b}}(A^e\text{-}\mathsf{Mod})$ is called a *dualising complex* over A if it satisfies the following conditions:

(a) R has finite injective dimension over A and over A^{op} respectively.
(b) R is homologically finite over A and over A^{op} respectively.
(c) The canonical morphisms $A \to \mathrm{RHom}_A(R, R)$ and $A \to \mathrm{RHom}_{A^{\mathsf{op}}}(R, R)$ are isomorphisms in $\mathsf{D}(A^e\text{-}\mathsf{Mod})$.

If A is $\mathbb{Z}$-graded, a graded dualising complex is defined similarly.

4.2. **Rigid dualising complexes.** Let R be a complex of A^e-modules, viewed as a complex of A-bimodules. Let R^{op} denote the "opposite complex" of R which is defined as follows: as a complex of k-modules $R^{\mathsf{op}} = R$, and the left and right A^{op}-module actions on R^{op} are given by

$$a \cdot r \cdot b \quad := \quad bra$$

for all $a, b \in A^{\mathsf{op}}$ and $r \in R^{\mathsf{op}} (= R)$. If $R \in \mathsf{D}(A^e\text{-}\mathsf{Mod})$ then $R^{\mathsf{op}} \in \mathsf{D}^{\mathrm{b}}((A^{\mathsf{op}})^e\text{-}\mathsf{Mod})$. The flip map

$$\tau : (A^{\mathsf{op}})^e = A^{\mathsf{op}} \otimes A \longrightarrow A \otimes A^{\mathsf{op}} = A^e$$

is an algebra isomorphism, so $(A^{\mathsf{op}})^e$ is isomorphic to A^e. Hence there is a natural isomorphism $\mathsf{D}^{\mathrm{b}}(A^e\text{-}\mathsf{Mod}) \cong \mathsf{D}^{\mathrm{b}}((A^{\mathsf{op}})^e\text{-}\mathsf{Mod})$.

An unfortunate failing of dualising complexes as defined in (4.1) is their lack of uniqueness. To remedy this defect Van den Bergh [VdB2] introduced the idea of a rigid complex:

Definition. [VdB2] Let A be a noetherian algebra. A dualising complex R over A is called *rigid* if there is an isomorphism

$$R \cong \mathrm{RHom}_{A^e}(A, R \otimes R^{\mathsf{op}})$$

in $\mathsf{D}(A^e\text{-}\mathsf{Mod})$. Here the left A^e-module structure of $R \otimes R^{\mathsf{op}}$ comes from the left A-module structure of R and the left A^{op}-module structure of R^{op}.

Earlier, Yekutieli [Ye1] defined the concept of a *balanced* dualising complex over a graded algebra A; a balanced dualising complex over a connected graded algebra is rigid, [VdB2, Proposition 8.2(2)]. If the algebra A has a rigid dualising complex R, then R is unique up to isomorphism, [VdB2, Proposition 8.2].

4.3. **Van den Bergh condition.**

Definition. Suppose that A has finite injective dimension d. Then A satisfies the *Van den Bergh condition* if

$$\mathrm{Ext}^i_{A^e}(A, A^e) = \begin{cases} 0 & i \neq d \\ U & i = d \end{cases}$$

where U is an invertible A-bimodule.

For us, the Van den Bergh condition will constitute a key hypothesis in deriving Poincaré duality between Hochschild cohomology and Hochschild homology; see Proposition 5.1. Our definition is motivated by [VdB2, Proposition 8.4], which we reformulate as the following

Proposition. [VdB2] *Let A be a noetherian algebra. Then the Van den Bergh condition holds if and only if A has a rigid dualising complex $R = V[s]$, where V is invertible and $s \in \mathbb{Z}$. In this case $U = V^{-1}$ and $s = d$.*

4.4. **Rigid Gorenstein algebras and Nakayama automorphisms.** The definition of the Van den Bergh condition leads us naturally to the following

Definition. Let A be an algebra with finite injective dimension d.

(a) A is *rigid Gorenstein* if there is an algebra automorphism ν such that

$$\mathrm{Ext}^i_{A^e}(A, A^e) = \begin{cases} 0 & i \neq d \\ {}^1A^\nu & i = d \end{cases}$$

as A-bimodules.

(b) The automorphism ν is called the *Nakayama automorphism* of A.

(c) The *Nakayama order* of A, denoted by $o(A)$, is the smallest positive integer n such that ν^n is inner, or ∞ if no such n exists.

By Proposition 4.3, A is rigid Gorenstein if and only ${}^\nu A^1[d]$ is a rigid dualising complex of A. The terminology in (b) generalises standard usage from the classical theory of Frobenius algebras, as - for example - in [Ya, Section 2.1]. For a Frobenius algebra A has as its rigid dualising complex the k-dual A^* of A [Ye3, Proposition 5.9], and $A^* \cong {}^\nu A^1$, by [Ya, Theorem 2.4.1].

The following observations are clear from the definition and (2.3.2).

Proposition. *(a) The Nakayama automorphism is determined up to multiplication by an inner automorphism of A.*

(b) The Nakayama automorphism acts trivially on $Z(A)$.

4.5. **AS-Gorenstein Hopf algebras are rigid Gorenstein.** In this subsection we prove the statement in the heading, and also that the automorphism ξS^2 is the Nakayama automorphism of A. (Recall that ξ is defined in (2.5.1).) Throughout this subsection we assume that S is bijective.

Lemma. *Let A be a Hopf algebra and let $\pi : A \to k$ be an algebra map. Then, as A-bimodules,*

$$k_\pi \otimes_A L(A^e) \;=\; k^{\Xi[\pi]} \otimes_A L(A^e) \;\cong\; {}^1A^{\Xi[\pi]S^2}.$$

Proof. The lemma follows from a direct computation after we understand the module structure on $L(A^e)$.

As a k-vector space, A^e is isomorphic to $A \otimes A$; and there are four different A-actions on A^e which commute with each other. To avoid possible confusions we will try to use different notations to indicate different places of A and different actions of A. Let $B = A$ as a Hopf algebra and $N = A$ as A-bimodule. We denote the four A-module structures as follows. The left A-action on A and the right B-action on A, both induced by the multiplication of A, are denoted by $*_1$ and $*_2$ respectively. The left B-action on N induced by the multiplication of A is denoted by $*_3$ and another left A-action on N defined by

$$a *_4 n \;=\; nS(a).$$

Hence $A \otimes N$ has a left $A \otimes A$-module structure induced by $*_1$ and $*_4$ and a B-bimodule structure induced by $*_3$ and $*_2$. The left A-action on $L(A^e)$ is defined via the coproduct Δ and the $A \otimes A$-action on $A \otimes N$. The assertion of the lemma is equivalent to the following statement: as B-bimodules,

$$k_\pi \otimes_A \mathrm{Res}_\Delta(A \otimes N) \cong {}^1B^{\Xi[\pi]S^2}. \tag{4.5.1}$$

Let $J = \ker \pi$. Then $A/J = k_\pi$ as right A-module and π induces a natural B-bimodule homomorphism

$$f \;:\; \mathrm{Res}_\Delta(A \otimes N) \;\to\; \mathrm{Res}_\Delta(A \otimes N)/J\,\mathrm{Res}_\Delta(A \otimes N) = k_\pi \otimes_A \mathrm{Res}_\Delta(A \otimes N).$$

To prove the assertion (4.5.1) it suffices to show the following:

$$\Xi[\pi]S^2(b) *_3 (1 \otimes 1) - (1 \otimes 1) *_2 b \;\; = \; 1 \otimes \Xi[\pi]S^2(b) - b \otimes 1 \in J\,\mathrm{Res}_\Delta(A \otimes N). \tag{4.5.2}$$

$$b *_3 (1 \otimes 1) = 1 \otimes b \notin J\,\mathrm{Res}_\Delta(A \otimes N). \tag{4.5.3}$$

By Lemma 2.1(b), there is an isomorphism $\Phi : \mathrm{Res}_\Delta(A \otimes N) \to A \otimes_k N$ where $\Phi(a \otimes n) = \sum a_1 \otimes S(a_2) *_4 n$. This isomorphism induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Res}_\Delta(A \otimes N) & \xrightarrow{\Phi} & A \otimes_k N \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ k_\pi \otimes_A \mathrm{Res}_\Delta(A \otimes N) & \xrightarrow{\cong} & A/J \otimes_k N \end{array}$$

where $f' : A \otimes N \to A \otimes N/J(A \otimes N) = A/J \otimes N = k \otimes N$ is the canonical map. Now the conditions in (4.5.2) and (4.5.3) are equivalent to the following two conditions

$$f'\Phi(1 \otimes \Xi[\pi]S^2(b) - b \otimes 1) = 0 \;\; \textit{for all} \;\; b \in B. \tag{4.5.4}$$

$$f'\Phi(1 \otimes b) \neq 0 \;\; \textit{for all} \;\; b \in B. \tag{4.5.5}$$

By definition,

$$f'\Phi(b\otimes 1) = f'(\sum b_1\otimes S(b_2)*_4 1) = f'(\sum b_1\otimes S^2(b_2)) = \sum \pi(b_1)\otimes S^2(b_2).$$

By Lemma 2.5(d),

$$\sum \pi(b_1)\otimes S^2(b_2) = \sum \pi(S^2(b_1))\otimes S^2(B_2) = \sum 1\otimes \pi(S^2(b_1))S^2(b_2).$$

Since S^2 commutes with Δ, we have

$$\sum 1\otimes \pi(S^2(b_1))S^2(b_2) = 1\otimes \Xi[\pi]S^2(b) = f'\Phi(1\otimes \Xi[\pi]S^2(b)).$$

Thus (4.5.4) follows. To see (4.5.5) we note, for every $0\neq b\in B$,

$$f'\Phi(1\otimes b) = 1\otimes b$$

which is a nonzero element in $k\otimes N$. □

Proposition. *Let A be a noetherian AS-Gorenstein Hopf algebra (with bijective antipode S). Let d be the injective dimension of A.*

(a) A is rigid Gorenstein with Nakayama automorphism ξS^2.
(b) The rigid dualising complex of A is ${}^{\xi S^2}A^1[d]$.
(c) $\int_A^l = {}^1k^{\xi S^2} = {}^1k^{\xi}$.

Proof. (a) By Lemma 2.4(b), for all $i\geq 0$,

$$\mathrm{Ext}^i_{A^e}(A, A^e)\cong \mathrm{Ext}^i_A(k, L(A^e)).$$

Since $L(A^e)$ is a free left A-module by Lemma 2.2(c), we have

$$\mathrm{Ext}^i_A(k, L(A^e))\cong \mathrm{Ext}^i_A(k, A)\otimes_A L(A^e).$$

By the AS-Gorenstein condition, for every $i<d$, $\mathrm{Ext}^i_A(k,A)=0$ and hence $\mathrm{Ext}^i_{A^e}(A,A^e)=0$. The only nonzero term is

$$\mathrm{Ext}^d_{A^e}(A, A^e)\cong \mathrm{Ext}^d_A(k, A)\otimes_A L(A^e)\cong k^{\xi}\otimes_A L(A^e)\cong {}^1A^{\xi S^2}$$

where the last isomorphism follows from the above lemma. Therefore the assertion follows.

(b) This follows from (a) and Proposition 4.3.

(c) This is immediate from the definition, since S^2 fixes all one-dimensional representations of A. □

Remark. The case of the above proposition where A has finite k-dimension (or, equivalently, $d=0$), is [Sc, Proposition 3.6]. Note also the earlier result of Oberst and Schneider [OS]: a finite dimensional Hopf algebra is *symmetric* - that is, its Nakayama automorphism is the identity - if and only if $S^2=\xi=\mathrm{id}_A$.

4.6. **The antipode.** If A is a Hopf algebra (with bijective antipode S as usual), then so is $\mathbf{A}' := (A, \Delta^{\mathsf{op}}, S^{-1}, \epsilon)$ [Mo, Lemma 1.5.11]. Suppose that A is noetherian and AS-Gorenstein. Then we can apply Proposition 4.5 to $\mathbf{A}'$. In particular we can conclude from (a) of Proposition 4.5 that A has Nakayama automorphism

$$\nu' \quad = \quad \phi S^{-2},$$

where ϕ is the *right* winding automorphism (with respect to Δ) of the left integral of A. That is, in the notation of (2.5),

$$\phi(a) = \sum a_1\pi_0(a_2)$$

for all $a \in A$.

However, as we noted in Proposition 4.4(a), the Nakayama automorphism of A is unique up to an inner automorphism. In view of Proposition 4.5(a) we have therefore proved the

Corollary. *Let A be a noetherian AS-Gorenstein Hopf algebra with bijective antipode S. Then*

$$S^4 \quad = \quad \gamma \circ \phi \circ \xi^{-1} \tag{4.6.1}$$

where ξ and ϕ are respectively the left and right winding automorphisms given by the left integral of A, and γ is an inner automorphism.

Naturally one immediately asks:

Question. What is the inner automorphism γ in Corollary 4.6?

The answer is known when A is finite dimensional, thanks to Radford's 1976 paper [Ra]. Suppose that A has finite dimension. Then Radford proved in [Ra, Proposition 6] a version of Corollary 4.6 with the added information that γ *is conjugation by the group-like element of A which is the character of the right structure on* $\int^l_{A^*}$. In particular, notice from this that - in general - γ is not trivial. It is tempting to suspect that, for noetherian A, the Hopf dual A° of A will play an important role in answering the above question.

Observe that the three maps composed to give S^4 in (4.6.1) commute with each other, so one deduces at once the main result of [Ra] - namely, that S has finite order when A is finite dimensional. We can generalise this result somewhat, as in the following proposition. Recall that $io(A)$ denotes the *integral order* of A which is (by definition) the order of ξ (see [LWZ, Definition 2.2]).

Proposition. *Let A be a noetherian AS-Gorenstein Hopf algebra with bijective antipode S.*

(a) The automorphisms γ, ϕ and ξ in the corollary commute with each other.
(b) The Nakayama order $o(A)$ is equal to either $io(A)$ or $2\, io(A)$.
(c) Suppose $io(A)$ is finite. Then $S^{4\, io(A)}$ is an inner automorphism.
(d) If A has only finitely many one-dimensional modules, then $io(A)$ is finite. In this case, some power of S and of ν is inner.

Proof. (a) It is easy to see from the definition that ϕ and ξ commute. By Lemma 2.5(d), S^2 commutes with ξ. Similarly, S^2 commutes with ϕ. Since $S^4 = \gamma \circ \phi \circ \xi^{-1}$, ϕ and ξ commute with γ.

(b,c) Let $n = io(A)$ and $m = o(A)$. By the definition of $o(A)$, ν^m is inner. Thus, by Proposition 4.5 and Lemma 2.5(d), $S^{2m}\,\xi^m$ is inner. Since, for any inner automorphism τ, $\epsilon\tau = \epsilon$ and since $\epsilon S^{2m} = \epsilon$, we have

$$\epsilon = \epsilon\nu^m = \epsilon S^{2m}\xi^m = \epsilon\xi^m = \pi_0^m.$$

This means that the order n of π_0 in $G(A^\circ)$ divides m.

On the other hand, $S^4 = \gamma \circ \phi \circ \xi^{-1}$. Then, by (a),

$$S^{4n} = \gamma^n \circ \phi^n \circ \xi^{-n} = \gamma^n.$$

The final equality follows from the fact that n is the order of ξ and ϕ. So S^{4n} is inner. This implies that

$$\nu^{2n} = (S^2\xi)^{2n} = S^{4n}\xi^{2n} = \gamma^n$$

which is inner. Hence m divides $2n$. Therefore m is either n or $2n$.

(d) Let A_{ab} be the Hopf algebra A/I, where I is the Hopf ideal generated by elements $xy - yx$ for all $x, y \in A$. If A has finitely many 1-dimensional modules, then A_{ab} is finite dimensional. By [LWZ, Lemma 4.5], $io(A)$ divides $\dim A_{ab}$, which is finite. The second statement follows from (b,c). □

Remarks. (1) The conclusion of Proposition 4.6(d) is also valid when A is AS-regular and PI (meaning that A satisfies a polynomial identity), as we'll show in (6.2).

(2) In general both $io(A)$ and therefore $o(A)$ can be infinite for an AS-Gorenstein noetherian Hopf algebra: consider, for example the enveloping algebra A of the two-dimensional complex non-abelian Lie algebra, for which $io(A)$ will be shown to be infinite in Proposition 6.3(c). Similarly, in general no power of S for such an A is inner, as illustrated, for example, by $A = \mathcal{O}_q(SL(2, \mathbb{C}))$, for a generic parameter q. The only inner automorphism in this case is the identity, by [Jo, Lemma 9.1.14], and so (6.6.1) confirms that S^2 has infinite order in the outer automorphism group.

5. Twisted Hochschild cohomology

In this section we will complete the proof of Theorem 0.2(b).

5.1. **Cohomology and duality.** We begin with some definitions parallel to those introduced for homology in (3.1). Let A be an algebra, and let σ be an algebra automorphism of A. The *twisted Hochschild cohomology groups* $HH^*_\sigma(A)$ of A with respect to σ were defined in [KMT] in a way exactly analogous to the twisted homology groups discussed in (3.1). When σ is diagonalisable the argument of [HK1, Proposition 2.1] applies to show that, for all $i \geq 0$,

$$HH^i_\sigma(A) \;\cong\; H^i(A, {}^\sigma A).$$

Moreover [We, Corollary 9.1.5] can be invoked to show that, for all $i \geq 0$,

$$H^i(A, {}^\sigma A) \;\cong\; \mathrm{Ext}^i_{A^e}(A, {}^\sigma A).$$

Definition. (a) The *Hochschild cohomological dimension* of A is

$$\mathrm{Hcdim}\, A = \max\{i \mid HH^i(A) = H^i(A, A) \neq 0\}.$$

(b) The *twisted Hochschild cohomological dimension* of A is

$$\mathrm{tHcdim}\, A = \max\{i \mid H^i(A, {}^\sigma A) \neq 0 \text{ for some automorphism } \sigma \text{ of } A\}.$$

(c) We say that A has *finite homological dimension* if there is an integer d such that $H^i(A, -) = 0$ for all $i > d$.

In the presence of the Van den Bergh condition introduced in (4.3) there is a beautiful twisted version of Poincaré duality for A.

Proposition. [VdB1] *Let A be an algebra of finite homological dimension d. Suppose that the Van den Bergh condition holds. Then for every A-bimodule N,*

$$H^i(A, N) = H_{d-i}(A, U \otimes_A N)$$

for all i, where U is the invertible A-bimodule $\mathrm{Ext}^d_{A^e}(A, A^e)$.

The above result provides further evidence that, at least for rigid Gorenstein algebras, twisted (co)homology is an important and necessary part of the study of Hochschild homology and cohomology.

Corollary. *Let A be a noetherian rigid Gorenstein algebra with Nakayama automorphism ν. Suppose that A has finite homological dimension d.*

(a)

$$H_d(A, A^\nu) \;\cong\; Z(A) \neq 0,$$

and so $\operatorname{tHdim} A = d$.

(b)

$$H^d(A, {}^\nu A) \;\cong\; A/[A, A].$$

If A has a non-zero module of finite k-dimension then $\operatorname{tHcdim} A = d$.

Proof. By definition $U = {}^1A^\nu$. Let $V = U^{-1} = {}^\nu A^1$.

(a) By Proposition 5.1,

$$H_d(A, {}^1A^\nu) \cong= H^0(A, A) \cong Z(A) \supset k.$$

(b) By Proposition 5.1,

$$H^d(A, {}^\nu A^1) \;\cong\; H_0(A, U \otimes_A {}^\nu A^1) \;\cong\; H_0(A, A) \;=\; A/[A, A].$$

Suppose that A has a non-zero finite dimensional module. We claim that $1 \notin [A, A]$, so that $A/[A, A] \neq 0$. To see this we can pass to a finite dimensional factor ring of A and assume that A is a simple finite dimensional algebra, hence a central simple algebra. In that case it is well-known that $1 \notin [A, A]$.

When $A/[A, A] \neq 0$, then $H^d(A, {}^\nu A^1) \neq 0$ and hence $\operatorname{tHcdim} A = d$. □

5.2. **Poincaré duality for Hopf algebras.** In parallel to Proposition 3.3 for Hochschild dimension, there is an easy upper bound for the twisted cohomological dimension of a Hopf algebra. Throughout this subsection we assume S is bijective.

Lemma. *Let A be a Hopf algebra of finite global dimension d. Then A has finite homological dimension, bounded by d. As a consequence,* $\operatorname{tHcdim} A \leq d$.

Proof. By [We, Corollary 9.1.5], for all $i \geq 0$,

$$H^i(A, M) \;\cong\; \operatorname{Ext}^i_{A^e}(A, M)$$

for every A-bimodule M. By Lemma 2.4(b),

$$\operatorname{Ext}^i_{A^e}(A, M) \;\cong\; \operatorname{Ext}^i_A(k, L(M)) \;=\; 0$$

for all $i > d$. □

The following corollary is an immediate consequence of Propositions 5.1 and 4.5 and the above lemma.

Corollary. *Let A be a noetherian AS-regular Hopf algebra of global dimension d. Suppose that S is bijective, and let ξ be the winding automorphism induced by the left integral, (2.5.1). Then for every A-bimodule M and for all i, $0 \leq i \leq d$,*

$$H^i(A, M) = H_{d-i}(A, {}^{\xi^{-1}S^{-2}}M).$$

5.3. **Proof of Theorem 0.2(b).** Given an A-bimodule M, we let $[A, M]$ denote the k-subspace of M spanned by $\{am - ma : a \in A,\ m \in M\}$. It is easy to see that

$$H_0(A, M) = A \otimes_{A^e} M = M/[A, M].$$

The following result follows from Corollary 5.2, and incorporates the remainder of Theorem 0.2(b).

Theorem. *Let A be a noetherian AS-regular Hopf algebra of global dimension d and having bijective antipode S. Define $\nu = \xi S^2$.*

(a) For every algebra automorphism σ of A,

$$H^d(A, {}^\sigma A) \cong A \otimes_{A^e} {}^{nu^{-1}\sigma}A \ = \ {}^{\nu^{-1}\sigma}A/[A, {}^{\nu^{-1}\sigma}A].$$

(b) In particular,

$$H^d(A, {}^\nu A) \cong A/[A, A] \neq 0,$$

and so $\mathrm{tHcdim}\, A = d$.

(c) $\mathrm{Hcdim}\, A = d$ *if and only if* $A^\nu/[A, A^\nu] \neq 0$.

6. Examples

6.1. **Some general properties.** To apply the results of Sections 3 and 5 to a particular Hopf algebra A, we need to check that

(1) A is AS-regular, and

(2) the antipode S of A is bijective.

In all examples in this section the antipode S is bijective, thanks either to known descriptions of the antipode, or to Skryabin's result Proposition 1.1, so (2) will present no difficulties. All the examples considered here are Auslander-Gorenstein, and other than the group rings of (6.7), all the examples are Cohen-Macaulay. (See for example [BG1] for the definitions of these concepts.) So AS-regularity follows from the next lemma.

Lemma. *Let A be a noetherian Hopf algebra. If A is Auslander-Gorenstein (respectively, Auslander regular) and Cohen-Macaulay, then it is AS-Gorenstein (respectively, AS-regular).*

Proof. Let d be the injective dimension of A. Since $\mathrm{GKdim}\, k = 0$, the Cohen-Macaulay condition forces $\mathrm{Ext}^i_A(k, A) = 0$ for all $i \neq d$, and then $\mathrm{Ext}^d_A(k, A) \neq 0$ by [ASZ1, Theorem 2.3(2)]. Now the Auslander condition and the Cohen-Macaulay conditions applied to this latter module show that $\mathrm{Ext}^d_A(k, A)$ has GK-dimension zero; that is, it is finite dimensional over k. The lemma now follows from Lemma 3.2. □

Let ν be the Nakayama automorphism of A. By (3.1.1), to connect twisted Hochschild homologies $HH^\nu_i(A)$ with $H_i(A, {}^\nu A)$, one has to check that

(3) ν is diagonalisable.

In general, this condition will fail - see, for instance, Proposition 6.3(d). We will review the situation for each class of algebras in turn.

Finally, even when ν is not the identity, we sometimes nevertheless find that

(4) $\mathrm{Hdim}\, A = \mathrm{Hcdim}\, A = \mathrm{gldim}\, A$.

We will discuss when this is true.

6.2. **Noetherian PI Hopf algebras.** Let A be a noetherian affine PI Hopf algebra, and consider conditions (1) and (2) of (6.1). By Skryabin's result Proposition 1.1(b), the antipode of A is bijective, so (2) is satisfied. As for (1), first note that A is Auslander-Gorenstein, of dimension d, say, by [WZ1, Theorem 0.1]. By [Br2, Theorem A] there is an irreducible left A-module W for which $\mathrm{Ext}^d_A(W, A) \neq 0$, so that [BG1, Lemma 1.11] confirms that $\mathrm{Ext}^d_A(V, A) \neq 0$ for every left irreducible A-module V. Hence [SZ, Theorem 3.10(iii)(a)] ensures that A is (Krull) Cohen-Macaulay. Since the Krull and GK-dimensions coincide for affine noetherian PI algebras by [MR, Proposition 13.10.6 and Theorem 6.4.8], we can conclude that A is Auslander-Gorenstein and Cohen-Macaulay. Thus, by Lemma 6.1, A is AS-Gorenstein.

The situation regarding (3) of (6.1) in the PI case is less straightforward, since it depends on the characteristic of k. The enveloping algebra of the two-dimensional solvable non-abelian Lie algebra in positive characteristic demonstrates that diagonalisability of ξ and thus of ν will fail in general for Hopf algebras which are finite modules over their centres. Nevertheless, in this setting ν has finite order up to an inner automorphism, and so Maschke's theorem comes into play in characteristic 0. Here is the required result:

Proposition. *Let A be a noetherian affine PI Hopf algebra of finite global dimension.*

(a) The winding automorphism ξ has finite order.
(b) Some power of the antipode S of A is inner.
(c) The Nakayama automorphism ν has finite order up to inner automorphisms.

Proof. The discussion in the opening paragraph of this subsection shows that A is homologically homogeneous, so that A is a finite module over its centre by [SZ, Theorem 5.6(iv)].

(a) By [LWZ, Lemma 5.3(g)], $io(A)$ is finite. This is saying that ξ has finite order.

(b,c) Follow from part (a) and Proposition 4.6(b,c). □

Notice that the proof of the proposition works without the global dimension hypothesis provided A is a finite module over its centre.

Given (b) of the proposition, it makes sense to ask:

Question. If A is a noetherian affine PI Hopf algebra, does the antipode of A have finite order?

For regular PI Hopf algebras the equalities (4) of (6.1) are valid. To see this, we need a lemma. Recall for its proof that the Nakayama automorphism acts trivially on the centre, by Proposition 4.4(b).

Lemma. *Let A be a noetherian affine PI Hopf algebra of finite global dimension, and let $\nu = \xi S^2$ be its Nakayama automorphism.*

(a) $\nu, \nu^{-1} \in N(A)$.
(b) If $\sigma \in N(A)$ and the restriction of σ to the centre is the identity, then

$$ {}^{\sigma}A/[A, {}^{\sigma}A] \neq 0. $$

Proof. (a) By [BG1, Corollary 1.8], A is a direct sum of prime rings. The central idempotents corresponding to the direct summands of A are fixed by ν. For this

reason we may assume A is a prime ring (and forget about the coalgebra structure for the rest of the proof).

Since A is a noetherian affine prime PI algebra, the Goldie quotient ring $Q(A)$ is isomorphic to the central localization AC^{-1}, where $C = Z(A) \setminus \{0\}$. Clearly $Q(A)$ is a central simple algebra with centre $Z(A)C^{-1}$ [MR, Theorem 13.6.5]. The automorphism ν extends to $Q(A)$ naturally, still denoted by ν. The restriction of ν to the centre $Z(Q)$ is the identity, and so, by the Skolem-Noether theorem, ν is an inner automorphism. Thus there is an invertible element $x \in Q(A)$ such that $\nu(b) = xbx^{-1}$ for all $b \in Q(A)$. Write $x = ms^{-1}$ where $m \in A$ is not a zero divisor and $s \in Z(A) \setminus \{0\}$. Then, for every $a \in A$,

$$\nu(a)m \;=\; xax^{-1}m \;=\; xas \;=\; xsa \;=\; ma.$$

Hence $\nu \in N(A)$. The proof works for any automorphism whose restriction to the centre is the identity, and in particular for ν^{-1}.

(b) As in (a) we may assume A is prime. Let $Q = Q(A)$. It suffices to show that there is an elememt $m \in A$ such that $m \notin [Q, {}^{\sigma}Q]$. By the proof of (a), σ is an inner automorphism, say $\sigma(b) = mbm^{-1}$ for some regular element $m \in A$. For all $a, b \in Q$,

$$\sigma(b)a - ab = mbm^{-1}a - ab \;=\; m(bm^{-1}a - m^{-1}ab) \;=\; m[b, m^{-1}a].$$

This means that $[Q, {}^{\sigma}Q] = m[Q, Q]$. Since $1 \notin [Q, Q]$, we deduce $m \notin [Q, {}^{\sigma}Q]$. □

Theorem. *Let A be a noetherian affine PI Hopf algebra of GK-dimension d.*

(a) A has rigid dualising complex ${}^{\nu}A^1[d]$ where $\nu = \xi S^2$ is the Nakayama automorphism of A.
(b) Suppose that A has finite global dimension. Then $\operatorname{Hdim} A = \operatorname{Hcdim} A = \operatorname{gldim} A = d$.

Proof. (a) By the opening paragraph of (6.2) A is AS-Gorenstein of dimension d, so this is immediate from Proposition 4.5.

(b) This follows from the above lemma and Theorem 3.4(c) and 5.3(c). □

6.3. **Enveloping algebras.** In this subsection we drop the hypothesis that k is algebraically closed.

Proposition. *Let $\mathfrak{g}$ be a Lie algebra over k of finite dimension d, and let A be its universal enveloping algebra $U(\mathfrak{g})$.*

(a) $S^2 = \mathrm{id}_A$.
(b) A is an AS-regular Hopf algebra with $\operatorname{gldim} A = d$.
(c) [Ye2] *The rigid dualising complex of A is $(A \otimes \bigwedge^d \mathfrak{g})[d]$. Thus the Nakayama automorphism $\nu = \xi$ of A is the inverse of the winding automorphism $\Xi[\chi]$, where χ is the representation of $\mathfrak{g}$ on $\bigwedge^d \mathfrak{g}$. That is,*

$$\nu(x) \;=\; x + \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(x))$$

for $x \in \mathfrak{g}$.
(d) If $\operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(x)) \neq 0$, then ν is not diagonalisable.
(e) If $\mathfrak{g}$ is semisimple, then ν is the identity and hence $\operatorname{Hdim} A = \operatorname{Hcdim} A = d$.
(f) $H_d(A, A^{\nu}) \cong Z(A) \neq 0$, and $H^d(A, {}^{\nu}A) \cong A/[A, A]$, so that $\operatorname{tHcdim} A = \operatorname{tHdim} A = d$. *The twisted Poincaré duality of Corollary 5.2 holds for A.*

Proof. (a) Recall that the antipode is given by $S(x) = -x$ for $x \in \mathfrak{g}$, so S^2 the identity.

(b) That the global dimension is d is [CE, Theorem XIII.8.2]. It is well-known that A is Auslander-regular for all $\mathfrak{g}$. The assertion follows from Lemma 6.1.

(c) The description of rigid dualising complex of A is the main result of [Ye2].

(d) This follows from (c).

(e) If $\mathfrak{g}$ is semisimple, then $\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)) = 0$ for all $x \in \mathfrak{g}$. By (c), ν is the identity. The assertion follows from Theorems 3.4(c) and 5.3(c).

(f) This follows from Corollaries 5.1 and 5.2. □

Remark. Twisted Poincaré duality at the level of ordinary Lie algebra homology and cohomology follows for $U(\mathfrak{g})$ from Lemma 3.3. This retrieves the classical duality for Lie algebra (co)homology which dates back to the work of Koszul [Ko] and Hazewinkel [Ha].

6.4. **Quantised enveloping algebras.** For the definition of the quantised enveloping algebra $U_q(\mathfrak{g})$ of the semisimple Lie algebra $\mathfrak{g}$, see for example [BG2, I.6.3] or [Ja, 4.3]. The results in this case are already known; the details are as follows.

Proposition. *Let ℓ be an odd positive integer greater than 2. Let q denote either a primitive ℓth root of one in $\mathbb{C}$, or an indeterminate over $\mathbb{C}$; let k be $\mathbb{C}$ in the first case, or $\mathbb{C}(q)$ in the second case. Let $\mathfrak{g}$ be a semisimple complex Lie algebra of dimension d, and let A be the corresponding quantised enveloping algebra $U_q(\mathfrak{g})$. (If $\mathfrak{g}$ has a summand of type G_2 assume that 3 does not divide ℓ.)*

(a) S^2 is inner.
(b) A is AS-regular of global dimension d.
(c) The left integral of A is trivial, i.e., $\int_A^l \cong k$ as A-bimodule. Thus $\xi = \nu = \mathrm{id}_A$.
(d) A has rigid dualising complex $A[d]$.
(e) $\mathrm{Hdim}\, A = \mathrm{Hcdim}\, A = d$ and Poincaré duality holds.

Proof. (a) See [KS, Proposition 6, p. 164] or [Ja, (4.9)(1)].

(b,c) That A has finite global dimension was proved in [BG1, Proposition 2.2]. Then AS-regularity follows from [Ch, Proposition], and her result shows also that $\mathrm{gldim}\, A$ is d for all values of the parameter q, and that $\int_A^l \cong k$. The triviality of ν now follows from (a) and Proposition 4.5.

(d) This is the main theorem of [Ch]. (Of course it follows also from (c) and Proposition 4.5(b).)

(e) This is immediate from Theorems 3.4(c) and 5.3(c), and Corollary 5.2, and the fact that $\nu = \mathrm{id}_A$. □

6.5. **Quantised function algebras.** Our hypotheses on q, ℓ and k in this paragraph will be the same as in Proposition 6.4. Let G be a connected complex semisimple algebraic group of dimension d. The definition of the *quantised function algebra* $\mathcal{O}_q(G)$ is given in the case where q is an indeterminate over $\mathbb{C}$ in [BG2, I.7.5], and in the case where q is a primitive ℓth root of 1 in $\mathbb{C}$ in [BG2, III.7.1].

Proposition. *Let G, d, q, ℓ and k be as stated above. Let $A = \mathcal{O}_q(G)$.*

(a) A is AS-regular of dimension d.
(b) $\nu = \xi S^2$, A has rigid dualising complex ${}^{\nu}A^1[d]$, $\mathrm{tHdim}\, A = \mathrm{tHcdim}\, A = d$, and twisted Poincaré duality holds.

Proof. (a) Suppose first that q is a root of unity. By [BG1, Theorem 2.8] A is noetherian PI, Auslander regular and Cohen Macaulay of global dimension d, and everything is proved in (6.2). Suppose now that q is an indeterminate. It is proved in [BG1, Proposition 2.7] that $\operatorname{gldim} A$ is finite, and indeed the global dimension is d by [GZ, Theorem 0.1]. By [GZ, Theorem 0.1], A is Auslander-regular and Cohen-Macaulay. By Lemma 6.1, A is AS-regular.

(b) This follows from Proposition 4.5 and Corollaries 5.1 and 5.2. □

6.6. **The special linear group.** In Proposition 6.5 we did not answer the following

Question. What is the precise form of the Nakayama automorphism $\nu = \xi S^2$ of $\mathcal{O}_q(G)$?

Here we answer the question for the case $G = SL(n, \mathbb{C})$, giving ν in a form which, we conjecture, remains valid for *all* semisimple groups G. First we need the following lemma, a straightforward exercise in the use of change of rings formulae proved in [LWZ, Lemma 2.6]. Recall the definition of $\tau-$normal elements in (2.2).

Lemma. *Let A be an AS-Gorenstein algebra of injective dimension d, and let x be a τ-normal non-zero-divisor in the augmentation ideal of A. Let $\bar{A} := A/\langle x\rangle$.*

(a) If M is an x-torsionfree left A-module and N is an x-torsionfree right A-module, then

$$\operatorname{Ext}^1_A(A/(x), M) = {}^{\tau}(M/xM), \quad \operatorname{Ext}^1_{A^{\mathrm{op}}}(A/(x), N) = (N/Nx)^{\tau^{-1}}.$$

(b) $\bar{A}$ is an AS-Gorenstein algebra of dimension $d-1$, with augmentation induced by that of A.

(c) $$\operatorname{Ext}^d_A(k^{\tau^{-1}}, A) \quad \cong \quad (\operatorname{Ext}^d_A(k, A))^{\tau} \quad \cong \quad \operatorname{Ext}^{d-1}_{\bar{A}}(k, \bar{A}).$$

(d) As bimodules, $$\textstyle\int_A^l \quad \cong \quad (\int_{\bar{A}}^l)^{\tau^{-1}}.$$

Recall, from [BG2, I.2.2-I.2.4] for example, that $\mathcal{O}_q(SL_n)$ is generated by elements X_{ij} for $i, j = 1, \ldots, n$, with relations as given there. (The key ones for the proof of the proposition are stated at the start of the proof below.)

Proposition. *Let n be an integer greater than 1, and let $A = \mathcal{O}_q(SL_n)$ where q is a nonzero scalar.*

(a) $\int_A^l \cong A/\langle X_{ii} - q^{2(n+1-2i)}, X_{tj} : 1 \le i, j, t \le n,\ t \ne j\rangle$, as right A-modules.

(b) The automorphism ξ is determined by

$$\xi(X_{ij}) = q^{2(n+1-2i)} X_{ij}$$

for all i, j.

(c) The Nakayama automorphism of A is determined by

$$\nu(X_{ij}) = q^{2(n+1-i-j)} X_{ij}$$

for all i, j. If q is indeterminate, then ν is unique.

(d) ν is diagonalisable.

Proof. (a) Recall that the relations involving X_{1n} and X_{n1} are

$$X_{1j}X_{1n} = qX_{1n}X_{1j} : j \ne n \quad \text{and} \quad X_{in}X_{1n} = q^{-1}X_{1n}X_{in} : i \ne 1,$$
$$X_{j1}X_{n1} = qX_{n1}X_{j1} : j \ne n \quad \text{and} \quad X_{ni}X_{n1} = q^{-1}X_{n1}X_{ni} : i \ne 1;$$

and

$$X_{1n}X_{ij} = X_{ij}X_{1n} \quad \text{and} \quad X_{n1}X_{ij} = X_{ij}X_{n1} : i \ne 1, j \ne n.$$

In particular, therefore, X_{1n} and X_{n1} are normal elements. Let $\bar{A} = A/(X_{1n})$ and $\hat{A} = \bar{A}/(X_{n1})$. Consider first X_{1n}. The relations show that this element is τ-normal, where $\tau : X_{11} \to q^{-1}X_{11}, X_{ii} \to X_{ii}$ for $i \neq 1, n$, $X_{nn} \to qX_{nn}$, and, when $i \neq j$, $X_{ij} \to b_{ij}X_{ij}$ for some non-zero scalars b_{ij} whose exact value need not concern us. By Lemma 6.6(d),

$$\textstyle\int_A^l \quad \cong \quad (\int_{\bar{A}}^l)^{\tau^{-1}}.$$

The argument for X_{n1} is similar: again, one finds that

$$\textstyle\int_{\bar{A}}^l \quad \cong \quad (\int_{\hat{A}}^l)^{\tau^{-1}}.$$

Notice now that in $\hat{A}$, the cosets represented by $\{X_{2n}, X_{n2}\}$ form a pair of normal elements similar to $\{X_{1n}, X_{n1}\}$. Proceeding as above, a similar relation between integrals holds. The next pair of normal elements to deal with is $\{X_{1n-1}, X_{n-11}\}$. Continuing in this way to deal with all off-diagonal generators in pairs, eventually all such X_{ij}s are factored out. Let θ denote the composition of two copies of the winding automorphisms associated to each X_{ij}, for $i < j$, and set $B = k[X_{11}^{\pm 1}, \ldots, X_{(n-1)(n-1)}^{\pm 1}]$. We reach the conclusion

$$\textstyle\int_A^l \quad = \quad \mathrm{Ext}_A^{n^2-1}(k, A) \quad \cong \quad (\mathrm{Ext}_B^{n-1}(k, B))^{\theta^{-1}}.$$

That is, as right $A-$modules,

$$\textstyle\int_A^l \quad \cong \quad A/\langle X_{ii} - q^{2(n+1-2i)}, X_{tj} : 1 \leq i, j, t \leq n,\ t \neq j\rangle,$$

as claimed.

(b) Since left winding automorphisms of $\mathcal{O}_q(SL_n)$ take a constant value on each column of the matrix (X_{ij}), this follows from (a) and the definition of ξ.

(c) This is immediate from (b) and Proposition 6.5(b), given that

$$S^2(X_{ij}) = q^{2(i-j)}X_{ij}, \tag{6.6.1}$$

[FRT, Theorem 4]. When q is indeterminate, $\mathcal{O}_q(SL_n)$ does not have any non-trivial units, [Jo, Lemma 9.1.14]. Hence ν is unique.

(d) This is clear by using a monomial basis of $\mathcal{O}_q(SL_n)$. □

Remarks. (a) The argument we've just used for Proposition 6.6 works also for $\mathcal{O}_q(GL_n)$, and one gets the identical conclusion.

(b) The same method applies to the multiparameter quantum group $\mathcal{O}_{\lambda,p_{ij}}(GL_n)$, defined, for example, in [BG2, I.2.2-I.2.4]. We leave the details to the interested reader.

(c) Hadfield and Krähmer in [HK2] obtain Theorem 0.4(a) for $\mathcal{O}_q(SL_n)$ with the same automorphism ν as in Proposition 6.5(b).

(d) We give now a more symmetric, coordinate free formula for the Nakayama automorphism ν which, we conjecture, remains valid for arbitrary G. Recall, from [BG2, I.9.21(a)] for example, that given a Hopf algebra H there are left and right module algebra actions of the Hopf dual H° on H, defined respectively by

$$u.h = \sum h_1\langle u, h_2\rangle \quad \text{and} \quad h.u = \sum \langle u, h_1\rangle h_2,$$

for $h \in H$ and $u \in H^\circ$. In particular, the right and left winding automorphisms of H occur in this way, when we take u to be a group-like element of H°. When $H = \mathcal{O}_q(G)$ and $\mathfrak{g}$ is the Lie algebra of G, $U_q(\mathfrak{g}) \subseteq H^\circ$, and the group-like elements contained in $U_q(\mathfrak{g})$ are precisely the group $\langle K_\alpha : \alpha \in P\rangle$, where P is the root lattice of $\mathfrak{g}$.

Now let $\rho \in P$ be half the sum of the positive roots, and consider the right action of $K_{4\rho}$ on $\mathcal{O}_q(SL(n,\mathbb{C}))$. We calculate that, for $i,j = 1,\dots,n$,

$$X_{ij}.K_{4\rho} = \sum_t X_{it}(K_{4\rho})X_{tj} = q^{(\omega_i-\omega_{i-1},4\rho)}X_{tj} = q^{2(n-2i+1)}X_{ij}.$$

Here, $\omega_1,\dots,\omega_{n-1}$ are the fundamental weights of $\mathfrak{sl}(n,\mathbb{C})$, and $\omega_0 = \omega_n = 0$. Comparing this with Proposition 6.6(b) and bearing in mind Theorem 6.5(b) and the notation of (2.5), we see that

$$\pi_0 = K_{4\rho} \quad \textit{for the case} \quad G = SL(n,\mathbb{C}). \tag{6.6.2}$$

Conjecture. *With the above notation, (6.6.2) is valid for all semisimple groups; that is, the Nakayama automorphism of $\mathcal{O}_q(G)$ is $S^2\Xi[K_{4\rho}]$ for all semisimple groups G and all values of q permitted in Proposition 6.4.*

6.7. **Noetherian group algebras.**

Theorem. *Let k be a field of characteristic $p \geq 0$, and let G be a polycyclic-by-finite group of Hirsch length d.*

(a) kG is noetherian Auslander Gorenstein and AS-Gorenstein of injective dimension d.
(b) kG is AS-regular if and only if it is Auslander regular if and only if G contains no elements of order p.

Proof. (a) First, kG is noetherian by [Pa, Corollary 10.2.8]. Since we can find a poly-(infinite cyclic) normal subgroup H of finite index in G [Pa, Lemma 10.2.5], it is sufficient to prove that (i) kH is AS-Gorenstein (respectively Auslander Gorenstein) when H is poly-(infinite cyclic), and (ii) kH AS-Gorenstein (respectively Auslander Gorenstein) implies that kG is AS-Gorenstein (respectively Auslander Gorenstein) when $\mid G : H \mid$ is finite.

(i) We prove this by induction on the Hirsch number d of H. When $d = 0$ there is nothing to prove, so assume the result is true for all poly-(infinite cyclic) groups of Hirsch number less than d. Choose a normal subgroup T of H with $H/T = \langle xT\rangle$ infinite cyclic.

AS-Gorenstein property: By the induction hypothesis, $\mathrm{Ext}^i_{kT}(k,kT) = \delta_{i,d-1}k$. Applying $kH \otimes_{kT} -$ to a projective $kT-$resolution of k, it is easy to prove that, for all $i \geq 0$,

$$\mathrm{Ext}^i_{kH}(kH \otimes_{kT} k, kH) \quad \cong \quad \mathrm{Ext}^i_{kT}(k,kT) \otimes_{kT} kH \tag{6.7.1}$$

as right $kH-$modules. In particular, therefore, as right $k\langle x\rangle-$modules,

$$\mathrm{Ext}^i_{kH}(kH \otimes_{kT} k, kH) \cong \delta_{i,d-1}k\langle x\rangle.$$

Let's write ${}^H_T\!\uparrow k$ for $kH \otimes_{kT} k$. The short exact sequence of left $kH-$modules

$$0 \longrightarrow {}^H_T\!\uparrow k \longrightarrow {}^H_T\!\uparrow k \longrightarrow k \longrightarrow 0,$$

where the left-hand embedding is given by right multiplication by $x-1$, induces exact sequences

$$\mathrm{Ext}^i_{kH}({}^H_T\!\uparrow k, kH) \longrightarrow \mathrm{Ext}^i_{kH}({}^H_T\!\uparrow k, kH) \longrightarrow \mathrm{Ext}^{i+1}_{kH}(k,kH) \longrightarrow \mathrm{Ext}^{i+1}_{kH}({}^H_T\!\uparrow k, kH),$$

where the left hand and right hand maps are left multiplication by $x-1$, [Br2, Lemma 2.1]. Taking $i < d-2$ here, and applying the induction hypothesis together

with (6.7.1), shows that $\mathrm{Ext}^t_{kH}(k, kH) = 0$ for $t \leq d-2$. Taking $i = d-2$, we find in view of (6.7.1) that $\mathrm{Ext}^{d-1}_{kH}(k, kH)$ is the kernel of the map

$$\mathrm{Ext}^{d-1}_{kH}({}^H_T\!\uparrow k, kH) \to \mathrm{Ext}^{d-1}_{kH}({}^H_T\!\uparrow k, kH),$$

a homomorphism ρ between free $k\langle x\rangle-$modules of rank one given by multiplication by $x-1$. Therefore, $\mathrm{Ext}^{d-1}_{kH}(k, kH)$ is 0. Moreover, since $\mathrm{Ext}^{d+1}_{kH}({}^H_T\!\uparrow k, kH)$ is 0, by (6.7.1), the cokernel of ρ is $\mathrm{Ext}^d_{kH}(k, kH)$. Thus $\mathrm{Ext}^d_{kH}(k, kH)$ has k-dimension one, as required.

Auslander Gorenstein property: Note that $kH = kT[t^{\pm 1}; \sigma]$ for some automorphism σ of kT. By the induction hypothesis, kT is Auslander Gorenstein. By [Ek, Theorem 4.2], $kT[t; \sigma]$ is Auslander Gorenstein. By [ASZ2, Proposition 2.1] the localization $kT[t^{\pm 1}; \sigma]$ is Auslander Gorenstein.

(ii) This follows because crossed products of finite groups are Frobenius extensions - see [AB, Lemma 5.4].

(b) This follows from (a) and [Pa, Theorem 10.3.13]. □

Remarks. (a) It is an open question whether all noetherian group algebras kG come from polycyclic-by-finite groups G.

(b) In general a noetherian group algebra kG could have infinite GK-dimension [KL, Corollary 11.15]. Even when $\mathrm{GKdim}\, kG$ is finite, it could be larger than the global dimension of kG. Hence in these cases, kG is not Cohen-Macaulay.

Poincaré duality of group homology and cohomology was investigated by Bieri in a series of papers beginning with [Bi1]. Given a commutative ring R, a non-negative integer n and a group G, he defined [Bi2, 5.1.1] G to be a *duality group of dimension n over R* if there exists an $RG-$module C such that, for each $RG-$module A and each $k \in \mathbb{Z}$, there are isomorphisms

$$H^k(G, A) \cong H_{n-k}(G, C \otimes_R A)$$

induced by the cap product. Moreover, [Bi2, Definition 3.2], G is a *Poincaré duality group of dimension n over R* if C above is R with *almost trivial* $G-$structure, where this means that the image of G in $\mathrm{Aut}_R(R)$ is $\{\pm \mathrm{Id}\}$.

It's clear that, for polycyclic-by-finite groups, Bieri's results describe the shadow at the level of group (co)homology of the existence and nature of the dualising complex for kG. On the other hand it is straightforward to apply the results of [Bi1], [Bi2] to determine the Nakayama automorphism for kG. In order to do this, we define the *adjoint trace* $\mathtt{T}$ of the polycyclic-by-finite group G as follows. Fix a series

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G \tag{6.7.2}$$

of normal subgroups G_i of G whose factors G_i/G_{i-1} are either torsion-free Abelian, or finite. Provide each torsion-free factor $A_i = G_i/G_{i-1}$ with a $\mathbb{Z}-$basis (of cardinality t_i, say), so that conjugation by $g \in G$ assigns a matrix $\tilde{g}_i \in \mathrm{GL}_{t_i}(\mathbb{Z})$ to the pair (g, A_i). Now define a map

$$\mathtt{T} : G \longrightarrow \{\pm 1\} : g \mapsto \prod_i \det(\tilde{g}_i),$$

where the product is taken over those i, $1 \leq i \leq n$, such that G_i/G_{i-1} is infinite. One checks easily that this definition doesn't depend on the choice of series (6.7.2),

and that $\mathtt{T}$ is a group homomorphism [Bi1, §3.2]. Finally, set $\mathtt{T}_k$ to be the composition of $\mathtt{T}$ with the canonical homomorphism from $\{\pm 1\}$ to k, and use the same symbol for the resulting representation of kG on k.

Proposition. *Let k be a field of characteristic $p \geq 0$, and let G be a polycyclic-by-finite group with Hirsch length d. If $p > 0$ assume that G contains no elements of order p.*

(a) [Bi2] *G is a Poincaré duality group over k. The dualising module C is $H^d(G, kG)$; that is, C is the left integral of kG.*

(b) The Nakayama automorphism ν of kG is the winding automorphism induced by the adjoint trace $\mathtt{T}_k$ of G in k. That is,

$$\nu(g) \quad = \quad \mathtt{T}_k(g)g$$

for all $g \in G$. In particular, ν is diagonalisable.

(c) $H_d(kG, kG^\nu) \cong Z(kG) \neq 0$, and $H^d(kG, {}^\nu kG) \cong kG/[kG, kG] \neq 0$.

(d) The integral order of kG is either 1 or 2.

Proof. (a) The first statement is proved in [Bi2, Theorems 5.6.2 and 5.6.4], and the second is [Bi2, Proposition 5.2.1]. (Of course, (a) also follows from Theorem 6.7 and Lemma 3.3.)

(b) By Theorem 6.7, kG is AS-regular. Since the square of the antipode of kG is the identity, Proposition 4.5(a) shows that the Nakayama automorphism of kG is the winding automorphism induced by the right module structure of the left integral. The calculations in [Bi1, §3.2] show that this right structure is given by $\mathtt{T}_k$.

(c) This follows from Corollary 5.1.

(d) This is clear from part (b). □

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