On a question of Andreas Weiermann

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1 Introduction

The goal of this paper is to give some information about the following question, posed by Andreas Weiermann (private communication). Is it true that for every $\beta, \gamma < \varepsilon_0$ there exist α so that whenever A is α–large, A satisfies some inessential assumption, say $\min(A) \geq 2$, and $G : A \to \beta$ is such that $\forall a \in A\, \mathrm{psn}(G(a)) \leq a$ there must exist γ–large $C \subseteq A$ on which G is nondecreasing. Here ε_0 is the smallest ordinal solution to the equation $\omega^\varepsilon = \varepsilon$, the notion of α–largeness is in the sense of the so–called Hardy hierarchy and $\mathrm{psn}(\alpha)$ is the greatest natural number which occurs in the full Cantor normal expansion of α. We answer this positively. We derive this result from a partition theorem of J. Ketonen and R. Solovay [10] as reworked for the Hardy hierarchy in [1, 2, 3] and [12], in particular from theorem 5 in [2]. Later we obtain much sharper upper bound for α in terms of β and γ for very small ordinals β.

Now, we write something on the motivations which stand behind this work. One of the open problems of proof theory is to determine the exact strength of the Ramsey's theorem for pairs (RT^2_2) stated as the sentence of second order arithmetic

For each X, if X codes a coloring of pairs of natural numbers into two colors, then there exists an infinite set Y such that Y is homogeneous for X.

The exact relation of this principle to the other second order principles is unknown, see [6] and [8] (for some background definitions see [15]). Even the first order proof theoretic strength of RT^2_2 is not fully described. It is known that RT^2_2 implies the first order Σ_2 collection principle, see [9]. On the other hand it is not known whether it is Π_2 conservative over $\mathrm{I}\Sigma_1$. In [4], the authors describe the set of Σ_1 definable functions $\{h_n : n \in \omega\}$ such that the theory $\mathrm{I}\Sigma_1 \cup \{\forall x \exists y h_n(x) = y : n \in \omega\}$ has the same first order Π_2–theorems as $\mathrm{RCA}_0 + \mathrm{RT}^2_2$. These functions are constructed in a natural way from an indicator for $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ and describe bounds for some Ramsey properties of sets (see [4]). Then, the Π_2 conservativity problem of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ over $\mathrm{I}\Sigma_1$ is reduced to the problem whether all h_n's are provably total in $\mathrm{I}\Sigma_1$. Since the functions which are provably total in $\mathrm{I}\Sigma_1$ are exactly primitive recursive functions, the problem of conservativity is reduced to the problem of giving (in $\mathrm{I}\Sigma_1$)

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primitive recursive upper bounds for some Ramsey properties. However, let us note that it is even not known whether all h_n's are primitive recursive in the Ackermann's function (see also Question 1 on page 126 of [12]).

The functions h_n's are constructed from the indicator for RT^2_2 but an apparently related family of functions may be constructed using combinatorics on α–large sets and their corresponding functions from Hardy hierarchy, for $\alpha < \varepsilon_0$ (for some background see [1] or the unpublished monograph [11]). Let $\alpha_0 = 0$ and let α_{n+1} be the least α such that for every set of natural numbers X which is α_n–large and for every 2–coloring $F\colon [X]^2 \longrightarrow 2$ of pairs of X there is an α_n–large $Y \subseteq X$ which is homogeneous for F. In [12] it is proved that such defined hierarchy of ordinals cannot be bounded in ω^ω. The rate of growth of this hierarchy is not known. If it would be possible to prove (in $\mathrm{I}\Sigma_1$) that it is included in ω^ω then it would prove that all h_n's are primitive recursive (provably in $\mathrm{I}\Sigma_1$). On the other hand, showing that α_n's go beyond ω^ω would suggest that also h_n's have not a primitive recursive a rate of growth.

Nevertheless, before attacking directly the problem of Π_2 conservativity of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ over $\mathrm{I}\Sigma_1$ Andreas Weiermann wanted to look closer at some other second order principles, weaker but somewhat related to RT^2_2, like Chain–Antichain Property (every infinite order has an infinite chain or antichain) or Tournament Property, see [4]. Here, we deal with combinatorics which is related to a principle that every function from natural numbers into natural number has a weakly monotone increasing subsequence.

Our paper can be seen as a research in combinatorics but with motivations which are not purely combinatorial but comes also from questions mentioned above.

We have organized the paper as follows. In section 2 we give the necessary preliminaries. In section 3 we derive the existence of the appropriate ordinals α. In section 4 we give an estimate of α below ω^ω which is much sharper than the estimate obtained directly from the Ramsey–style result as in section 3. In section 5 we go up to ω^{ω^ω}.

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2 Preliminaries

Let $\alpha < \varepsilon_0$. Then α may be written in the Cantor normal form

$$\alpha = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_s} \cdot a_s \tag{1}$$

for some $\alpha_0 > \alpha_1 > \cdots > \alpha_s$ with $\alpha > \alpha_0$ and $a_0, \cdots, a_s \in \mathbb{N} \setminus \{0\}$.

By $\mathrm{LM}(\alpha)$ we denote the leftmost exponent of α, that is, if α is written in the form (1), then $\mathrm{LM}(\alpha) = \alpha_0$. Similarly, by $\mathrm{RM}(\alpha)$ we denote the rightmost exponent of α, i.e. α_s in (1). (In [1] we denoted $\mathrm{RM}(\alpha)$ by $\varrho(\alpha)$.)

As in the literature we write $\beta \gg \alpha$ if either $\alpha = 0$ or $\beta = 0$ or all the exponents in the Cantor normal form of β are $\geq$ all the exponents in the normal form of α. We write $\beta \ggg \alpha$ for the same but with strict inequality.

For the notion of the Hardy hierarchy and the notion of largeness determined by it we refer the reader to [1]. In fact, in order to avoid repetition we assume the reader to have a copy of [1] in hand.

In [1] we followed Ketonen and Solovay [10] and used their *norm* function. Here we shall use the *pseudonorm* function defined as the greatest natural number which occurs in the *full* Cantor normal form of α, which is obtained from (1) by writing all exponents $\alpha_0, \cdots, \alpha_s$ in their Cantor normal forms and repeating till we get an expression in which only natural numbers occur. Technically we define the function psn sending ordinals below ε_0 into $\mathbb{N}$ by putting $\mathrm{psn}(n) = n$ for $n < \omega$ and for $\alpha \geq \omega$

$$\mathrm{psn}(\alpha) = \max(\mathrm{psn}(\alpha_0), \cdots, \mathrm{psn}(\alpha_s), a_0, \cdots, a_s)$$

where α is written in its Cantor normal form (1). See [5] for more about the use of some version of norm to get an information about the Hardy hierarchy and related topics.

Lemma 2.1 *If β is limit, $\alpha < \beta$ and $A \subset \mathbb{N}$ is β–large and satisfies $\min(A) > \mathrm{psn}(\alpha) > 1$, then A is α–large.*

Proof: See [3], corollary 2.2. □

We shall use the following function $F\colon \varepsilon_0 \to \varepsilon_0$ (essentially it adds to α the order type of the set of limit ordinals smaller than α). It is formally defined as follows. We write

$$\alpha = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_s} \cdot a_s + \omega^n \cdot m_n + \cdots + \omega^0 \cdot m_0,$$

where $\alpha > \alpha_0 > \cdots > \alpha_s \geq \omega$ and $n < \omega$ (we allow some m_i's to be zero, moreover we write ω^0 rather than 1 to increase readability). Then $F(\alpha)$ is equal to

$$\begin{gathered} \omega^{\alpha_0} \cdot 2a_0 + \cdots + \omega^{\alpha_s} \cdot 2a_s + \\ \omega^n \cdot m_n + \omega^{n-1} \cdot (m_n + m_{n-1}) + \cdots + \omega^0 \cdot (m_n + \cdots + m_0) + \\ (a_0 + \cdots + a_s). \end{gathered} \tag{2}$$

Observe that $F(\alpha)$ is about $\alpha(+)\alpha$, where $\alpha(+)\beta$ denotes the so–called *natural sum* of α and β.

Lemma 2.2 (The estimation lemma) *For every $\alpha < \varepsilon_0$ we have: for every $A \subseteq \mathbb{N}$ with $\min(A) > 0$, if there exists a strictly decreasing function $G : A \to (\leq \alpha)$ such that $\forall a \in A\, \mathrm{psn}(G(a)) \leq a$, then A is at most $F(\alpha)$–large.*

Proof: See section 3 in [3]. □

Let us use the following notation, taken from Ramsey theory (cf. [7]):

> $\alpha \to (\beta)^n_c$ iff for every α–large set A with $\min(A) > c$ and every partition $P : [A]^n \to (< c)$ there exists a β–large homogeneous set.

We shall need the following result of Ketonen and Solovay [10] as reworked for the Hardy hierarchy in [2].

Theorem 2.3 *Let A be an $\omega^{\omega^{\alpha} \cdot c}$–large set and let $P : [A]^2 \to (< c)$ be a partition of $[A]^2$ into c parts as indicated. Assume also $\min(A) > c$. Then there exists an ω^{α}–large homogeneous set for this partition.*

Proof: This is theorem 5 in [2]. □

It turns out that if h satisfies the usual assumptions, that is it is increasing and increases the argument and α is fixed, then h_α may be also iterated in the Hardy style. Thus we have: $(h_\alpha)_0 = \mathrm{id}$, $(h_\alpha)_{\beta+1} = (h_\alpha)_\beta \circ h_\alpha$ and $(h_\alpha)_\lambda(b) = (h_\alpha)_{\{\lambda\}(b)}(b)$.

Lemma 2.4 $\forall\beta\,\forall\alpha \gg \mathrm{LM}(\beta)\ (h_{\omega^\alpha})_\beta = h_{\omega^\alpha\cdot\beta}$.

Proof: See [12], lemma 5.3. □

We shall also need the following fact.

Lemma 2.5 *For all α, all $\beta \geq \alpha$ and all $a \geq 1$*

$$\{h_{\omega^\beta+\omega^\alpha}(a)\downarrow \Rightarrow [h_{\omega^\alpha}\circ h_{\omega^\beta}(a)\downarrow \ \&\ h_{\omega^\alpha}\circ h_{\omega^\beta}(a) \leq h_{\omega^\beta}\circ h_{\omega^\alpha}(a)]\}.$$

Proof: See [3], lemma 3.2. □

We also need the following lemma which can be easily verified by induction up to ε_0.

Lemma 2.6 *For $\alpha,\beta < \varepsilon_0$ such that $\beta \gg \alpha$, $h_{\beta+\alpha} = h_\beta \circ h_\alpha$.*

Another lemma which we shall need later is as follows. In the lemma below we use the usual notation $a_0 = \min(A)$. By S_α we denote the Hardy iterations of the usual successor: $S(x) = x + 1$.

Lemma 2.7 *Assume that $\xi < \varepsilon_0$, $k \geq 2$, $\min(A) \geq 2$ and A is $\omega^{\omega^{\xi+1}k}$–large. Then A is $\omega^{\omega^\xi\cdot s}$–large, where $s = S_{\omega^2\cdot 7\cdot(k-1)}(a_0)$. In particular, A is $\omega^{\omega^\xi k 4^{\min(A)}}$–large.*

Proof: Let ξ, k and A satisfy assumptions of the lemma, let h denote the successor in the sense of A. For a function f, by $f^{(i)}(x)$ we denote the i-th iterate of f on x.

So let A be $\omega^{\omega^{\xi+1}k}$–large. Then A is $\omega^{\omega^{\xi+1}(k-1)+\omega^{\xi+1}}$–large, hence it is $\omega^{\omega^{\xi+1}(k-1)+\omega}$–large, so it is $\omega^{\omega^{\xi+1}(k-1)+a_0}$–large. Thus A is $\omega^{\omega^{\xi+1}(k-1)+(a_0-1)}a_0$–large and so $\omega^{\omega^{\xi+1}(k-1)+(a_0-1)} + \omega^2$–large. Hence A is $\omega^{\omega^{\xi+1}(k-1)}h_{\omega^2}(a_0)$–large what gives that A is, at least, $\omega^{\omega^{\xi+1}(k-1)}8$–large (because $\min(A) \geq 2$). Thus A is $\omega^{\omega^{\xi+1}(k-1)} + \omega^2\cdot 7$–large and $A - [a_0, h_{\omega^2.7}(a_0))$ is $\omega^{\omega^{\xi+1}(k-1)}$–large.

Applying this procedure $k-1$ times we infer that $A - [a_0, h^{(k-1)}_{\omega^2.7}(a_0))$ is $\omega^{\omega^{\xi+1}}$–large. We simply repeat the same procedure but instead of a_0 we take $h_{\omega^2.7}(a_0)$, then $h_{\omega^2.7}(h_{\omega^2.7}(a_0))$, …, and finally $h^{(k-1)}_{\omega^2.7}(a_0)$. But $h^{(i)}_{\omega^2.7}(x)$ is just $h_{\omega^2.7i}(x)$ so we infer that A is $\omega^{\omega^{\xi+1}} + \omega^2\cdot 7(k-1)$–large. Thus, A is $\omega^{\omega^\xi h_{\omega^2.7(k-1)}(a_0)}$–large. The last part of the lemma follows from the fact that $s = h_{\omega^2\cdot 7(k-1)}(a_0) \geq 4^{a_0}k$ and hence A is $\omega^{\omega^\xi k 4^{\min(A)}}$–large. □

3 The existence proof

The preliminaries collected in section 2 allow us to give a very simple proof of the existence of the required α.

Theorem 3.1 *Given $\beta,\gamma < \varepsilon_0$ there exists α such that for every α–large A and every $G\colon A \to (\le \beta)$ such that $\forall a \in A\, \mathrm{psn}(G(a)) \le a$ there exists a γ–large $C \subseteq A$ on which G is nondecreasing.*

Proof: Let β,γ be given. Let $F\colon (< \varepsilon_0) \to (< \varepsilon_0)$ be defined as on page 3 after lemma 2.1. Let $m = \max(\mathrm{psn}(F(\beta)) + 1, \mathrm{psn}(\gamma))$. Pick also ϱ such that $\omega^\varrho \ge \max(F(\beta)+1,\gamma)$. Let $\alpha = \omega^{\omega^{\varrho}\cdot 2} + m + 1$. We assert that α has the desired property.

So let $A, G\colon A \longrightarrow (\le \beta)$ satisfy the assumption. Let $u = h^A_{m+1}(\min(A))$. Then $u > m$. Let $A^* = \{x \in A : u \le x\}$. Then A^* is $\omega^{\omega^{\varrho}\cdot 2}$–large. Let $G^* = G\restriction A^*$. Let a partition $P\colon [A^*]^2 \to 2$ be defined as follows. Put $P(x,y) = 0$ if $G^*(x) \le G^*(y)$ and $P(x,y) = 1$ otherwise. By theorem 2.3 there exist an ω^ϱ–large $C \subseteq A^* \subseteq A$ homogeneous for P. Obviously $\min(C) \ge \min(A^*) > m$. By lemma 2.1 C is both $F(\beta) + 1$–large and γ–large. Assume that $[C]^2$ is colored by 1. Then G^* is strictly decreasing on C. But this is impossible by the estimation lemma (i.e. lemma 2.2). It follows that $[C]^2$ is colored by 0, so G is nondecreasing on C. But as observed above this set is γ–large. □

Observe that the estimate given by this proof is very weak. The essence is that we pick a large enough ω^ϱ and must go down from $\omega^{\omega^{\varrho}\cdot 2}$ to ω^ϱ, that is we loose one exponent. But observe that the question has much more to do with the so–called *monotone subsequence theorem* than with Ramsey theorem. The monotone subsequence theorem is only mentioned on page 17 in [7], but see [14]. In § 4.1 of [14] the authors give a proof via Dilworth theorem and in problem 1 to chapter 4 they sketch a direct proof. Therefore it is by no means surprising that we may obtain much stronger estimates, at least for very small ordinals. This will be done in subsequent sections.

4 Below ω^ω

In order to get a better estimate it will be more convenient to work with functions sending A to $(< \beta)$ rather than to $(\le \beta)$. Moreover we assume that $\min(A) > 1$. Let us also strengthen the conclusion slightly. Let $\mathrm{WR}(\alpha,\beta,\gamma)$ be an abbreviation for

> whenever A is α–large, $G\colon A \to \beta$, where $\min(A) \ge 2$ and G is such that for all $a \in A$ $\mathrm{psn}(G(a)) \le a$, then there exists a γ–large $C \subseteq A$ such that G is either strictly increasing or constant on C.

Moreover we shall assume that γ is of some very special form: $\gamma = \omega^{\omega^{\xi}\cdot k}$. Indeed, most of the argument given below depends on this assumption.

The goal of this section are the following two theorems.

Theorem 4.1 *Let γ be of the form $\gamma = \omega^{\omega^{\xi}\cdot k}$. Given $\beta < \omega^\omega$ write $\beta = \omega^m \cdot n_m + \omega^{m-1} \cdot n_{m-1} + \cdots + \omega^0 \cdot n_0$ in the Cantor normal form. Let*

$$\alpha = \omega^{\omega^{\xi}\cdot k\cdot 4^m} \cdot n_m + \omega^{\omega^{\xi}\cdot k\cdot 4^{(m-1)}} \cdot n_{m-1} + \cdots + \omega^{\omega^{\xi}\cdot k\cdot 4^0} \cdot n_0. \qquad (3)$$

Then $\mathrm{WR}(\alpha, \beta, \gamma)$.

In the following result we need the assumption that $k > 1$. If $k = 1$ then we may require the existence of an $\omega^{\omega^{\xi}\cdot 2}$–large set on which the appropriate function is constant or increasing, so we need a slightly stronger assumption. We could work equally well assuming $\xi > 0$.

Theorem 4.2 *Assume that* $k > 1$. *Let* $\gamma = \omega^{\omega^{\xi}\cdot k}$ *and* $\beta = \omega^{\omega}$. *Let* $\alpha = \omega^{\omega^{\xi+1}\cdot 16\cdot k}$. *Then* $\mathrm{WR}(\alpha, \beta, \gamma)$.

We begin with some simple observations. The first one is as follows. Let $\beta = 1$. Then let $\alpha = \gamma$ and we are done, indeed, G has only one value, so is constant. Thus we have $\mathrm{WR}(\gamma, 1, \gamma)$.

Let $G\colon A \to \beta + 1$ satisfy the assumption. Let $B = \{a \in A\colon G(a) = \beta\}$. If this set is γ–large, then we are done, so assume that B is γ-small. Thus we must ensure that $A \setminus B$ is $\alpha(\beta, \gamma)$–large. Using theorem 12 in [1] we see that

$$\text{if } \mathrm{WR}(\alpha, \beta, \gamma) \text{ and } \alpha \gg \gamma \text{ then } \mathrm{WR}(\alpha + \gamma, \beta + 1, \gamma) \tag{4}$$

Let us generalize this trick. Assume that $\beta = \mu + \nu$, where $\mu \gg \nu$ and both μ, ν are greater than 0. Let $G\colon A \to \mu + \nu$ satisfy the assumption. Let $B = \{a \in A\colon G(a) \geq \mu\}$. Then for every $a \in B$ there exists $\zeta(a) < \nu$ such that $G(a) = \mu + \zeta(a)$. This follows easily by considering the Cantor normal form of the ordinal $G(a)$. Thus, write it as

$$G(a) = \omega^{\eta_0} \cdot w_0 + \ldots + \omega^{\eta_{t-1}} \cdot w_{t-1}$$

and see that the initial part of this expansion must equal μ and let the further part be $\zeta(a)$. Thus, we are given a function $\zeta\colon B \to \nu$, so if B is α–large, where $\mathrm{WR}(\alpha, \nu, \gamma)$, then we are done. This is so because in this case, by our inductive assumption, there would be a γ–large $C \subseteq B$ such that $\zeta(x)$ is constant or strictly increasing on C. Then, of course, $G(x) = \mu + \zeta(x)$ would be constant or strictly increasing on C, too. It follows that we are done (with the following Lemma) if the remainder $A \setminus B$ is α'–large, where $\mathrm{WR}(\alpha', \mu, \gamma)$. Indeed, in this case, G sends this remainder $A \setminus B$ to μ and there would be a γ–large $C \subseteq A \setminus B$ such that G is constant or strictly increasing on C. Summing up, using the same theorem 12 in [1] we get that

Lemma 4.3 *If* $\beta = \mu + \nu$, $\mu \gg \nu$, α, α' *are such that* $\mathrm{WR}(\alpha', \nu, \gamma)$, $\mathrm{WR}(\alpha, \mu, \gamma)$ *and* $\alpha \gg \alpha'$, *then* $\mathrm{WR}(\alpha + \alpha', \mu + \nu, \gamma)$.

In the following lemma the assumption that γ is of the form $\omega^{\omega^{\xi}\cdot k}$ is essential. (What is needed is that γ is of the form ω^{δ}, where $\delta \gg \delta$, so δ is of the form $\omega^{\xi} \cdot k$ and, hence, $\gamma = \omega^{\omega^{\xi}\cdot k}$.) In the lemma the square almost suffices, we take the fourth power rather than square just to handle a minor tail that occurs.

Lemma 4.4 $\mathrm{WR}(\omega^{\omega^{\xi}\cdot k\cdot 4}, \omega, \omega^{\omega^{\xi}\cdot k})$

Proof: Let $G\colon A \to \omega$ satisfy the assumption. Thus we have: for all $a \in A$ $G(a) \leq a$. In order to work out this case assume that there exists a $(\gamma + 1)$–large $B \subseteq A$, say $B = \{b_0, \cdots, b_{r-1}\}$ in increasing order, such that for every $j < r - 1$ the interval $A \cap [b_j, b_{j+1})$ of A is $\omega^{\delta+1}$–large. We assert that under this assumption each G as above is either increasing or constant on some γ–large $C \subseteq A$. Indeed, consider two cases.

CASE 1. For every $j < r-1$ there exists $c \in A \cap [b_j, b_{j+1})$ such that $G(c) \geq b_j$. In this case choose for each $j < r-1$ one such c_j and let C be the set just chosen c's. Then it is easy to check that G is strictly increasing on C. The reason is as follows. Let $c_1 < c_2$, both in C. Then $b_{j_1} \leq c_1 < b_{j_1+1} \leq b_{j_2} \leq c_2$ for the elements b_{j_1}, b_{j_2} as above. But then $G(c_2) \geq b_{j_2} \geq b_{j_1+1} > c_1 \geq G(c_1)$ by the assumption on G. Moreover C is γ–large. The reason is lemma 5(ii) in [1]. Indeed, C has the same cardinality as $B \setminus \{\min(B)\}$ and the consecutive elements of C are smaller than consecutive elements of $B \setminus \{\min(B)\}$, which is γ–large.

CASE 2. There exists $j < r-1$ such that for all $a \in A \cap [b_j, b_{j+1})$ $G(a) < b_j$. In this case consider the partition $A \cap [b_j, b_{j+1}) = \cup_{i<b_j} A_i$, where $A_i = \{a \in [b_j, b_{j+1}) \cap A : G(a) = i\}$ and apply theorem 1 in [1] to see that G is constant on some ω^δ–large C, namely $C = A_i$ for some i.

Thus what is needed is to see how large A is supposed to be in order to ensure the existence of B as above. To see this, observe at first that it suffices that the intervals $A \cap [b_j, b_{j+1}]$ are $\omega^{\delta+1}+1$–large, this follows from lemma 2.5. Indeed, we want that $A \cap [b_j, b_{j+1})$ is $\omega^{\delta+1}$–large what is equivalent to $h_{\omega^0} \circ h_{\omega^{\delta+1}}(b_j) \leq b_{j+1}$. Then, by lemma 2.5, it suffices that $h_{\omega^{\delta+1}} \circ h_{\omega^0}(b_j) \leq b_{j+1}$ what is equivalent to $A \cap [b_j, b_{j+1}]$ being $\omega^{\delta+1}+1$–large. Of course it suffices to ensure that these intervals are $\omega^{\delta \cdot 2}$–large because $\min(A) > 1$. Using the specific form of δ and lemma 2.4 we see that every set which is $\omega^{\delta \cdot 3}$–large almost has the required property, that is it is the union of its intervals $A \cap [b_j, b_{j+1})$, but then the set B is ω^δ–large. In order to ensure that it is $\omega^\delta + 1$–large as required, it suffices to have one more $\omega^{\delta \cdot 2}$–large set at the beginning of A, but clearly it suffices to assume that A is $\omega^{\delta \cdot 4}$–large. □

Lemma 4.5 *If* $\alpha = \omega^{\omega^\xi \cdot 4k} \cdot m$, *then* $\mathrm{WR}(\alpha, \omega \cdot m, \omega^{\omega^\xi \cdot k})$.

Proof: Immediate by lemmas 4.4 and 4.3 by induction on m. □

Lemma 4.6 *For all* $m > 0$ *if* $\alpha = \omega^{\omega^\xi \cdot k \cdot 4^m}$, *then* $\mathrm{WR}(\alpha, \omega^m, \omega^{\omega^\xi \cdot k})$.

Proof: By induction on m, case $m = 1$ is just lemma 4.4. Assume the assertion $\mathrm{WR}(\alpha, \omega^m, \omega^{\omega^\xi \cdot k})$ for m. Let $G\colon A \to \omega^{m+1}$. We write $G(a) = \omega^m \cdot \mathrm{wsp}(a) + \zeta(a)$ with $\zeta(a) < \omega^m$. Let $B \subseteq A$ be such that the function $a \mapsto \mathrm{wsp}(a)$ is either constant or strictly increasing on B. If it is strictly increasing on B, so is G, so we need merely to know that B is $\omega^{\omega^\xi \cdot k}$–large, so in this case it suffices that A is $\omega^{\omega^\xi \cdot k \cdot 4}$–large. So assume that wsp is constant on B. Then the function $b \mapsto \zeta(b)$ sends B to ω^m, so we shall be done if we knew that B is ϱ–large for some ϱ such that $\mathrm{WR}(\varrho, \omega^m, \omega^{\omega^\xi \cdot k})$. By the inductive assumption it is enough if B is $\omega^{\omega^\xi \cdot k \cdot 4^m}$–large. Together, by Lemma 4.4, it suffices to require that A is $\omega^{\omega^\xi \cdot k \cdot 4^{m+1}}$–large. □

Proof of theorem 4.1: Immediate by lemmas 4.6 and 4.3. □

Proof of theorem 4.2: Assume that $k > 1$. Let $G\colon A \to \omega^\omega$ satisfy the assumption. Every $G(a)$ is of the form $G(a) = \omega^{\ell(a)} \cdot \mathrm{wsp}(a) + \zeta(a)$, where $\ell(a) \leq a$, $0 < \mathrm{wsp}(a) \leq a$ and $\zeta(a) < \omega^{\ell(a)}$. (We assume here that if $\mathrm{wsp}(a) = 0$ then

$\ell(a) = 0$, too.) In particular, if $\ell(a) = 0$ then $G(a) = \mathrm{wsp}(a)$. Let $B \subseteq A$ be such that the function $a \mapsto \ell(a)$ is either constant or increasing on B. As usual, if it is increasing, so is G, so this case causes no problems. So assume that this function is constant on B. Pick $C \subseteq B$ on which the function $b \mapsto \mathrm{wsp}(b)$ is either increasing or constant. Finally, pick $D \subseteq C$ on which the function $c \mapsto \zeta(c)$ is either increasing or constant. We want to ensure that we are able to find an $\omega^{\omega^{\xi} \cdot k}$–large D. Observe that ℓ, the constant value of $\ell(b)$ on B, satisfies $\ell \leq b_0$. This follows from the assumption about G applied to b_0. Moreover, $b_0 \leq c_0$ because $C \subseteq B$. It follows that the function $c \mapsto \zeta(c)$ sends C to ω^{c_0}. By lemma 4.6 it suffices to make sure that C is $\omega^{\omega^{\xi} \cdot k \cdot 4^{c_0}}$–large. By lemma 2.7 it is enough if C is $\omega^{\omega^{\xi+1} k}$–large. Thus it suffices to observe that when passing from A to C we used two functions into ω, so it suffices to multiply the exponent by 4 twice. □

5 Below $\omega^{\omega^{\omega}}$

The direct method used above, when applied to ordinals β above ω^{ω}, gives only a very weak estimate, comparable with the one given by the proof of theorem 3.1. In order to see this consider the case $\beta = \omega^{\omega \cdot 2}$. So let $G \colon A \to \omega^{\omega \cdot 2}$. Then every $G(a)$ is of the form $\omega^{\omega+n} \cdot k_n + \omega^{\omega+n-1} \cdot k_{n-1} + \cdots + \omega^{\omega} \cdot k_0 + \zeta$, where $\zeta < \omega^{\omega}$. We must consider the functions $a \mapsto n$, $a \mapsto k_n, \cdots, a \mapsto k_0, a \mapsto \zeta$ and the argument as above yields: if $\alpha = \omega^{\omega^{\xi+2} \cdot k \cdot 16^2}$, then $\mathrm{WR}(\alpha, \omega^{\omega \cdot 2}, \omega^{\omega^{\xi} \cdot k})$. Thus adding ω in the exponent of β needs addition of 1 at the second level of exponent in γ. Similarly, if $\beta = \omega^{\omega^2}$, then we need α of order $\omega^{\omega^{\xi+\omega}}$. In order to get a better estimate we need some additional work. It will be based on a slightly another expansion of ordinals below $\omega^{\omega^{\omega}}$ than the usual Cantor normal form. This will be used to get an estimate of the length of the expansion. Below, if we write $\varrho = \sum_j \omega^{\varrho_j} \cdot r_j$ we assume that the sum is written in such a way that the sequence of exponents ϱ_j is decreasing.

Lemma 5.1 *Let $\varrho < \omega^{\omega^{n+1}}$. Then for every $k \leq n$ we may represent ϱ in the form*

$$\varrho = \sum_{i \leq m_k} \omega^{\omega^n \cdot w_{n,i} + \omega^{n-1} \cdot w_{n-1,i} + \cdots + \omega^{n-k} \cdot w_{n-k,i}} \cdot \zeta_i, \qquad (*)$$

where the coefficients ζ_i are strictly smaller than $\omega^{\omega^{n-k}}$ and the length $m_k + 1$ of the k–th sum is estimated as follows. If $k = 0$ set $w = \max w : \omega^{\omega^n \cdot w} \leq \varrho$ and $m_0 \leq w$. For $k > 0$ let $\zeta_i : i \leq m_{k-1}$ be the coefficients in the $(k-1)$–st sum $(*)$. Let $u_i = \max u : \omega^{\omega^{n-k} \cdot u} \leq \zeta_i$ and then $m_k \leq (u_0 + 1) + \cdots + (u_{m_{k-1}} + 1)$.

If $\varrho = 0$ then we represent ϱ as the empty sum.

For $k \leq n$ we shall refer to the expansion $(*)$ as to the k–th Cantor normal form of ϱ.

Proof: Fix ϱ and proceed by induction on k. Let $k = 0$. Set $w = \max w : \omega^{\omega^n \cdot w} \leq \varrho$. By the theorem on division with remainder (cf. [13]) there exist $\zeta = \zeta_w$ and $\eta = \eta_w < \omega^{\omega^n \cdot w}$ such that $\varrho = \omega^{\omega^n \cdot w} \cdot \zeta_w + \eta_w$. We have also $\zeta_w < \omega^{\omega^n}$. Indeed, if $\zeta_w \geq \omega^{\omega^n}$, then $\varrho \geq \omega^{\omega^n \cdot w} \cdot \zeta_w \geq \omega^{\omega^n \cdot w} \cdot \omega^{\omega^n} = \omega^{\omega^n \cdot (w+1)}$ contrary to maximality of w. Now we divide η_w with remainder by $\omega^{\omega^n \cdot (w-1)}$ and obtain ζ_{w-1}, η_{w-1} such that $\varrho = \omega^{\omega^n \cdot w} \cdot \zeta_w + \omega^{\omega^n \cdot (w-1)} \cdot \zeta_{w-1} + \eta_{w-1}$ where

$\eta_{w-1} < \omega^{\omega^n \cdot (w-1)}$ and check that $\zeta_{w-1} < \omega^{\omega^n}$. We continue in the same fashion, that is divide η_{w-1} by $\omega^{\omega^n \cdot (w-2)}$ and obtain ζ_{w-2}, η_{w-2} and so on. Together we obtain $\varrho = \sum_{j=w}^{0} \omega^{\omega^n \cdot j} \cdot \zeta_j$ as required in the case $k = 0$. In particular the desired inequality $m_0 \leq w$ is obvious.

Assume the lemma for $k-1$ and let $\zeta_{m_{k-1}}, \cdots, \zeta_0$ be the coefficients of the sum $(*)$ for $k-1$. For each j we let $u_j = \max u : \omega^{\omega^{n-k} \cdot u} \leq \zeta_j$. By the initial step applied to each ζ_j we have a representation $\zeta_j = \omega^{\omega^{n-k} \cdot u_j} \cdot \zeta_{j,u_j} + \omega^{\omega^{n-k} \cdot (u_j - 1)} \cdot \zeta_{j,u_j-1} + \cdots + \omega^{\omega^{n-k} \cdot 0} \cdot \zeta_{j,0}$, where all $\zeta_{j,i}$ are smaller than $\omega^{\omega^{n-k}}$ for $i = u_j, u_j - 1, \cdots, 0$. We substitute these representations to the sum $(*)$ and obtain the representation $(*)$ for k, of course after a re–enumeration of the ζ's. Obviously, it has the required properties. $\square$

Let $k \leq n$. For $(a_n, \cdots, a_k)$ by $\omega(a_n, \cdots, a_k)$ we denote the ordinal $\omega^n \cdot a_n + \cdots + \omega^k \cdot a_k$ (these will be used as abbreviated versions of exponents occurring in the k–th expansion from lemma 5.1). If the sequence is empty we put $\omega(\emptyset) = 0$.

For a set X, X^i is the set of sequences of the length i of elements from X. In particular, $X^0 = \{\emptyset\}$ since there is only one sequence of the length 0. We implicitly use this fact in the next lemma which is the main tool in obtaining an upper bound for functions into $\omega^{\omega^{n+1}}$. In the lemma we use an inequality $h_{\omega^2 4(n+1)}(x) \geq 4^{(x+1)^n}$ for $x \geq 1$. It can be easily verified by the fact that for a function h being a successor function (what is the worst case) we have $h_{\omega^2}(x) = 2^x x$ and that, by lemma 2.6, $h_{\omega^2 r}(x) = (h_{\omega^2})^r(x)$, where $(h_{\omega^2})^r(x)$ is the r–times iterated h_{ω^2} on x.

Lemma 5.2 *Let $\mu \gg \omega^0$, let A be $\omega^{\omega^{\mu+n+1} k 4^2 (n+1)^2}$–large and let $G\colon A \longrightarrow \omega^{\omega^{n+1}}$ such that for all $a \in A$, $\mathrm{psn}(G(a)) \leq a$. Then there exist a sequence of sets: $B_{n+1} \supseteq B_n \supseteq \ldots \supseteq B_0$; a sequence of functions $G_i\colon B_i \longrightarrow \omega^{\omega^{n+1}}$ for $i \leq n+1$; and there exist a sequence of sets $C_i \subseteq [0, \min(B_i)]^{n-i+1}$ and a sequence of tuples $\bar{c}_i \in [0, \min(B_{i+1})]^{n-i}$, for $i \leq n$, such that the following holds for all $i = n, \ldots, 0$:*

1. *$B_{n+1} = A$ and $G_{n+1} = G$;*
2. *B_i is $\omega^{\omega^{\mu+i+1} k 4 (n+1)(i+1)}$–large;*
3. *the $(n-i)$–th Cantor normal form of $G_i(x)$ can be written, independently of $x \in B_i$, either as*
$$G_i(x) = G_{i+1}(x) = \sum_{\bar{c} \in C_i} \omega^{\omega(\bar{c})} \xi_{\bar{c}}(x)$$
or as
$$G_i(x) = \sum_{\bar{c} \in C_i} \omega^{\omega(\bar{c})} \xi_{\bar{c}}(x) + \omega^{\omega(\bar{c}_i) + \omega^d \delta(x)} \eta(x),$$
where $i \leq d \leq n$ and $\bar{c}_i$ is the sequence $(c_{i,n}, \ldots, c_{i,d+1}, 0, \ldots, 0)$ of the length $n-i$ and where $\delta\colon B_i \longrightarrow \omega$ is strictly increasing on B_i and for all $x \in B_i$, $\eta(x) < \omega^{\omega(\bar{c}_i) + \omega^d}$ and for all $x \in B_i \setminus \{\min(B_i)\}$, $\eta(x) \neq 0$.
Here, the choice of d, $\bar{c}_i$, δ, η and $\xi_{\bar{c}}$ for $\bar{c} \in C_i$ is done independently of $x \in B_i$.
4. *for all $D \subseteq B_i$, if G_i is strictly increasing on D then G_{i+1} is strictly increasing on D;*

5. *either $G_i = G_{i+1}$ on B_i or there is no subset $D \subseteq B_i$ such that card$(D) \geq 2$ and G_i is constant on D.*

Moreover, we assume that for all $0 \leq i \leq n$, C_i is not empty. If, for some $i \leq n$, $C_i = \emptyset$ we stop the construction. If, for some $i \geq 0$, $C_i = \emptyset$ then for $x \in B_i$, $G_i(x) = \omega^{\omega(\bar{c}_i)+\omega^i\delta(x)}\eta(x)$, where $\delta\colon B_i \longrightarrow \omega$ is strictly increasing on B_i and for all $x \in B_i$, $\eta(x) < \omega^{\omega(\bar{c}_i)+\omega^i}$ and for all $x \in B_i \setminus \{\min(B_i)\}$, $\eta(x) \neq 0$ (this is the second case in point 3).

Proof: Before we begin the proof of the lemma let us comment on point 3. From our assumption that the ordering of exponents in the sum $\sum_j \omega^{\varrho_j} \cdot r_j$ is always decreasing if follows that if

$$G_i(x) = \sum_{\bar{c}\in C_i} \omega^{\omega(\bar{c})}\xi_{\bar{c}}(x) + \omega^{\omega(\bar{c}_i)+\omega^d\delta(x)}\eta(x)$$

and $\bar{a}_0$ is a minimal member of C_i in the lexicographic ordering then there is no sequence $\bar{b}$ of length $n-i$ such that $\omega^{\omega(\bar{b})}$ appears in the $(n-i)$-th Cantor normal form of $G_i(x)$ and $\omega(\bar{a}_0) > \omega(\bar{b}) > \omega(\bar{c}_i) + \omega^d$ (for $\bar{c}_i = (c_{i,n}, \ldots, c_{i,d+1}, 0, \ldots, 0)$ we think of $\omega(\bar{c}_i) + \omega^d$ as $\omega^n c_{i,n} + \ldots + \omega^{d+1}c_{i,d+1} + \omega^d$).

Now, to construct B_n and G_n we define the function $w\colon B_{n+1} \longrightarrow \omega$ as

$$w(a) = \max(\{u : \omega^{\omega^{n}\cdot u} \leq G_{n+1}(a)\} \cup \{0\}).$$

We take the set $B_n \subseteq B_{n+1}$ such that $w(x)$ is constant or strictly increasing on B_n. Since we have to deal with only one function into ω, B_n can be chosen to be $\omega^{\omega^{\mu+n+1}k4(n+1)^2}$–large.

If $w(a)$ is strictly increasing on B_n we take $C_n = \emptyset$ and $\bar{c}_n = \emptyset$. The function G_n is defined as

$$G_n(a) = \omega^{\omega^n\cdot w(a)}\eta(a) = \omega^{\omega(\emptyset)+\omega^n w(a)}\eta(a),$$

where η is such that for $a \in B_n$,

$$G_{n+1}(a) = \omega^{\omega^n w(a)}\eta(a) + \xi(a)$$

for some $\xi(a) < \omega^{\omega^n w(a)}$. It is straightforward to check that points 3–5 are satisfied and that G_n is strictly increasing on B_n.

Now, let us assume that $w(x)$ is constant on B_n and equal to some $w \leq \mathrm{psn}(G(\min(B_n))) \leq \min(B_n)$. Then, we set $C_n = \{0, \ldots, w\}$ and we set

$$G_n(x) = G_{n+1}(x) = \sum_{i\in C_n} \omega^{\omega^n\cdot i}\xi_i(x),$$

when we write $G_n(x)$ in the 0-th Cantor normal form. Again, the sequence $\bar{c}_n = \emptyset$. It is easy to check that points 3–5 are satisfied.

Now, let us assume that for some $i + 1 \leq n$ we constructed B_{i+1}, G_{i+1}, $C_{i+1} \neq \emptyset$ and $\bar{c}_{i+1}$. Then, for $a \in B_{i+1}$, $G_{i+1}(a)$ can be written as

$$\sum_{\bar{c}\in C_{i+1}} \omega^{\omega(\bar{c})}\xi_{\bar{c}}(a) + \xi(a),$$

where $\xi(a)$ depends on the case in point 3 of the lemma. Now, for $\bar{c} \in C_{i+1}$, we define the function $w_{\bar{c}} \colon B_{i+1} \longrightarrow \omega$ as

$$w_{\bar{c}}(a) = \max(\{u : \omega^{\omega(\bar{c})+\omega^i u} \leq \xi_{\bar{c}}(a)\} \cup \{0\}). \tag{5}$$

We choose B_i such that all $w_{\bar{c}}$ are constant or strictly increasing on B_i. Firstly, let us assure that B_i is suitably large. The set B_{i+1} is

$$\omega^{\omega^{\mu+i+2}k4(n+1)(i+2)}\text{–large.}$$

Thus, it is also

$$\omega^{\omega^{\mu+i+2}k4(n+1)(i+1)+\omega^2 4(n+1)}\text{–large.}$$

Since, by the remark before the lemma, $h_{\omega^2 4(n+1)}(x) \geq 4^{(x+1)^n}$, the set B_{i+1} is also

$$\omega^{\omega^{\mu+i+2}k4(n+1)(i+1)+4^{(\min(B_{i+1})+1)^n}}\text{–large}$$

and consequently B_{i+1} is

$$\omega^{\omega^{\mu+i+1}k4(n+1)(i+1)\cdot 4^{(\min(B_{i+1})+1)^n}}\text{–large.}$$

But the cardinality of C_{i+1} is less than $(\min(B_{i+1})+1)^n$, thus B_i can be chosen to be

$$\omega^{\omega^{\mu+i+1}k4(n+1)(i+1)}\text{–large.}$$

Now, to define G_i, C_i and $\bar{c}_i$ we consider two cases. The first one is when all $w_{\bar{c}}$ are constant on B_i. Then,

$$C_i = \{(c_n, \ldots, c_{i+1}, c) : (c_n, \ldots, c_{i+1}) \in C_{i+1} \wedge c \leq w_{c_n,\ldots,c_{i+1}}(\min(B_i))\}.$$

Then, we choose $\bar{c}_i$ as $(\bar{c}_{i+1}, 0)$. We define $G_i = G_{i+1}$ and we write G_i, for $a \in B_i$, as

$$G_i(a) = \sum_{\bar{c} \in C_i} \omega^{\omega(\bar{c})} \xi_{\bar{c}}(a) + \gamma(a),$$

where $\sum_{\bar{c} \in C_i} \omega^{\omega(\bar{c})} \xi_{\bar{c}}(a)$ is the beginning of the $(n-i)$-th Cantor normal form of $G_{i+1}(a)$ and $\gamma(a)$ is just the zero function or $\omega^{\omega(\bar{c}_i)+\omega^d \delta(a)} \eta(a)$ in the case when

$$G_{i+1}(a) = \sum_{\bar{c} \in C_{i+1}} \omega^{\omega(\bar{c})} \xi'_{\bar{c}}(a) + \omega^{\omega(\bar{c}_{i+1})+\omega^d \delta(a)} \eta(a),$$

for some $n - i \leq d \leq n$, δ being strictly increasing on B_i and $\eta(a) < \omega^{\omega^d}$. It is straightforward to check that points 3–5 are satisfied.

Now, let us consider the case when there exists $\bar{c} \in C_{i+1}$ such that $w_{\bar{c}}$ is strictly increasing on B_i. Then we choose $\bar{c}_i$ as the greatest such sequence in the lexicographic ordering and we set C_i as

$$\begin{aligned} C_i = \{&(c_n, \ldots, c_{i+1}, c) : \\ &(c_n, \ldots, c_{i+1}) \in C_{i+1} \text{ is greater than } \bar{c}_i \text{ and } c \leq w_{c_n,\ldots,c_{i+1}}(\min(B_i))\}. \end{aligned}$$

For each $\tilde{c} \in C_{i+1}$ greater than $\bar{c}_i$, the function $w_{\tilde{c}}$ is constant on B_i. Then, by the definition of $w_{\tilde{c}}$ (see (5)), in the $(n-i)$-th Cantor normal form of $G_{i+1}(a)$

for $a \in B_i$ there are no elements of the sum of the form $\omega^{\omega(\tilde{c})+\omega^i c}\xi_{\tilde{c},c}(a)$ for $\tilde{c} \in C_{i+1}$, $\tilde{c}$ greater than $\bar{c}_i$ and $c > w_{\tilde{c}}(a) = w_{\tilde{c}}(\min(B_i))$. This is why, in defining C_i, we restrict our our attention only to $c \leq w_{c_n,\ldots,c_{i+1}}(\min(B_i))$.

Let us observe that it may happen that C_i is empty.

Now, we define G_i. Firstly, let us write $G_{i+1}(a)$ for $a \in B_i$ in the following form

$$G_{i+1}(a) = \sum_{\bar{c}\in C_i} \omega^{\omega(\bar{c})}\xi_{\bar{c}}(a) + \omega^{\omega(\bar{c}_i)+\omega^i w_{\bar{c}_i}(a)}\eta(a) + \xi(a),$$

where $\sum_{\bar{c}\in C_i} \omega^{\omega(\bar{c})}\xi_{\bar{c}}(a) + \omega^{\omega(\bar{c}_i)+\omega^i w_{\bar{c}_i}(a)}\eta(a)$ is the beginning of the $(n-i)$-th Cantor normal form of $G_{i+1}(a)$, $\eta(a) < \omega^{\omega^i}$ and $\xi(a) < \omega^{\omega(\bar{c}_i)+\omega^i}$. Let us observe that since $w_{\bar{c}_i}$ is strictly increasing on B_i and is defined as $\max(\{u : \omega^{\omega(\bar{c}_i)+\omega^i u} \leq \xi_{\bar{c}_i}(a)\} \cup \{0\})$ it follows that it may happen that $\eta(a) = 0$ only if $w_{\bar{c}_i}(a) = 0$. Thus, for all $a \in B_i \setminus \{\min(B_i)\}$, $\eta(x) \neq 0$. Now, we define $G_i(a)$ as the first elements of the $(n-i)$-th Cantor normal form of $G_{i+1}(a)$, that is

$$G_i(a) = \sum_{\bar{c}\in C_i} \omega^{\omega(\bar{c})}\xi_{\bar{c}}(a) + \omega^{\omega(\bar{c}_i)+\omega^i w_{\bar{c}_i}(a)}\eta(a).$$

Now, the point 3 is obviously satisfied. Similarly, since $w_{\bar{c}_i}$ is strictly increasing on B_i and $\eta(x) \neq 0$ for $x \in B_i \setminus \{\min(B_i)\}$, the point 5 is satisfied. Let us now verify the point 4. Let $A \subseteq B_i$ be such that G_i is strictly increasing on A. Then, $G_{i+1}(a)$ differs from $G_i(a)$ only by a factor $\xi(a)$ which is smaller than $\omega^{\omega(\bar{c}_i)+\omega^i}$. And moreover the factor $\omega^{\omega(\bar{c}_i)+\omega^i w_{\bar{c}_i}(a)}\eta(a)$ occurs in all $G_{i+1}(a)$, for $a \in B_i \setminus \{\min(B_i)\}$. Thus, the factor $\xi(a)$ is inessential and G_{i+1} is strictly increasing on A, too. Thus, we have proved the lemma. □

Now, we can give an estimation for α in $\mathrm{WR}(\alpha, \omega^{\omega^{n+1}}, \omega^{\omega^\mu k})$, where $\mu \gg \omega^0$.

Lemma 5.3 *Let $\mu \gg \omega^0$, $k \geq 1$.* $\mathrm{WR}(\omega^{\omega^{\mu+n+1}(k+1)4^2(n+1)^2}, \omega^{\omega^{n+1}}, \omega^{\omega^\mu k})$.

Proof: Let A be $\omega^{\omega^{\mu+n+1}(k+1)4^2(n+1)^2}$–large and let $G\colon A \longrightarrow \omega^{\omega^{n+1}}$. Then, we take the sequences $A = B_{n+1} \supseteq B_n \supseteq \ldots \supseteq B_i$; $G_{n+1}, \ldots, G_i$; $C_n, \ldots, C_i$ and $\bar{c}_n, \ldots, \bar{c}_i$ from lemma 5.2. We have that either $i = 0$ or $C_i = \emptyset$. By point 2 of lemma 5.2 we have that B_i is $\omega^{\omega^{\mu+1}(k+1)4(n+1)}$–large. We consider two cases. Firstly, let us assume that $C_i = \emptyset$. Then, by the second case of point 3 in Lemma 5.2, for all $a \in B_i$, $G_i(a)$ can be written as

$$G_i(a) = \omega^{\omega(\bar{c}_i)+\omega^d\delta(a)}\eta(a),$$

for some $i \leq d \leq n$, $\bar{c}_i = (c_{i,n}, \ldots, c_{i,d+1}, 0, \ldots, 0)$, for $\delta(a)$ being strictly increasing on B_i and for $\eta(a) < \omega^{\omega^d}$ and $\eta(a) \neq 0$ for all $a \in B_i \setminus \{\min(B_i)\}$. (This is the only case when C_i may be empty.) It follows that G_i is strictly increasing on B_i. Then, by point 4 of lemma 5.2 applied $n-i+1$ times, G is strictly increasing on B_i, too.

Now, let us assume that $C_i \neq \emptyset$. Then, it has to be the case that $i = 0$. According to point 3 of lemma 5.2, we again have two cases.

In the first case, for all $a \in B_0$,

$$G_0(a) = \sum_{\bar{c}\in C_0} \omega^{\omega(\bar{c})}\xi_{\bar{c}}(a),$$

where all $\xi_{\bar{c}}$ are functions into ω. Let us observe that, by point 3 of lemma 5.2, in this case for $x \in B_0$,

$$G_0(x) = G_1(x) = \ldots = G_{n+1}(x) = G(x).$$

This is so because if we once use the second option in point 3 of lemma 5.2 then we will never leave it later to the first option. Then, the cardinality of C_0 is not greater than $(\min(B_0)+1)^{n+1}$. So, if we find $B \subseteq B_0$ such that for all $\bar{c} \in C_0$ functions $\xi_{\bar{c}}$ are constant or strictly increasing on B than G will be constant or strictly increasing on B, too. Thus, we need to take care of, at most, $(\min(B_0)+1)^{n+1}$ functions into ω. But B_0 is

$$\omega^{\omega^{\mu+1}(k+1)4(n+1)}\text{–large.}$$

Since $k \geq 1$, it is also

$$\omega^{\omega^{\mu+1}k+\omega^2 4(n+1)}\text{–large.}$$

Since, $h_{\omega^2 4(n+1)}(x) \geq 4^{(x+1)^{n+1}}$, it follows that B_0 is

$$\omega^{\omega^{\mu}k4^{(\min(B_0)+1)^{n+1}}}\text{–large.}$$

Thus, we can find a suitable B such that it is

$$\omega^{\omega^{\mu}k}\text{–large.}$$

In the second case, for all $a \in B_0$,

$$G_0(a) = \sum_{\bar{c} \in C_0} \omega^{\omega(\bar{c})}\xi_{\bar{c}}(a) + \omega^{\omega(\bar{c}_0)+\omega^d\delta(x)}\eta(x),$$

for some $0 \leq d \leq n$ and where $\delta\colon B_0 \longrightarrow \omega$ is strictly increasing on B_0, $\eta(a) < \omega^{\omega^d}$ and for all $x \in B_0 \setminus \{\min(B_0)\}$, $\eta(x) \neq 0$. Then, for all $\bar{c} \in C_0$, $\xi_{\bar{c}}$ is a function into ω and we need to find the set $B \subseteq B_0$ such that all $\xi_{\bar{c}}$ are constant or strictly increasing on B. For such a set B, G_0 is strictly increasing on it. Consequently, by point 4 of lemma 5.2 applied $n-1$ times, G is strictly increasing on B, too. By the same analysis of largeness as in the first case we can show that B can be chosen $\omega^{\omega^{\mu}k}$–large. This finishes the proof of the lemma. □

Theorem 5.4 *Let $\mu \gg \omega$ and $k \geq 1$.* $\mathrm{WR}(\omega^{\omega^{\mu+\omega}k4^2}, \omega^{\omega^{\omega}}, \omega^{\omega^{\mu}k})$.

Proof: Let A be $\omega^{\omega^{\mu+\omega}k4^2}$–large and let $G\colon A \longrightarrow \omega^{\omega^{\omega}}$. Then, let us consider the function

$$w(a) = \max(\{u : \omega^{\omega^u} \leq G(a)\} \cup \{0\})$$

and take $B \subseteq A$ such that B is $\omega^{\omega^{\mu+\omega}k4}$–large and w is constant or strictly increasing on B. In the latter case G is strictly increasing on B too so we consider only the former one. Since for all $a \in B$ $\mathrm{psn}(G(a)) \leq a$, G is on B a function into $\omega^{\omega^{b_0+1}}$, where $b_0 = \min(B)$. It follows, by lemma 5.3, that if B is

$$\omega^{\omega^{\mu+b_0+1}(k+1)4^2(b_0+2)^2}\text{–large}$$

then there is $C \subseteq B$ such that C is $\omega^{\omega^{\mu}k}$–large and G is constant or strictly increasing on C. But since B is $\omega^{\omega^{\mu+\omega}k4}$–large it is also $\omega^{\omega^{\mu+\omega}k+\omega^2 k3}$–large and, since $h_{\omega^2 k3}(x) \geq 4^2(k+1)(x+1)^2 + x + 1$ for $x \geq 1$, we finally obtain that B is

$$\omega^{\omega^{\mu+b_0+1}(k+1)4^2(b_0+1)^2}\text{–large.}$$

This is so because

$$\begin{aligned} h_{\omega^{\omega^{\mu+\omega}k+\omega^2 k3}}(x) &\geq h_{\omega^{(\omega^{\mu+\omega}k+h_{\omega^2 k3}(x))}}(x) \\ &\geq h_{\omega^{\omega^{\mu+\omega}k+(4^2(k+1)(x+1)^2+x+1)}}(x) \\ &\geq h_{\omega^{\omega^{\mu+x+1+4^2(k+1)(x+1)^2}k}}(x) \\ &\geq h_{\omega^{\omega^{\mu+x+1}4^2(k+1)(x+1)^2}}(x). \end{aligned}$$

This finishes the proof of the theorem. □

We conjecture that also for larger ordinals the similar estimations to the given above holds. That is, if $\mu \gg \omega_n$, for $n \geq 1$, then there is $b \in \omega$ such that $\mathrm{WR}(\omega^{\omega^{\mu+\omega_n}k \cdot b}, \omega^{\omega^{\omega_n}}, \omega^{\omega^{\mu}k})$. However, it seems that to extend our results to the case of $n > 1$ one needs to develop a new approach which would reduce the complexity of some properties of considered objects.

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