A survey of Lie groupoid methods in mechanics and classical field theory

XXIIth Workshop, Bedlewo

Joris Vankerschaver
Joris.Vankerschaver@UGent.be

Ghent University

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Disclaimer

1. Survey talk;
2. Based on previous work by many different groups (to name but a few: Cortés, Grabowski, Iglesias, de León, Marrero, Martín de Diego, Martínez, Moser, Saunders, Urbanski, Veselov, Weinstein);
3. Field theories: joint work with F. Cantrijn.

Lie groupoids

- Used since the 1920s to describe equivalence relations (Brandt, Grothendieck, Connes, etc.). Have rejoiced in increasing attention over the years:

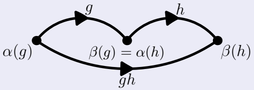
  > "*... that groupoids should perhaps be renamed 'groups', and those special groupoids with just one base point given a new name to reflect their singular nature.*"
  > (F. Lawvere, quoted by A. Weinstein)

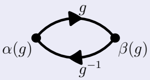
- Other uses: analysis on manifolds with corners (Melrose), categorical physics (Baez), etc.
- Our aims will be much more modest...

Introductory definitions

Definition

For the purpose of this talk, it suffices to know that a groupoid is a set G with anchor mappings $\alpha, \beta : G \to Q$ and a *partial* multiplication $m : G_2 \to G$.





- Categorical view: small category with all arrows invertible;
- Lie groupoids: "everything" suitably smooth.

(Cf. the talk of David Iglesias)

Mechanics on groupoids

Motivation: Moser-Veselov discretization of the rigid body

- Continuous Lagrangian: $L = \frac{1}{2}\Omega^T \cdot \mathbb{I} \cdot \Omega$, where $\Omega = R^T \dot{R}$ and $R \in SO(3)$. Discretize by setting

$$\Omega \approx \frac{1}{h} R_k^T (R_{k+1} - R_k).$$

- Discrete Lagrangian:

$$L(R_k, R_{k+1}) = \mathrm{Tr}(R_{k+1}^T \mathbb{I} R_k) - \frac{1}{2}\mathrm{Tr}(\Lambda_k (R_k R_k^T - \mathbf{1})).$$

- To obtain the discrete equations of motion: vary the discrete action sum $S = \sum_{k=1}^{N-1} L(R_k, R_{k+1})$ to obtain

$$R_{k+1}\mathbb{I} + R_{k-1}\mathbb{I} = \Lambda_k R_k.$$

where Λ_k are Lagrange multipliers ensuring that $R_k \in SO(3)$.

Moser-Veselov: reduction

- Recall the equations $R_{k+1}\mathbb{I} + R_{k-1}\mathbb{I} = \Lambda_k R_k$.
- Multiply by the left with R_{k+1}^T and use the fact that $\Lambda_k^T = \Lambda_k$ to write

$$\text{Moser-Veselov: } \begin{cases} M_{k+1} &= \omega_k M_k \omega_k^T \\ M_k &= \omega_k^T \mathbb{I} - \mathbb{I}\omega_k \end{cases}$$

where $\omega_k = R_k^T R_{k-1} \in SO(3)$, and $M_k \in so(3)^*$.

Relation with groupoids

- Original system lives on $SO(3) \times SO(3)$, the Moser-Veselov algorithm takes place on $SO(3)$: groupoids.
- Derivation of Moser-Veselov: example of groupoid reduction. Morphism: $\Phi : SO(3) \times SO(3) \to SO(3)$, with

$$\Phi(R_k, R_{k+1}) = \omega_{k+1} = R_{k+1}^T R_k.$$

(Starting point for the work of Alan Weinstein in 1993).

Intermezzo: the importance of the MV equations

Advantages

- Theoretical: integrable, symplectic, conservation laws, etc.
- Practical: easy to implement, 2nd order, etc.

Illustration

A free rigid body rotates stably around its shortest and longest axes, but unstably around the middle axis.

- Animation: stable motion
- Animation: unstable motion

Groupoid mechanics

Weinstein 1995; Marrero, Martín de Diego, Martínez 2006

- Configuration space: a Lie groupoid G.
- Lagrangian: a function $L : G \to \mathbb{R}$.
- A sequence $(g_1, \ldots, g_N) \in G^N$ is admissible if (g_i, g_{i+1}) is composable (for $i = 1, \ldots, N-1$). Example: if $G = Q \times Q$, then

$$(x_1, x_2), (x_2, x_3), (x_3, x_4), \ldots, (x_{n-1}, x_n)$$

is admissible.

Aim

Find admissible sequences which extremize the following discrete action sum:

$$S(g_1, \ldots, g_N) = \sum_{i=1}^{N} L(g_i).$$

The sequences will satisfy the discrete Euler-Lagrange equations.

The Euler-Lagrange equations

- Form the discrete action sum S as follows:

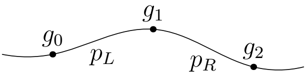
$$S : (g_1, g_2, \ldots, g_N) \mapsto \sum_{k=1}^{N} L(g_k),$$

  where $(g_1, g_2, \ldots, g_N)$ is a sequence of composable pairs, *i.e.* (g_i, g_{i+1}) is composable for $i = 1, \ldots, N-1$.

- The extremals of this sum (under variations that keep the end points fixed) satisfy the discrete Euler-Lagrange equations:

$$\overleftarrow{X}(g_1)(L) - \overrightarrow{X}(g_2)(L) = 0,$$

  for all sections X of AG.



Further geometry: the prolongation groupoid PG

Saunders 2004

Discrete version of the iterated tangent bundle $T(TQ)$.

$$
\begin{array}{ccc}
P^\pi G & \longrightarrow & AG \times AG \\
\downarrow & & \downarrow {\scriptstyle \pi\times\pi} \\
G & \xrightarrow[(\alpha,\beta)]{} & Q \times Q
\end{array}
$$

The Poincaré-Cartan sections are sections of PG^*.

Theorem

- *PG has both groupoid and algebroid structures;*
- *PG is isomorphic to $V\beta \oplus V\alpha$. The isomorphism:*

$$(g; u_{\alpha(g)}, v_{\beta(g)}) \mapsto (TL_g(u_{\alpha(g)}), TR_g(v_{\beta(g)})).$$

The Poincaré-Cartan forms

Considered as sections of $V\beta \oplus V\alpha$, the Poincaré-Cartan sections associated to L are defined as

$$\theta_L^-(g)(X_g, Y_g) = -X_g(L) \quad \text{and} \quad \theta_L^+(g)(X_g, Y_g) = Y_g(L).$$

On PG, they satisfy

$$\theta_L^-(g)(X^{(1,0)}(g)) = -\overrightarrow{X}(g)(L) \quad \text{and} \quad \theta_L^+(g)(X^{(1,0)}(g)) = \overleftarrow{X}(g)(L)$$

Remark

- θ_L^- and θ_L^+ can also be derived from the variational principle;
- they also arise as pullbacks along the discrete Legendre transformation of a canonical Liouville form;
- The symplectic section $\Omega_L = \mathrm{d}\theta_L^+ = -\mathrm{d}\theta_L^-$ is preserved under the discrete flow (symplecticity).

Example: the heavy top

The continuous theory:

- Configuration space: the Lie algebroid $\tau : S^2 \times so(3) \to S^2$;
- Lagrangian:

$$L(\Gamma, \Omega) = \frac{1}{2}\Omega \cdot \mathbf{I} \cdot \Omega - mgl\Gamma \cdot \mathrm{e}.$$

The discrete theory:

- Configuration space: the transformation groupoid $S^2 \times SO(3)$, with following anchor maps:

$$\alpha(p, A) = p, \quad \beta(p, A) = pA,$$

and multiplication defined by

$$(p, A) \cdot (q, B) = (p, AB) \quad (\text{if } q = pA).$$

Example: the heavy top

- Lagrangian: approximate $\hat{\Omega} \in so(3)$ as follows:

$$\hat{\Omega} = R^T \dot{R} \approx \frac{1}{h} R_k^T (R_{k+1} - R_k) =: \frac{1}{h}(W_k - \mathbf{1}),$$

and substitute this in the continuous Lagrangian. The result:

$$L_{\text{discrete}}(\Gamma_k, W_k) = -\frac{1}{h}\mathrm{Tr}(\mathbf{I} W_k) - hmgl\Gamma_k \cdot \mathrm{e}.$$

- Equations of motion:

$$\Pi_{k+1} = W_k^T \Pi_k + mglh^2 \Gamma_{k+1} \times \mathrm{e},$$

together with $\Gamma_{k+1} = W_k^T \Gamma_k$ (composability) and $\hat{\Pi}_k = W_k \mathbf{I} - \mathbf{I} W_k^T$. (See Bobenko and Suris)

Further topics: nonholonomic constraints

D. Iglesias, J. C. Marrero, D. Martín de Diego, E. Martínez 2007

Ingredients

- A discrete Lagrangian $L : G \to \mathbb{R}$;
- a constraint distribution $D_c \subset AG$;
- a discrete constraint submanifold $\mathcal{C} \hookrightarrow G$, such that $\dim \mathcal{C} = \dim D_c$.

Discrete Hölder principle

A sequence of $(g_1, \ldots, g_N)$ of admissible elements such that $g_1 \cdots g_N = g$ and $g_i \in \mathcal{C}$ for $i = 1, \ldots, N$ is a solution of the discrete nonholonomic Lagrangian system if and only if its a Hölder critical point of S, *i.e.* if $\delta S = 0$ for all variations taking values in D_c.

Discrete nonholonomic systems

Equations of motion

The discrete Euler-Lagrange equations for the nonholonomic system $(L, D_c, \mathcal{C})$ are given by

$$\overleftarrow{X}(g_1)(L) - \overrightarrow{X}(g_2)(L) = 0 \quad \text{for } X \in \mathrm{Sec}(D_c),$$

or alternatively (λ_α are Lagrange multipliers)

$$\overleftarrow{Y}(g_1)(L) - \overrightarrow{Y}(g_2)(L) = \lambda_\alpha X^\alpha(Y)|_{\beta(g_1)}$$

for all $Y \in \mathrm{Sec}(AG)$ and $(g_1, g_2) \in G_2 \cap (\mathcal{C} \times \mathcal{C})$.

Example: rolling ball on a rotating table

Continuous model

- Configuration space: $Q = \mathbb{R}^2 \times SO(3)$;
- L on $TQ/SO(3)$:

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{4} \mathrm{Tr}(\omega^2) \quad (\omega \in so(3)).$$

- Constraints: $\dot{x} + R\omega_2/2 = -\Omega y$ and $\dot{y} - R\omega_1/2 = \Omega x$.

This is a system on a Lie algebroid.

Rolling ball on a rotating table: discretisation

- Configuration space: Atiyah groupoid: $\mathbb{R}^2 \times \mathbb{R}^2 \times SO(3)$.
- Discrete Lagrangian:

$$L_d(x_0, y_0, x_1, y_1; W) = L(x_0, y_0; \frac{x_1 - x_0}{h}, \frac{y_1 - y_0}{h}; \frac{1}{h} \log W),$$

where $\log = \exp^{-1} : SO(3) \to so(3)$, approximated as $(\log W)/h = (W - \mathbf{1})/h$. After simplification:

$$L_d = \frac{m}{2}\left(\frac{x_1 - x_0}{h}\right)^2 + \frac{m}{2}\left(\frac{y_1 - y_0}{h}\right)^2 - \frac{I}{(2h)^2}\mathrm{Tr}(W).$$

- Discrete constraints:

$$\frac{x_1 - x_0}{h} + \frac{R}{2h} W_1 = -\Omega \frac{y_0 + y_1}{2}$$

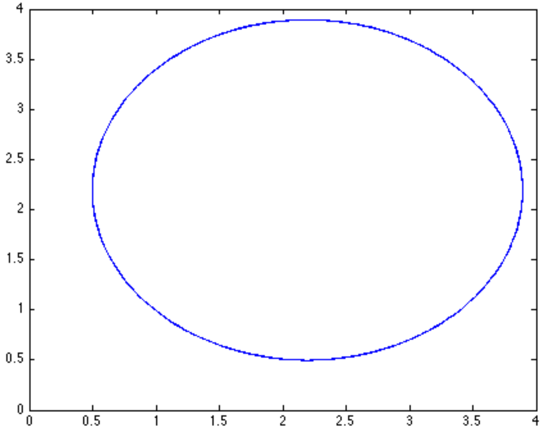
$+$ the other one.
- D_c: affine subbundle of $AG = T\mathbb{R}^2 \times so(3)$ induced by the continuous constraints.

Rolling ball on a rotating table: discretisation

- Equations of motion:

$$
\begin{aligned}
\frac{x_2 - 2x_1 + x_0}{h^2} + \frac{I\Omega}{I + mR^2}\frac{y_2 - y_0}{2h} &= 0 \\
\frac{y_2 - 2y_1 + y_0}{h^2} - \frac{I\Omega}{I + mR^2}\frac{x_2 - x_0}{2h} &= 0 \\
(W_2)_3 - (W_1)_3 &= 0 \\
\frac{x_2 - x_1}{h} + \frac{R}{2h}(W_2)_2 + \Omega\frac{y_1 + y_2}{2} &= 0 \\
\frac{y_2 - y_1}{h} - \frac{R}{2h}(W_2)_1 + \Omega\frac{x_1 + x_2}{2} &= 0
\end{aligned}
$$

- Point of contact describes a circle on the plate: accurately captured by this discrete system.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4

Groupoids in field theory

Discretising a field theory: overview

Cantrijn, Vankerschaver 2007

New elements

For a discretisation of a field theory, we need:

- A cell complex in the *base space*;
- A suitable discretisation of the *fibre*: $\mathbb{G}^k$.

Simplifications

We consider only the following kind of field theories:

- The base space is $\mathbb{R}^2$;
- The fibre bundle is trivial.

$\Rightarrow$ k-symplectic approach.

Fibre discretisation

k-symplectic approach (Günther, de León, Salgado, etc.)

- Fields are maps from $\mathbb{R}^n$ to a manifold Q, or sections of the bundle $\mathbb{R}^n \times Q \to \mathbb{R}^n$;
- The jet bundle $J^1\pi$ is isomorphic to

$$\mathbb{R}^n \times \underbrace{[TQ \oplus \cdots \oplus TQ]}_{n \text{ times}}.$$

Discretization

Apply the "Moser-Veselov procedure" n times:

$$TQ \oplus \cdots \oplus TQ \rightsquigarrow \underbrace{Q \times \cdots \times Q}_{n \text{ times}}.$$

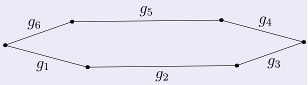
The manifold of k-gons $\mathbb{G}^k$

From now on, let G be an arbitrary Lie groupoid.

The manifold of k-gons $\mathbb{G}^k$

Elements of $\mathbb{G}^k$: k-tuples $(g_1, g_2, \ldots, g_k)$ of composable elements in G such that

$$g_1 \cdot g_2 \cdots g_k = e_{\alpha(g_1)}.$$



Why study such a manifold?

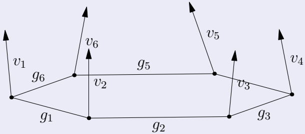
- $\mathbb{G}^k$ is a discrete counterpart to $J^1\pi$;
- the discrete Lagrangian L is defined on $\mathbb{G}^k$.

The prolongation $P^k\mathbb{G}$

Definition

The elements of $P^k\mathbb{G}$ are of the form

$$([g]; v_1, \ldots, v_k) \in \mathbb{G}^k \times AG \times \cdots \times AG.$$



Commutative diagram:

$$\begin{array}{ccc} P^k\mathbb{G} & \longrightarrow & AG \times \cdots \times AG \\ \downarrow & & \downarrow \\ \mathbb{G}^k & \longrightarrow & Q \times \cdots \times Q \end{array}$$

Notice the similarity with PG:

- $P^k\mathbb{G}$ is a Lie algebroid;
- $P^k\mathbb{G}$ is *not* a groupoid, as it has k anchors.

Lie algebroid structure of $P^k\mathbb{G}$

Define the projection maps $P^i : P^k\mathbb{G} \to PG$ $(i = 1, \ldots, k)$ as follows:

$$P^i([g]; v_1, \ldots, v_k) = (g_i; v_i, v_{i+1}).$$

Theorem

There exists a unique Lie algebroid structure on $P^k\mathbb{G}$ such that P^i, $i = 1, \ldots, k$, are Lie algebroid morphisms.

Hence, the dual $P^k\mathbb{G}^*$ is equipped with

- a linear Poisson structure;
- an exterior differential d_k.

Poincaré-Cartan sections $\theta_L^{(i)}$, $i = 1, \ldots, k$: sections of $P^k\mathbb{G}^*$, and

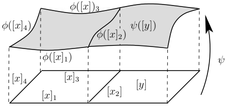
$$\theta_L^{(1)} + \cdots + \theta_L^{(k)} = \mathrm{d}_k L.$$

Discrete field theories

Consider a planar graph (V, E, F) in $\mathbb{R}^2$. The set of edges can be extended to the discrete groupoid $V \times V$.

Definition

Discrete fields are groupoid morphisms from $V \times V$ to G. Equivalently, they are maps from F to $\mathbb{G}^k$ satisfying an additional morphism property.



Morphism property

If $[x]$ and $[y]$ are elements of $\mathbb{X}^k$ having an edge in common, then the images of $[x]$ and $[y]$ under ψ have the corresponding edge in $\mathbb{G}^k$ in common. Explicitly:

$$[x]_l = ([y]_m)^{-1} \quad \text{implies that} \quad \psi([x])_l = (\psi([y])_m)^{-1}. \tag{1}$$

The Euler-Lagrange equations

Theorem

The extremals of the discrete action sum satisfy the following discrete Euler-Lagrange equations: for all $v \in A_q G$,

$$v^{(1)}_{[g_1]}(L) + v^{(2)}_{[g_2]}(L) + v^{(3)}_{[g_3]}(L) + v^{(4)}_{[g_4]}(L) = 0.$$

The solutions of the discrete Euler-Lagrange equations are multisymplectic (this is important in the construction of numerical integrators).

$$\sum_{[x]\cap\partial U\neq\varnothing}\left(\sum_{l;\alpha^{(l)}([x])\in\partial U}\Big(\Omega^{(l)}_L(\psi([x]))(V_1, V_2)\Big)\right) = 0.$$

Symmetry and reduction

- $\frac{1}{2}$ Noether theorem: every continuous symmetry gives rise to a conservation law;
- Reduction and (sometimes) reconstruction.

Remark

Lie groupoids are indispensable in doing symmetry reduction: take the unreduced Lie groupoid $Q \times Q$ and a symmetry group $\mathcal{G}$:

$$\Phi : Q \times Q \longrightarrow (Q \times Q)/\mathcal{G} \quad \text{(Atiyah groupoid).}$$

Reduced space cannot be described classically!

An important special case: G is a Lie group

Discrete differential geometry

- Discrete $\mathcal{G}$-connections, curvature, flatness (similar to continuous case).
- If $\mathcal{G}$ is Abelian, discrete differential forms.

Definition

A discrete $\mathcal{G}$-connection is a map $\omega : E \to \mathcal{G}$ such that

$$\omega(\mathfrak{e}^{-1}) = \omega(\mathfrak{e})^{-1}. \quad \text{for all } \mathfrak{e}.$$

Curvature: the map $\Omega : F \to \mathcal{G}$, defined as $\Omega(\mathfrak{f}) = \omega(\mathfrak{e}_1) \cdots \omega(\mathfrak{e}_k)$.

A connection is flat if $\Omega(\mathfrak{f}) = e$ for all $\mathfrak{f} \in F$.

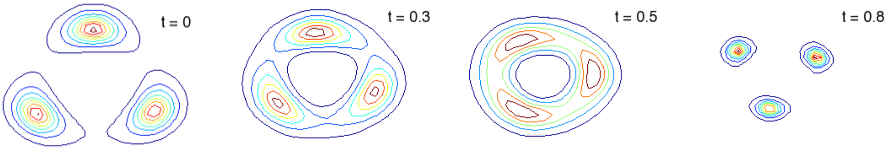
Euler-Poincaré reduction

The fields take values in a Lie group $\mathcal{G}$, which is also the symmetry group of the theory.

Discrete version of a theorem by M. Castrillón-López

The reduced fields are discrete $\mathcal{G}$-connections. The reconstruction procedure can be carried out iff the discrete curvature vanishes.

Prime example: harmonic maps, nonlinear σ-models.



Outlook

Future directions

- Adiabatic groupoids (have both continuous and discrete aspects);
- Higher-order integrators;
- Better theoretical justification.

Just as deduction should be supplemented by intuition, so the impulse to progressive generalization must be tempered and balanced by respect and love for colorful detail. The individual problem should not be degraded to the rank of special illustion of lofty general theories. In fact, general theories emerge from consideration of the specific, and they are meaningless if they do not serve to clarify and order the more particular substance below. (Richard Courant, quoted by G. Patrick)

Thank you for your time!

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